1. Practice sheet for the lecture: Combinatorics (DS I) — **Felsner/ Hoffmann**, 8. April

Delivery date: 17. April

`http://www.math.tu-berlin.de/~felsner/Lehre/dsI13.html`

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(1) How many different words occur as permutation of the letters of ABRA-KADABRA?

(2) The squares of a 4×7 chessboard are coloured arbitrarily black and white (i.e. there are $2^{4 \cdot 7}$ colourings). Show that there is a $i \times j$-rectangle with $i \geq 2$ and $j \geq 2$, such that all four of its corners have the same colour. Is the same true for a 4×6 chessboard?

(3) Calculate the first values of the series $\sum_k \binom{n-k}{k}$. Try to find a recursive formula for it and prove its correctness.

(4) Let $L_1, \ldots, L_t$ be a system of pairwise orthogonal latin squares of order n. Show that $t \leq n-1$.

(5) A sequence of numbers $a_1, a_2, \ldots, a_n$ is called *unimodal,* if there exists an $m \in [n]$, such that $a_i \leq a_{i+1}$ for all $i < m$ and $a_i \geq a_{i+1}$ for all $i \geq m$. Give three different proofs of the unimodality of the sequence $\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \ldots, \binom{n}{n}$ for all $n \in \mathbb{N}$, based on the three given hints:

(a) Use the definition

$$\binom{n}{k} := \frac{n!}{k! \cdot (n-k)!}.$$

(b) Consider the recursive definition

$$\binom{n}{k} := \binom{n-1}{k-1} + \binom{n-1}{k} \text{ and } \binom{n}{0} = \binom{n}{n} = 1,$$

based on Pascale's triangle.

(c) Use the bijection between $\binom{n}{k}$ and the number of subsets of $[n]$, having k elements.

(6) There are nine caves in a forest, each being the home of one animal. Between any to caves there is a path, but they do not intersect. Since it is election time, the candidating animals travel through the forest to advertise themselves. Each candidate visits every cave, starting from their one cave and arriving there at the end of the tour again. Also each path is only used once in total, since nobody wants to look at the embarrassing posters (not even their own), put there on the first traverse. How many candidates are there at most?