8. Practice sheet for the lecture: Combinatorics (DS I) — **Felsner/Heldt, Knauer**, 30. May

Delivery date: 6. – 10. June
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI11.html`

(1) *Please hand in your solution of this exercise:* Take a standard deck of 52 playing cards. Split them into 13 piles S_i, each containing 4 cards. Show that for any such splitting you can choose one card $a_i \in S_i$ from each set, such that the set $\{a_1, \ldots, a_{13}\}$ contains one card of each of the ranks $\{2, 3, 4, \ldots, 10, \text{jack}, \text{queen}, \text{king}, \text{ace}\}$.

(2)

(a) A graph $G = (V, E)$ is bipartite iff V can be partitioned into X and Y, such that all edges have one endpoint in X and one in Y. Prove that G is bipartite iff there is no odd cycle in G. An odd cycle is a sequence of vertices and edges $v_1 e_1 v_2 e_2 \ldots e_{2k} v_n e_{2k+1}$, such that $e_i = (v_{i-1}, v_i)$ and $e_{2k+1} = (v_{2k+1}, v_1)$ with $e_i \in E$ and $v_i \in V$.

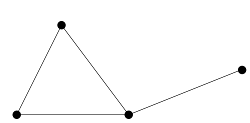
(b) Show that every regular, bipartite graph has a perfect matching. A bipartite graph is called *regular* if every vertex has degree d and a matching is a *perfect matching* if every vertex is incident to a matching edge. Does every bipartite graph have a perfect matching? Give a lower bound for the number of perfect matchings of a regular bipartite graph.

(c) Show that a regular bipartite graph can be covered with perfect matchings, i.e. that the set of edges can be partitioned into perfect matchings. Give a lower bound for the number of covers with perfect matchings.

(3) Let G be a graph and M be a matching of G. Color all edges e of G blue if $e \in M$ and red otherwise. The vertex v is *exposed* if all adjacent edges are red (i.e. do not belong to the matching). Furthermore a path between two vertices is *alternating colored,* if the path's edges are alternating red and blue. Show, that a matching is maximum (i.e. there is no matching, containing more edges) if and only if for all pairs of exposed vertices v, w there is no alternating path.

(4) The following is an incorrect proof of Dilworth's Theorem. Find the mistake:
Induction on $n := |P|$; $n = 1$ is obvious. For the induction step $n \to (n+1)$ let us assume the theorem holds for posets of n elements. Let $m \in P$ be a maxima of P. Apply the hypothesis to $P \setminus \{m\}$ and gain a decomposition into chains $C_1, \ldots, C_w$ of $P \setminus \{m\}$, with $w = \text{width}(P \setminus \{m\})$. If $w < \text{width}(P)$ then add $C_{w+1} = \{m\}$ as additional chain and we have a chain decomposition of P. If $w = \text{width}(P)$ the set $\{\max(C_i) \mid i = 1, \ldots, w\} \cup \{m\}$ can not be an antichain. Therefore $m \geq \max(C_i)$ for some $i = 1, \ldots, w$. Now $C_i \cup \{m\}$ is a chain, so $C_1, \ldots C_{i-1}, C_i \cup \{m\}, C_{i+1}, \ldots C_w$ is a chain decomposition of P. Thus in any case we have a chain decomposition of P with width(P) chains.

(5) Consider the graph G in the picture. Let $P := \left\{x \in \mathbb{R}^4_{\geq 0} \mid A \cdot x \leq \mathbb{1}\right\} \subseteq \mathbb{R}^4$, where A is the incidence matrix of G. Let $Q \subseteq [0, 1]^4$ be the convex hull of the characteristic vectors of matchings of G. Visualize and compare P and Q.



the graph G.

What can you say about the relation between these polytopes for general graphs?