7. Practice sheet for the lecture: Combinatorics (DS I) **Felsner/ Heldt, Knauer**
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http://www.math.tu-berlin.de/~felsner/Lehre/dsI11.html

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(1)

(a) Let $(P, \leq)$ be a poset, consisting of n disjoint chains of length $a_1, a_2, \ldots, a_n$. How many linear extensions does P have?

(b) Solve again exercise 1 of sheet 1. Use part (a) of this exercise to do so. The corresponding exercise was: *A spider has a sock and a shoe for each of his eight feet. In how many different ways can he put on his shoes and socks, assuming that on each foot he has to put on the sock first?*

(c) Suppose the largest chain in a finite poset P contains m elements. Show that P can be partitioned into m antichains.

(d) Let $(P, \leq)$ be a poset and $\max((P, \leq)) := \{x \in P \mid x \leq y \Rightarrow y = x\}$ be the set of its maxima. Let $e((P, \leq))$ be the number of linear extensions of $(P, \leq)$.Prove

$$e((P, \leq)) = \sum_{x \in \max(P)} e((P \setminus \{x\}, \leq')),$$

where $\leq'$ is the restriction of $\leq$ to $P \setminus \{x\}$, i.e. $\leq' := \leq \cap (P \setminus \{x\}) \times (P \setminus \{x\}) \subseteq P \times P$.

(2) A family of k-sets is *compressed* if it is an initial segment in the reverse lexicographic order of k-sets.

(a) Prove that the shadow of a compressed family is compressed.

(b) Is any compressed set the shadow of a compressed set?

(3) Let $(P, \leq)$ be a poset. An *up set* is a set $U \subseteq P$, such that for all $x \in U$ and $y \in P$ with $x \leq y$ we have $y \in U$. A *down set* is a set $D \subseteq P$, such that for all $x \in D$ and $y \in P$ with $y \leq x$ we have $y \in D$. Let A, B be up sets and C, D down sets of the boolean lattice $\mathcal{B}_n$.

(a) Prove $|A| \cdot |B| \leq 2^n \cdot |A \cap B|$ (Hint: Induction on n).

(b) Prove $|C| \cdot |D| \leq 2^n \cdot |C \cap D|$ (Hint: Apply (a)).

(c) Prove $|A| \cdot |C| \geq 2^n \cdot |A \cap C|$ (Hint: Apply (a)).

(This is known as Kleitman's Lemma)

(4) Let $(P, \leq)$ be a *tree–shaped poset,* i.e. a poset such that for each $x \in P \setminus \min(P)$ there is a parent $y \in P$ with $z \leq y < x$ for all $z \in P$ with $z < x$. Consider a recursive procedure to count the linear extensions of $(P, \leq)$. Use this procedure to derive an explicit formula involving the hook $h(x) := \#\{z \in P : x \leq z\}$ of the elements of $(P, \leq)$.

(5) *Please hand in your solution of this exercise:* Let $k \leq \frac{n}{2}$. Describe a bijection

$$f : \binom{[n]}{k} \longrightarrow \binom{[n]}{n-k}$$

with the property $A \subseteq f(A)$ for all $A \in \binom{[n]}{k}$.