6. Practice sheet for the lecture: **Felsner/ Heldt, Knauer**
Combinatorics (DS I) 12. Mai

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`http://www.math.tu-berlin.de/~felsner/Lehre/dsI11.html`

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(1) The q–binomials fullfill the equation

$$\begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n-k \\ m-k \end{bmatrix}$$

for all $n \geq m \geq k \geq 0$. Prove this via the model on $\mathbb{F}_q$ subspaces for q–binomials. Can you give other proofs?

(2) The q–binomials fullfill the equation

$$\sum_{i=0}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \cdot q^{(k+1)(n-i)} = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}$$

for all $n \geq m \geq k \geq 0$. Prove this via the lattice path model for q–binomials. Can you give other proofs?

(3) *Please hand in your solution of this exercise:* A permutation $\pi \in S_n$ is *alternating* if $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \ldots$ holds. Let $\mathrm{Alt}_n \subseteq S_n$ be the set of alternating permutations. A permutation σ is *reverse alternating* if $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \ldots$ holds. Let $\mathrm{RAlt}_n \subseteq S_n$ be the set of reverse alternating permutations.

(a) Prove $|\mathrm{Alt}_n| = |\mathrm{RAlt}_n|$.

(b) Let $E_n := |\mathrm{Alt}_n|$ and prove $2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}$ for all $n \geq 1$ (Hint: Apply (a)).

(c) Let $E_n(q) := \sum_{\pi \in \mathrm{RAlt}_n} q^{inv(\pi)}$ and $E_n^\star(q) := \sum_{\pi \in \mathrm{Alt}_n} q^{inv(\pi)}$. Prove

$$E_n^\star(q) = q^{\binom{n}{2}} E_n\left(\tfrac{1}{q}\right).$$

(4) Let $\mathrm{des}(\sigma) := |\{i \in [n-1] \mid \sigma_i > \sigma_{i+1}\}|$ be the number of *descents* of $\sigma \in S_n$ and

$$A_n(x) := \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)+1} = \sum_{k=1}^{n} a_{n,k} x^k.$$

(a) Compute $A_1(x), \ldots, A_4(x)$.

(b) Prove $a_{n,k} = a_{n,n-k}$.

(c) Prove the linear recursion $a_{n,k+1} = (k+1)a_{n-1,k+1} + (n-k)a_{n-1,k}$.

(d) Use (c) to deduce an (differential) equation for $A_n(x)$.

(5) A *linear extension* of a poset $(\{m_1, \ldots, m_k\}, \leq)$ is a total ordering $m_1 <' m_2 <' \cdots <' m_k$ such that there is no $i > j$ with $m_i \leq m_j$, i.e. the total order $\leq'$ respects the poset's partial order $\leq$. Now consider the poset P_n on the set $\left\{a_1, \ldots, a_{\lceil \frac{n}{2} \rceil}, b_1, \ldots, b_{\lfloor \frac{n}{2} \rfloor}\right\}$ with the cover relations $a_i < a_{i+1}$, $b_i < b_{i+1}$, and $b_i > a_{i-1}$ as well as $a_i > b_{i-2}$ for all i. Count the linear extensions of P_n (Hint: There are pictures of the Hasse diagrams of P_8, P_9 and P_{10} on the homepage of this course available at `http://www.math.tu-berlin.de/~felsner/Lehre/DSI11/ladder-n.pdf`).