5. Practice sheet for the lecture: Combinatorics (DS I) **Felsner/ Heldt, Knauer**

9. Mai

Delivery date: 16. – 20. May

http://www.math.tu-berlin.de/~felsner/Lehre/dsI11.html

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(1) In how many ways can you pay n Dollar with 1\$, 5\$ and 10\$ notes? Find a generating function and compute the number of ways to pay 50 Dollar.
(Hint: find three distinct generating functions, each for one type of notes only, and put them together in the right way)

(2) *Please hand in your solution of this exercise in written form:* Let a_k be the number of words of length k over the alphabet $\{n, w, e\}$ with no w next to an e (i.e. no substring of the form we or ew). These words can be interpreted as lattice paths of length k which go north, west, or east and never intersect themselves. Find

(a) a recurrence equation of fixed depth for the numbers a_k.

(b) a generating function for them, computed from the linear recursion.

(c) a closed form for a_k.

(3)

(a) Count words of length k over the alphabet $\{n, w, e\}$ with no we or ew with respect to their last step, i.e. let n_k be the number of these words, ending with n, w_k the number of words ending with w and e_k the number of remaining words. So we have $a_k = n_k + w_k + e_k$ (with a_k from exercise 2). Find a matrix A such that

$$\begin{pmatrix} n_k \\ w_k \\ e_k \end{pmatrix} = A \cdot \begin{pmatrix} n_{k-1} \\ w_{k-1} \\ e_{k-1} \end{pmatrix}$$

and use the characteristic polynomial of A to derive a linear recursion for A^n and therefore one for a_k (Hint: Look at the derivation of the explicit form for Fibonacci numbers in the lecture; the same techniques are applied there). Can you also directly derive a closed form for a_k from A?

(b) Let $A(z) = \sum_{k=0}^{\infty} a_k z^k$ be the generating function for the words. Give a combinatoric proof for the equation

$$A(z) = \left(1 + 2\sum_{k=1}^{\infty} z^k\right) \cdot (z \cdot A(z) + 1)$$

and directly deduce a closed form for $A(z)$ without using exercise (1).

(4) Use the recursion for the Bell numbers $B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(n-k)$ to find a exponential generating function $F(z) = \sum \frac{B(n)}{n!} z^n$ for the bell numbers (Hint: Look at $\frac{d}{dz} B(z)$).

(5) Let $k \in \mathbb{N}$ be fixed. Prove, that for each $n \in \mathbb{N}$ there are unique $a_k > a_{k-1} > \ldots > a_t \geq t \geq 1$ with $a_i \in \mathbb{N}$, such that

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_t}{t}.$$