4. Practice sheet for the lecture: Combinatorics (DS I) | **Felsner/Heldt, Knauer** 29. April

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http://www.math.tu-berlin.de/~felsner/Lehre/dsI11.html

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(1)

(a) How many subsets of the set $[n]$ contain at least one odd integer?

(b) How many sequences $(T_1, T_2, \ldots, T_k)$ with $T_1 \subset T_2 \subset \ldots \subset T_k \subseteq [n]$ are there?

(2) *Please hand your solution of this exercise in a written form in:*
A composition of n is an ordered set of numbers $(a_1, \ldots, a_k)$ with $a_i \in \mathbb{N}$ and $0 < a_i < n$ such that $a_1 + \cdots + a_k = n$. For example, 4 has the compositions $1+1+1+1$, $1+1+2$, $1+2+1$, $2+1+1$, $2+2$, $3+1$, $1+3$ and 4.

(a) Prove that n has $\binom{n-1}{k-1}$ compositions into exactly k parts/numbers.

(b) Use (a) to show, that n has in total 2^{n-1} compositions.

(c) Let $c(n)$ be the number of compositions of n into an even number of even parts. For $n = 8$ the compositions (with above's additional constraints) are $2+2+2+2$, $4+4$, $6+2$ and $2+6$. Give a closed formula for $c(n)$.

(3)

(a) The *Durfee square* of a partition P is the largest square fitting in the top left corner of P's Ferrers shape. How can you determine the square's size directly from the partition without considering the Ferrers diagram?

(b) Show that the number of partitions of n into at most k parts equals the number of partitions of n, such that each part is $\leq k$.

(c) Show that the number of partitions of n into at most k parts is as big as the number of partitions of $n+k$ into exactly k parts.

(d) Show that the number of partitions of n into exactly k different odd parts equals the number of self-conjugated partitions of n into k parts.

(e) Show, that each partition of n has either at least $\sqrt{n}$ parts or the biggest part is $\geq \sqrt{n}$.

(4) How many functions $f : [n] \to [n]$ exist, such that at most two elements are mapped to the same image, i.e. $|f^{-1}(a)| \leq 2$ for all $a \in [n]$? (Hint: Do not expect to find a nice closed form. A sum is fine)

(5)

(a) The Bell number $B(n) = \sum_{k=1}^{n} S(n,k)$ is the number of all partitions of $[n]$. Give a combinatorial proof for the recursion $B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k)$.

(b) Give a combinatorical argument, which proves that the number of partitions of $[n]$, such that no two consecutive numbers appear in the same block, is $B(n-1)$.