3. Practice sheet for the lecture: Combinatorics (DS I) — **Felsner/ Heldt, Knauer**

21. April

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`http://www.math.tu-berlin.de/~felsner/Lehre/dsI11.html`

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(1) Let $F(n)$ be the number of walks in $\mathbb{Z}^2$, moving either one unit up, down, left or right, which start in $(0,0)$ and return to $(0,0)$ aftern n steps. Give a closed form for $F(n)$. There are many correct solutions; give a nice one!

(2) Complete the lecture's proof of the binomial theorem. To do so, consider polynomials $g(x,y), h(x,y) \in \mathbb{C}[x,y]$ with $\deg(g) \leq \deg(h) \leq m$. Here the degree is $\deg(x^i y^j) := i + j$ and $\deg(\sum_{i,k} c_{i,k} x^i y^k) := \max_{i,k:c_{ik} \neq 0} \deg(x^i y^k)$. Show, if there are $x_1, \ldots, x_{m+1} \in \mathbb{C}$, such that $x_i \neq x_j$ for $i \neq j$ and $g(x_i, y) = h(x_i, y) \in \mathbb{C}[y]$ we have $g(x,y) = h(x,y)$.
Can you even weaken the requirements?

(3)

(a) What is the expected number E_ℓ of cycles of length ℓ of permutations $\sigma \in S_n$? Use the equation

$$E_\ell = \frac{1}{n!} \sum_{\sigma \in S_n} \#(\ell\text{–cycles of } \sigma).$$

(b) Another (more complicated) way to establish E_ℓ is, to start with the probability $P(k,\ell)$, that a permutation $\sigma \in S_n$ has exactly k cycles of length ℓ and write E_ℓ as

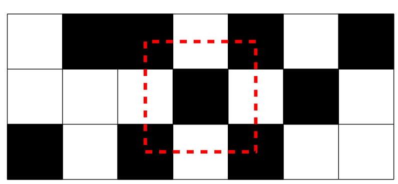
$$E_\ell = \sum_{k-1}^{n} k \cdot P(k,\ell).$$

Compute

$$P(k,\ell) = \frac{|\{\sigma \in S_n | \sigma \text{ has exactly } k \text{ cycles of length } \ell\}|}{n!}$$

for all k and $\frac{1}{2}n < \ell \leq n$ as well as $\frac{1}{3}n < \ell \leq \frac{1}{2}n$.

(4) Show, that any coloring of the fields of an 3×7 chess–board with colors black and white contain a rectangle, such that all four corners have the same color. Is the same claim true for a 4×6 board?



one coloring of a 3×7 board with a (highlighted) black rectangle

(5) *Please hand your solution of this exercise in a written form in:* In the parliament of some country there are $2n + 1$ seats filled by 3 parties. How many possible distributions (i,j,k), (i.e. party 1 has i, party 2 has j, and party 3 has k seats) are there, such that no party has an absolute majority?
(Hint: look at small numbers and make a good guess.)