2. Practice sheet for the lecture: Combinatorics (DS I)

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`http://www.math.tu-berlin.de/~felsner/Lehre/dsI11.html`

(1) Let $d(n)$ be the number of derangements of S_n. Prove $d(n) = n \cdot d(n-1) + (-1)^n$ for all $n \in \mathbb{N}$.

(2)

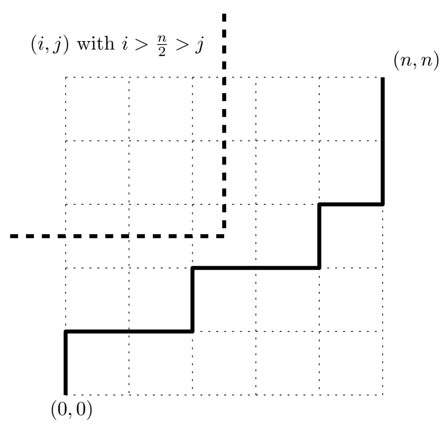
(a) Prove $f_n + f_{n-1} + \sum_{k=0}^{n-2} 2^{n-k-2} f_k = 2^n$ for the Fibonacci numbers f_n, using a bijection.

(b) Decompose the generating function of the Fibonacci numbers, such that

$$\sum_{k=0}^{\infty} f_k z^k = \frac{1}{1-z-z^2} = \frac{A}{1-\alpha z} + \frac{B}{1-\beta z}$$

with partial fractial decomposition (Partialbruchzerlegung) and find an explizit formula for its coefficients based on geometric progression (Geometrische Reihe).

(3) Let $n \in \mathbb{N}$ be odd. Count the number of lattice paths from $(0,0)$ to (n,n), moving only up and right, which avoid all lattice points (i,j), such that $i > \frac{n}{2} > j$ (the figure below shows one of theses paths for $n = 5$).



(4) Please hand your solution of this exercise in a written form in:

(a) Let $x^{\underline{n}} := (x)_n$ denote the falling factorials and $x^{\overline{n}} := x \cdot (x+1) \cdots (x+n-1)$ the *raising factorials.* Deduce the following equation from Vandermonde's identity:

$$(x+y)^{\overline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\overline{k}} y^{\overline{n-k}} \quad \text{(Hint: } (-x)^{\overline{n}} = (-1)^n x^{\underline{n}})$$

(b) The Stirling numbers of first kind $s(n,k)$ count the number of permutations in S_n consisting of k disjoint cycles. Compute all values of $s(n,k)$ for $n \leq 4$ and give a (bijective) proof of the equation $s(n,k) = (n-1)s(n-1,k) + s(n-1,k-1)$.

(c) Show that the identities $x^{\overline{n}} = \sum_{k=0}^{n} s(n,k) x^k$ and $x^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k} s(n,k) x^k$ hold (Hint: prove the first one inductively and deduce the second from the first).

(5)

$$\text{Let } Q_n := \sum_{k=0}^{k=2^n} \binom{2^n - k}{k} (-1)^k. \text{ What is } Q_{100000}?$$