11. Practice sheet for the lecture: Combinatorics (DS I) — **Felsner/Heldt, Knauer**, 21. June

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http://www.math.tu-berlin.de/~felsner/Lehre/dsI11.html

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(1) *Please hand in your solution of this exercise:* Let $P := (M, \leq)$ be a finite poset with $|M| = n$. A relation matrix $(a_{ij}) = A \in \{0,1\}^{n\times n}$ is matrix, such that $a_{ij} = 1 \Leftrightarrow m_i \leq m_j$ for some order $M = (m_1, m_2, \dots, m_n)$.

- (a) Which conditions on $(m_1, \dots, m_n)$ suffice to ensure, that A is an upper triangle matrix?
- (b) Show that $\#\{(i,j)|a_{ij} = 1\} = \frac{n^2+n}{2} \Leftrightarrow P$ is a total order .
- (c) Which conditions on a $\{0,1\}$-matrix B imply that B represents a partial order relation?
- (d) Let k be the length of P's biggest chain. Show that A has the minimal polynomial $\mu_A(x) = (x-1)^k$.

(2) Consider the following algorithm:

Input: Poset $(P, \leq)$, linear exension L of $(P, \leq)$.
$\tilde{L} := []$.
while $P \neq \emptyset$ do:
  $x := \max_L(\mathrm{Min}_{\leq}(P))$.
  $\tilde{L} := \tilde{L} + x$ and $P = P - x$.
Output: $\tilde{L}$

Let $(P, \leq)$ be a poset, L a linear extension of $(P, \leq)$ and $\tilde{L}$ the output of the algorithm above w.r.t. $(P, \leq)$ and L.

- (a) Show that $\tilde{L}$ is a linear extension of $(P, \leq)$.
- (b) Show that $L = \tilde{L}$ if and only if $\dim(P) = 1$.
- (c) Let L be a non-separating linear extension of P. Show that $\{L, \tilde{L}\}$ is a realizer of $(P, \leq)$.
- (d) Give an example of a 2-dimensional poset $(P, \leq)$ and a linear extension L such that $\{L, \tilde{L}\}$ is not a realizer of $(P, \leq)$.

(3) Let $(P, \leq)$ be a *tree–shaped poset,* i.e. a poset such that for each $x \in P \setminus \min(P)$ there is a parent $y \in P$ with $z \leq y < x$ for all $z \in P$ with $z < x$. What is the dimension of P?

(4) For $n \in \mathbb{N}$, the *divisor–poset* P_n is the set of all divisors of n, ordered by divisibility, i.e. $P_n := \big\{\{x \in \mathbb{N} : x \mid n\}, \leq\big\}$, such that $x \leq y \Leftrightarrow x \mid y$.

- (a) Sketch the Hasse diagramms of P_{60} and P_{1001} (Hint: $1001 = 7 \cdot 11 \cdot 13$).
- (b) What ist the dimension of P_n (Hint: Use the dimension of $B_{n'}$ for a well-chosen n')?

(5)

- (a) Let $(P, \leq)$ be a poset, c_k the maximal size of a k-chain of $(P, \leq)$ and $\mathcal{A}$ an anti-chain decomposition of $(P, \leq)$. Prove
$$c_k \leq \sum_{A \in \mathcal{A}} \min\{|A|, k\}.$$
- (b) Let $\lambda(B_n)$ be the Ferrer's diagram of the partition of 2^n corresponding to B_n by the Greene-Kleitman theorem. Find the shape of $\lambda(B_n)$.