10. Practice sheet for the lecture: Combinatorics (DS I) **Felsner/Heldt, Knauer**

10. June

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http://www.math.tu-berlin.de/~felsner/Lehre/dsI11.html

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(1)

(a) Compute the cycle index of the symmetry group of the solid dodecahedron, acting on the dodecahedron's faces. (Hint: Use flags to identify group elements; solid means,that there are no mirror symmetries)

(b) Let D_n be the Dihedral group, i.e. the symmetry group of a regular n-gon. Compute the cycle index of D_n.

(2)

(a) Let S_n act on $\binom{[n]}{k}$ by $\pi(\{c_1,\dots,c_k\}) := \{\pi(c_1),\dots,\pi(c_k)\}$ with $\pi \in S_n$ and $\{c_1,\dots,c_k\} \in \binom{[n]}{k}$. For $\sigma \in S_n$ let a_i be the number of cycles of σ of length i and b_j be the number of cycles of σ acting as a permutation on $\binom{[n]}{k}$ of length j. Express b_j in terms of $a_i, i = 1,\dots,n$.

(b) Let $g_{n,k}$ be the number of non-isomorphic graphs on n vertices with k edges. Let G be the symmetric group on the vertices, which acts on $\binom{[n]}{2}$ by $\pi(\{i,j\}) := \{\pi(i),\pi(j)\}$. Prove

$$\sum_{k=0}^{\binom{n}{2}} g_{n,k}x^k = P_G(1+x, 1+x^2, \dots 1+x^{\binom{n}{2}}).$$

(3) *Please hand in your solution of this exercise:* Let G be a group acting on the set M and H be a group acting on the set N with $N \cap M = \emptyset$. Let P_G be the cycle index of G with respect to M and P_H be the cycle index of H with respect to N. Consider $G \cdot H := \{g \cdot h \mid g \in G \text{ and } h \in H\}$, where

$$(g \cdot h)(a) := \begin{cases} g(a) & \text{if} \quad a \in M \\ h(a) & \text{if} \quad a \in N \end{cases}$$

Prove that $G \cdot H$ is a group acting on $M \cup N$ and that $P_{G \cdot H} = P_G \cdot P_H$, where $P_{G \cdot H}$ is the cycle index of $G \cdot H$ with respect to $M \cup N$.

(4) Let G be a group, acting on the set D. Let P_G be the corresponding cycle index. Then $F(t) = \sum_{k=1}^{|D|} f_k t^k := P_G(1+t, 1+t^2, \dots, 1+t^{|D|})$ is a polynomial in t.

(a) Give an interpretation for f_k. (Hint: Consider the action of G on $\binom{D}{k}$ and the CFB Lemma)

(b) Interpret $F(t)$ in terms of Polya's fundamental Theorem (weighted case).

(5) Consider an $n \times n$ chess–board for even $n \in \mathbb{N}$. How many configurations (up to the symmetries of D_4) of n rooks (Türme) on the board are there, such that no rook can attack another one? (Hint: Use the CFB Lemma) How many distinct configurations exist, if you are only considering the symmetries of D_4, which map black fields to black fields?