Discrete Structures I – Combinatorics

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Chapter 1

Introductory Examples

1.1 Derangements

Definition (permutation)**.** *A permutation is a bijection from a finite set X into itself.*
$[n] := \{1, \ldots, n\}$ *is the generic set of cardinality n. The set of permutations of the generic set of cardinality n is called S_n.*

S_n has some properties that we want to keep in mind:

- S_n has group structure
- $|S_n| = n!$
- two line notation for $\pi \in S_n$:

$$\pi = \begin{pmatrix} 1 & 2 & \ldots & n \\ \pi(1) & \pi(2) & \ldots & \pi(n) \end{pmatrix}$$

Example.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 5 & 2 & 7 & 6 & 1 \end{pmatrix}$$

Definition (derangement)**.** *A derangement is a permutation without fixed points, i.e. $\pi \in S_n$ with $\pi(i) \neq i \quad \forall i \in [n]$. The set of derangements in S_n is called D_n.*

Problem (Montmort, 1708)**.** *How many derangements of n letters exist?*

Let $d(n)$ be this number. We will list some possible answers for Montmorts problem and prove them later on.

$$\begin{array}{c|ccccccc} n & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ d(n) & 0 & 1 & 2 & 9 & 44 & 265 & 1858 \end{array} \tag{values}$$

$$d(n) = (n-1)\left(d(n-1) + d(n-2)\right) \tag{recursion 1}$$

$$n! = \sum_{k=0}^{n} \binom{n}{k} d(k) \tag{recursion 2}$$

$$d(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n+k} k! \tag{summation}$$

$$d(n) = \left\lceil \frac{n!}{e} \right\rfloor = \left\lfloor \frac{n!}{e} + \frac{1}{2} \right\rfloor \tag{explicit}$$

$$d(n) \sim \sqrt{2\pi n}\frac{n^n}{e^{n+1}} \sim c^{n \log n} \tag{asymptotic}$$

$$\sum_{n \geq 0} \frac{d(n)}{n!} z^n = \frac{e^{-z}}{1-z} \tag{generating function}$$

For the table of values and more information have a look at the encyclopedia of integer sequences[1].

Proof. recursion 1 (proof by bijection) Let $\pi \in D_n$.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \dots & x & \dots & n-1 & n \\ & & & & n & & & y \end{pmatrix}$$

Now we construct a shorter permutation by shrinking column x:

$$\pi' = \begin{pmatrix} 1 & 2 & 3 & \dots & x & \dots & n-1 \\ & & & & y & & \end{pmatrix}$$

If $x \neq y$ then it follows that $\pi' \in D_{n-1}$ else we remove the fixed point x and compactify the permutation to an even shorter permutation π'' and it therefore is $\pi'' \in D_{n-2}$. □

Example.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 7 & 1 & 3 & 10 & 2 & 8 & 9 & 6 & 5 \end{pmatrix}$$
$$\pi' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 7 & 1 & 3 & 5 & 2 & 8 & 9 & 6 \end{pmatrix}$$
$$\pi'' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 6 & 1 & 3 & 2 & 7 & 8 & 5 \end{pmatrix} \tag{compactification}$$

Claim 1.1.1. *Every element of $D_{n-1} \cup D_{n-2}$ is exactly $(n-1)$ times in the image of this map.*

Proof. Case 1: $\pi' \in D_{n-1}$. Choose a column $\begin{bmatrix} x \\ y \end{bmatrix}$ ($(n-1)$ possibilities) and break it for n. We obtain $\begin{pmatrix} \dots & x & \dots & n \\ \dots & n & \dots & y \end{pmatrix}$

Case 2: $\pi'' \in D_{n-2}$. Choose $b \in [n-1]$ and make a permutation $\widetilde{\pi}''$ by increasing all elements $\geq b$ (this permutes $\{1, \dots, b-1\} \cup \{b+1, \dots, n-1\}$). Add the fixed point b to get π'. Use b for the column involving n.

[1] `http://oeis.org`

Bijection:

$$D_n \longleftrightarrow [n-1] \times D_{n-1} \cup [n-1] \times D_{n-2}$$
$$\pi \longleftrightarrow \begin{cases} (c, \pi') & \text{c column} \\ (b, \pi'') & \text{b break element} \end{cases}$$

□

Proof of recursion 2.

$$\pi \in S_n \Rightarrow \mathrm{Fix}(\pi) = \{i | \pi(i) = i\} \subseteq [n]$$

π "is" (i. e. induces) a fixed point free permutation on $[n] \setminus \mathrm{Fix}(\pi)$. It follows:

$$\pi'' \in \mathrm{Der}\left([n] \setminus \mathrm{Fix}(\pi)\right)$$

We can now define a bijection between S_n and $\bigcup_{A \subseteq [n]} \mathrm{Der}(A)$ which adds a fixed point to any element of S_n or deletes fixed points and restricts the derangement. □

Example (S_3)**.**

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \in \mathrm{Der}(\emptyset) \qquad \pi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in \mathrm{Der}\left(\{1,2\}\right)$$
$$\pi_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \in \mathrm{Der}\left(\{2,3\}\right) \qquad \pi_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \in \mathrm{Der}\left(\{1,2,3\}\right)$$
$$\pi_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \in \mathrm{Der}\left(\{1,2,3\}\right) \qquad \pi_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \in \mathrm{Der}\left(\{1,3\}\right)$$

Now we can see the following equation.

$$\begin{aligned} n! &= |S_n| \\ &= \sum_{A \subseteq [n]} |\mathrm{Der}(A)| \\ &= \sum_{A \subseteq [n]} d\left(|A|\right) \\ &= \sum_{k=0}^{n} \binom{n}{k} d(k) \end{aligned}$$

To get an explicit formula for $d(n)$ we need an inversion formula, which is a simplified version of the Möbius inversion formula.

Proposition 1.1.2. *Let $f, g : \mathbb{N} \longrightarrow \mathbb{R}$. The following equivalence holds:*

$$g(n) = \sum_k \binom{n}{k} (-1)^k f(k) \Leftrightarrow f(n) = \sum_k \binom{n}{k} (-1)^k g(k)$$

The fundamental lemma helps us proof this proposition.

Lemma 1.1.3.

$$\sum_{A \subseteq B} (-1)^{|A|} = \begin{cases} 0 & B \neq \emptyset \\ 1 & B = \emptyset \end{cases}$$

Proof of 1.1.3. We know that $|A|$ even contributes a $+1$ and an uneven A contributes a -1, so we show that there are as many even as odd subsets of B. Choose $b \in B$. Then we can form pairs of subsets by removing b if $b \in A$ or by adding b if $b \notin A$, we call the new set $A \oplus b$. Then one of the sets A and $A \oplus b$ is even and the other one is odd so the cardinalities add up to 0. □

Proof of 1.1.2. For notational convenience let $f(A) := f(|A|)$ and $g(A) := g(|A|)$.
The proposition is equivalent to

$$g(B) = \sum_{A\subseteq B} (-1)^{|A|} f(A) \Leftrightarrow f(C) = \sum_{B\subseteq C} (-1)^{|B|} g(B)$$

We proof this by

$$\begin{aligned}
\sum_{B\subseteq C} (-1)^{|B|} g(B) &= \sum_{B\subseteq C} (-1)^{|B|} \sum_{A\subseteq B} (-1)^{|A|} f(A) \\
&= \sum_{A\subseteq C} (-1)^{|A|} f(A) \sum_{B:A\subseteq B\subseteq C} (-1)^{|B|} \\
&= \sum_{A\subseteq C} (-1)^{|A|} f(A)(-1)^{|A|} \underbrace{\sum_{X\subseteq C\setminus A} (-1)^{|X|}}_{=0 \text{ unless } A=C} \\
&= f(C)(-1)^{|C|}(-1)^{|C|} \qquad (1.1.3) \\
&= f(C)
\end{aligned}$$

□

Proof of summation. We define $g(k) := (-1)^k d(k)$ and $f(k) := k!$. So because of recursion 2 we get

$$f(n) = \sum \binom{n}{k} (-1)^k g(k) \Rightarrow g(n) = \sum \binom{n}{k} (-1)^k f(k)$$
$$\Rightarrow d(n) = (-1)^n g(n) = \sum \binom{n}{k} (-1)^k f(k) = \sum \binom{n}{k} (-1)^{n+k} k!$$

□

To proof the explicit version, you have to use the following formulas

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
$$e^z = \sum_{k\geq 0} \frac{z^k}{k!}$$
$$\frac{n!}{e} = n! e^{-1}$$

Together with some estimates for the speed of convergence of the series of e^{-1} you get the result.

For the asymptotic formulas you have to use the Stirling approximation of $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$ together with the explicit formula.

Proof of generating function. We define a new function and consider its derivative.

$$D(z) := \sum_{n\geq 0} \frac{d(n)}{n!} z^n \qquad D'(z) = \sum_{n\geq 1} \frac{d(n)}{(n-1)!} z^{n-1}$$

From recursion 1 we obtain

$$\begin{aligned}
\frac{d(n+1)}{n!} - \frac{d(n)}{(n-1)!} &= \frac{d(n-1)}{(n-1)!} \\
\Rightarrow (1-z)D'(z) &= \sum_{n\geq 1} \frac{d(n)}{(n-1)!}(z^{n-1} - z^n) \\
&= \sum_{n\geq 1} \left[-\frac{d(n+1)}{n!} + \frac{d(n)}{(n-1)!} \right] z^n \\
\text{recursion 1} &= -\sum_{n\geq 1} \frac{d(n-1)}{(n-1)!} z^n \\
&= -zD(z) \Rightarrow (1-z)D'(z) = -zD(z)
\end{aligned}$$

The solution $D(z) = \frac{e^{-z}}{1-z}$ can be verified by computation. □

1.2 The Officers Problem

Problem (Euler, 1782)**.** *36 officers from 6 different countries (c) inherit 6 different ranks (r) such that every combination* (c, r) *appears exactly once. Can they be arranged in an* 6×6 *array such that each* c *and each* r *appears in every row and every column?*

First of all we generalize the problem to n^2 objects with two types of attributes, each with n variants.

Example $(n = 3)$**.**

A_a	B_c	C_b
C_c	A_b	B_a
B_b	C_a	A_c

As it worked out to fill in the array we now try it with $n = 4$.

Example $(n = 4)$**.**

A_a	B	C	D
D	A	B	C
C	D	A	B
B	C	D	A

In this case we fail with placing the a's. We say that this array is not extendable (no transversal).

Definition. *A Latin square of order* n *is a filling of an* $n \times n$ *array with elements from* $[n]$ *such that each row and each column contains each element.*

As you see by following the scheme of the examples above a Latin square of order n exists for every n.

Definition. *Two Latin squares* L_1, L_2 *of order* n *are called orthogonal if each of the pairs* $(i, j) \in [n]^2$ *appears in a cell of the array.*

We will now give a solution for $n = 4$ with three mutually orthogonal Latin squares:

A_a^α	C_d^β	D_b^γ	B_c^δ
B_b^β	D_c^α	C_a^δ	A_d^γ
C_c^γ	A_b^δ	B_d^α	D_a^β
D_d^δ	B_a^γ	A_c^β	C_b^α

In 1779 Euler conjectured that there is no pair of orthogonal Latin squares when $n \equiv 2 \mod 4$ but this was disproved in 1960.

Theorem 1.2.1. *For all $n \geq 3, n \neq 6$ there exists a pair of orthogonal Latin squares.*

for odd n. Let G be a group of order n. Identify rows and columns of the array with elements of G as follows:

$$L_1(g,k) := gk \qquad L_2(g,k) := g^{-1}k$$

We have to verify that L_1 and L_2 are Latin and orthogonal. If we fix a $g \in G$ we see that the following functions are bijections because G is a group:

$$k \longmapsto gk \qquad k \longmapsto kg$$

For orthogonality let $a, b \in G$. We have to show that (a, b) appears that is we have to find $x, y \in G$ such that

$$(a,b) = (L_1(x,y), L_2(x,y)) = (xy, x^{-1}y)$$

But that implies $ab^{-1} = (xy)(x^{-1}y)^{-1} = xyy^{-1}x = x^2$. To verify that $x \longmapsto x^2$ is a bijection it is enough to show that every z is a square. We write n to $n = 2s+1$ and see $z^{2s+1} = e$ where e is the neutral element of G. Out of that we get

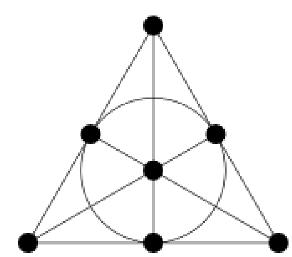
$$\left(z^{s+1}\right)^2 = z \Rightarrow z \text{ is square}$$

The solution x of $ab^{-1} = x^2$ is uniquely determined. Given x we can determine y as $y_1 = x^{-1}a$ or $y_2 = xb$ but since $x^{-1}a = xb$ y is also welldefined. □

We will now take a look at projective planes as they will help us get some interesting results about Latin squares.

Example. *The following picture[2] shows the smallest projective plane, it is of order 2. Let us collect some properties:*

- *7 points*
- *7 lines*
- *every pair of points determines a line*
- *every pair of lines determines a point*
- *there exists a quadrangle.*



Claim 1.2.2 (axioms for projective planes)**.** *If $\mathfrak{P}$ is a projective plane then there is an n such that*

$$\#\textit{points} = n^2+n+1 \qquad \#\textit{points on a line} = n+1$$
$$\#\textit{lines} = n^2+n+1 \qquad \#\textit{lines through a point} = n+1$$

n is the order of the plane.

Theorem 1.2.3.
There exists a system of $n-1$ mutually orthogonal Latin squares of order n $\Leftrightarrow$ *There exists a projective plane of order n*

Corollary. *For all q prime or a prime power there exists a system of $q-1$ pairwise orthogonal Latin squares of order q.*

In 1989 it was shown by computer search that there is no projective plane of order 10.

[2] http://de.academic.ru/dic.nsf/dewiki/367688

Proposition 1.2.4. *If $L_1, \ldots, L_t$ is a system of pairwise orthogonal Latin squares of order n then $t \leq n-1$.*

Proof. Permute the columns such that in each square the first row becomes $\begin{bmatrix} 1 & 2 & 3 & \ldots & n \end{bmatrix}$ Consider the entries at position $(2,1)$ in every Latin square. These entries are different because $\begin{bmatrix} i & i \end{bmatrix}$ already occurs in position $(1,i)$ and they are $\neq 1$. So there are only $n-1$ choices left. $\square$

Chapter 2

Basic Counting

Theorem 2.0.5 (Rule of the sum)**.** *A, B finite sets, $A \cap B = \emptyset$. Then:*

$$|A \cup B| = |A| + |B|$$

Theorem 2.0.6 (Rule of the product)**.** *A, B finite sets. Then:*

$$|A \times B| = |A||B|$$

Theorem 2.0.7 (Rule of bijection)**.** *A, B sets. Then:*

$$\exists bij. : A \to B \Leftrightarrow |A| = |B|$$

Combinatorial proof of identities: count elements of a set in two ways

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \tag{2.1}$$

$$\begin{aligned}
\binom{n}{k} &= \#k \text{ subsets of } [n] \\
&\to \begin{cases} k \text{ subsets of } [n] \text{ containing } n \\ k \text{ subsets of } [n] \text{ not containing } n \end{cases} \\
&\Leftrightarrow \begin{cases} \#k-1 \text{ subsets of } [n-1] \\ \#k \text{ subsets of } [n-1] \end{cases} \\
&= \begin{cases} \binom{n-1}{k-1} \\ \binom{n-1}{k} \end{cases}
\end{aligned} \tag{2.2}$$

$$\binom{n}{k}k = n\binom{n-1}{k-1} \tag{2.3}$$
$$\binom{n}{k}k = \#\text{ pairs } (A,a) : A \in \binom{[n]}{k}, a \in A$$
$$n\binom{n-1}{k-1} = \#\text{ pairs } (a,B) : a \in [n], B \in \binom{[n]\setminus a}{k-1}$$
$$(A,a) \to (a, A\setminus a) \qquad \text{(bijection)}$$
$$(B \cup a, a) \leftarrow (a, B)$$
$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!} \qquad \text{(explicit formula)}$$
$$\binom{n}{0} = 1$$

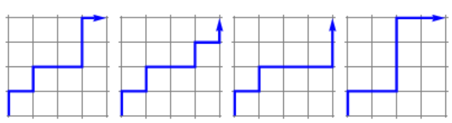
Sketch. Pick distinct elements $a_1, a_2, \dots, a_k$ from $[n]$. For a_i we take any element from $[n] \setminus \{a_1, \dots a_{i-1}\}$, i. e. we have $n-(i-1)$ possibilities. Make these a set $\{a_1, \dots, a_k\}$. This set has $k!$ orderings and we therefore obtain

$$k!\binom{n}{k} = n(n-1)\dots(n-(k-1))$$

□

2.1 Models for $\binom{n}{k}$

- k subsets of S with $|S| = n$
- $0,1$ vectors of length n with k entries 1
- path from $(0,0)$ to $(n-k,k)$ with unit steps right (0) and up (1)[1]



- Identities:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{2.4}$$
$$\binom{n}{k} = \binom{n}{n-k} \qquad \text{(symmetry)}$$
$$\underbrace{\binom{n}{m}\binom{m}{k}}_{=\#\text{ pairs } (A,B), A\in\binom{[n]}{m}, B\in\binom{A}{k}} = \underbrace{\binom{n}{k}\binom{n-k}{m-k}}_{=\#\text{ pairs } (B,C), B\in\binom{[n]}{k}, C\in\binom{[n]\setminus B}{m-k}} \tag{2.5}$$

Bijection

$$(A,B) \to (B, A\setminus B)$$
$$(B\cup C, B) \leftarrow (B,C)$$

[1]picture from `http://www.robertdickau.com/manhattan.html`

- The well known Pascal triangle shows the binomial coefficients[2]

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

- Vandermonde identity

$$\underbrace{\sum_{l=0}^{k}\binom{n}{l}\binom{m}{k-l}}_{\text{disjoint union of families of pairs } \bigcup P_l} = \underbrace{\binom{n+m}{k}}_{\#k \text{ subsets of } S\cup T:|S|=n,|T|=m,S\cap T=\emptyset} \tag{2.6}$$

P_l consists of pairs (A,B) with $A\in\binom{S}{l}, B\in\binom{T}{k-l}$. We can now see the bijection:

$$\begin{aligned}(A,B) &\to A\cup B\\ (S\cap C, T\cap C) &\leftarrow C\end{aligned}$$

- $$\sum_{k=0}^{m}\binom{n+k}{k} = \binom{n+m+1}{m}$$

Proof. Given a $0,1$-vector (string) of length $n+m+1$ with m 1's remove the final substring of the form $[0,1,\dots,1]$. The rest is a string of length $n+k$ with k 1's for some k. □

2.2 Extending binomial coefficients

Definition. *The extended binomial coefficient is defined for complex numbers $r\in\mathbb{C}$ and $k\in\mathbb{N}$. First consider the falling factorials:*

$$(r)_k := r(r-1)\dots(r-k+1)$$

This can be viewed as a polynomial in r and it is natural to define $\binom{r}{k} := \frac{(r)_k}{k!}$. Especially we obtain

$$\begin{aligned}\binom{-r}{k} &= \frac{(-r)(-r-1)\dots(-r-k+1)}{k!}\\ &= (-1)^k\frac{(r+k-1)\dots(r+1)r}{k!}\\ &= (-1)^k\binom{r+k-1}{k}\\ \binom{-1}{k} &= (-1)^k\end{aligned}$$

What about the identities in the extended version? Are they still valid?

- symmetry: only valid for $n,k\in\mathbb{N}$

[2] `http://wiki.verkata.com/de/wiki/Binomialkoeffizient`

- In 2.5 both sides are polynomials of degree m in x. They agree for all $x = n \in \mathbb{N}$ so they are the same
$$\binom{x}{m}\binom{m}{k} = \binom{x}{k}\binom{x-k}{m-k}$$

- The Vandermonde identity 2.6 represents polynomials in two variables x, y:
$$\sum_{l=0}^{k}\binom{x}{l}\binom{y}{k-l} = \binom{x+y}{k}$$
They agree on $S \times T$ with $|S|, |T| \geq k+1$ so they agree. From multiplication with $k!$ we obtain
$$\sum_{l=0}^{k}\binom{k}{l}(x)_l(y)_{k-l} = (x+y)_k$$
For the special case $y = 1$ we change the names of the variables: $k \hookrightarrow n, l \hookrightarrow k$ and obtain
$$\sum_{k=0}^{n}\binom{n}{k}(x)_k = ($$

This is wrong! (1)_k = 0 for all k>1The

Theorem 2.2.1 (Binomial theorem)**.**
$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} \tag{2.7}$$

Several proofs for the binomial theorem 2.2.1. The binomial theorem can be proven in many different ways. First of all it can be proven by induction using the recurrence for binomial coefficients.

The second option is to expand the left hand side. It will have 2^n summands which can be written in standardized form with x's preceeding y's because of the commutativity of x and y. We observe that $x^k y^{n-k}$ appears $\binom{n}{k}$ times.

For the third proof we have a look at A^B which is the set of functions $B \to A$ with sets $A, B : |A| = a, |B| = b$. We already know $|A^B| = a^b$ and see that
$$(X+Y)^N = \bigcup_{K \subseteq N} X^K Y^{N \setminus K} \qquad X \cap Y = \emptyset \tag{2.8}$$
This yields instances of the binomial theorem with $x, y \in \mathbb{N}$. Both sides are polynomials of degree $\leq n$ in x and y so they are equal as polynomials. $\square$

Chapter 3

Fibonacci Numbers

The Fibonacci numbers are a very famous sequence of natural numbers because they appear in many examples from nature.

n	0	1	2	3	4	5	6	7	8	9	10	...
F_n	0	1	1	2	3	5	8	13	21	34	55	...

The first numbers are given in the tabular above. They can be computed by the recursive formula

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ F_0 = 0 & \qquad F_1 = 1 \end{aligned} \tag{3.1}$$

First of all we give some examples where the Fibonacci numbers appear.

Example (rabbits and drones)**.** *In the first example we look at a population of pairs of rabbits.*

The children can have children on their own when they are two years old. After two years they get children every year. This is shown in figure 3.1 [1]*.*

In the drone example we watch a population of bees. Queen bees have two parents whereas drones (male) just have one queen (female) parent. Also see 3.2 [2] *for illustration.*

[1] `http://www.morrischia.com/david/portfolio/boozy/research/fibonacci's_20rabbits.html`
[2] `http://www.microscopy-uk.org.uk/mag//artsep98/fibonac.html`

Figure 3.1: Population of rabbits

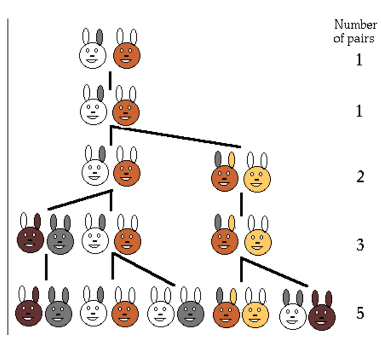
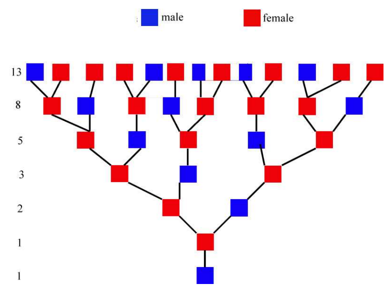


Figure 3.2: Population of bees



3.1 Models for F-numbers

There are several ways to represent the Fibonacci numbers and to link theses models with each other. First of all there is the sum representation.

Definition.

$$f_n := \#\ \textit{ways of writing } n \textit{ as ordered sum of 1 and 2}$$

Example.

$$\begin{aligned}
&1 = 1\\
&2 = 1 + 1 = 2\\
&3 = 1 + 1 + 1 = 1 + 2 = 2 + 1\\
&4 = 1 + 1 + 1 + 1 = 1 + 1 + 2 = 1 + 2 + 1 = 2 + 1 + 1 = 2 + 2
\end{aligned}$$

To get a recursive formula for computing f_n we split up the set of $1, 2$-sums into two disjoint subsets

$$\text{last summand of } 1, 2\text{-sum } = \begin{cases} 1 \\ 2 \end{cases}$$

$$\Rightarrow 1, 2\text{-sum of } n - \text{ last summand} = \begin{cases} 1, 2\text{-sum of } n - 1 \\ 1, 2\text{-sum of } n - 2 \end{cases}$$

$$f_n = f_{n-1} + f_{n-2}$$

$$\Rightarrow f_n = F_{n+1}$$

One variant of the sum model above is the tiling model. We look at a $1 \times n$ board with tiled dominos and monominos. There are monominos for 1s and dominos for 2s. Another option is to write down a $0, 1$-sequence of length n, ending in 1 without consecutive 0s (bijection with tilings: 1 for monominos, 01 for dominos). An example is shown in the following tabular for $n = 8$.

1	0 1	0 1	1	1	1
1+	2+	2+	1+	1+	1

Another nice interpretation is the reflection of a light ray on two layers of glass. The light ray is reflected at the border.

$$g_n := \#\text{ possible paths with } n \text{ reflections}$$

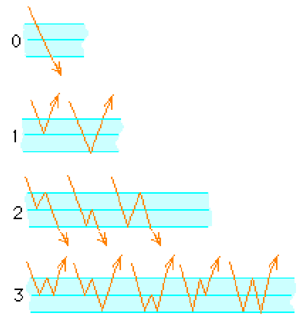


Figure 3.3: Reflection of light between two layers of glass

In figure 3.3[3] the first reflections are shown for $n = 0 \dots 3$. To see the recursion we look at the last reflection. If the light was reflected in the middle we remove the last two reflections and get g_{n-2}. Otherwise we just remove the last one and obtain g_{n-1}.

3.2 Formulas for Fibonacci Numbers

Lemma 3.2.1.

$$f_0 + f_1 + \cdots + f_n = f_{n+2} - 1 \tag{3.2}$$

Sketch 3.2. There is exactly one $1,2$-sum of $n+2$ that has no 2. Out of another one we pick the last 2:

$$\underbrace{\qquad}_{\sum=n-k} + \underbrace{2+1+\cdots+1}_{\sum=k+2}, \qquad k \in \{0,\dots,n\}$$

□

Lemma 3.2.2.

$$f_0 + f_2 + \cdots + f_{2n} = f_{2n+1} \tag{3.3}$$

Sketch 3.3. Pick the last 1 in a $1,2$-sum of $2n+1$ (there is always a last one):

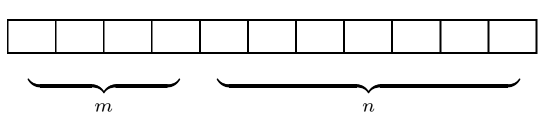
$$\underbrace{\qquad}_{\sum=2(n-k)} + \underbrace{1+2+\cdots+2}_{\sum=2k+1}, \qquad k \in \{0,\dots,n\}$$

□

Lemma 3.2.3.

$$f_{m+n} = f_m f_n + f_{m-1} f_{n-1} \tag{3.4}$$

Sketch 3.4. Look at the tilings for f_{m+n}



The parts of the tiling of m and n cells are either seperable (break line) or the border is covered by a domino. □

[3] `http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibpuzzles.html`

Lemma 3.2.4.

$$f_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \tag{3.5}$$

Sketch 3.5. $\mathcal{F}_n$ is the set of tilings of a $1 \times n$ box and $\mathcal{F}_{n,k}$ is the set of tilings of $\mathcal{F}_n$ consisting of $n-k$ parts, i. e. they have k dominos. The number of possibilities to choose k dominos out of $n-k$ parts is $|\mathcal{F}_{n,k}| = \binom{n-k}{k}$ and this yields

$$|\mathcal{F}_n| = |\dot{\bigcup_{k\geq 0}} \mathcal{F}_{n,k}| = \sum_{k\geq 0} |\mathcal{F}_{n,k}| = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}$$

□

Lemma 3.2.5.

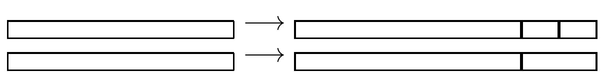
$$f_{2n-1} = \sum_{k=1}^{n} \binom{n}{k} f_{k-1} \tag{3.6}$$

Proof. If k of the first n pieces of a tiling are monominos they cover $2n-k$ cells and this yields the result. □

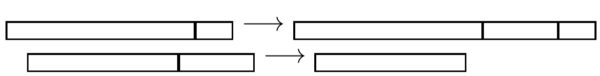
Lemma 3.2.6.

$$3f_n = f_{n+2} + f_{n-2} \tag{3.7}$$

Sketch 3.7. We take three copies of each tiling of length n and extend them to tilings of length $n+2$ and $n-2$. For the first tiling we add a domino or two monominos.



For the third tiling we have two cases: either it ends on a monomino or a domino. In the first case we move the monomino out (to the end) and insert a domino. In the second case we strip the domino off and get a tiling of length $n-2$.


(move monomino out)
(strip off domino)

This construction yields all tilings of length $n+2$ and $n-2$ and therefore proves the lemma. □

Lemma 3.2.7.

$${f_n}^2 = f_{n-1}f_{n+1} + (-1)^n \tag{3.8}$$

Proof.

Between the blocks we see in the picture we take the last breakline and then switch them:

□

There is a unique pattern (the brick wall) without break line.

3.3 The $\mathcal{F}$-number System

First we introduce an ad-hoc notation: we use $k \longrightarrow j$ to say that $k \geq j+2$.

Theorem 3.3.1. *For all $n \in \mathbb{N}$ there exists a unique representation*

$$n = F_{k_1} + F_{k_2} + \cdots + F_{k_r} \tag{3.9}$$
$$\textit{with } k_1 \longrightarrow k_2 \longrightarrow \ldots \longrightarrow k_r \longrightarrow 0 \tag{3.10}$$

Example.

$$\begin{aligned} n &= 20 \\ &= 13+5+2 \\ &= F_7 + F_5 + F_3 \\ &= (101010)_F \end{aligned}$$

Proof. Existence: Choose k such that $F_k \leq n < F_{k+1} = F_k + F_{k-1}$. We then get the existence from

$$n = F_k + \underbrace{(n - F_k)}_{<F_{k-1}}$$

by induction.

Uniqueness: For all n with $0 \leq n < F_m$ we have

$$n = \sum_{i=2}^{m-1} c_i F_i \qquad c_i \in \{0,1\}$$

Therefore n equals a $0,1$-sequence of length $m-2$ without consecutive 1s. How many sequences with this property do we have?

$$f_m = \#0,1\text{-sequences without consecutive 0s, ending in 1}$$

So we have $f_{m-1} = F_m$ sequences with the requested property. □

This theorem implies that we cannot use more than one $0,1$-sequence for each n.

Theorem 3.3.2 (Binet's formula)**.**

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

The explicit formula uses the golden number $\Phi = \frac{1+\sqrt{5}}{2} \approx 1,618\ldots$

An idea of proving Binet's formula via a generating function $F(z)$:

$$\begin{aligned} F(z) &= F_0 + F_1 z + F_2 z^2 + \ldots \\ zF(z) &= \phantom{F_0 + {}} F_0 z + F_1 z^2 + \ldots \\ z^2 F(z) &= \phantom{F_0 + F_0 z + {}} F_0 z^2 + \ldots \end{aligned}$$

If we bring these equations in a relation to each other we get:

$$\begin{aligned} F(z) &= zF(z) + z^2 F(z) + \underbrace{F_1}_{=1} z \\ \Leftrightarrow \quad F(z)(1 - z - z^2) &= z \\ \Leftrightarrow \quad F(z) &= \frac{z}{1 - z - z^2} = z\frac{1}{1-(z+z^2)} \end{aligned}$$

By partial fraction decomposition we obtain

$$F(z) = \frac{A}{1-\alpha z} + \frac{B}{1-\beta z}$$

expand both summands as geometric series and compare coefficients.

As a second approach we use Linear Algebra:

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} \Rightarrow \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The characteristic polynomial of A is the following:

$$p_A(z) = (1-z)(-z) - 1 = z^2 - z - 1$$

It has roots $\Phi, \widehat{\Phi}$ and eigenvectors $r_\Phi, r_{\widehat{\Phi}}$:

$$\Phi = \frac{1+\sqrt{5}}{2} \qquad \widehat{\Phi} = \frac{1-\sqrt{5}}{2}$$

$$r_\Phi = \begin{pmatrix} 2 \\ -1+\sqrt{5} \end{pmatrix} \qquad r_{\widehat{\Phi}} = \begin{pmatrix} 2 \\ -1-\sqrt{5} \end{pmatrix}$$

We therefore define

$$B = \left[r_\Phi, r_{\widehat{\Phi}} \right] \qquad D = \begin{pmatrix} \Phi & 0 \\ 0 & \widehat{\Phi} \end{pmatrix}$$

Now we can combine the matrices and vectors and get to Binet's formula:

$$\begin{aligned} A &= BDB^{-1} \\ \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} &= (BDB^{-1})^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= BD^nB^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

With $B^{-1} = \frac{1}{4\sqrt{5}} \begin{pmatrix} 1+\sqrt{5} & 2 \\ -1-\sqrt{5} & -2 \end{pmatrix}$ we can write down the coefficients of $A^n = BD^nB^{-1}$. This yields the result.

3.4 Continued Fractions and Continuants

Definition (Continued fractions, continuants)**.** *A continued fraction is a fraction of the specific form*

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \dots}}}} = [a_0 a_1 a_2 \dots]$$

The continuants are polynomials related to continued fractions:

$$\begin{aligned} K_0 &= 1 \\ K_1(x_1) &= x_1 \\ K_n(x_1 \dots x_n) &= x_n K_{n-1}(x_1 \dots x_{n-1}) + K_{n-2}(x_1 \dots x_{n-2}) \end{aligned}$$

Lemma 3.4.1.

$$\frac{K_n(x_1 \dots x_n)}{K_{n-1}(x_1 \dots x_{n-1})} = [x_n x_{n-1} \dots x_1]$$

Proof by induction.

$$
\begin{aligned}
\frac{K_1(x_1)}{K_0} &= x_1 = [x_1] \\
\frac{K_n(x_1 \dots x_n)}{K_{n-1}(x_1 \dots x_{n-1})} &= \frac{x_n K_{n-1}(x_1 \dots x_{n-1}) + K_{n-2}(x_1 \dots x_{n-2})}{K_{n-1}(x_1 \dots x_{n-1})} \\
&= x_n + \frac{1}{\frac{K_{n-1}(x_1 \dots x_{n-1})}{K_{n-2}(x_1 \dots x_{n-2})}} \\
&= x_n + \frac{1}{[x_{n-1} \dots x_1]} \\
&= [x_n \dots x_1]
\end{aligned}
$$

□

Lemma 3.4.2. *$K_n(1 \dots 1)$ equals the number of summands in $K_n(x_1 \dots x_n)$ which is equal to F_{n+1}.*

Proof by induction.

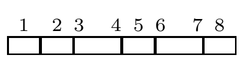
$$
\begin{aligned}
K_0 &= F_1 \\
K_1(x_1) &= x_1 \\
\Rightarrow K_1(1) &= 1 = F_2 \\
K_n(x_1 \dots x_n) &= \underbrace{x_n K_{n-1}(x_1 \dots x_{n-1})}_{\text{terms containing } x_n} + \underbrace{K_{n-2}(x_1 \dots x_{n-2})}_{\text{terms not containing } x_n} \\
\overset{\text{no cancellation}}{\Rightarrow} K_n(1 \dots 1) &= 1K_{n-1}(1 \dots 1) + K_{n-2}(1 \dots 1) \\
&= F_n + F_{n-1} \\
&= F_{n+1}
\end{aligned}
$$

□

It follows that the continued fraction of 1s converges to the golden number:

$$[1 \dots 1] = \frac{F_{n+1}}{F_n} \longrightarrow \Phi$$

From the lemma 3.4.2 we also get that the summands of $K_n(x_1 \dots x_n)$ can be set in correspondence with the monomino/domino tilings as in the following example:



$$\longleftrightarrow \qquad x_1 x_2 x_5 x_8$$

Chapter 4

The Twelvefold Way

In this chapter we want to have a look on counting functions $f : N \to M$ with $|N| = n = \#$ balls $< \infty$, $|M| = m =\#$ boxes $< \infty$ where both balls and boxes can be either distinguishable (D) or indistinguishable (I).

We define a relation for counting functions for indistinguishable boxes:

$$f \sim g \Leftrightarrow \exists \pi \in S_M \text{ such that } \pi \circ f = g$$

For indistinguishable balls we have a similar relation:

$$f \sim g \Leftrightarrow \delta \in S_N \text{ such that } f \circ \delta = g$$

In the following table we summarize twelve counting problems.

balls	**boxes**	**injective** $(m \geq n)$	**surjective** $(n \geq m)$	**arbitrary**
D	D	$(m)_n$	$m!S(n,m)$	m^n
I	D	$\binom{m}{n}$	$\binom{n-1}{m-1}$	$\binom{n+m-1}{m-1}$
D	I	1	$S(n,m)$	$\sum_{t\leq m} S(n,t)$
I	I	1	$P_m(n)$	$\sum_{t\leq m} P_t(n)$

To proof the statements conjectured in the table above we need the following definitions.

Definition (Stirling number of second kind)**.** *The Stirling number $S(n,m)$ of second kind describes the number of partitions of a n-set into m parts. It is constructed in a recursive way:*

$$\begin{aligned}
S(0,0) &= 1 \\
S(n,0) &= 0 \\
S(n,1) &= 1 \\
S(n,2) &= \frac{1}{2}\left(\underbrace{2^n}_{0,1\text{-}vectors} - \underbrace{2}_{S_i=\emptyset, S_i=N}\right) \\
&= 2^{n-1} - 1 \\
S(n,n-1) &= \binom{n}{2} \\
S(n,n) &= 1 \\
S(n,m) &= \underbrace{mS(n-1,m)}_{n\ can\ be\ element\ of\ each\ part\ of\ the\ remaining\ partition} + \underbrace{S(n-1,m-1)}_{n\ is\ in\ a\ set\ on\ its\ own}
\end{aligned}$$

By $P_m(n)$ we mean the set of partitions of n into m parts.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 3 & & 1 & & & \\
 & & 1 & & 7 & & 6 & & 1 & & \\
 & 1 & & 15 & & 25 & & 10 & & 1 & \\
1 & & 31 & & 90 & & 65 & & 15 & & 1
\end{array}$$

Table 4.1: Stirling numbers of second kind

The first Stirling numbers are shown in table 4.1. The m^{th} entry in row n is $S(n, m)$.

I, D, injective Out of m boxes we choose n that contain a ball $\Rightarrow \binom{m}{n}$

I, D, surjective In every box that contains at least one ball we mark the last ball in this box. Especially the very last ball is the last one in its box. Therefore we just have to mark $m-1$ balls in a row of $n-1$. $\Rightarrow \binom{n-1}{m-1}$

I, D, arbitrary To compute the number of possible distributions we add m virtual balls, one at the end of each box. So we have to place these $m-1$ (because the last one has a fixed position) virtual balls in $n+m-1$ places. They then mark the end of each box. $\Rightarrow \binom{n+m-1}{m-1}$

D, I, surjective We now have distinguishable balls and indistinguishable boxes. We build a partition $P = \{S_1, \ldots, S_k\}, \emptyset \neq S_i \subseteq N, S_i \cap S_j = \emptyset \forall i \neq j, \bigcup_{i=1}^k S_i = N$

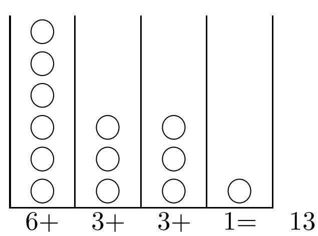
Proposition 4.0.3.

$$\underbrace{x^n}_{\#\textit{ functions from n-set to x-set}} = \sum_{k=0}^{n} \underbrace{\binom{x}{k}}_{\textit{choose k-set from x-set}} \underbrace{k!S(n,k)}_{\#\textit{ surjective functions from n-set to k-set}}$$

The counting argument yields the formula for all $x \in \mathbb{N}$. Both sides are polynomials in x and equal on integer evaluations so they are equal. Polynomials of degree $\leq n$ are a vector space $\cong \mathbb{R}^{n-1}$ or $\cong \mathbb{C}^{n-1}$. The set $\mathcal{B}_1 = \left\{1, x, x^2, \ldots, x^n\right\}$ is a basis. Another basis is $\mathcal{B}_2 = \{1, (x)_1, (x)_2, \ldots, (x)_n\}$. The proposition yields to a transformation matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$

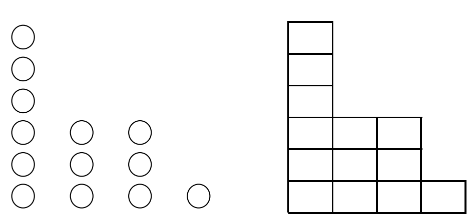
D, I, arbitrary This case follows from the sujective case by sorting the boxes.

I, I, surjective We sort the boxes by the number of balls in it.



$$\Rightarrow n = \lambda_1 + \lambda_2 + \ldots + \lambda_m, \qquad \lambda_1 \geq \ldots \lambda_m \geq 1$$

This is a partition of n into m parts. Every partition can be represented in a Ferrer's diagram:



When we have a look at the diagrams we can rebuild the recursive formula for

$$\begin{aligned} p_k(n) &:= \#\text{ partitions of } n \text{ into } k \text{ parts} \\ p_k(n) &= p_{k-1}(n-1) + p_k(n-k) \end{aligned}$$

The set of partitions of n can be divided into two parts (for a fixed k): one set contains all partitions with $\lambda_k = 1$. For these we delete the last column in the Ferrer diagram and get $p_{k-1}(n-1)$. The other set contains partitions with $\lambda_k > 1$. By deleting the last row we get $p_k(n-k)$ and therefore the recursion for $p_k(n)$ yields a way of evaluating $p(n)$.

$$\#\text{ partitions of } n = p(n) = \sum_{k=1}^{n} p_k(n)$$

Example. $p(5) = 7$

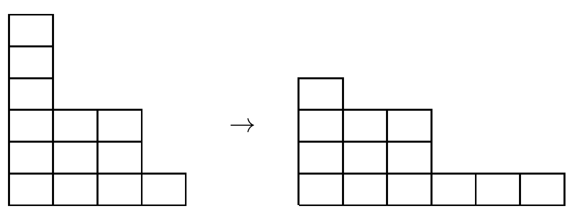
$5^1 \quad 4^1 1^1 \quad 3^1 2^1 \quad 3^1 1^2 \quad 2^2 1^1 \quad 2^1 1^3 \quad 1^5$

$$\Rightarrow p_1(5) = 1 \quad p_2(5) = 2 \quad p_3(5) = 2 \quad p_4(5) = 1 \quad p_5(5) = 1$$

Proposition 4.0.4.

$$\textit{\# partitions of } n \textit{ into pieces } \leq k = \textit{\# partitions of } n \textit{ into } \leq k \textit{ pieces}$$

Proof. Flip the Ferrer diagram and take the conjugate partition:



Let $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$ be a partition of n. Then $\lambda_i^* = |\{j : \lambda_j \geq i\}|$ yields the partition $\lambda^* = (\lambda_1^*, \dots, \lambda_t^*) \vdash n, t = \lambda_1$ □

Theorem 4.0.5 (Euler)**.**

$$\sum_{n=0}^{\infty} p(n) x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

Proof.

$$\frac{1}{1-x^k} = 1 + x^k + x^{2k} + x^{3k} + \dots$$

What is the coefficient of x^n in $\prod \frac{1}{1-x^k} = (1+x+\dots)(1+x^2+\dots)(1+x^3+\dots)\dots$? To get x^n we only have to look at the first n summands from the first n factors (finite number of summands).

$$x^n = x^{1a_1} x^{2a_2} \dots x^{na_n}$$

Take the a_i^{th} term from the i^{th} factor:

$$\sum i a_i = n \equiv \text{ partition of } n \text{ taking } a_i \text{ parts of size } i$$

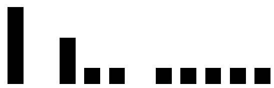
□

We got a paper from Prof. Felsner with some information about history and the details for the explicit expression for $p(n)$. An approximation is

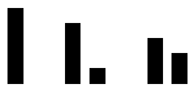
$$p(n) \sim ce^{\pi\sqrt{\frac{2n}{3}}}$$

Theorem 4.0.6.

 # partitions of n into different parts = # partitions of n into odd parts

Example.  *only odd parts*

5^1 $3^1 1^2$ 1^5

 *only different parts*

5^1 $4^1 1^1$ $3^1 2^1$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} p_{\mathrm{diff}(n)} x^n &= \prod_{k\geq 1}(1+x^k) \\
\sum_{n=1}^{\infty} p_{\mathrm{odd}(n)} x^n &= \prod_{k\geq 0}\frac{1}{1-x^{2k+1}} \\
(1-x^k)(1+x^k) &= 1-x^{2k} \\
\Rightarrow 1+x^k &= \frac{1-x^{2k}}{1-x^k} \\
\Rightarrow \prod_{k\geq 1}(1+x^k) &= \prod_{k\geq 1}\frac{1-x^{2k}}{1-x^k} \\
&= \frac{\cancel{1-x^2}}{1-x^k}\frac{\cancel{1-x^4}}{\cancel{1-x^2}}\frac{\cancel{1-x^6}}{1-x^3}\cdots \\
&= \frac{1}{1-x}\frac{1}{1-x^3}\cdots \\
&= \prod_{k\geq 0}\frac{1}{1-x^{2k+1}}
\end{aligned}$$

□

A bijective proof of 4.0.6. Let $(d_1, d_2, \ldots, d_k)$ be a partition with distinct parts, i.e.

$$\begin{aligned}
n &= d_1 + d_2 + \ldots + d_k \\
&= 2^{a_1}u_1 + 2^{a_2}u_2 + \ldots + 2^{a_k}u_k \\
&= \mu_1(2^{\alpha_1} + 2^{\alpha_2} + \ldots + 2^{\alpha_{l_1}}) + \mu_2(2^{\beta_1} + 2^{\beta_2} + \ldots + 2^{\beta_{l_2}}) + \ldots \\
&= \mu_1 r_1 + \mu_2 r_2 + \ldots + \mu_s r_s
\end{aligned}$$

where

$$\begin{aligned}
u_i \text{ odd}, 2^{a_i}u_i &= d_i \\
\{u_1, \ldots, u_k\} = \{\mu_1, \ldots, \mu_s\}, s \leq k, \sum_{i=1}^{s} l_i &= k \\
2^{\omega_1} + \ldots + 2^{\omega_{l_i}} = \text{ unique binary expansion of } r_i
\end{aligned}$$

This yields the partition $\mu_1^{r_1}\mu_2^{r_2}\ldots\mu_s^{r_s}$ of n into odd parts. □

Example.

$$
\begin{aligned}
6 &= 3+2+1 \\
&= 1\cdot 3 + 2\cdot 1 + 1\cdot 1 \\
&= 3\cdot(2^0) + 1\cdot(2^1+2^0) \\
&= 3\cdot 1 + 1\cdot 3 \\
&= 3+1+1+1 \\
6 &= 1+1+1+1+1+1 \\
&= 1\cdot 6 \\
&= 1\cdot(2^2+2^1) \\
&= 1\cdot 2^2 + 1\cdot 2^1 \\
&= 4+2
\end{aligned}
$$

Theorem 4.0.7 (Euler's Pentagonal Number Theorem)**.**

$$
\begin{aligned}
\prod_{k\geq 1}(1-x^k) &= \sum_{k=-\infty}^{\infty}(-1)^k x^{\frac{(3k-1)k}{2}} \qquad (4.1)\\
&= 1+\sum_{k\geq 1}(-1)^k\left[x^{\frac{(3k-1)k}{2}}+x^{\frac{(3k+1)k}{2}}\right]
\end{aligned}
$$

The Pentagonal Number Theorem 4.0.7 is a special case of the Jacobi Triple Product Identity:

$$\prod_{k\geq 1}(1-x^{2k})(1+zx^{2k-1})(1+z^{-1}x^{2k-1}) = \sum_{k=-\infty}^{\infty} x^{k^2}z^k \qquad (4.2)$$

To get the Pentagonal Theorem we replace x by $x^{\frac{3}{2}}$ and z by $-x^{-\frac{1}{2}}$

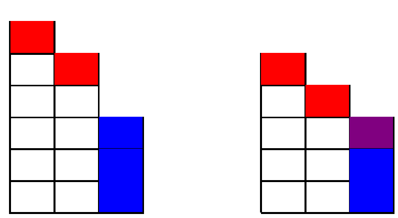
A combinatorial proof of Euler's 4.0.7.

$$
\begin{aligned}
&\prod_{k\geq 1}(1-x^k) \\
&= \sum_{n\geq 0} a_n x^n \\
&= \sum_{n\geq 0}\left[\underbrace{p_d^{\text{even}}(n)}_{\text{\# even number of distinct parts}} - \underbrace{p_d^{\text{odd}}(n)}_{\text{\# odd number of distinct parts}}\right]x^n
\end{aligned}
$$

Claim 4.0.8.

$$p_d^{even}(n) - p_d^{odd(n)} = (-1)^k\delta_{\left[n=\frac{(3k\pm 1)k}{2}\right]}$$

Proof. Use slope and intersection of Ferrer diagram.



The number of red cells is called "slope" S and the blue ones build the "front" F of the Ferrer diagram. The purple one is the "intersection" Δ of slope and front. In the given example we have $S = 2, F = 3, \Delta = 0$ in the left diagram and $S = 3, F = 3, \Delta = 1$ in the right diagram. We recognize that there are three types of these diagrams:

Type I $S \geq F + \Delta$

Type II $S < F - \Delta$

Type III otherwise: $F - \Delta \leq S < F + \Delta$

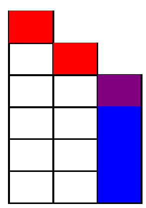
$$\Rightarrow \Delta = 1, \text{ i.e. } F - 1 \leq S \leq F + 1 \qquad \Rightarrow S \in \{F - 1, F\}$$

□

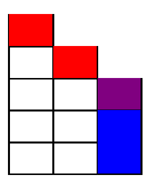
□

Lemma 4.0.9. *If $n = \frac{(3k\pm1)k}{2}$ there is a unique Type III partition and these are all Type III partitions.*

Proof. Assume $S = F - 1, k := S$, i.e. $F = k + 1$ with $\Delta = 1$

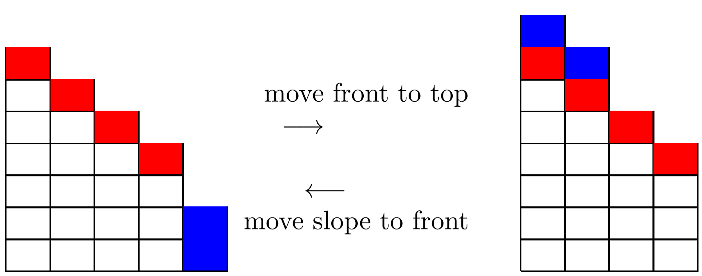


The Ferrer diagram above shows the partition of $n = \frac{(k+1)k}{2} + k^2$.
Now we assume $k = S = F$, with $\Delta = 1$ and have another look at the Ferrer diagram:



This diagram now shows the partition of $n = k(k-1) + \frac{k(k+1)}{2}$. □

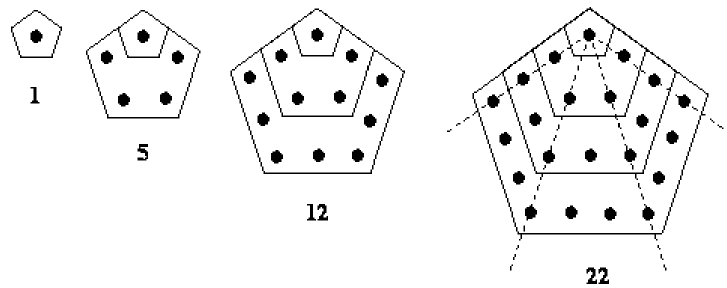
To show the bijection between Type I and Type II diagrams we have another look at slope and front.



The bijection shown is clearly a bijection between Type I and Type II partitions.

Why is the Euler theorem called "Pentagonal"? The idea behind this is to count the number of corners recursively boxed pentagons have[1]:

[1] image below from `http://www.mathpages.com/home/kmath623/kmath623.htm`



Intuitively we get the recursive formula

$$\begin{aligned} c_k &= c_{k-1} + 3k + 1 \\ \Leftrightarrow c_k - c_{k-1} &= 3k + 1 \\ \Rightarrow c_k &= \frac{(3k-1)k}{2} \end{aligned}$$

Together with the following values we obtain a fast way to evaluate $p(n)$:

k	0	1	2	3	4	...
$\frac{(3k-1)}{2}k$	0	1	5	12	22	...
$\frac{(3k+1)}{2}k$	0	2	7	15	26	...

Claim 4.0.10.

$$\begin{aligned} p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) \\ + p(n-12) + p(n-15) - p(n-22) - p(n-26) \pm \ldots \end{aligned} \tag{4.3}$$

Proof.

$$\begin{aligned} \sum p(n)x^n &= \prod_{k\geq 1} \frac{1}{1-x^k} \\ \left(\sum p(n)x^n\right)\left(\prod_{k\geq 1} 1 - x^k\right) &= \prod \frac{1-x^k}{1-x^k} = 1 \\ \text{Euler } \Rightarrow \left(\sum p(n)x^n\right)\left(\sum (-1)^k x^{\frac{(3k-1)k}{2}}\right) &= 1 \end{aligned}$$

The claim follows by multiplying out this relation and comparing coefficients. $\square$

Chapter 5

Formal Power Series

Any sequence of numbers $\{a_n\}_{n=0}^{\infty}$ can be represented by a formal power series (FPS) $\sum_{n=0}^{\infty} a_n x^n =: A(x)$. This is just a formal way of looking at the sequence.

Example.

$$\{n!\}_{n=0}^{\infty} \quad \longrightarrow \quad \sum_{n=0}^{\infty} n! x^n$$

This FPS is not converging but it is representing the sequence.

We use the operations $\cdot, +, -, /$ as shortcut notations for manipulation of the sequences. To do this all coefficients need to be finitely computable. Notations:

$$\alpha A(x) = \sum_{n\geq 0} \alpha a_n x^n$$
$$A(x) + B(x) = \sum_{n\geq 0} (a_n + b_n) x^n$$
$$A(x)B(x) = \sum_{n\geq 0} \left(\sum_{k=0}^{n} a_n b_{n-k} \right) x^n$$

Important formal power series are the following:

$$\frac{1}{1-x} = \sum_{k\geq 0} x^k \qquad\qquad e^x = \sum_{n\geq 0} \frac{x^n}{n!}$$

Proposition 5.0.11. *$A(x)$ has a multiplikative inverse iff. $a_0 \neq 0$*

Proof.

$$A(x)B(x) = \sum_{n\geq 0} \left(\sum_{k=0}^{n} a_n b_{n-k} \right) x^n = 1$$

By comparing the coefficients we can inductively compute $b_k \quad \forall k = 0, \ldots, n$ under $a_0 \neq 0$

$$a_0 b_0 = 1 \Rightarrow b_0 = \frac{1}{a_0}$$
$$a_0 b_1 + a_1 b_0 = 0 \Rightarrow b_1 = -\frac{a_1 b_0}{a_0}$$
$$\Rightarrow b_k = -\frac{1}{a_0} \sum_{k=1}^{n} a_k b_{n-k}$$

$B(x)$ with the given coefficients b_k is the inverse of $A(x)$. $\square$

Example. *The FPS of the factorial sequence* $A(x) = \sum_{n\geq 0} n!x^n$ *has the inverse*

$$B(x) = 1 - x - x^2 - 3x^3 - 13x^4 - 71x^5 - \dots$$

5.1 Bernoulli numbers and summation

$$\begin{aligned}
f(x) &= \frac{e^x - 1}{x} \\
&= \left[\sum_{n\geq 0} \frac{x^n}{n!} - 1\right] \frac{1}{x} \\
&= \sum_{n\geq 0} \frac{x^n}{(n+1)!} \\
B(x) &= \frac{1}{f(x)} = \sum_{n\geq 0} b_n \frac{x^n}{n!}
\end{aligned}$$

The inverse of $f(x)$ is the exponential generating function for the sequence $\{b_n\}_{n\geq 0}$ which are called Bernoulli numbers (often referred to as ”B-numbers”). The initial values are

$$b_0 = 1 \quad b_1 = -\frac{1}{2} \quad b_2 = \frac{1}{6} \quad b_3 = 0 \quad \dots$$

The defining recurrence is given by

$$\sum_{k=0}^{n} \frac{b_k}{k!} \frac{1}{(n-k+1)!} = 0 \quad n \geq 1 \qquad \text{(defining recurrence)}$$

$$\sum_{k=0}^{n} \binom{n+1}{k} b_k = 0 \qquad \text{(multiply by } (n+1)!\text{)}$$

$$\sum_{k=0}^{n} \binom{n}{k} b_k = b_n \quad b_0 = 1 \qquad \text{(add } b_{n+1}\text{, shift } n \to n+1\text{)}$$

The B-numbers are useful for summations of the form $\sum_{k=0}^{n} k^s$ because these can be written in terms of B-numbers.

Claim 5.1.1.

$$P(x, n) := \sum_{s\geq 0} \left(\sum_{k=0}^{n-1} k^s\right) \frac{x^s}{s!} = \frac{e^{xn} - 1}{e^x - 1}$$

Proof.

$$\begin{aligned}
P(x, n) &= \sum_{s\geq 0} \left(\sum_{k=0}^{n-1} k^s\right) \frac{x^s}{s!} \\
&= \sum_{k=0}^{n-1} \left(\sum_{s\geq 0} \frac{(kx)^s}{s!}\right) \\
&= \sum_{k=0}^{n-1} e^{kx} \\
&= \frac{e^{xn} - 1}{e^x - 1}
\end{aligned}$$

The last step follows from $\sum_{k=0}^{n-1} y^k = \frac{y^n - 1}{y - 1}$. □

This can be related to the exponential generating function for the Bernoulli numbers

$$
\begin{aligned}
B(x) &= \frac{x}{e^x - 1} \\
\Rightarrow B(x)(e^{xn} - 1) &= xP(x,n) \\
\Rightarrow \left(\sum_{l\geq 0} \frac{b_l}{l!}x^l\right)\left(\sum_{m\geq 1} \frac{(xn)^m}{m!}\right) &= \sum_{s\geq 0}\left(\sum_{k=0}^{n-1} k^s\right)\frac{x^{s+1}}{s!}
\end{aligned}
$$

The coefficient of x^{s+1} is therefore

$$
\begin{aligned}
s!\sum_{l=0}^{s} \frac{b_l}{l!}\frac{n^{s+1-l}}{(s+1-l)!} &= \sum_{k=0}^{n-1} k^s \\
\Rightarrow \frac{1}{s+1}\sum_{l=0}^{s}\binom{s+1}{l} b_l n^{s+1-l} &= \sum_{k=0}^{n-1} k^s
\end{aligned}
$$

That means that we got a closed form for $\sum_{k=0}^{n-1} k^s$ as a polynomial of degree s+1.

Example.

$$
\begin{aligned}
\sum_{k=0}^{n-1} k &= \frac{1}{2}\sum_{l=0}^{1}\binom{2}{l} b_l n^{2-l} \\
&= \frac{1}{2}(n^2 - n) = \binom{n}{2} && (s=1) \\
\sum_{k=0}^{n-1} k^3 &= \frac{1}{4}\sum_{l=0}^{3}\binom{4}{l} b_l n^{4-l} \\
&= \frac{1}{4}\left(b_0 n^4 + 4b_1 n^3 + 6b_2 n^2 + 4b_3 n\right) = \frac{1}{4}(n^4 - 2n^3 + n^2) && (s=3)
\end{aligned}
$$

5.2 Composition of series

Proposition 5.2.1. *If $f(x)$ and $g(x)$ are FPS and $g(0) = 0$ (i.e. $g(x) = \sum_{n\geq 1} g_n x^n$) then there is a well defined composition $f(g(x))$*

Proof.

$$
\begin{aligned}
f(x) &= \sum_{n\geq 0} a_n x^n \\
g(x) &= \sum_{m\geq 1} b_m x^m \\
f(g(x)) &= \sum_{n\geq 0} a_n g(x)^n \\
&= \sum_{n\geq 0} a_n \left(\sum_{m\geq 1} b_m x^m\right)^n
\end{aligned}
$$

The coefficient of x^n is:

$$
\sum_{k=0}^{n} a_k \left(\sum_{i_1+i_2+\ldots+i_k=n, i_j\geq 1} b_{i_1} b_{i_2} \ldots b_{i_k}\right)
$$

This is finite because of the condition $g(0) = 0$ and the coefficient is well defined. ∎

Example. positive

$$\begin{aligned} f(x) &= e^x - 1 \\ g(x) &= \log(x+1) \\ &= \sum_{k\geq 1} (-1)^{k+1} \frac{x^k}{k} \\ \Rightarrow f(g(x)) &= x = g(f(x)) \end{aligned}$$

negative

$$\begin{aligned} f(x) &= e^x \\ g(x) &= x+1 \\ \Rightarrow f(g(x)) &= e^{x+1} \end{aligned}$$

The composition is invalid as FPS but e^{x+y} makes sense as FPS in two variables.

5.3 Roots of FPS

Definition. *A FPS $g(x)$ is called root of the FPS $f(x)$*

$$\begin{aligned} :\Leftrightarrow f(x) &= \sqrt{g(x)} \\ \Leftrightarrow f(x)^2 &= g(x) \end{aligned}$$

Example.

$$\begin{aligned} f(x) &= \sqrt{\frac{1}{1+x}} = (1+x)^{-\frac{1}{2}} \\ f(x)^2(1+x) = 1 & \end{aligned}$$

Claim 5.3.1. *The binomial theorem holds*

$$(1+x)^{-\frac{1}{2}} = \sum_{k\geq 0} \binom{-\frac{1}{2}}{k} x^k$$

The binomial expansion is a FPS.

Proof.

$$\begin{aligned} \left((1+x)^{-\frac{1}{2}}\right)^2 &= \left(\sum_{k\geq 0} \binom{-\frac{1}{2}}{k} x^k\right)^2 \\ &= \sum_{n\geq 0} \left[\sum_k \binom{-\frac{1}{2}}{k}\binom{-\frac{1}{2}}{n-k}\right] x^n && \text{(evaluation of Vandermonde)} \\ &= \sum_{n\geq 0} \binom{-1}{n} x^n \\ &= \sum_{n\geq 0} (-1)^n x^n \\ &= \frac{1}{1+x} && \text{(geometric series)} \end{aligned}$$

□

Similarly one can proof $f(x) = (1+x)^{\frac{1}{2}} = \sum \binom{\frac{1}{2}}{k} x^k$
Indeed the binomial theorem holds for all rational exponents.

5.4 Catalan Numbers

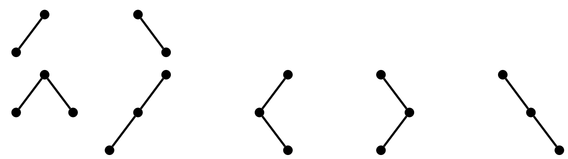
Definition. *The number of rooted binary trees with n nodes is called c_n. The first values are*

$c_0 = 1$ $\emptyset$

$c_1 = 1$ •

$c_2 = 2$

$c_3 = 5$



We want to find a generating function $T(x)$ for rooted binary trees. Every binary tree is either empty or it consists of a root with two subtrees T_l and T_r. Therefore we need $T(x) = 1 + xT(x)^2$.

A recursion for c_n is given by

$$c_{n+1} = \sum_{k=0}^{n} c_k c_{n-k}$$

Let $C(x) = \sum_{n\geq 0} c_n x^n$. This yields

$$\begin{aligned}
C(x)^2 &= \sum_{n\geq 0} \left(\sum_{k=0}^{n} c_k c_{n-k} \right) x^n \\
&= \sum_{n\geq 0} c_{n+1} x^n \\
&= \sum_{n\geq 0} c_n x^{n-1} \\
\Rightarrow xC(x)^2 &= C(x) - 1 \\
\Rightarrow C(x)^2 - \frac{C(x)}{x} + \frac{1}{x} &= 0 \\
\Rightarrow C(x) &= \frac{1 \pm \sqrt{1-4x}}{2x} \\
\overset{x=0}{\Rightarrow} 2xC(x) &= 1 - \sqrt{1-4x} \\
\Rightarrow 2xC(x) &= 1 - \sum_{k\geq 0} \binom{\frac{1}{2}}{k} (-4x)^k
\end{aligned}$$

By comparing coefficients at x^{n+1} we get

$$\begin{aligned}
c_n &= \frac{1}{2}(-1)\binom{\frac{1}{2}}{n+1}(-4)^{n+1} \\
&= 2(-4)^n \binom{\frac{1}{2}}{n+1} \\
&= \frac{1}{n+1}\binom{2n}{n} && \text{(some extra computation)}
\end{aligned}$$

Chapter 6

Solving linear recurrences via generating functions

We are given a sequence $(f_n)_{n\geq 0}$ with a linear recursion

$$f_{n+k} + a_1 f_{n+k-1} + a_2 f_{n+k-2} + \ldots + a_k f_n = 0$$

and initial values $f_0 = c_0, f_1 = c_1, \ldots f_{k-1} = c_{k-1}$. The aim is to find an explicit expression for f_n.

Step 1

$$F(x) = \sum_{n\geq 0} f_n x^n$$

First we turn the recurrence into an identity for the FPS. The goal is to write $F(x)$ as a rational function

$$F(x) + a_1 x F(x) + a_2 x^2 F(x) + \ldots + a_k x^k F(x) = Q(x)$$

where $Q(x)$ is a polynomial of degree $\leq k-1$, depending on the initial values. The coefficient of x^{n+k} is

$$f_{n+k} + a_1 f_{n+k-1} + a_2 f_{n+k-2} + \ldots + a_k f_n$$

For $n < k$ we get some linear combination of $f_0, \ldots, f_{k-1}$ which need not be 0 but can be expressed in terms of the initial values $f_i = c_i$.

$$\begin{aligned} Q(x) &= c_0 + (c_1 + a_1 c_0)x + (c_2 + a_1 c_1 + a_2 c_0)x^2 + \ldots + (\ldots)\, x^{k-1} \\ F(x)&\left(1 + a_1 x + a_2 x^2 + \ldots + a_k x^k\right) = Q(x) \\ F(x) &= \frac{Q(x)}{1 + a_1 x + a_2 x^2 + \ldots + a_k x^k} \end{aligned}$$

Step 2 Define $P(x) := 1 + a_1 x + a_2 x^2 + \ldots + a_k x^k$ and find the roots of the reflected polynomial

$$\begin{aligned} \widehat{P}(x) &= x^k + a_1 x^{k-1} + \ldots + a_k \\ &= (x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\ldots(x-\alpha_s)^{m_s} \end{aligned}$$

α_i are roots of $\widehat{P}(x)$ with multiplicities m_i.

$$\begin{aligned} x^k \widehat{P}\left(\frac{1}{x}\right) &= P(x) \\ &= (1-\alpha_1 x)^{m_1}(1-\alpha_2 x)^{m_2}\ldots(1-\alpha_s x)^{m_s} \end{aligned}$$

Step 3 Partial fraction decomposition:

$$\frac{1}{P(x)} = \sum_{j=1}^{s} \sum_{i=1}^{m_j} \frac{\gamma_{ji}}{(1-\alpha_j x)^i}$$

This holds for some coefficients γ_{ji}. Due to the binomial theorem we get

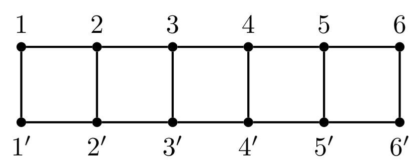
$$\begin{aligned}\frac{1}{(1-\alpha x)^i} &= (1-\alpha x)^{-i}\\ &= \sum_{n\geq 0} \binom{-i}{n} (-\alpha x)^n\\ &= \sum_{n\geq 0} \binom{i+n-1}{n} (\alpha x)^n && ((-i)_n = (i+n-1)_n(-1)^n)\end{aligned}$$

Step 4 Evaluate coefficients:

$$\begin{aligned}f_n &= [x^n]\, Q(x) \sum_{j=1}^{s} \sum_{i=1}^{m_j} \sum_{t\geq 0} \gamma_{ji} \binom{i+t-1}{t} (\alpha_j x)^t\\ &= \sum_{j=1}^{s} g_j(n) \alpha_j^{n-k+1}\end{aligned}$$

when $g_j(n)$ is a polynomial of degree $\leq k-1$ in α_j.

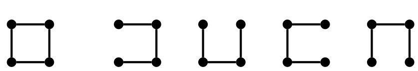
Example. *Let G_n be the following graph with $2n$ vertices.*

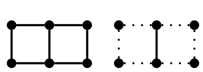


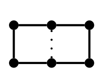
We want to count spanning trees, i. e. minimal connecting sets of edges.
$g_n :=$ # spanning trees of G_n.

G_k Spanning trees g_k

G_1  $g_1 = 1$

G_2  $g_2 = 4$

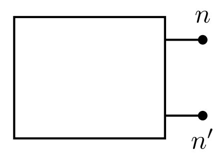
G_3  $\Rightarrow$ we need to leave out one of the three edges on each side of the middle edge $\Rightarrow 3 \cdot 3 = 9$ possibilities $g_3 = 15$

 $\Rightarrow$ we need to leave out one of the remaining 6 edges $\Rightarrow$ 6 possibilities

6.1 How to compute spanning trees of G_n

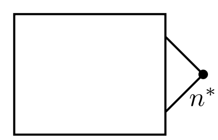
We first divide the set of all spanning trees of G_n into two groups: those containing the edge n–n' and those not containing n–n'.

not containing n–n'**:** The spanning tree looks like



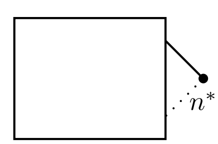
The spanning trees of this form are in bijection to spanning trees of G_{n-1}.

containing n–n'**:** We build a new pseudo node n^* by contracting the edge n–n' so we get a new graph H_n.



The spanning trees of this type are in bijection to spanning trees of H_n. How many spanning trees does H_n have? Again we define a partition into two parts:

containing either $(n-1)$–n^* **or** $(n-1)'$–n^***:** There is a $1-2$-correspondence with spanning trees of G_{n-1}.



containing $(n-1)$–n^* **and** $(n-1)'$–n^***:** Then the spanning tree does not contain $(n-1)$–$(n-1)'$ and therefore corresponds to a spanning tree of H_{n-1}. At this point we define $h_n :=$ # spanning trees of H_n.

From the partitions and bijections above we get the following recursive relations:

$$g_n = g_{n-1} + h_n \quad \forall n > 1 \tag{6.1}$$
$$h_n = 2g_{n-1} + h_{n-1} \quad \forall n > 1 \tag{6.2}$$
$$g_1 = 1 \tag{6.3}$$
$$h_1 = 1 \tag{6.4}$$

This yields a way of computing g_n as $g_n = g_{n-1} + 2\sum_{k=1}^{n-2} g_k$. This is not a linear recurrence because the length of summation depends on n. We define two generating functions.

$$G(z) = \sum_{n\geq 1} g_n z^n$$
$$H(z) = \sum_{n\geq 1} h_n z^n$$

From the recursive formulas we also get recursions for the generating functions:

$$\begin{aligned} G(z) &= zG(z) + H(z) \qquad (6.5)\\ H(z) &= 2zG(z) + zH(z) + z \qquad (6.6)\\ \Rightarrow H(z) &= \frac{2zG(z) + z}{1 - z}\\ \Rightarrow G(z) &= \frac{z}{1 - 4z + z^2} \end{aligned}$$

We note that $4g_n = g_{n-1} + g_{n+1}$ by comparing coefficients. A direct proof of this recurrence is left to the reader as an exercise. By using the reflected polynomial and initial values we now compute γ_1 and γ_2.

$$\begin{aligned} \widehat{P}(z) &= z^2 - 4z + 1\\ &= \left(z - (2 + \sqrt{3})\right)\left(z - (2 - \sqrt{3})\right)\\ \Rightarrow G(z) &= z\left(\gamma_1 \sum_{n\geq 0} (2 + \sqrt{3})^n z^n + \gamma_2 \sum_{n\geq 0} (2 - \sqrt{3})^n z^n\right)\\ \gamma_1 + \gamma_2 &= g_1 = 1\\ \gamma_1(2 + \sqrt{3}) + \gamma_2(2 - \sqrt{3}) &= g_2 = 4\\ \Rightarrow \gamma_1 &= \frac{3 + 2\sqrt{3}}{6}\\ \gamma_2 &= \frac{3 - 2\sqrt{3}}{6} \end{aligned}$$

6.2 An example with exponential generating function

Again we look at the Fibonacci numbers:

$$F_0 = 0 \qquad F_1 = 1 \qquad F_{n+1} = F_n + F_{n-1}$$

The generating function is $G(z) = \sum_{n\geq 0} \frac{F_n}{n!} z^n$ and by using the recursion we get

$$\begin{aligned} \underbrace{\sum_{n\geq 0} \frac{F_{n+2}}{n!} z^n}_{=G''(z)} &= \underbrace{\sum_{n\geq 0} \frac{F_{n+1}}{n!} z^n}_{=G'(z)} + \underbrace{\sum_{n\geq 0} \frac{F_n}{n!} z^n}_{=G(z)}\\ G(0) &= 0\\ G'(0) &= 1 \end{aligned}$$

The solution of this linear differential equation is

$$G(z) = \frac{1}{\sqrt{5}}\left(e^{\frac{1+\sqrt{5}}{2}z} - e^{\frac{1-\sqrt{5}}{2}z}\right)$$

Extracting the coefficients of $G(z)$ we again obtain Binet's formula for F_n.

Chapter 7

q-Enumeration

We want to refine counting and learn how this yields more general identities (q-binomial theorems). In this chapter $[n]$ will denote the sum of q-powers, i. e.

$$[n] := q^0 + q^1 + \ldots + q^{n-1} = \frac{1-q^n}{1-q}$$

(It should always be clear from the context whether $[n]$ denotes this q-sum or the set of n elements.)

Definition. *Let $\pi \in S_n$ be a permutation. By $inv(\pi)$ we denote the number of inversions of π where an inversion of π is a pair $i < j$ with $\pi(i) > \pi(j)$*

Example.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 5 & 1 & 7 & 2 & 6 \end{pmatrix}$$

Inversions: $14, 16, 12, 57, 24, 26, 34, 36, 56 \Rightarrow inv(\pi) = 9$

$$\sum_{\pi \in S_n} q^{inv(\pi)} = a_1 + a_2 q + \ldots + a_{\binom{n}{2}} q^{\binom{n}{2}}$$
$$a_k = \#\ \pi \text{ with } inv(\pi) = k$$
$$\sum a_k = n!$$
$$q = 1 \text{ gives the count of permutations}$$

Theorem 7.0.1.

$$\sum_{\pi \in S_n} q^{inv(\pi)} = [1]\,[2] \ldots [n] =: [n]!$$

$[n]!$ is called the q-factorial.

Proof.

$$S_n \ni (\pi_1 \ldots \pi_n) \leftrightarrow (a_1 a_2 \ldots a_n) : 0 \le a_i < i \qquad \text{(bijection)}$$
$$inv(\varphi(a_1, \ldots, a_n)) = a_1 + a_2 + \ldots + a_n$$
$$a_k(\pi) = \#\text{ inversions } (k,l) \text{ with } k \text{ left of } l \text{ in } \pi \text{ but } l < k$$
$$\Rightarrow 0 \le a_k(\pi) < k \quad \forall \pi$$

□

Claim 7.0.2. *From the inversion sequence we can uniquely reconstruct the corresponding permutation π.*

Example. *Inversion sequence* $a = (0\ 1\ 1\ 0\ 4\ 3\ 6\ 2)$

$$\begin{array}{l}
\to 1 \\
\to 21 \\
\to 231 \\
\to 2314 \\
\to 52314 \\
\to 526314 \\
\to 7526314 \\
\to 75263814 = \pi
\end{array}$$

$$\begin{aligned}
\sum_{\pi \in S_n} q^{inv(\pi)} &= \sum_{(a_1,\ldots,a_n):0\leq a_k<k} q^{a_1+\ldots+a_n} \\
&= \left(\sum_{i=0}^{0} q^i\right)\left(\sum_{i=0}^{1} q^i\right)\cdots\left(\sum_{i=0}^{n-1} q^i\right) \\
&= [1]\,[2]\ldots[n]
\end{aligned}$$

Now we consider a second statistic for permutations:

$$\begin{aligned}
\pi &= (\pi_1, \ldots, \pi_n) \in S_n \\
D(\pi) &:= \{j : \pi_j > \pi_{j+1}\} = \text{ set of down steps} \\
maj(\pi) &:= \sum_{j\in D(\pi)} j
\end{aligned}$$

Example.

$$\begin{aligned}
\pi &= (\underline{7}\,\underline{5}2\underline{6}3\underline{8}14) \\
D(\pi) &= \{1, 2, 4, 6\} \\
maj(\pi) &= 1 + 2 + 4 + 6 = 13 \\
inv(\pi) &= 1 + 1 + 4 + 3 + 6 + 2 = 17
\end{aligned}$$

Theorem 7.0.3 (MacMahon, 1910)**.**

$$\sum_{\pi\in S_n} q^{maj(\pi)} = \sum_{\pi\in S_n} q^{inv(\pi)}$$

We say that the permutation statistics in $maj(\pi)$ and $inv(\pi)$ are equidistributed.

Example. $n = 3$

π	$inv(\pi)$	$maj(\pi)$
123	0	0
132	1	2
213	1	1
231	2	2
312	2	1
321	3	3

Idea of proof. Let $M_n(z) = \sum_{\pi\in S_n} z^{maj(\pi)}$.

Claim 7.0.4.

$$\left(\frac{1}{1-z}\right)^n = M_n(z)\prod_{k=1}^{n}\left(\frac{1}{1-z^k}\right)$$

From the claim we get

$$\begin{aligned} M_n(z) &= \prod_{k=1}^{n} \frac{1-z^k}{1-z} \\ &= [1]\,[2]\dots[n] \end{aligned}$$

$$\mathbb{N}^n \ni (q_1 \dots q_n) \overset{\phi\text{ bij.}}{\leftrightarrow} \begin{cases} (\pi_1 \dots \pi_n) \in S_n \\ (p_1 \dots p_n) \in \mathbb{N}^n, p_1 \geq p_2 \geq \dots \geq p_n \geq 0 \end{cases}$$

ϕ is a bijection with $\sum q_i = maj(\pi) + \sum p_i$. Starting from $(q_1, \dots, q_n)$ there is a unique "stable" sorting $(q_{\pi_1}, \dots, q_{\pi_n})$ such that

$$q_{\pi_1} \geq q_{\pi_2} \geq \dots \geq q_{\pi_n} \tag{sorted}$$

$$q_{\pi_i} = q_{\pi_{i+1}} \Rightarrow \pi_i < \pi_{i+1} \tag{stability}$$

If $j \in D(\pi)$ then $\pi_j > \pi_{j+1} \Rightarrow q_{\pi_j} > q_{\pi_{j+1}}$. We then subtract 1 from $q_{\pi_1} \dots q_{\pi_j}$. After doing this for all $j \in D(\pi)$ we have $p_1, \dots, p_n$ still with $p_1 \geq \dots \geq p_n$ and

$$\sum q_i = \underbrace{\sum_{j \in D(\pi)} j}_{=maj(\pi)} + \sum p_i$$

$\square$

Claim 7.0.5.

$$\prod_{k=1}^{n} \frac{1}{1-z^k} = \sum_{p_1 \geq \dots \geq p_n \geq 0} z^{p_1+p_2+\dots+p_n}$$

Proof.

$$\prod_{k=1}^{n} (1 + z^k + z^{2k} + \dots) \leftarrow (b_1, \dots, b_n) \overset{\text{conjugate partition}}{\quad} (p_1, \dots, p_n)$$

$\square$

Similar to the q-factorial we define $\begin{bmatrix} n \\ k \end{bmatrix}$ as the q-binomial coefficient. By $\{0,1\}^n_k$ we denote the set of words $w = w_1 w_2 \dots w_n$ of length n over $\{0,1\}$ with k 1's.
$inv(w) = |\{(i,j)\text{with } w_i w_j = 10 \text{ with } i < j\}|$

Proposition 7.0.6.

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{w \in \{0,1\}^n_k} q^{inv(w)}$$

Proof. We will show

$$[n]! = \left[\sum q^{inv(w)}\right] [k]!\,[n-k]!$$

Find a bijection

$$\begin{array}{ccccc} \pi & \longleftrightarrow & (w, & \sigma_1, & \sigma_2) \\ | & & | & | & | \\ \in S_n & & & \in S_k & \\ & & \in \{0,1\}^n_k & & \in S_{n-k} \end{array}$$

$$inv(\pi) = inv(w) + inv(\sigma_1) + inv(\sigma_2)$$

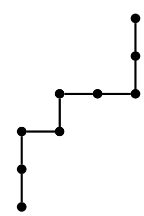
∎

Example (k=4)**.**

$$\begin{aligned} &(4\underline{6}235\underline{78}1\underline{9}) = S_9 \\ \Rightarrow\ & w = 010001101 \\ &\sigma_1 = (1234)\ \textit{induced by}\ (6789) \\ &\sigma_2 = (42351) \end{aligned}$$

Observation: $w \in \{0,1\}^n_k$ corresponds to a path w. Any inversion of w corresponds to a square below the path and therefore the set of all inversions corresponds to the area below the path.

$$w = 11010011, inv(w) = 8$$



Claim 7.0.7.

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix} \qquad \text{(symmetry)}$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \qquad \text{(recursion)}$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \qquad \text{(recursion2)}$$

Proof. For the recursion look at the first element of w. If it is a 0 it can just be deleted without changing the number of inversions. If it is a 1 it corresponds to $n-k$ inversions.

$$\sum_{w \in \{0,1\}^n_k} q^{inv(w)} = \underbrace{\sum_{w : w_1 = 0} q^{inv(w)}}_{= \begin{bmatrix} n-1 \\ k \end{bmatrix}} + \underbrace{\sum_{w : w_1 = 1} q^{inv(w)}}_{= q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}}$$

For the second recursion we look at the last element of w and see that it corresponds to k or 0 inversions. ∎

Theorem 7.0.8 (q-binomial theorem)**.**

$$(1+qx)(1+q^2x)\dots(1+q^nx) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k+1}{2}} x^k$$

Proof. Look at the left side:

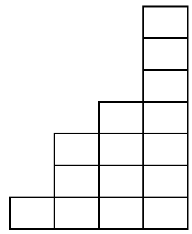
$$\prod_{k=1}^{n} (1+q^kx) = \sum_{k=0}^{n} b_k(q) x^k$$

$$b_k(q) = \sum_{\substack{|\lambda| = \lambda_1 + \ldots + \lambda_k \\ \text{all pieces distinct}}} q^{|\lambda|}$$

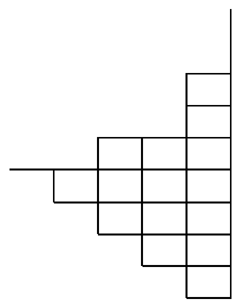
Let $P(n,k) =$ set of partitions of n into k distinct parts $\leq n$. Then there is a bijection between $\lambda \in P(n,k)$ and $w \in \{0,1\}^n_k \equiv \{0,1\}^n_{n-k}$ such that $|\lambda| = inv(w) + \binom{k+1}{2}$.

This implies $b_k(q) = \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k+1}{2}}$ ∎

Example. *The example shows the bijection for* $\lambda = (1,3,4,7) \in P(9,4)$



shift diagram down:



In the lower section of this diagram there are $\binom{4+1}{2}$ *cells,* $w = 010011011$.

7.1 Another model for the q-binomial

First of all there are some analogies between objects on sets and corresponding objects in vector spaces which are shown in the following table.

sets	vector spaces
maps $f : S \to T$	linear maps $f : V \to W$
subsets $S \cap T = \emptyset$	subspaces $U \cap W = \{0\}$
cardinalities:	dimensions
$\lvert S \cup T\rvert + \lvert S \cap T\rvert$	$\dim(U \oplus W) + \dim(U \cup W)$
$= \lvert S\rvert + \lvert T\rvert$	$= \dim(U) + \dim(W)$

Now let $V_n(q)$ be a n-dimensional space over $GF(q)$ (field with q elements).

$$\Rightarrow |V_n(q)| = |GF(q)|^n = q^n$$

How many k-dimensional subspaces does $V_n(q)$ have?

Let M be a k-dimensional subspace of $V_n(q)$. How many bases does M have?

If $B = (x_1, \dots, x_k)$ is a sorted basis of M we have $q^k - 1$ choices for x_1 (every vector is valid except the null vector). Then we have $q^k - q$ choices for x_2 (every vector that is neither zero nor a multiple of x_1) and so on. So we get the following lemma.

Lemma 7.1.1.
- *A k-dimensional subspace M of $V_n(q)$ has* $\alpha_n = (q^k-1)(q^k-q)\dots(q^k-q^{k-1})$ *sorted bases.*
- *In $V_n(q)$ we have* $\beta_k = (q^n-1)(q^n-q)\dots(q^n-q^{k-1})$ *sorted sequences of k linear independent vectors.*

Proposition 7.1.2. *$V_n(q)$ has* $\begin{bmatrix} n \\ k \end{bmatrix}$ *k-dimensional subspaces.*

Proof.

$$\begin{aligned}
\#\ k\text{-dimensional subspaces of } V_n(q) &= \frac{\beta_k}{\alpha_k} \\
&= \frac{\beta_k}{(q-1)^k}\frac{(q-1)^k}{\alpha_k} \\
&= \frac{[n]\,[n-1]\ldots[n-k+1]}{[k]\,[k-1]\ldots[1]} \\
&= \begin{bmatrix} n \\ k \end{bmatrix}
\end{aligned}$$

□

Theorem 7.1.3 (Second binomial theorem)**.**

$$x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (x-1)(x-q)\ldots(x-q^{n-k-1})$$

Proof. We count linear maps from $V_n(q)$ to some q-space X.

1. Fix a basis $v_1, \ldots v_n$ of $V_n(q)$ and pick images for the v_i. There are x^n ways to do this.
2. Let U be a subspace of $V_n(q)$. How many linear maps have U as kernel?
   Assume $\dim(U) = k$. Fix a basis $u_1 \ldots u_k$ of U. Extend this basis to a basis $u_1 \ldots u_k v_1 \ldots v_{n-k}$ of $V_n(q)$ if $\varphi : V_n(q) \to X, \ker(\varphi) = U$
   $\Rightarrow \varphi$ maps $v_1 \ldots v_{n-k}$ to linear independent vectors $w_1 \ldots w_{n-k}$ in X
   $\Rightarrow$ To pick $w_1 \ldots w_{n-k}$ as needed we have $(x-1)(x-q)\ldots(x-q^{n-k-1})$ possibilities

$$\begin{aligned}
x^n &= \sum_{U \text{ subspace}} |\{\varphi : V_n(q) \to X : \ker(\varphi) = U\}| \\
&= \sum_{U \text{ subspace}} (x-1)(x-q)\ldots(x-q^{n-\dim(U)-1}) \\
&= \sum_{k=0}^{n} \underbrace{[\#k\text{-dimensional subspaces of } V_n(q)]}_{=\begin{bmatrix} n \\ k \end{bmatrix}}(x-1)(x-q)\ldots(x-q^{n-\dim(U)-1})
\end{aligned}$$

□

Remark.
- *The proof only shows the identity for $q = p^l$ prime power and $x = q^m$. But it holds for general q, x because polynomials only share finitely many values if they are not the same.*
- *For $q \to 1$ we get the classical formula $x^n = ((x-1)+1)^n = \sum \binom{n}{k}(x-1)^{n-k}$*
- *There are q-analogs like q-Fibonacci and q-Catalans.*

Chapter 8

Finite Sets And Posets

In this chapter we will look on several questions about finite sets and posets.

- Intersecting families:

  Example. *Base set* $N, |N| = m$, $\mathcal{A}$ *an intersecting family of subsets of* N, *i.e.* $\mathcal{A} \subseteq \mathcal{P}(N) \quad \forall A, B \in \mathcal{A}, A \cap B \neq \emptyset$. *How big can* $\mathcal{A}$ *be?*

  $$|\mathcal{A}| \leq \frac{1}{2} |\mathcal{P}(N)| = 2^{n-1}$$

  Proof. At most 1 of each complementary pair (S, S') of subsets can be in $\mathcal{A}$. This boundary is tight: Fix $x \in N$.

  $$\mathcal{A}_x := \{A \subseteq N : x \in A\}$$
  $$\Rightarrow |\mathcal{A}_x| = 2^{n-1}$$

  because all sets intersect in x. □

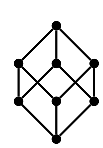
- Structure of maximizing families: An intersecting family $\mathcal{A}$ with $|\mathcal{A}| = 2^{n-1}$ contains S or $S' \quad \forall S \subseteq N$

  $\Rightarrow$if $S \in \mathcal{A}$ and $T \supseteq S$
  $\Rightarrow T \in \mathcal{A}$
  $\Rightarrow \mathcal{A}$ is upward closed, $\mathcal{A}$ is a filter

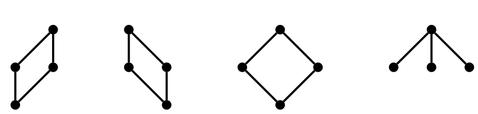
- How many maximizing families are there?

n	2	3	4	5	6	...
# sol	2	4	12	81	2646	...

Example ($n = 3$)**.** *Look at the boolean lattice* $\mathcal{B}_3$



The maximizing families of this lattice consist of the four subsets shown below:



8.1 Posets – Partially ordered sets

Definition. *$P = (X, \leq)$ is a poset if $\leq \subseteq X \times X$ with the properties*

- *reflexive: $x \leq x \quad \forall x \in X$*
- *transitive: $x \leq y, y \leq z \Rightarrow x \leq z$*
- *antisymmetric: $x \leq y, y \leq x \Rightarrow x = y$*

$P = (X, <)$ is a strict ordered set. It is

- *irreflexive: $x \not< x$*
- *transitive: $x < y, y < z \Rightarrow x < z$*

Example.
- *$X = \{(\pi_1, \dots \pi_n)\}$ with left-to-right order or total order*
- *$(X, \emptyset)$ is a strict order (called emptyorder or antichain order)*
- *$\mathcal{A} \subseteq \mathcal{P}(N)$ a family, containment order: $A, B \in \mathcal{A}, A \leq B \Leftrightarrow A \subseteq B$*
- *$I_1, \dots, I_n$ intervals in $\mathbb{R}$, containment order*
- *$I_1, \dots, I_n$ intervals in $\mathbb{R}$ with $I_i < I_j \Leftrightarrow \sup(I_i) < \inf(I_j)$*
- *S_n symmetric group. Possible orders when taking the transitive closure:*
  - *$\pi < \sigma$ iff. π precedes σ in lexicographic order: $\pi = (\pi_1, \dots, \pi_n), \sigma = (\sigma_1, \dots, \sigma_n)$*
  $$\pi < \sigma \Leftrightarrow i = \pi_i < \sigma_i \text{ for } \min\{j \in \{1, \dots, n\} : \pi_j \neq \sigma_j\}$$
  - *$\pi < \sigma$ iff. $Inv(\pi) \subseteq Inv(\sigma)$*
  - *$\pi < \pi'$ iff. $\pi = (\pi_1, \dots, \pi_n), \pi' = (\pi_1, \dots, \pi_{i-1}, \pi_{i+1}, \pi_i, \pi_{i+2}, \dots, \pi_n)$ and $\pi_{i+1} > \pi_i$*

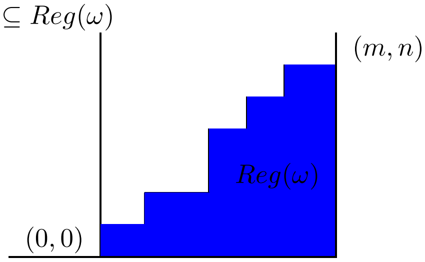
Remark. *The transitive closure of relations always exists.*

Definition. *A poset $P = (X, \leq)$ is a lattice iff.*

- *supremum/join of x and y exists: $(x \vee y)$*
$$\forall x, y \in X \quad \exists z : x \leq z, y \leq z \text{ and for all } z' : x \leq z', y \leq z' \Rightarrow z \leq z'$$
- *infimum/meet of x and y exists: $(x \wedge y)$*
$$\forall x, y \in X \quad \exists z : x \geq z, y \geq z \text{ and for all } z' : x \geq z', y \geq z' \Rightarrow z \geq z'$$

Example (Boolean lattice $\mathcal{B}_n$)**.**
$$A, B \subseteq [n] : \sup(A, B) = A \cup B$$
$$\inf(A, B) = A \cap B$$

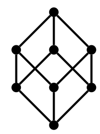
Example (grid path)**.** *Let X be a grid path from $(0,0)$ to (n, m). Order relation: $\omega' \leq \omega$ iff. $\omega' \subseteq Reg(\omega)$*



Remark. *This defines a lattice:*
$$\omega \vee \omega' = \sup(\omega, \omega') = \text{ boundary of union of the regions } Reg(\omega) \cup Reg(\omega')$$
If X is finite then the lattice properties imply that P has a 0 and a 1 : $0 \leq x \quad \forall x \in X$ and $x \leq 1 \quad \forall x \in X$

8.2 The diagram of a poset

Example. $\mathcal{B}_3$



Nodes correspond to elements of P.
Edges correspond to cover relation of P, i. e. there is an edge xy iff. $x < y$ and $\nexists z : x < z < y$
The diagram shows the cover relations drawn as increasing lines.

Definition. *Important substructures in posets:*

- F *filter/upset if* $x \in F$ *and* $y \geq x \Rightarrow y \in F$
- I *ideal/downset if* $x \in I$ *and* $y \leq x \Rightarrow y \in I$
- A *an antichain if* $x, y \in A \Rightarrow x \parallel y$ *incomparable, i. e.* $x \nleq y, y \nleq x$
- C *chain if* $x, y \in C \Rightarrow x \leq y$ *or* $y \leq x$

Remark. • *The complement of a filter is an ideal. Prove it as an exercise.*

- A *antichain,* C *chain* $\Rightarrow |A \cup C| \leq 1$
- C *chain in* $\mathcal{B}_n \Rightarrow |C| \leq n+1$. $\mathcal{B}_n$ *contains antichains of cardinality* $\binom{n}{\lfloor \frac{n}{2} \rfloor}$

Theorem 8.2.1 (Sperner)**.** *$\mathcal{A}$ antichain in $\mathcal{B}_n$ then $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.*

Proof. $\pi \in S_n$ meets A iff. A is an initial segment of π, i. e. $A = \{\pi_1, \dots, \pi_k\}$ for some $k, \pi = (\pi_1, \dots \pi_n)$.

If $|A| = k$ there are $k!(n-k)!$ permutations that meet A.

If $\mathcal{A}$ is an antichain and π a permutation then there is at most one element $A \in \mathcal{A}$ that is met by π because the elements met by π form a chain in $\mathcal{B}_n$.

$$\Rightarrow \sum_{A \in \mathcal{A}} |A|!\,(n - |A|)! \leq n!$$

Let $p_k(\mathcal{A}) = \#$ k-sets in $\mathcal{A}$.

$$\Rightarrow \sum_{k=0}^{n} p_k(\mathcal{A}) k!(n-k)! \leq n!$$

$$\Rightarrow \sum_{k=0}^{n} \frac{p_k(\mathcal{A})}{\binom{n}{k}} \leq 1 \qquad \text{(LYM inequality)}$$

Since $\binom{n}{k} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$ we have

$$\sum_{k=0}^{n} p_k(\mathcal{A}) \binom{n}{\lfloor \frac{n}{2} \rfloor}^{-1} \leq 1$$

and get

$$\underbrace{\sum_{k=0}^{n} p_k(\mathcal{A})}_{=|\mathcal{A}|} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

□

Characterizing the case of equality: $|\mathcal{A}| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$

$$\begin{aligned} &\Rightarrow p_k(\mathcal{A}) = 0 \quad \forall k : \binom{n}{k} < \binom{n}{\lfloor \frac{n}{2} \rfloor} \\ &\Rightarrow \mathcal{A} \subseteq \binom{[n]}{\lfloor \frac{n}{2} \rfloor} \cup \binom{[n]}{\lceil \frac{n}{2} \rceil} \end{aligned}$$

$$\begin{aligned} n \text{ even } &\Rightarrow \mathcal{A} \text{ is unique} \\ n \text{ odd } &\Rightarrow \text{ there are exactly two maximum antichains} \end{aligned}$$

The last case is not trivial to show but it can be shown that if $\mathcal{A}$ has mixing ranks there exists a π that meets no $A \in \mathcal{A}$.

Example (smallest odd case: $n = 3$). *The two maximum antichains are* $\{\{1\}, \{2\}, \{3\}\}$ *and* $\{\{1,2\}, \{2,3\}, \{1,3\}\}$

8.3 k-intersecting families

Theorem 8.3.1 (Erdös-Ko-Rado). *Let* $\mathcal{F} \subseteq \binom{[n]}{k}$. *Intersecting* $A, B \in \mathcal{F} \Rightarrow A \cap B \neq \emptyset$ *and* $n \geq 2k$ *then* $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

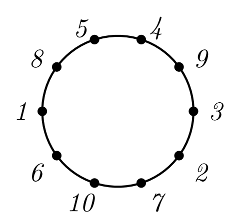
Proof.

Observation.

- *If $n < 2k$ then any two k-element sets intersect*
- *Tight: star families attain the bound:*

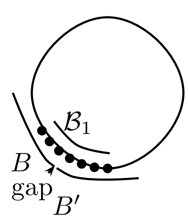
$$\mathcal{F}_x = \left\{ A \in \binom{[n]}{k} : x \in A \right\}, \quad x \in [n]$$

For the proof we use cyclic permutations. A cyclic permutation C meets a k-set if the set is consecutive on the cycle. Think of the k-set as an arc of C.



Lemma 8.3.2. *If $\mathcal{B}_1, \ldots \mathcal{B}_t$ are k-arcs of C with $|C| = n \geq 2k$ and they are intersecting. $\mathcal{B}_i \cap \mathcal{B}_j \neq \emptyset$ then $t \leq k$.*

Proof. We have a look at the picture below.



$\Rightarrow B \cap B' = \emptyset \Rightarrow$ at most one of the sets B, B' is in $\mathcal{B}_1, \ldots, \mathcal{B}_t$. Each $\mathcal{B}_i$ is a member of a pair B, B' for some gap in $\mathcal{B}_1$. $\mathcal{B}_1$ has $k-1$ gaps $\Rightarrow t \leq k$.

□

Count pairs (A, C), C cyclic permutation, $A \in \mathcal{F}$, A is an arc of C. Do some double counting:

$$\begin{aligned}
\sum_{C \text{ cycl. perm.}} (\ \#A'\text{s in } \mathcal{F} : A \text{ arc of } C) &\leq \sum_{C} k = k(n-1)! \\
\sum_{A \in \mathcal{F}} (\ \#C \text{ cyclic permutation with arc } A) &= \sum_{A \in \mathcal{F}} k!(n-k)! = |\mathcal{F}|\, k!(n-k)! \\
\Rightarrow |\mathcal{F}| &\leq \frac{k(n-1)!}{k!(n-k)!} = \binom{n-1}{k-1}
\end{aligned}$$

□

Question: If $\mathcal{F}$ is k-intersecting (i. e. $|A| = k \quad \forall A \in \mathcal{F}$ and intersecting) and maximal (i. e. $A \in \binom{[n]}{k}, A \notin \mathcal{F} \Rightarrow \mathcal{F} \cup \{A\}$ is not intersecting). How big is $\mathcal{F}$?

Answer: $\mathcal{F}$ can be quite sparse.

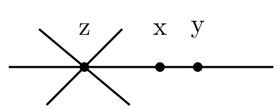
Construction: Take the lines of a projective plane. These form a maximal $k+1$-intersecting family.

A projective plane $\mathcal{P}$ consists of points and lines:

$$\begin{aligned}
\#\text{ points} &= n = k^2 + k + 1 \\
&= \#\text{ lines} \\
\#\text{ points on a line} &= k + 1 \\
&= \#\text{ lines through a point}
\end{aligned}$$

Claim 8.3.3. *The family of lines is maximal.*

Proof. Suppose $\mathcal{L} \cup \{E\}$ is intersecting. Let $x, y \in E$. Look at the line L of $\mathcal{P}$ with $x, y \in L$. Let $z \neq x, y$ be a point of L. E intersects all lines containing $z \Rightarrow z \in E$ because otherwise E would be too big $\Rightarrow E = L$.



□

8.4 Shadows – Another proof for Sperner's theorem

Definition. *Let $\mathcal{B} \subseteq \binom{[n]}{k}$. The down shadow of $\mathcal{B}$ is $\Delta\mathcal{B}$ and the up shadow is $\nabla\mathcal{B}$ with the given definition:*

$$\begin{aligned}
\Delta\mathcal{B} &= \left\{ D \in \binom{[n]}{k-1} : \exists B \in \mathcal{B} : D \subseteq B \right\} \\
\nabla\mathcal{B} &= \left\{ C \in \binom{[n]}{k+1} : \exists B \in \mathcal{B} : B \subseteq C \right\}
\end{aligned}$$

Lemma 8.4.1. *If $k < n$ we have a relation on the cardinalitites of $\mathcal{B}$ and its up shadow:*

$$|\nabla\mathcal{B}| \geq \frac{n-k}{k+1}\, |\mathcal{B}|$$

Proof. Double counting of pairs (B, C) with $B \in \mathcal{B}, C \in \nabla\mathcal{B}$ and $B \subseteq C$.

$$\begin{aligned}
\# \text{ pairs } &= \sum_{B\in\mathcal{B}} \#C : (B,C) \text{ is a pair} \\
&= \sum_{B\in\mathcal{B}} (n-k) \\
&= |\mathcal{B}|\,(n-k) \\
\# \text{ pairs } &= \sum_{C\in\nabla\mathcal{B}} \#B : (B,C) \text{ is a pair} \\
&\leq \sum_{C\in\nabla\mathcal{B}} (k+1) \\
&= |\nabla\mathcal{B}|\,(k+1) \\
\Rightarrow |\nabla\mathcal{B}|\,(k+1) &\geq |\mathcal{B}|\,(n-k)
\end{aligned}$$

□

There is a symmetric result for the down shadow:

Lemma 8.4.2.

$$|\Delta\mathcal{B}| \geq \frac{k}{n-k+1}\,|\mathcal{B}|$$

Again the proof is done by double counting pairs (D, B) with $D \in \Delta\mathcal{B}, B \in \mathcal{B}, D \subseteq B$.
Consequences:

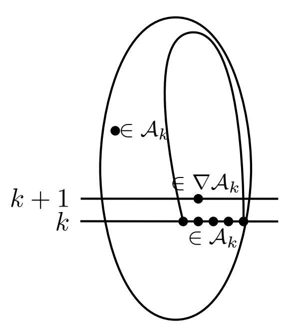
$$k \leq \frac{1}{2}(n-1) \Rightarrow |\nabla\mathcal{B}| \geq |\mathcal{B}| \tag{8.1}$$

$$k \geq \frac{1}{2}(n+1) \Rightarrow |\Delta\mathcal{B}| \geq |\mathcal{B}| \tag{8.2}$$

Proof of Sperner theorem 8.2.1. Let $(p_0 p_1 \dots p_n)$ be a profile of $\mathcal{A}$.

$$p_i = |\mathcal{A}_i|\,, \mathcal{A}_i = \mathcal{A} \cap \binom{[n]}{i}, |\mathcal{A}| = \sum p_i$$

Suppose there exists $k \leq \frac{1}{2}(n-1)$ with $p_k \neq 0$. Take the smallest one. Consider $\mathcal{A}' = \mathcal{A} \backslash \mathcal{A}_k \cup \nabla\mathcal{A}_k$. This is an antichain:



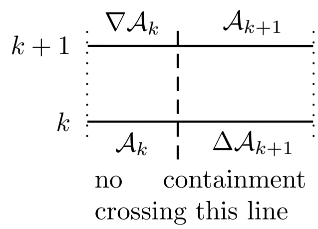
Now we have $|\mathcal{A}'| \geq |\mathcal{A}|$ and repeat this construction until $p_k = 0 \quad \forall k \leq \frac{1}{2}(n-1)$.
Dually we replace $\mathcal{A}_k$ with $k \geq \frac{1}{2}(n+1)$ by $\Delta\mathcal{A}_k$, the size of $\mathcal{A}$ non-decreasing.
Finally $\mathcal{A}$ has profile $(0, \dots, 0, p_{\lfloor \frac{n}{2} \rfloor}, 0, \dots, 0)$ and this yields the result in the result. □

We therefore can sharpen our observations 8.2 and 8.1 to strict inequalities:

$$k < \frac{1}{2}(n-1) \Rightarrow |\nabla\mathcal{B}| > |\mathcal{B}|$$
$$k > \frac{1}{2}(n+1) \Rightarrow |\Delta\mathcal{B}| > |\mathcal{B}|$$

This stronger form yields uniqueness in the even case. In the odd case we obtain that all maximal antichains of $\mathcal{B}_n$ are contained in the two middle levels.

Let $n = 2k+1$. $\mathcal{A}$ contains elements on both levels, $\mathcal{A}$ maximal. If $|\mathcal{A}_k| = |\nabla\mathcal{A}_k|$ then there is no excess in the double counting, i. e. every k-subset of a $C \in \nabla\mathcal{A}_k$ is in $\mathcal{A}_k$.



This contradicts the fact that between any two k-element sets there exists a zig-zag-path in the graph because the path then has to cross the dashed line.

Corollary. *For n odd there are exactly two maximal antichains, each consists of one of the two middle levels.*

Theorem 8.4.3 (Kruskal-Katona)**.** *Let $\mathcal{A} \subseteq \binom{[n]}{k}$ and $|\mathcal{A}| = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_t}{t}$ with $a_k > a_{k-1} > \ldots > a_t \geq t \geq 1$. From exercise sheet 5 we know that there is a unique respresentation of $|\mathcal{A}|$ with these properties. Then it follows:*

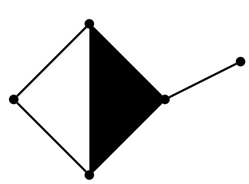
$$|\Delta\mathcal{A}| \geq \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \ldots + \binom{a_t}{t-1}$$

Remark. *The theorem gives a complete characterization of f-vectors of abstract simplicial complexes.*

$(f_0 \ldots f_{d-1})$ is the f-vector of some $(d-1)$-dimensional simplicial complex (i. e. it is generated by d-element sets). $f_i = \#$ elements on level $i+1$ if and only if $0 \leq f_k$ and $f_k \geq f_{k+1}^{\Delta}$.

f_{k+1}^{Δ} is the size of the down shadow of a family of $(k+1)$-sets of size f_{k+1}, guaranteed by Kruskal-Katona 8.4.3.

Example.



1 triangle ∼ 2-simplex
6 edges ∼ 1-simplices
5 vertices ∼ 0-simplices

The family is downward closed, i. e. any downset in $\mathcal{B}_n$ is an abstract simplicial complex.

8.5 Erdös-Ko-Rado from Kruskal-Katona

Erdös-Ko-Rado 8.3.1: $n \geq 2k, \mathcal{A} \subseteq \binom{[n]}{k}$ an intersecting family $\Rightarrow |\mathcal{A}| \leq \binom{n-1}{k-1}$

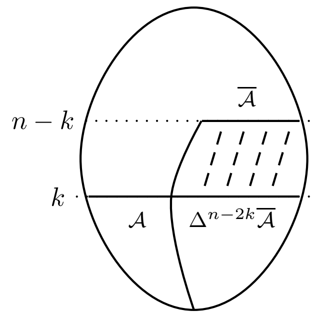


Figure 8.1: Illustration of Erdös-Ko-Rado

Proof. Let $\mathcal{A}$ be an intersecting family.

$$\overline{\mathcal{A}} = \{\overline{A} : A \in \mathcal{A}\} \subseteq \binom{[n]}{n-k}$$

By overlined sets we mean the complement. Note: $A \cap B = \emptyset \Leftrightarrow A \subseteq \overline{B}$. Therefore we get

$$\begin{aligned} A \nsubseteq \overline{B} \quad \forall A \in \mathcal{A}, \overline{B} \in \overline{\mathcal{A}} \\ \Rightarrow \mathcal{A} \cap \Delta^{n-2k}\overline{\mathcal{A}} = \emptyset \\ \Rightarrow |\mathcal{A}| + \left|\Delta^{n-2k}\overline{\mathcal{A}}\right| \leq \binom{n}{k} && (8.3) \\ && (8.4) \end{aligned}$$

Now assume $|\mathcal{A}| > \binom{n-1}{k-1} = \binom{n-1}{n-k}$.

$$|\overline{\mathcal{A}}| > \binom{n-1}{n-k} \overset{8.4.3}{\Rightarrow} \left|\Delta^{n-2k}\overline{\mathcal{A}}\right| > \binom{n-1}{(n-k)-(n-2k)} = \binom{n-1}{k}$$

$$\Rightarrow |\overline{\mathcal{A}}| + \left|\Delta^{n-2k}\overline{\mathcal{A}}\right| > \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k} \tag{8.5}$$

8.3 and 8.5 are in a contradiction which then proves the theorem of Erdös-Ko-Rado. An illustration of the proof is given in 8.1 □

We do not give the proofs but there are some interesting remarks regarding the Kruskal-Katona theorem 8.4.3.

- We can always list all k-subsets in reverse lexicographic order of the incidence vectors (see 8.2 for $k = 3$)
- $\binom{4}{3} + \binom{3}{2} + \binom{1}{1} = 8$. This connects the list of subsets to the Kruskal-Katona number system.
- A family of k-sets is compressed if it is an initial sequence of the reverse lexicographic order on k-sets.
- The shadow of a compressed family is compressed, its size is as in the theorem 8.4.3. In the example 8.2 the shadow of $\mathcal{A}$ is

$$\Delta\mathcal{A} = \{12, 13, 23, 14, 24, 34, 15, 25, 35, 45\} \Rightarrow |\Delta\mathcal{A}| = 10 = \binom{4}{2} + \binom{3}{1} + \binom{1}{0}$$

$$\begin{array}{l}
\left.\begin{array}{ll}
 & 1\,1\,1\,0\,0\,0\cdots \\
\binom{4}{3} & 1\,1\,0\,1\,0\,0\cdots \\
 & 1\,0\,1\,1\,0\,0\cdots \\
 & 0\,1\,1\,1\,0\,0\cdots \\
 & 1\,1\,0\,0\,1\,0\cdots \\
\binom{3}{2} & 1\,0\,1\,0\,1\,0\cdots \\
 & 0\,1\,1\,0\,1\,0\cdots \\
\binom{1}{1} & 1\,0\,0\,1\,1\,0\cdots
\end{array}\right\}\text{compressed family } \mathcal{A} \text{ of size 8 on 3-sets} \\
\begin{array}{ll}
\phantom{\binom{4}{3}} & 0\,1\,0\,1\,1\,0\cdots \\
 & 0\,0\,1\,1\,1\,0\cdots \\
 & 1\,1\,0\,0\,0\,1\cdots \\
 & \vdots
\end{array}
\end{array}$$

Figure 8.2: Infinite listing of 3-subsets

- Shifting (compression operator): For $i < j$ do

$$S_{ij}(A) = \begin{cases} A \setminus \{j\} \cup \{i\} & i \notin A, j \in A, A \setminus \{j\} \cup \{i\} \notin \mathcal{A} \\ A & \text{otherwise} \end{cases}$$

  S_{ij} depends on $\mathcal{A}$ and $S_{ij}(\mathcal{A}) = \{S_{ij}(A) : A \in \mathcal{A}\}$.

  Lemma 8.5.1.

$$\Delta S_{ij}(\mathcal{A}) \subseteq S_{ij}(\Delta\mathcal{A})$$

  Consequence: Shifting cannot increase the size of the shadow: $|S_{ij}(\Delta\mathcal{A})| = |\Delta\mathcal{A}|$

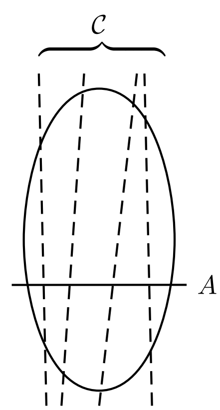
- If $\mathcal{A}$ is stable under shifting it is compressed.

8.6 Symmetric chain decompositions

Remember the Sperner theorem 8.2.1:

$$|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

If there exists a chain partition $\mathcal{C}$ such that $|\mathcal{C}| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ we can conclude that $|\mathcal{A}| \leq |\mathcal{C}| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ for all antichains $\mathcal{A}$.



Let $\mathcal{C}$ be a chain partition and A be an antichain.

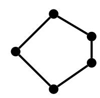
$$\begin{aligned} &|A \cap C| \leq 1 \\ \Rightarrow &\forall x \in A \quad \exists C \in \mathcal{C}, x \in C \\ \Rightarrow &|\mathcal{C}| \geq |A| \end{aligned}$$

Definition (rank function on a poset)**.** *Let $P = (X, <)$ be a poset.*

$$rank(x) = \max\{|C| : C \text{ a chain with maximal element } x\}$$

P is ranked if all maximal chains in P have the same length.

Example. $rank(\mathcal{B}_3) = 4$



is not ranked.

Example. *Let B_x be the lattice of a multisubset of X, X a multiset. Then B_x is ranked with*

$$rank(M) = \text{ sum of multiplicities } + 1 \text{ for any multiset } M$$

Definition (unrefinable chain)**.** *Let P be a ranked poset. C is an unrefinable chain in P iff. $C = (x_1 < x_2 < \ldots < x_k)$, no ranks are skipped.*

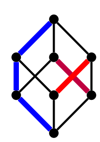
An unrefinable chain C in P is symmetric iff.

$$rank(\min(C)) + rank(\max(C)) = rank(P) + 1$$
$$rank(x_{i+1}) = rank(x_i) + 1 \quad \forall i$$

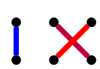
$x_i < x_{i+1}$ is a cover relation.

A chain decomposition $\mathcal{C}$ is symmetric iff. each chain in $\mathcal{C}$ is symmetric.

Example ($\mathcal{B}_3$)**.** *The three colored symmetric chains build a chain decomposition of $\mathcal{B}_3$.*



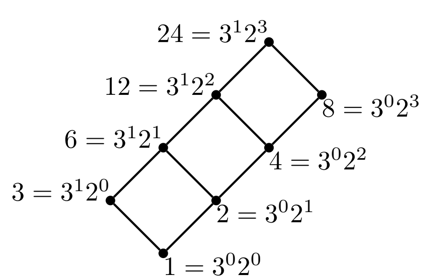
Proposition 8.6.1. *If P is ranked and it admits a symmetric chain decomposition then the middle rank is a maximal antichain:*



We will apply this proposition in the case of multiset lattices, also known as divisor posets.

$$N = p_1^{m_1} p_2^{m_2} \ldots p_k^{m_k}, \text{ divisors: } p_1^{l_1} p_2^{l_2} \ldots p_k^{l_k}, l_i \leq m_i$$

Example ($N = 24 = 3 \cdot 2^3$)**.** *The divisor poset of $N = 24$ has eight nodes and five different ranks.*



Remark. *Proofs with the LYM inequality or shadows do not work in divisor posets.*

Let $B(m_1 m_2 \dots m_k)$ be the divisor poset of $p_1^{m_1} p_2^{m_2} \dots p_k^{m_k}$ resp. multiset lattice resp. the product $C_1 \times C_2 \times \dots \times C_k$ when C_i is a chain of size $m_i + 1$.

Definition (Product of posets)**.** *Let $P = (X, \leq_P), Q = (Y, \leq_Q)$ be two posets. The product of P and Q is*

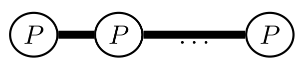
$$\begin{aligned} P \times Q &= (X \times Y, \leq_\times) \\ X \times Y &= \{(x, y) : x \in X, y \in Y\} \\ (x, y) \leq_\times (x', y') &\Leftrightarrow x \leq_P x', y \leq_Q y' \end{aligned}$$

Theorem 8.6.2. *$B(m_1 m_2 \dots m_k)$ admits a symmetric chain decomposition.*

Proof. Induction on k

$k = 1$: Is clearly true.

$k - 1 \to k$: Let $\mathcal{C}$ be a symmetric chain decomposition of $P = B(m_1, \dots, m_{k-1})$. To do the induction step we have to consider the product of a chain of size $m_k + 1$ and P:



The chain decomposition $\mathcal{C}$ of P.

Let $S_1 \subset \dots \subset S_l$ be a chain of multisets in $\mathcal{C}$. We then build the required chain decomposition out of the smaller one given by induction hypothesis:

S_l	$S_l + k$	$S_l + k^2$	$\dots$	$S_l + k^{m_k}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_2	$S_2 + k$	$S_2 + k^2$	$\dots$	$S_2 + k^{m_k}$
S_1	$S_1 + k$	$S_1 + k^2$	$\dots$	$S_1 + k^{m_k}$

Now the first row and first column (framed) build a symmetric chain. If we cut these off the next row builds another symmetric chain together with the next column and so on. This makes $\min(l, k + 1)$ new symmetric chains out of the old one.

For $r = rank(P)$ we know $rank(S_1) + rank(S_2) = r + 1$, $rank(B(m_1, \dots, m_k)) = r + m_k$.

Claim 8.6.3.

$$\begin{aligned} rank(S_1) + \underbrace{rank(S_l + k^{m_k})}_{=rank(S_l)+m_k} = r + m_k + 1 \\ \Rightarrow (S_1 \dots S_l, S_l + k, \dots, S_l + k^{m_k}) \textit{ is symmtric} \end{aligned}$$

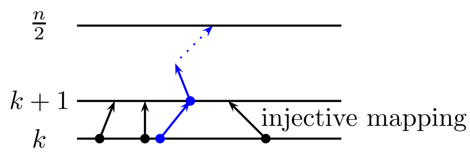
The next chain (second row + second column) is also symmetric because the rank of the minimum increases and the rank of the maximum decreases by 1. Therefore the construction yields a symmetric chain decomposition and every element is covered.

□

Since $B(1, \dots, 1) = \mathcal{B}_k$ we have a symmetric chain decomposition of $\mathcal{B}_k$ which we can construct directly. Advantages: we can explicitly construct the chain containing $A \subseteq [n]$ and we can give an injective mapping from k-subsets to $k + 1$-subsets for $k < \frac{n}{2}$.

n	0	1	2	3	4	5	6	7	8
D_n	1	3	6	20	168	7581	$\sim 10^{6.8}$	$\sim 10^{12.3}$	$\sim 10^{22}$
antichains	$\emptyset$	$\emptyset$	$\emptyset$	$\dots$					
		$\{\emptyset\}$	$\{\emptyset\}$						
		$\{\{1\}\}$	$\{\{1\}\}$						
			$\{\{2\}\}$						
			$\{\{1,2\}\}$						
			$\{\{1\}\{2\}\}$						

Table 8.1: First Dedekind numbers



We start with the given set $A \subseteq [n]$ (blue arrows) and get the characteristic bracketing.

Example ($n = 12$)**.**

$$A = \{1, 2, 7, 8, 10, 11\}$$

Bracketing:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
	)	)	(	(	(	(	)	)	(	)	)	(

This bracketing yields a set C of matched brackets. Forget about these and the rest is a sequence which looks like this $)) \dots) (\dots (($

Example.

1	*2*	*3*	*12*
)	)	(	(

Adding these elements one by one to the elements defined by the matched brackets gives us a chain starting at $)) \dots)$ and ending at $((\dots ($

The chain of A is $\emptyset + C, 1 + C, 1 + 2 + C, 1 + 2 + 3 + C, 1 + 2 + 3 + 12 + C$. C is the core of $A : C = \{7, 8, 10, 11\}$

Symmetry conditions:

$$\begin{aligned} rank(\min(C)) &= |\text{core}| + 1 \\ rank(\max(C)) &= |\text{core}| + (n - 2\,|\text{core}|) + 1 \\ \Rightarrow rank(\min(C)) + rank(\max(C)) &= (n+1) + 1 = rank(\mathcal{B}_n) + 1 \end{aligned}$$

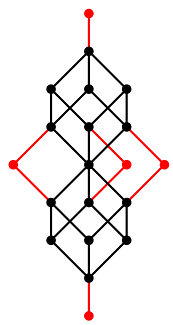
If $B \in \text{chain}(A)$ then $\text{chain}(B) = \text{chain}(A)$ (because they have the same matched brackets) and therefore the decomposition is well-defined.

8.7 An application: Dedekinds problem

Dedekind asked: how many antichains does $\mathcal{B}_n$ have? The first Dedekind numbers are given in 8.1 and it is obvious that they increase pretty fast.

The down sets I_A of antichains can be ordered by inclusion. The poset belonging to the inclusion of down sets in $\mathcal{B}_n$ is called the free distributed lattice ($FDL(n)$).

Example ($FDL(3)$)**.** *The free distributed lattice for $n = 3$ has 20 nodes and $D_3 = 20$.*



Bounds: $|\mathcal{B}_n| = 2^n \Rightarrow D_n \leq 2^{2^n}$

All subsets of an antichain count $D_n \geq 2^{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$

$\Rightarrow \frac{1}{\sqrt{\pi n}} 2^n \overset{Stirling}{\sim} \binom{n}{\frac{n}{2}} \leq \log_2 D_n \leq 2^n$

It is also known that $\log_2 D_n \leq (1 + \mathcal{O}\left(\frac{\log n}{n}\right))\binom{n}{\frac{n}{2}}$.

The easy upper bound 2^{2^n} can be improved by using symmetric chains. Idea: Let $\mathcal{C} = (C_1, \ldots, C_{\binom{n}{\lfloor \frac{n}{2} \rfloor}})$ be the symmetric chain decomposition ordered by the size of the chains. The idea is to encode a downset by the position where a chain C_i is leaving it. To keep the number of bits that are needed to encode this position small we have a lemma.[1]

Lemma 8.7.1. *If $A_0 \subset A_1 \subset \ldots \subset A_k$ is a $k+1$-chain of the symmetric chain decomposition of $\mathcal{B}_n$ then for all $A_i, i = 1, \ldots, k-1$ there is a set B_i such that $A_{i-1} \subset B_i \subset A_{i+1}$ and $|\,chain(B_i)\,| < k$*

Proof.

$$\begin{array}{cc|c|c|c|c} A_{i+1} & \ldots &) & \ldots &) & \ldots \\ A_i & \ldots &) & \ldots & (& \ldots \\ A_{i-1} & \ldots & (& \ldots & (& \ldots \\ B_i & \ldots & (& \ldots &) & \ldots \end{array}$$

Now we have $A_{i-1} \subset B_i \subset A_{i+1}$ and B_i has more matching brackets than A_i has. □

When it comes to decide which elements of the chain go into the down set we use the scheme by comparing colors. Elements in the down set are marked red, elements which are not in the downset are marked blue.

$$\begin{array}{cccccccc} A_0 & A_1 & A_2 & \ldots & A_{i-1} & A_i & A_{i+1} & \ldots & A_k \\ & & & & & B_i & & & \end{array}$$

If $B_i \in$ down set then $A_0, \ldots, A_{i-1}$ are in the down set. Else if $B_i \notin$ down set then $A_{i+1}, \ldots, A_k$ are also not in the down set.

Similarly we know that if all B_i are in the downset then all $A_i, i < k-1$ are also in the down set. If no B_i is in the downset then clearly there is also no $A_i, i > 1$ in the down set.

If B_i is followed by B_{i+1} then $A_1, \ldots, A_{i-1}, \ldots, A_{i+2}, \ldots, A_k$.

If B_i is followed by B_{i+1} then we do not have any choice:

$$\begin{array}{ccccccccc} A_0 & A_1 & A_2 & \ldots & A_{i-1} & A_i & A_{i+1} & \ldots & A_k \\ & & & & B_i & B_{i+1} & & & \end{array}$$

We now found out that at most two consecutive A_i's are free for a choice and we have three possibilities for that:

(• •) (• •) (• •)

Since we search a down set the combination (• •) is not valid.

The number of possibilities to choose a down set has to be $\leq 3^{\binom{n}{\frac{n}{2}}}$ and we get the result:

$$\log_2 D_n \leq \binom{n}{\frac{n}{2}} \log_2 3$$

[1] Also see "'Combinatorics of finite sets"' by Ian Anderson, pp.36–38, for more information

Chapter 9

Duality Theorems

Theorem 9.0.2 (Dilworth's theorem)**.** *Let P be a finite poset. Then*

$$\max_{A \text{ antichain}} |A| = \min_{\mathcal{C} \text{ chain partition}} |\mathcal{C}|$$

Proof. **"'$\leq$"'** This is clearly true because all elements from A have to be in different chains as they are incomparable.

"'$\geq$"' Induction on $|P|$:

We split up our poset at a maximal antichain A. Then we have a chain partition of the upset of A and a chain partition of the downset of A by induction hypothesis and we can glue these partitions together to yield a chain partition of P. See 9.1 for explanation.

This analysis does not respect the case that the up set or the down set of all maximal antichains A has the same size as P so we need to pay special attention to these cases. Let $A = \min(P)$ or $A = \max(P)$.

(1) the unique maximum antichain is $A = \min(P)$

- delete one $m \in \min(P)$
- get chain partition on $P - m$ of size $|A| - 1$
- add m as singleton chain

(2) $\max(P)$ and $\min(P)$ are maximal antichains

- $m \in \min(P), m \leq m' \in \max(P)$

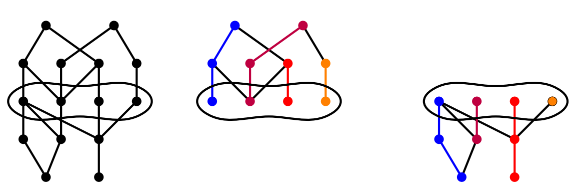


Figure 9.1: chain partition vs. antichain

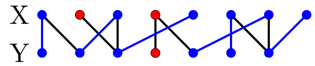


Figure 9.2: Bipartite graph as a poset

- delete $\{m, m'\}$
- $P - \{m, m'\}$ has $\mathcal{C}$ of size $|A| - 1$
- add $C = \{m, m'\}$

$\square$

Definition (Vertex cover, matching)**.** *Let $G = (V, E)$ be a graph, V the vertex set, $E \subseteq V \times V$ the set of edges.*

$V' \subseteq V$ is called a vertex cover iff.

$$\forall e \in E \quad \exists v \in V' : v \in e$$

$M \subseteq E$ is called a matching iff.

$$\forall e, f \in M : e \cap f = \emptyset$$

There are simple inequalities for the size of minimal vertex covers and maximal matchings.

$$\max_{M \text{ matching}} |M| \leq \min_{V' \text{ vertex cover}} |V'| \leq 2 \max_{M \text{ matching}} |M|$$

The first inequality is because we have to take one vertex from every edge in a given matching into every minimal vertex cover.

The second inequality can be seen through $V' \supseteq \{v : \exists e \in M : v \in e\}$

There are no better bounds. For an example think of a triangle.

Theorem 9.0.3 (König-Egervary)**.** *Let $G = (X \cup Y, E)$ be a bipartite graph. Then*

$$\max_{M \textit{ matching}} |M| = \min_{V \textit{ vertex cover}} |V|$$

Proof. See G as a poset (9.2). Observations:

I A bijection between antichains and vertex covers $A \longleftrightarrow X + Y - A = V \quad |A| = |X + Y| - |V|$.

$$\begin{aligned}
&\text{every comparability contains } \leq 1 \text{ vertex of } X + Y - V \\
\Leftrightarrow &X + Y - V \text{ is an antichain} \\
\Leftrightarrow &V \text{ is a vertex cover} \\
\Leftrightarrow &\text{ every edge contains } \geq 1 \text{ vertex}
\end{aligned}$$

II A bijection between matchings and chain partitions

$M \longleftrightarrow M+$ singletons $\quad |\mathcal{C}| = |X + Y| - |M|$

$$\begin{aligned}
&M \text{ matching} && \text{(blue edges in 9.2)} \\
\Leftrightarrow &M + \text{singletons (red vertices in 9.2) is a chain partition} \\
\Rightarrow &\text{ size of chain partition } = |X + Y| - |M|
\end{aligned}$$

Due to Dilworth's theorem 9.0.2 and by building complements we get the result.

$$\begin{aligned}
&\max_{A \text{ antichain}} |A| = \min_{\mathcal{C} \text{ chain partition}} |\mathcal{C}| \\
\Leftrightarrow &\min_{V \text{ vertex cover}} |V| = \max_{M \text{ matching}} |M|
\end{aligned}$$

□

Definition (Neighbourhood). *The neighbourhood $N(S)$ of a vertex set $S \subseteq V$ is the set of all adjacent vertice:*

$$N(S) := \{y \in V \setminus S | (s,y) \in E \text{ for some } s \in S\}$$

Theorem 9.0.4 (Hall's theorem/Marriage theorem). *Let $G = (X \cup Y, E)$ be a bipartite graph. G has a matching of size $|X|$ if and only if*

$$\forall S \subseteq X : |N(S)| \geq |S| \qquad \text{(Hall condition)}$$

Proof. $\Rightarrow$ G has a matching of size $|X|$.

$$\begin{aligned} S \subseteq X \Rightarrow & N(S) \supseteq \{y \in Y | (y,s) \in M\} \\ |N(S)| \geq |\{y \in Y | (y,s) \in M\}| = |S| \end{aligned}$$

$\Leftarrow$ Say G has no matching of size $|X|$. By 9.0.3 we know that there is a vertex cover V with $|V| < |X|$, say $V = X' \cup Y', X' \subseteq X, Y' \subseteq Y$ and there is no edge between $X \setminus X'$ and $Y \setminus Y'$, i.e. all edges from $X \setminus X'$ go to $Y \setminus Y'$.

$$\begin{aligned} \Rightarrow & |Y'| \geq |N(X \setminus X'| \\ & |Y'| = |V \setminus X'| < |X \setminus X'| \\ \Rightarrow & |X \setminus X'| > |N(X \setminus X')| \end{aligned}$$

This violates the Hall condition and therefore leads to a contradiction.

□

Definition (SDR). *Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a family of finite sets. Then $(x_1, \ldots, x_n)$ is a SDR (system of distinct representatives) of $\mathcal{A}$ iff.*

$$x_i \in A_i \quad \forall i \in [n] \text{ and } x_i \neq x_j \quad \forall i \neq j$$

Any family $\mathcal{A}$ of sets can be seen as a bipartite graph $G = (X \cup Y, E)$ with $X = \mathcal{A}$ and $Y = \bigcup_{A_i \in \mathcal{A}} A_i$ with $(x_i, A_i) \in E \Leftrightarrow x_i \in A_i$. Finding a matching in this bipartition equals finding a SDR.

Theorem 9.0.5 (Quantified Hall). *Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a family of sets such that*

$$\forall J \subseteq [n] : |J| \leq \left| \bigcup_{j \in J} A_j \right| \qquad \text{(Hall condition)}$$

$$|A_i| \geq r \quad \forall i \in [n]$$

If $n \geq r$ then there are at least $r!$ many different SDRs for $\mathcal{A}$

Proof. Induction on n

$n = r$ take $x_1 \in A_1, x_2 \in A_2 \setminus \{x_1\}, \ldots, x_i \in A_i \setminus \{x_1, \ldots, x_{i-1}\}, \ldots, x_r \in A_r \setminus \{x_1, \ldots, x_{r-1}\}$. From this and the condition that $|A_i| \geq r \forall i \in [n]$ we have obviously at least $r!$ many different SDRs.

$n > r$ We have to do a case distinction:

Case 1 $\forall J \subset [n]$: $|J| < \left| \bigcup_{j \in J} A_j \right|$

Choose some $x_n \in A_n$, we have $\geq r$ choices for that. Consider

$$\begin{aligned}
&\mathcal{A}' = (A_1 - x_n, \ldots, A_{n-1} - x_n) \\
&\Rightarrow |A_i - x_n| \geq r - 1 \\
&\Rightarrow |J| \leq \left| \bigcup_{j \in J} A_j - x_n \right| \forall J \subseteq [n] && (|J| < \left|\textstyle\bigcup_{j\in J} A_j\right|) \\
&\Rightarrow \mathcal{A}' \text{ is Hall family} \\
&\Rightarrow \mathcal{A}' \text{ has } \geq (r-1)! \text{ many SDRs} && \text{(Induction hypothesis)} \\
&\Rightarrow \geq r! \text{ many SDRs of } \mathcal{A} && \text{(represent } A_n \text{ by } x_n\text{)}
\end{aligned}$$

Case 2 $\exists J \subset [n] : |J| = \left|\bigcup_{j\in J} A_j\right| \geq r$

$$\begin{aligned}
B &:= \bigcup_{j \in J} A_j \\
\mathcal{A}_1 &:= \{A_j : j \in J\} \\
\mathcal{A}_2 &:= \{A_i - B : i \in [n] - J\}
\end{aligned}$$

Now we show that both $\mathcal{A}_1$ and $\mathcal{A}_2$ are Hall.
$\mathcal{A}_1$ is Hall and all A_j have $|A_j| \geq r, r \leq |J| < n$. By induction hypothesis $\mathcal{A}_1$ has then at least $r!$ many SDRs. So we just need to show the Hall condition for $\mathcal{A}_2$:

$$\begin{aligned}
\left| \bigcup_{i \in I \subseteq [n]-J} A_i - B \right| &= \left| \bigcup_{i \in I \cup J} A_i - B \right| \\
&= \left| \bigcup_{i \in I \cup J} A_i \right| - |B| \\
&\geq |I| + |J| - |B| && \text{(Hall condition)} \\
&= |I| && (|J| = |B|) \\
&\Rightarrow \mathcal{A}_2 \text{ fulfills the Hall condition}
\end{aligned}$$

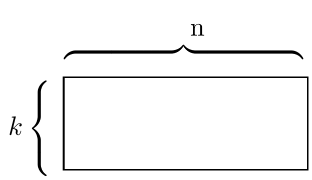
If we now combine the SDRs of $\mathcal{A}_1$ with the SDR of $\mathcal{A}_2$ (this works because the ground sets are disjoint) we get the result.

□

Application: Number of Latin Squares

1	2	3	4
2	3	4	1
4	1	2	3
3	4	1	2

In a $n \times n$ Latin Square each number appears exactly once in every row and every column.



In a $n \times k$ Partial Latin Square (PLS) every number appears at most once per row and column.

Theorem 9.0.6. *There are at least $(n-k)!$ ways to extend a $k \times n$ Partial Latin Square to a $(k+1) \times n$ PLS.*

Proof. We extend the PLS to a bigger one by adding an extra row. Let A_i denote the set of possible entries in column i, respecting the column condition. $|A_i| = n - k$.

Now it is enough to show that $\mathcal{A} = (A_1, \ldots, A_n)$ is Hall (with $|A_i| \geq n - r$ the quantified Hall theorem implies the result)

$$\begin{aligned} J &\subseteq [n] \\ S &= \{(j,x) | x \in A_j, j \in J\} \\ |S| &= (n-k)\,|J| \end{aligned} \tag{*}$$

Let $S^* = \{x : \exists j \text{ such that } (j,x) \in S\}$ since each x is in exactly $n-k$ of the A_i's we get that $|S^*| \geq \frac{|S|}{n-k} = |J|$ and this is the Hall condition. ☐

Definition. *By $L(n)$ we will denote the number of $n \times n$ Latin Squares.*

Corollary.

$$L(n) \geq \prod_{k=1}^{n} k!$$

9.1 The matching polytope

Definition. *Let $G = (V,E)$ be a graph. $E \ni e = \{u,v\}, u, v \in V$.*

The incidence vector of $F \subseteq E$ is called $\chi(F) \in \mathbb{R}^E$. Properties:

$$\chi(F)(e) = \begin{cases} 1 & e \in F \\ 0 & e \notin F \end{cases}$$

$$|F| = \langle \mathbb{1}, \chi(F) \rangle$$

$$|F \cap G| = \langle \chi(G), \chi(F) \rangle$$

We want to characterize $\chi(M)$ for matchings M of G. For every vertex $v \in V$ define $\delta_v \in \mathbb{R}^E$ such that

$$\delta_v := \chi\left(\{e \in E | v \in e\}\right)$$

$$\delta_v(e) = \begin{cases} 1 & e = \{u,v\} \in E \text{ for some } u \in V \\ 0 & \text{otherwise} \end{cases}$$

$M \subseteq E$ is a matching iff. every vertex is incident to at most one edge in M. A_G is the incidence matrix of G.

$$\Leftrightarrow \langle \chi(M), \delta_v \rangle \leq 1 \forall v \in V$$

$$\Leftrightarrow A_G \chi(M) = \begin{bmatrix} \delta_{v_1} \\ \delta_{v_2} \\ \vdots \\ \delta_{v_n} \end{bmatrix} \cdot \chi(M) \leq \mathbb{1}$$

$$A_G(v,e) = \begin{cases} 1 & v \in e \\ 0 & \text{otherwise} \end{cases}$$

If A_G is the incidence matrix of G then x is the incidence vector of a matching iff.

$$A_G \cdot x \leq \mathbb{1}, x \in \{0,1\}^E$$

A polyhedron is the intersection of bounded half-spaces ($\mathcal{H}$-description). For example

$$P_G = \left\{x \in \mathbb{R}^E | A_G x \leq \mathbb{1}, x \geq 0\right\}$$

A polyhedron P is called a polytope iff. P is a bounded polyhedron.

Theorem 9.1.1 (Main theorem for polytopes)**.** *Every polytope P has an $\mathcal{H}$-description and a $\mathcal{V}$-description*

Definition ($\mathcal{V}$-description)**.** *P has a $\mathcal{V}$-description iff. there is a set $V \subseteq P$ such that $P = conv(V)$ where $conv(V)$ is the convex hull, i. e. the smallest convex set that contains all $v \in V$.*

$$conv(V) = \left\{ \sum_{v_i \in V} \lambda_i v_i | \lambda_i \geq 0, \sum_i \lambda_i = 1 \right\}$$

The line segment with endpoints u and v is

$$conv\left(\{u, w\}\right) = \{\lambda u + (1 - \lambda) w | 1 \geq \lambda \geq 0\}$$

The polytope P_G just contains vectors $x \geq 0$ and the incidence matrix $A_G \geq 0$ has no 0-columns.

Definition (Linear program)**.** *A linear program is a function to maximize under certain constraints as follows:*

$$\begin{aligned} &\max \langle c, x \rangle \\ \textit{subject to } &Ax \leq b \\ &x \geq 0 \end{aligned} \tag{general LP}$$

In the case of our matching polytope the linear program looks like this:

$$\begin{aligned} &\max \langle \mathbb{1}, x \rangle \\ \text{subject to } &A_G x \leq \mathbb{1} \\ &x \geq 0 \end{aligned}$$

Remark.
- *If the optimum is attained by a $\{0, 1\}$-vector x then $x = \chi(M)$ and M is a maximum matching.*
- *If P is a polytope then for every c the maximum $\max_{x \in P} \langle x, c \rangle$ is attained at a vertex $v_0 \in P$*

Theorem 9.1.2. *Let $G = (V_1 \cup V_2, E)$ be a bipartite graph.*
Then the vertices of $P_G = \left\{ x \in \mathbb{R}^E | A_G x \leq \mathbb{1}, x \geq 0 \right\}$ are $\{0, 1\}$-vectors.

Proof.
- All vertices of P_G correspond to matchings. We say they are matchings.
- All matchings are vertices. Suppose a matching $\chi(M)$ is no vertex. We know that any set of $\{0, 1\}$-vectors is convexly independent, i. e.

$$\begin{aligned} x &= \lambda_1 x_1 + \ldots + \lambda_k x_k, \quad \lambda_i > 0 \\ 1 = x(i) &\overset{\sum \lambda_i = 1}{\Rightarrow} x_j(i) = 1 \forall j \\ 0 = x(i) &\overset{\sum \lambda_i = 1}{\Rightarrow} x_j(i) = 0 \forall j \end{aligned}$$

Hence all of the x_j are identical to x.

□

When $P \subseteq \mathbb{R}^m$ is a polytope with vertex v_0, then v_0 is the intersection of m bounding halfspaces of P. More formally:

$$P = \{x \in \mathbb{R}^m | Mx \leq b\}, v_0 \in P \text{ vertex}$$

Then there is a $m \times m$ non-singular submatrix M' of M such that $M' v_0 = b'$ with b' the corresponding subvector of b.

$$P_G = \left\{ x \in \mathbb{R}^E | \begin{bmatrix} A \\ -I \end{bmatrix} x \leq \begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix} \right\}$$

$$-Ix \leq 0 \Leftrightarrow x \geq 0$$

We want to show that every vertex of P_G is integer because then it is a $\{0,1\}$-vector as P_G is a subset of the unit cube. Let v_0 be a vertex.

$$\Rightarrow M'v_0 = \begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix} \text{ and } v_0 \text{ is the unique solution}$$

Recall Cramer's rule from linear algebra

$$v_0(e) = \frac{\det\left(M'|_e \begin{matrix}\mathbb{1}\\0\end{matrix}\right)}{\det(M')}$$

We are now going to show that v_0 is a $\{0,1\}$-vector. Because of the Leibniz formula for computation of determinantes $\det\left(M'|_e \begin{matrix}\mathbb{1}\\0\end{matrix}\right)$ is integer. So we just need to show that the determinante of every square submatrix M' of $\begin{bmatrix} A_G \\ -I \end{bmatrix}$ is in $\{0, \pm 1\}$. Proof by induction on k:

$k = 1$

$$M' \in \{0, \pm 1\} \Rightarrow \det(M') = M'$$

$k > 1$ **Case 1** M' has a row with a -1-entry and the rest of the row is 0 (i. e. it is a row from $-I$). Then we apply Laplace and see

$$\det(M') = \pm \det(M'_{\text{red}}) \in \{0, \pm 1\}$$

Case 2 M' has a 0-column.

$$\Rightarrow \det(M') = 0$$

Case 3 M' has a column with one 1-entry and the rest is 0. We then can also apply Laplace and get

$$\det(M') = \det(M'_{\text{red}}) \in \{0, \pm 1\}$$

Case 4 Every column contains exactly two 1s. Remember that $G = V_1 \cup V_2, E)$ is bipartite. Therefore every column has exactly one 1 in the rows belonging to V_1 ans another 1 in the rows belonging to V_2. That means that the rows of V_1 sum up to $\mathbb{1}$ and also do the rows belonging to V_2.

$\Rightarrow$ the rows are linearly independent and $\det(M') = 0$

We saw that all vertices of P_G are integer and since $0 \leq P_G x \leq \mathbb{1}$ all vertices of P_G are $\{0,1\}$-vectors.

Definition. *For a given linear program*

$$\begin{aligned} &\max \langle c, x\rangle && \text{(primal LP)} \\ &\textit{subject to } Ax \leq b && (9.1) \\ &x \geq 0 \end{aligned}$$

there is a corresponding dual linear program defined as

$$\begin{aligned} &\min \langle b, y\rangle && \text{(dual LP)} \\ &\textit{subject to } A^T y \geq c && (9.2) \\ &y \geq 0 \end{aligned}$$

Proposition 9.1.3 (weak duality)**.**

$$\langle c, x\rangle \leq \langle b, y\rangle$$

for all x primal feasible and y dual feasible.

Proof.

$$\begin{aligned}\langle c, x\rangle &\leq \left\langle A^T y, x\right\rangle && (9.2 \text{ and } x \geq 0)\\ &= \langle y, Ax\rangle \\ &\leq \langle y, b\rangle && (9.1 \text{ and } y \geq 0)\end{aligned}$$

□

The following theorem is given without any proof. The interested reader can find it in the scriptum of Prof. Möhring from ADM2[1].

Theorem 9.1.4 (strong duality)**.** *If the primal LP is bounded and has a solution then*

$$\max_x \langle c, x\rangle = \min_y \langle b, y\rangle$$

In our case we have the primal LP for the matching.

$$\begin{aligned}&\max \langle \mathbb{1}, x\rangle \\ \text{subject to } &A_G x \leq \mathbb{1} \\ &x \geq 0\end{aligned}$$

The dual LP is given by

$$\begin{aligned}&\min \langle \mathbb{1}, y\rangle \\ \text{subject to } &A_G^T y \geq \mathbb{1} \\ &y \geq 0\end{aligned}$$

If $y \in \{0,1\}^V$ is a feasible vertex for the dual then it corresponds to a vertex cover. Looking at optimalizing points $x \in \{0,1\}^E, y \in \{0,1\}^V$ this gives us

$$\max_{M \text{ matching}} |M| \overset{9.1.4}{=} \min_{V' \text{ vertex cover}} |V'|$$

This again yields a proof for the theorem of König-Egervary 9.0.3.

For that it has to be shown that y is integer. As all square submatrices of A_G^T have det $\in \{0, \pm 1\}$ by the same arguments as for the matching we can show that $y \in \mathbb{Z}_{\geq 0}^V$ and therefore the minimum is attained by some $y^* \in \{0,1\}^V$

Definition (unimodularity)**.** *A matrix A is called totally unimodular iff.*

$$\det(A') \in \{0, \pm 1\} \, \forall \textit{ square submatrices } A' \textit{ of } A$$

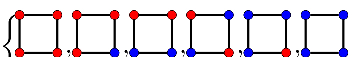
$$\begin{aligned}P(b) := \left\{x \in \mathbb{R}^d | Ax \leq b\right\} &\text{ has all integer vertices for all } b \in \mathbb{Z}^e \\ \Leftrightarrow A &\text{ is totally unimodular}\end{aligned}$$

[1] http://www.math.tu-berlin.de/~moehring/adm2/Chapter/chap4.pdf

Chapter 10

Pólya Theory – Counting With Symmetries

Example (Necklace)**.** *A necklace is a cyclic sequence of beads which can be flipped over. Let $N(n, m) = \#$ necklaces of length n with m colors for the beads.*

$N(4,2) = \left|\left\{\right.\right.$  $\left.\left.\right\}\right| = 6$

$$N(5,2) = 1+1+2+2+1+1 = 8$$
$$N(6,2) = 13$$
$$N(4,3) = 21$$

The group of symmetries is the dihedral group D_n.

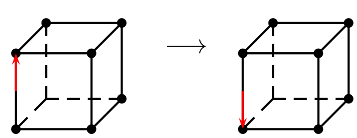
Example (Cube)**.** *How many essentially different colorings of the faces of a cube with m colors exist?*

$m = 2$

# red faces	*0*	*1*	*2*	*3*	*4*	*5*	*6*
# colorings	*1*	*1*	*2*	*2*	*2*	*1*	*1*

In total there are 10 *colorings of the cube with* 2 *colors.*

To get more informations about the group of symmetries we have a look at flags. A flag is a pair, consisting of an edge and an incident vertex. A pair of flags uniquely describes a group element.



The group of automorphisms of the group has 24 elements. As an abstract group $G_\sharp$ is isomorphic to S_4.

Here we consider actions of $G_\sharp$ on sets of size 6 (faces), 8 (vertices) and 12 (edges). So the same group can have different actions.

10.1 Permutation groups and the cycle index

G acting on m elements, G subgroup of S_m. $\pi \in G$ can be written as permutations, e. g. two row notation.

elements of $G_\sharp$		#	face type	vertex type	edge type
identity		1	$b_1 = 6$	$b_1 = 8$	$b_1 = 12$
fix a face	order 4	6	$b_1 = 2, b_4 = 1$	$b_4 = 2$	$b_4 = 3$
	order 2	3	$b_1 = 2, b_2 = 2$	$b_2 = 4$	$b_2 = 6$
fix a vertex	order 3	8	$b_3 = 2$	$b_1 = 2, b_3 = 2$	$b_3 = 4$
fix an edge		6	$b_2 = 3$	$b_2 = 4$	$b_1 = 2, b_2 = 5$

Table 10.1: Elements of $G_\sharp$

Example.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 6 & 7 & 1 & 5 & 2 & 4 \end{pmatrix}$$

This permutation has the following cycles:

$$\begin{tikzcd} 1 \arrow[r] & 3 \arrow[d] & & 6 \arrow[dl, shift left] & \\ 4 \arrow[u] & 7 \arrow[l] & 2 \arrow[ur, shift left] & & 5 \arrow[loop above] \end{tikzcd}$$

$$\Rightarrow \pi = (1374)(26)(5)$$

Definition (Type of a permutation)**.** *Let $\pi \in S_m$ be a permutation and b_i the number of cycles of length i in π. Then*

$$\mathit{type}(\pi) := (b_1, \ldots, b_m)$$

is called the type of π.

Example. *In our example the type is*

$$\mathit{type}(\pi) = (1, 1, 0, 1, 0, 0, 0)$$

Note that $\sum_{k=1}^{m} k b_k = m$.

Definition (Cycle Index of G)**.**

$$P_G(x_1 \ldots x_m) = \frac{1}{|G|} \sum_{\substack{g \in G \\ \mathit{type}(g) = (b_1, \ldots, b_m)}} x_1^{b_1} \ldots x_m^{b_m}$$

is called cycle index of G. It is a polynomial in the indeterminantes $x_1, \ldots, x_m$.

Example (Cycle index of $G_\sharp$)**.** *In 10.1 the types of the elements of $G_\sharp$ are listed. Therefore we get the cycle index for the action on the faces.*

$$P_{G_\sharp}(x_1, \ldots, x_6) = \frac{1}{24}(x_1^6 + 6x_1^2 x_4 + 3x_1^2 x_2^2 + 8x_3^2 + 6x_2^3)$$

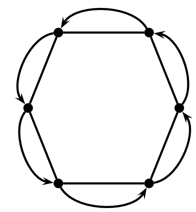
Next we look at S_m with its natural action on $[m]$ and use the notation $\overrightarrow{x}^b = x_1^{b_1} \ldots x_m^{b_m}$

$$\begin{aligned} P_{S_m}(\overrightarrow{x}) &= \frac{1}{m!} \sum_{b = (b_1 \ldots b_m) : \sum k b_k = m} (\#\text{ permutations of type } b)\, \overrightarrow{x}^b \\ &= \frac{1}{m!} \sum_{b = (b_1 \ldots b_m) : \sum k b_k = m} \frac{m!}{b_1! 1^{b_1} b_2! 2^{b_2} \ldots b_m! m^{b_m}} \overrightarrow{x}^b \end{aligned}$$

The intuition behind this is that we fill the elements into a frame of type b. Then $b_k!$ permutes cycles of length k and k^{b_k} chooses cycle leader in each cycle of length k. Now this is the coefficient of z^m in

$$\exp(x_1 z + \frac{x_2}{2} z^2 + \ldots + \ldots) = \prod_{k \geq 1} \sum_{n \geq 0} \frac{x_k^n}{n! k^n} z^{kn} = \sum_{m \geq 0} P_{S_m}(\overrightarrow{x}) z^m$$

Example (Cycle index of cyclic group of order n)**.**



The order of each group element is a divisor of n. The type is $b_d = \frac{n}{d}, d|n$

$$\Rightarrow P_{C_n}(\overrightarrow{x}) = \sum_{d|n} \varphi(d) x_d^{\frac{n}{d}}$$

is the cycle index of C_n. φ is Euler's Phi function:

$$\varphi(d) = |\{s < d : \gcd(s,d) = 1\}|$$

Some exercises:

1. Find $P_{D_n}(\overrightarrow{x})$, the cycle index of the dihedral group with $2n$ elements

2. Let G act on itself by right multiplication. Determine $P_G(\overrightarrow{x})$.

10.2 Lemma of Cauchy – Frobenius – Burnside

G acting on D induces an equivalence relation on D:

$$x \sim y \Leftrightarrow \exists g : g(x) = y$$

The obvious proof is left as an exercise.

The classes of this relation are called orbits. Let $O_G(D) = \{\text{orbits of the action}\}$.

Remark. *If $G = \langle \pi \rangle, \pi \in S_m$ then the orbits of G are just the cycles of π.*

For each $x \in D$ there is the stabilizer subgroup G_x of G:

$$G_x = \{g \in G : g(x) = x\}$$

Exercise: Show that this is a group.

For $g \in G$ there is the fixed set $Fix(g) = \{x \in D : g(x) = x\}$

Lemma 10.2.1 (Cauchy-Frobenius-Burnside)**.**

$$|O_G(D)| = \frac{1}{|G|} \sum_{g \in G} |Fix(g)|$$

Proof. Double counting of pairs (g, x) with $g(x) = x$ yields

$$\sum_{x \in D} |G_x| = \sum_{g \in G} |Fix(g)| \tag{10.1}$$

Let $G_{x \to y} = \{g \in G : g(x) = y\}$. Note:

$$\text{either } G_{x \to y} = \emptyset \text{ or } |G_{x \to y}| = |G_x| \text{ when } y \in orbit(x) \tag{10.2}$$

$$G = \dot{\bigcup_{y \in orbit(x)}} G_{x \to y} \tag{10.3}$$

10.1, 10.2, 10.3 together yield the result:

$$\begin{aligned}\sum_{g\in G}|Fix(g)| &\overset{10.1}{=} \sum_{x\in D}|G_x| \\ &\overset{10.3}{=} \sum_{x\in D}\frac{1}{|orbit(x)|}|G| \\ &= |G|\sum_{x\in D}\frac{1}{|orbit(x)|} \\ &= |G|\,|O_G\,(D)| \\ \Rightarrow |O_G\,(D)| &= \frac{1}{|G|}\sum_{g\in G}|Fix(g)|\end{aligned}$$

□

Example (Application: necklaces with two colors and rotational symmetry)**.** *A necklace is a sequence of 0's and 1's of length n and with rotational symmetry, i. e. it can be seen as an orbit of the action of C_n on $[2]^{[n]}$. Note that for each $d|n$ there are $\varphi(d)$ group elements of order d in C_n. From the lemma of Cauchy-Frobenius-Burnside 10.2.1 we then know:*

$$\begin{aligned}\#\ necklaces &= \frac{1}{n}\sum_{g\in C_n}|Fix(g)| \\ &= \frac{1}{n}\sum_{d|n}\varphi(d)\,(\#\ \text{0,1-sequences fixed by such a permutation}) \\ &= \frac{1}{n}\sum_{d|n}\varphi(d)2^{\frac{n}{d}}\end{aligned}$$

$2^{\frac{n}{d}}$ *is the number of possibilities to choose the value 0 or 1 for each cycle of the permutation.*

Example. *Colorings of the edges of a cube with three colors, for example red, green and blue. Let G be the group of the cube acting on edges. D is the set of $3-$colorings of the edges $[3]^{[12]}$ From the lemma we know:*

$$\begin{aligned}\#\ colorings &= |O_G(D)| \\ &= \frac{1}{24}\sum_{g\in G}|Fix(g)| \\ &= \frac{1}{24}\sum_{type\ t}(\#\text{permutations of type } t)\underbrace{(\#\text{colorings fixed by such a permutation})}_{=3^{\#\text{ cycles of the permutation}}} \\ &= \frac{1}{24}\underbrace{\left(1\cdot 3^{12}+6\cdot 3^3+3\cdot 3^6+8\cdot 3^4+6\cdot 3^7\right)}_{\text{data from table 10.1}} \\ &= 22815\end{aligned}$$

In general: consider a group acting on some set D. $f_1, f_2 \in R^D$ (R may be a set of colors) are G-indistinguishable if there exists some g such that $f_1 = f_2 \circ g$.

How many classes (orbits) are there?

$$\begin{aligned}\#\text{ classes } &= \frac{1}{|G|}\sum_{g\in G}|Fix(g)| \\ &= \frac{1}{|G|}\sum_{g\in G}|R|^{\#\text{ cycles}(g)} \\ &= \frac{1}{|G|}\sum_{g\in G}|R|^{\sum b_i} \\ &= P_G\left(|R|,|R|,\dots,|R|\right)\end{aligned}$$

Theorem 10.2.2 (Pólya I). *Let G be acting on D.*

$$\begin{aligned}&\#\ G\text{-equivalence classes of functions } D\to R, |R|=r \\ =&\#\ G\text{-indistinguishable } r\text{-colorings of } D \\ =&P_G(r,r,\dots,r)\end{aligned}$$

Example. *n indistinguishable balls and m distinguishable boxes (colors).*

$$P_{S_n}(m,\dots,m)=\frac{1}{n!}\sum_{(b_1\dots b_n):\sum kb_k=n}\frac{n!}{b_1!1^{b_1}b_2!2^{b_2}\dots b_n!n^{b_n}}m^{\sum b_k}=\binom{n+m-1}{m-1}$$

Definition (Pólya counting with weights).

- *$w(r)$ a weight for each $r\in R$*
- $w(R):=\sum_{r\in R}w(r)$
- *$f:D\to R$ then $w(f):=\prod_{x\in D}w\left(f(x)\right)$*

Lemma 10.2.3 (weight distribution on R^D).

$$\sum_{f\in R^D}w(f)=w(R)^{|D|}$$

Proof.

$$\begin{aligned}w(R)^{|D|}&=\left(\sum_{r\in R}w(R)\right)^{|D|} \\ &=\prod_{d\in D}\left(\sum_{r\in R}w(r)\right)\end{aligned}$$

Multiplying this out makes it clear that each summand corresponds to a function and has value $w(f)=\prod_{x\in D}w(f(x))$. ☐

Claim 10.2.4. *If f_1,f_2 belong to the same orbit of the G-action then $w(f_1)=w(f_2)$.*

Proof.

$$\begin{aligned}f_1\sim f_2 &\Leftrightarrow \exists g\in G: f_1=f_2\circ g \\ w(f_1)&=\prod_{x\in D}w\left(f_1(x)\right) \\ &=\prod_{x\in D}w\left(f_2\left(g(x)\right)\right) \\ &=\prod_{y=g(x),x\in D}w\left(f_2(y)\right) \\ &=\prod_{y\in D}w\left(f_2(y)\right)=w(f_2)\end{aligned}$$

☐

Hence for an orbit F of the G-action on R^D we can define $w(F) = w(f)$ for arbitrary $f \in F$. The goal is to find $\sum_{F \text{ orbit}} w(F)$

Let Λ be a partition of D. Consider the set $S \subseteq R^D$ of all f that are constant on blocks of Λ.

Lemma 10.2.5 (Partition lemma)**.** *If Λ consists of k blocks with sizes $\lambda_1, \dots, \lambda_k$ then*

$$\sum_{f \in S} w(f) = \prod_{i=1}^{k} \left(\sum_{r \in R} w(r)^{\lambda_i} \right)$$

Proof.

1. there exists a bijection between $f \in S$ and $R^{[k]} \ni g$. $g(i)$ is the value taken by f on each element of the block B_i of Λ, $i = 1, \dots, k$
2. From the lemma 10.2.3 we obtain

$$\begin{aligned}
\sum_{g \in R^{[k]}} w(g) &= w(R)^k \\
&= \prod_{i=1}^{k} \left(\sum_{r \in R} w(r) \right) \\
\Rightarrow \sum_{f \in S} w(f) &= \prod_{i=1}^{k} \left(\sum_{r \in R} w(r)^{|B_i|} \right) \\
&= \prod_{i=1}^{k} \left(\sum_{r \in R} w(R)^{\lambda_i} \right)
\end{aligned}$$

□

Definition.

$$w^{\lambda}(R) := \sum_{r \in R} w(r)^{\lambda}$$

Theorem 10.2.6 (Pólya II – Fundamental theorem)**.**

$$\sum_{F \text{ orbit}} w(F) = P_G\left(w(R), w^2(R), \dots, w^{|D|}(R) \right)$$

Remark. *If $w(r) = 1 \forall r \in R$ then $w^{\lambda}(R) = |R|$. Also*

$$w(F) = w(f) = \prod_{x \in D} w(f(x)) = 1$$

for some $f \in F$ and

$$\sum_{F \text{ orbit}} w(F) = \#\ \textit{orbits} = P_G\left(|R|, \dots, |R|\right)$$

that is we get 10.2.2 in this case.

Proof. Consider $f \in R^D$ and let $w = w(f)$.

$$S_w := \left\{ f \in R^D | w(f) = w \right\}$$

We collect some properties of S_w:

- S_w contains all functions f' from the orbit of f
- S_w is a union of orbits
- S_w is invariant under the G-action on R^D

Consider G acting on S_w. From the lemma 10.2.1 we obtain that

$$\begin{aligned}\#\text{ orbits of } G\text{-action on } S_w &= \frac{1}{|G|}\sum_{g\in G}|Fix(g)|\\ &= \frac{1}{|G|}\sum_{g\in G}\#f\in S_w \text{ fixed by } g\end{aligned}$$

Hence

$$\begin{aligned}\sum_{\text{F orbit}} w(F) &= \sum_{w \text{ weights of functions}} w\left(\#\text{ orbits of } R^D \text{ with weight } w\right)\\ &= \sum_w w(\#\text{ orbits in } S_w)\\ &= \sum_w w\frac{1}{|G|}\sum_g \#f\in S_w \text{ fixed by } g\\ &= \frac{1}{|G|}\sum_g \sum_{f \text{ fixed by } g} w(f)\end{aligned}$$

$$f \text{ fixed by } g \Leftrightarrow f \text{ is constant on each cycle of } g$$

If the type of g is $(b_1b_2\dots b_m)$ we can find an explicit expression for $\sum_{f \text{ fixed by } g} w(f)$ by applying the partition lemma 10.2.5.

$$\begin{aligned}\sum_{f \text{ fixed by } g} w(f) &= \prod_{i=1}^m\left(\sum_{r\in R} w(r)^i\right)^{b_i}\\ &= \prod_{i=1}^m w^i(R)^{b_i}\\ \Rightarrow \sum_{F \text{ orbit}} w(F) &= \frac{1}{|G|}\sum_g\left(\prod_{i=1}^m w^i(R)^{b_i}\right)\\ &= P_G(w(R), w^2(R), \dots, w^{|D|}(R))\end{aligned}$$

□

Example (Group actions).

- *Any permutation π can be written as a product of cycles, where the cycles are the orbits of the permutation.*
- *$G = \mathbb{Z}^2$ acting on $D = \mathbb{Z}^2$ by translation:*

$$g_{(x,y)}(a,b) := (a+x, b+y)$$

  if $gcd(x,y) = 1$ an orbit of $g(x,y)$ consists of all points on a straight line with slope x, y in $\mathbb{Z}^2$.

- *Edge colorings of the cube. Let $w(red) = x, w(blue) = y, w(green) = z$. Then*

$$\sum_{F\ orbit} w(F) \overset{10.2.6}{=} P_G(x+y+z, x^2+y^2+z^2, \dots, x^{12}+y^{12}+z^{12})$$

$$P_G(x_1,\dots,x_{12}) = \frac{1}{24}(x_1^{12}+6x_4^3+3x_2^6+8x_3^4+6x_1^2x_2^5)$$

$$\begin{aligned}\sum_{F\ orbit} w(F) =& \frac{1}{24}\big((x+y+z)^{12}+6(x^4+y^4+z^4)^3+3(x^2+y^2+z^2)^6\\ &+8(x^3+y^3+z^3)^4+6(x+y+z)^2(x^2+y^2+z^2)^5\big)\end{aligned}$$

  The coefficient of $x^4y^4z^4$ in the last expression is the number of edge colorings with 4 edges of each color.

Chapter 11

Linear Extensions And Dimension Of Posets

Definition. *$P = (X, <_P)$ a poset. A total order $L = (X, <_L)$ is a linear extension of P iff.*

$$x <_P y \Rightarrow x <_L y$$

Claim 11.0.7. *Every poset has a linear extension*

Proof. We give a generic algorithm LinExt(P) for linear extensions

Input: Poset $(P, <_P)$.
Output: Linear extension $(L, <_L)$.
$L := []$.
while $P \neq \emptyset$ do:
  choose $x \in \min(P)$.
  $L := L + x$.
  $P := P - x$.
return L.

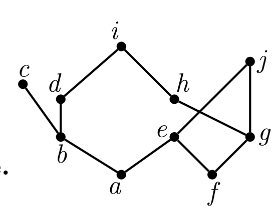
□

The output of LinExt(P) is a linear extension of P. We have to show two facts:

- it is a total order. This is clearly true.
- it extends P.

  Proof. If $x <_P y$ then x has to be taken before y becomes a minimum of P. Therefore we get $x <_L y$. □

Example.



Which solution the algorithm gives depends on the choice of the minimum. One possible solution is shown in the tabular below.

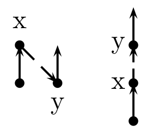
Step	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
$\min(P)$	*a,f*	*a,g*	*b,e,g*	*b,g*	*c,d,g*	*c,g*	*c,h,j*	*h,j*	*i,j*	*j*
L	*f*	*a*	*e*	*b*	*d*	*g*	*c*	*h*	*i*	*j*

Proposition 11.0.8. *If $x||_P y$ (i. e. x and y are incomparable) then there exists a linear extension with $x <_L y$.*

Proof. In the choose line of the algorithm avoid taking y when possible. When y is taken $\min(P) = \{y\}$. Therefore $P = Up(y) = \{z \in X : y \leq_P z\}$. Since $x \ni Up(y)$ it has been taken before. □

Second proof. $x||_P y \Rightarrow P + (x, y)$ is acyclic because otherwise $y <_P z_1 <_P z_2 <_p \ldots <_P z_k <_P x$ implies that y and x are comparable by transitivity $y <_P x$.

$trans(P + (x, y))$ denotes the transitive closure of $P + (x, y)$. The transitive closure is also a poset so it has a linear extension L. In $L, x <_L y$ and L is also a linear extension of P.

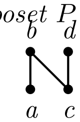


□

Consequence: Every poset P is the intersection of its linear extensions:

$$P = \bigcap_{L \in \mathcal{L}(P)} L$$

Example. *The given poset P has 5 different linear extension:*



acbd	*cabd*	*acdb*	*cadb*	*cdab*
L				L'

P is the intersection of two linear extensions: $P = L \cap L', \dim(N) = 2$. It is clear that if P is not a total order we do not need all linear extensions to get the poset as their intersection.

Definition (Realizer)**.** *A subset R of the set $\mathcal{L}(P)$ of all linear extensions of P is called a realizer of P iff. $P = \bigcap_{L \in R} L$*

Definition (Dimension)**.** *The dimension of P is the minimum cardinality of a realizer of P.*

Let $P = (X, <_P), Q = (Y, <_Q)$ be posets.

Definition (Order homomorphism, order embedding)**.** *A map $\varphi : X \longrightarrow Y$ is an order homomorphism iff.*

$$x <_P y \Rightarrow \varphi(x) <_Q \varphi(y)$$

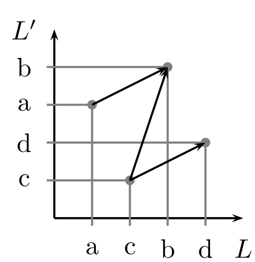
An order embedding is an order homomorphism with

$$x||_P y \Rightarrow \varphi(x)||_Q \varphi(y)$$

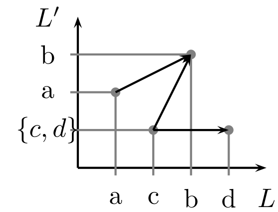
The dimension $\dim(P) = t$ if t is the smallest integer such that there exists an order embedding $P \longrightarrow \mathbb{R}^t$. $\mathbb{R}^t$ is taken with the dominance order

$$(x_1, \ldots, x_t) \leq_{dom} (y_1, \ldots, y_t) \Leftrightarrow x_i \leq y_i \forall i = 1 \ldots t$$

An order embedding of N into $\mathbb{R}^2$ is given in the picture on the right. A point in the picture is connected to another one when the hook is completely under the other one, i. e. if they do not intersect.



Any realizer yields an order embedding. Order embedding projections in the fiber can have sets instead of single elements. Some extra care for the order in the corresponding linear order is required so that they form a realizer.



11.1 Dimension of $\mathcal{B}_n$

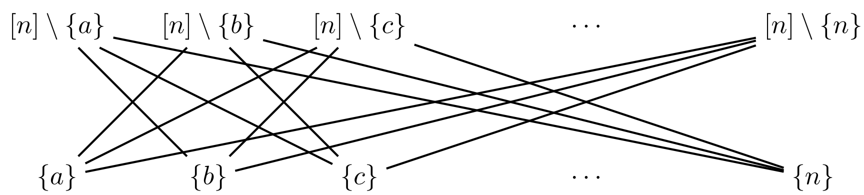
Let $\varphi : \mathcal{B}_n \longrightarrow \{$ vertices of the n-cube H_n $\}$ be an order embedding of $\mathcal{B}_n \longrightarrow H_n \longrightarrow \mathbb{R}^n$.

$$\dim(\mathcal{B}_n) \leq n$$

Our goal is to show equality $\dim(\mathcal{B}_n) = n$.

Lemma 11.1.1 (Monotonicity of dimension)**.** *Let $P' = (X', <_{P'})$ be an induced subposet of $P = (X, <_P)$, i. e. $X' \subset X$ and $x' <_{P'} y' \Leftrightarrow x' <_P y'$. Then* $\dim(P') \leq \dim(P)$

Proof. Restrict a realizer or an order embedding of P to P'. Restrict $\mathcal{B}_n$ to 1- and $(n-1)$-subsets of $[n]$.



This is S_n, the standard example for n-dimensional posets. Consider the incomparable pairs $(i, \bar{i})$ for $i = 1, \ldots, n$.

Claim 11.1.2. *For all linear extensions L of S_n there is at most one $i \in [n]$ such that $\bar{i} <_L i$.*

Proof. Suppose $\bar{i} <_L i$ and $\bar{j} <_L i$. Since $i \in \bar{j}$ we have $i <_{S_n} \bar{j}$ and $j \in \bar{i}$ yields $j <_{S_n} \bar{i}$.
Therefore we have $\bar{i} <_L i <_L \bar{j} <_L j <_L \bar{i}$ which is a contradiction. $\square$

A realizer must contain an L_i with $\bar{i} <_{L_i} i \forall i$. From the claim 11.1.2 we know that these realizers are different.

$$\Rightarrow |R| \geq n, R \text{ is a realizer of } S_n$$
$$\Rightarrow \dim(S_n) \geq n$$

Monotonicity gives us $\dim(\mathcal{B}_n) \geq \dim(S_n) \geq n$ and as we already know $\dim(\mathcal{B}_n) \leq n$ this gives us equality. $\square$

Proposition 11.1.3. *Let $P = (X, <_P)$ be a poset. Then*

$$\dim(P) \leq width(P) = \max |A|$$

This proposition gives us another bound for the dimension of posets. Some results are already known and can be sharpened:

$$width(S_n) = n \qquad width(\mathcal{B}_n) = \binom{n}{\frac{n}{2}} \text{ (Sperner 8.2.1)}$$

Proof. Dilworth's theorem (9.0.2) gives us a chain partition $C_1, \dots, C_w, w = width(P)$. If x and y are incomparable ($x||y$) then they appear in different chains. For each C_i construct a linear extension L_i with the property that $x||y$ and $x \in C_i$ implies $y <_{L_i} x$.

Such a set $L_1, \dots, L_w$ is a realizer $\bigcap_{i=1}^{w} L_i = P$.

To get the linear extension for C_i we run the generic algorithm for L_i and avoid taking elements of C_i whenever possible. □

Remark. *Computing the dimension is $\mathcal{NP}$-complete, i. e. it is hard to decide if* $\dim(P) \leq k$ *for* $k \geq 3$.

Having a small realizer can help in computational tasks (e. g. efficient embedding of P).

11.2 2-dimensional posets

Definition. *A conjugate of $P = (X, <_P)$ is a poset $Q = (X, <_Q)$ such that $x||_P y \Leftrightarrow x \sim_Q y$, i. e. $x <_Q y$ or $y <_Q x \forall x||_P y$*

Definition. *A linear extension L of P is non-separable iff.*

$$L = \dots u \dots v \dots w \dots \text{ and } u <_P w \Rightarrow u <_P v \text{ or } v <_P w \text{ or both}$$

Theorem 11.2.1. *For $P = (X, <_P)$ a poset is equivalent:*

1. $\dim(P) \leq 2$
2. *there exists a conjugate Q of P*
3. *P has a non-separating linear extension*

Proof. **1 → 2** L, L' realizer, $Q = L' \cap \overline{L}$ where $\overline{L}$ is the reverse of L.

Claim 11.2.2. *Q is a conjugate.*

Proof. $x||_P y$ and $x <_{L'} y$ then $x <_L y$ since $y <_{\overline{L}} x \Rightarrow x <_Q y$. $x <_P y$ then $x <_{L'} y$ and $x <_L y$. Hence $y <_{\overline{L}} x \Rightarrow x||_Q y$ □

2 → 3 Consider $P \cup Q$ (two different edge colors) as orientation on X. Observations:

- $\forall i, j \exists$ arrow $i \rightarrow j$ or $j \rightarrow i$
- the directed graph is acyclic because otherwise we can reduce a cycle down to a triangle and ≥ 2 of the arrows have the same color. A third one would then follow by transitivity $\Rightarrow$ contradiction.
- a directed complete acyclic graph is a total order which is non-separating because $L = \dots u \rightarrow v \rightarrow w \dots$ together with an arrow $u \rightarrow w$ in the other color can not be because that one would also be in Q (by transitivity) but Q is the conjugate of P.

3 → 1 Idea: See $\overline{L}$ as a preference list in the generic algorithm. The resulting linear extension L' forms a realizer together with L.

□

Corollary.

$$\dim(P) = 2 \Leftrightarrow \textit{cocomparability graph of } P \textit{ admits a transitive order}$$

11.3 Algorithms to find antichain partitions/chain partitions

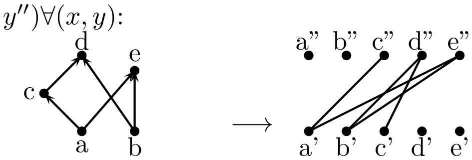
Let P be general.

minimum antichain partition Algorithm: identify minimal elements and remove them. This gives us a layered sequence $(A_i)_i$ of antichains $A_i = \min\left(P \setminus \bigcup_{j<i} A_j\right)$

$\Rightarrow$ the sequence is minimal because backwards we find a chain intersceting each A_i.

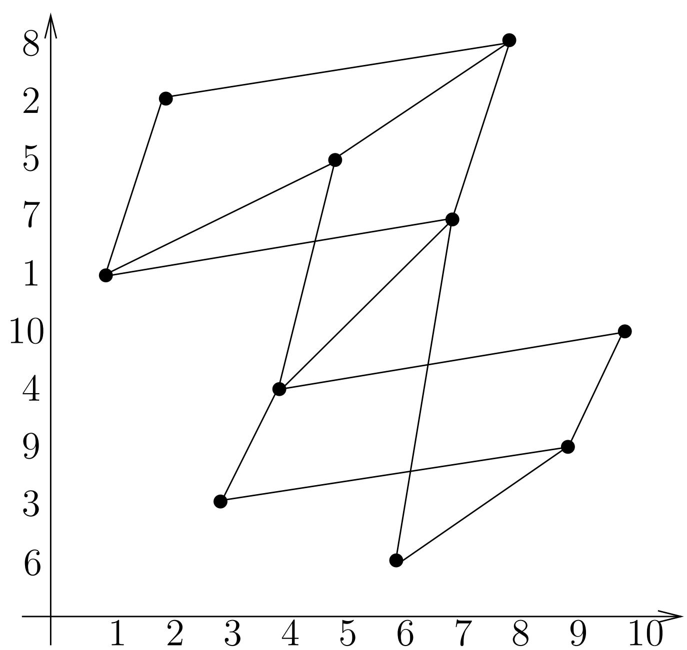
Complexity of the algorithm is $O(m+n)$ where m is the number of edges and n the number of elements in P.

minimum chain partition Convert the poset into a bipartite graph by doubling every element and inserting $(x', y'')\forall(x, y)$:



Finding a maximum matching has complexity $O(\sqrt{n}m)$, can be improved to $O(wn^2)$ where w is the width of the poset.

11.4 2-dimensional picture for 2-dimensional posets



Sweep through the picture from left to right. Every line corresponds to an antichain. How do we do the sweep? Insert a new point in the lowest of the available antichains. In the area to the left bottom of x there are all predecessors of x. This algorithm finds the canonical antichain partition $A_i = \min\left(P \setminus \bigcup_{j<i} A_j\right)$.

Complexity: insert each of n points at correct place in a list of $\leq height(P) =: h$ values.

$$O(n \log h) \leq O(n \log n)$$

Reuse the picture above to get a minimum antichain partition of P. This is possible as this is a minimum antichain partition of the conjugate Q. Therefore the algorithm for antichain partitions can be reused.

11.5 The Robinson-Shensted-Correspondence

This chapter is related to the representation theory for the symmetric group.

Definition (Young Tableaux)**.** *A young tableaux is a shape of a partition of n together with a filling of the numbers $1, \dots, n$.*

Example. *Young tableaux for $n = 15$*

$$\begin{array}{llll} 1 & 2 & 3 & 10 \\ 4 & 5 & 11 & \\ 6 & 8 & 14 & \\ 7 & 13 & & \\ 9 & 15 & & \\ 12 & & & \end{array}$$

Theorem 11.5.1.

$$\exists \textit{bijection } : \Pi \in S_n \longleftrightarrow (P, Q) \textit{ pair of young tableaus of same shape}$$

Example.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 3 & 9 & 4 & 10 & 1 & 7 & 5 & 2 & 8 \end{pmatrix}$$

corresponds to $P_\pi = \begin{array}{llll} 1 & 2 & 5 & 8 \\ 3 & 4 & 10 & \\ 6 & 7 & & \\ 9 & & & \end{array} \qquad Q_\pi = \begin{array}{llll} 1 & 3 & 5 & 10 \\ 2 & 4 & 7 & \\ 6 & 8 & & \\ 9 & & & \end{array}$

How does the bijection work? R-S-bumping: Insert elements of the permuatation from left to right.

$$\boxed{6} \xrightarrow{\text{3 bumps 6}} \begin{array}{l} 3 \\ 6 \end{array} \xrightarrow{9} \begin{array}{ll} 3 & 9 \\ 6 & \end{array} \xrightarrow{\text{4 bumps 9}} \begin{array}{ll} 3 & 4 \\ 6 & 9 \end{array} \xrightarrow{10} \begin{array}{lll} 3 & 4 & 10 \\ 6 & 9 & \end{array}$$

$$\xrightarrow{\text{1 bumps 3 bumps 6}} \begin{array}{lll} 1 & 4 & 10 \\ 3 & 9 & \\ 6 & & \end{array} \xrightarrow{\text{7 bumps 10}} \begin{array}{lll} 1 & 4 & 7 \\ 3 & 9 & 10 \\ 6 & & \end{array} \xrightarrow{\text{5 bumps 7 bumps 9}} \begin{array}{lll} 1 & 4 & 5 \\ 3 & 7 & 10 \\ 6 & 9 & \end{array}$$

$$\xrightarrow{\text{2 bumps 4 bumps 7 bumps 9}} \begin{array}{lll} 1 & 2 & 5 \\ 3 & 4 & 10 \\ 6 & 7 & \\ 9 & & \end{array} \xrightarrow{8} \begin{array}{llll} 1 & 2 & 5 & 8 \\ 3 & 4 & 10 & \\ 6 & 7 & & \\ 9 & & & \end{array}$$

Q_π is the growth diagram of P_π, i. e. it tells us in which order the cells were added.

$$\begin{array}{llll} 1 & 3 & 5 & 10 \\ 2 & 4 & 7 & \\ 6 & 8 & & \\ 9 & & & \end{array}$$

How do we get from P and Q to $\Pi_{P,Q}$?

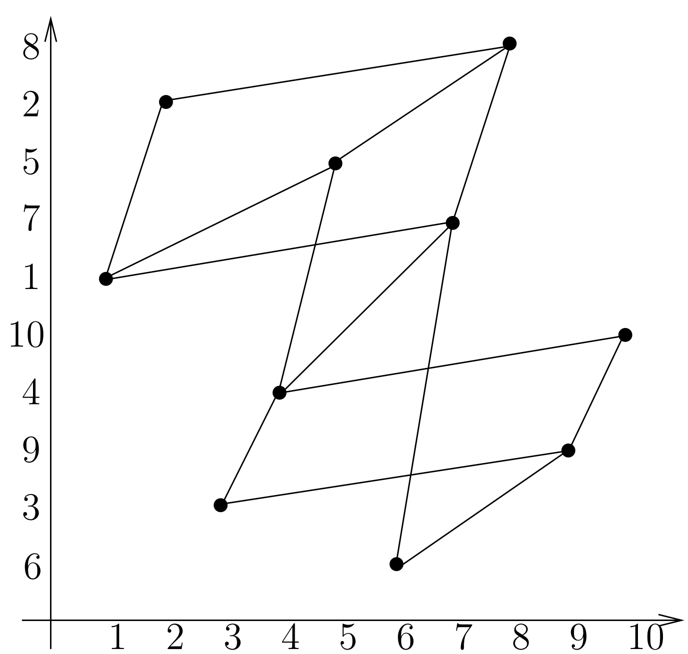
- *The 8 was the last number to be inserted.*
  - *It was in the first row of P*
  - $\Rightarrow \pi = (\dots 8)$
- *The 9 was inserted after the 8*
  - *It was bumped out by the biggest number in the row before that is smaller than 9*
  - $\Rightarrow$ *the 9 was bumped out by the 7*

  - *The 7 was bumped out by the 4*
  - *The 4 was bumped out by the 2*
  - $\Rightarrow \pi = (\ldots 2, 8)$

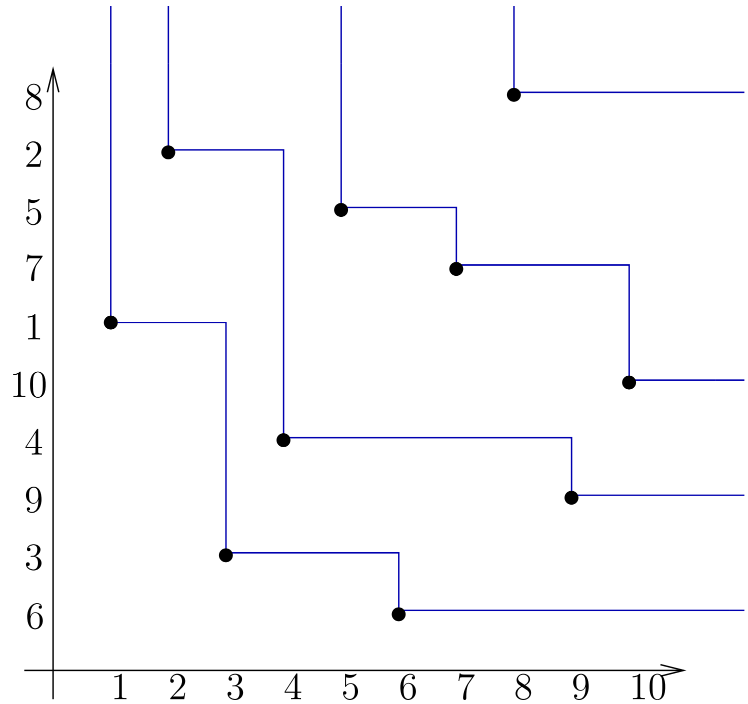
- *go on like this to construct the whole permutation.*

11.6 Viennot's skeletons

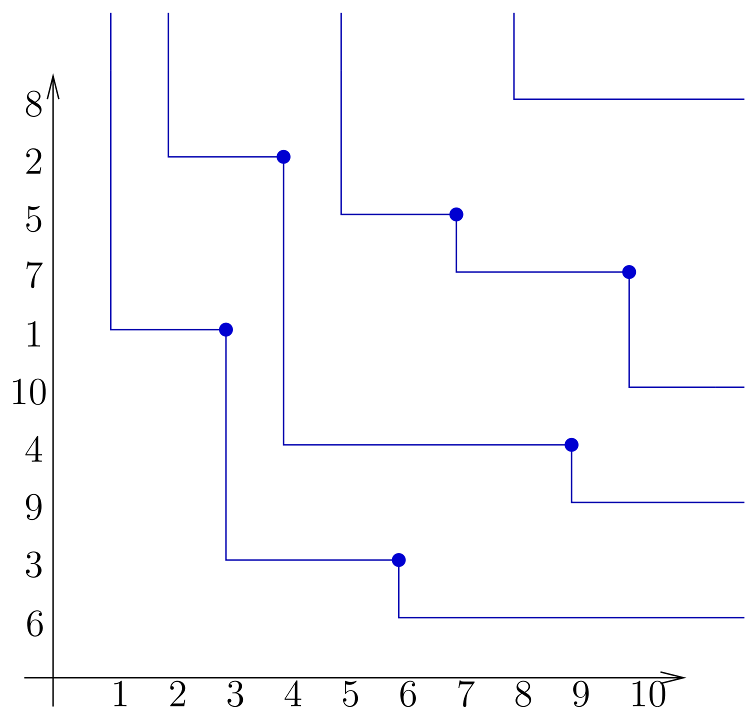
We can embed a 2-dimensional poset P into the 2-dimensional plane like in the following picture:



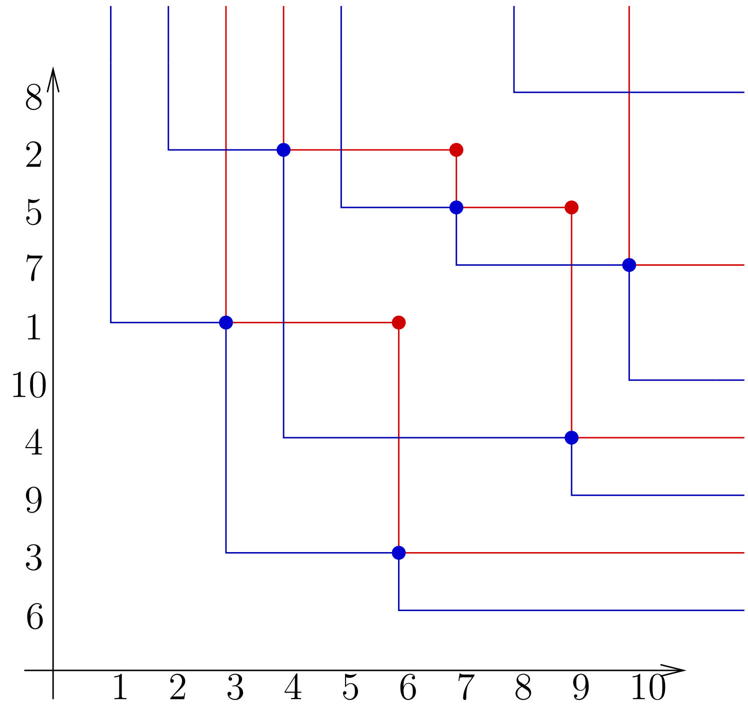
Via sweeping from left to right we get the skeleton $S(P)$:



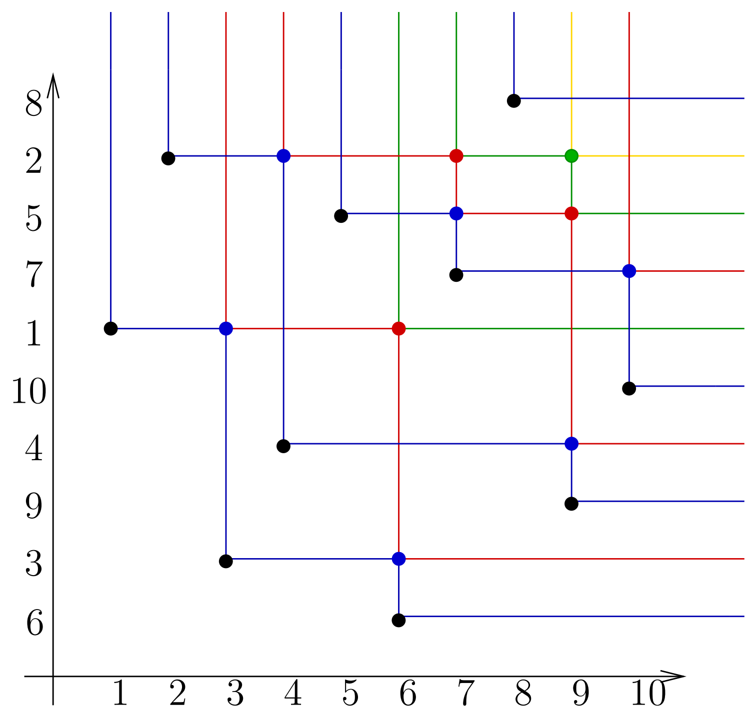
We recognize that $height(S(P)) \leq height(P)$. The canonical antichains of P yield an antichain partition of $S(P)$). After this we can mark the steps in the blue lines with dots (socalled staircases).



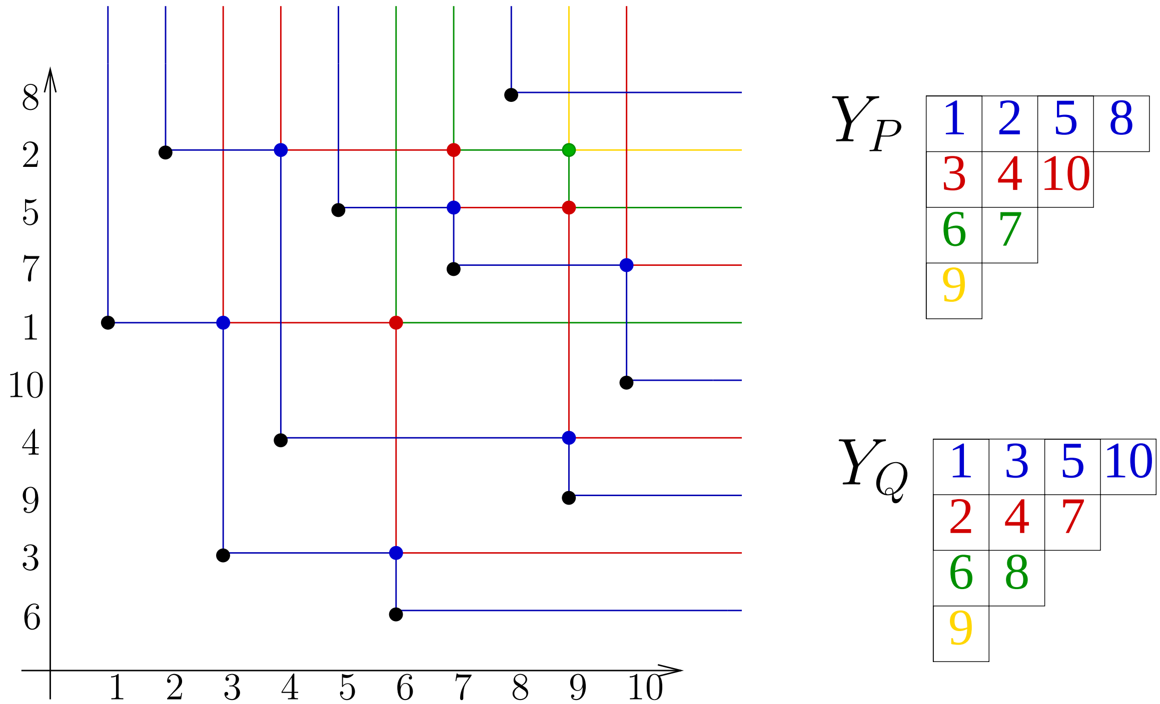
And then we continue the procedure to get the skeleton of the skeleton $S(S(P))$.



After doing this several times we get the picture below with four different skeletons:

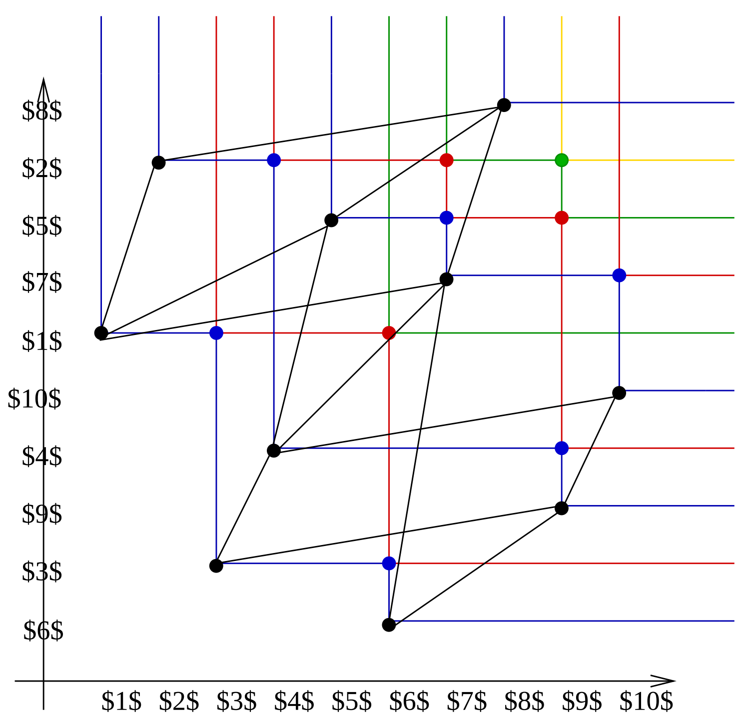


On the top of this picture we can read P_π from left to right and Q_π on the right from the bottom to the top.



The two young tableaus have the property that their entries are monotone increasing in rows (rather obvious) and monotone increasing in columns (which is slightly more complicated to show)

The other way round we can construct the skeletons from the given young tableaus as they give us the colorings on the top and on the right. Therefore we can obtain the original poset backwards. The complete picture is shown below.



Given P define

$$\begin{aligned}
c_i(P) &= \text{maximal \# elements that can be covered by} i \text{ chains}\\
c_1(P) &= height(P)\\
c_1(P) &< c_2(P) < \ldots < c_w(P) = n, w = width(P)\\
a_j(P) &= \text{ \# elements that can be covered by } j \text{ antichains}\\
a_1(P) &< a_2(P) < \ldots < a_h(P) = n
\end{aligned}$$

Example.

c_1	c_2	c_3	c_4	a_1	a_2	a_3	a_4
4	*7*	*9*	*10*	*4*	*7*	*9*	*10*

These are the values for the running example of this chapter.

Definition (k-chain)**.** *Let $P = (X, <)$ be a poset. A k-chain of P is a disjoint union of k chains in P.*

Example ($\mathcal{B}_3$)**.**

- 1-*chains are "'normal"' chains, they have size* ≤ 4
- 2-*chains have size* ≤ 6
- 3-*chains have size* ≤ 7

Theorem 11.6.1 (Greene-Kleitman)**.** *There exists a partition $\lambda(P)$ such that*

$$\begin{aligned}
a_j &= \# \textit{ cells in the first } j \textit{ rows of } \lambda(P)\\
c_i &= \# \textit{ cells in the first } i \textit{ leftmost columns of } \lambda(P)
\end{aligned}$$

Theorem 11.6.2. *If $\dim(P) = 2$ and P is relabeled such that (id, π) is a realizer of P than $\lambda(P)$ and P_π have the same shape.*

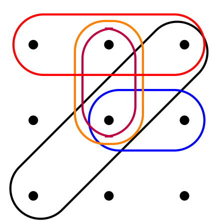
Chapter 12

Design Theory

Definition (Hypergraph)**.** *A hypergraph is a tuple $\mathcal{H} = (V, \mathcal{B})$ where V is a set of vertices and $\mathcal{B}$ is a set of hyperedges (multisubset of $Pot(V)$).*

Remark. *As an alternative we can see a hypergraph as an incidence structure $(\mathcal{P}, I, \mathcal{B})$ where $\mathcal{P}$ equals the set of points, $I \subseteq \mathcal{P} \times \mathcal{B}$ is an incidence relation and $\mathcal{B}$ is the set of blocks (lines)*

Example. $|V| = 9, |\mathcal{B}| = 5$



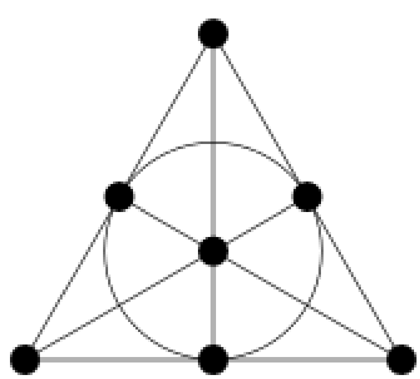
Definition (design)**.** *$\mathcal{S} = (\mathcal{P}, I, \mathcal{B})$ is a design with parameters λ, t, k, v iff.*

- $|\mathcal{P}| = v$
- $\forall B \in \mathcal{B} : |B| = k$
- $\forall T \subseteq \mathcal{P}$ *with* $|T| = t$ *there are exactly* λ *blocks* $B \in \mathcal{B}$ *with* $T \subseteq B$

We write $S_\lambda(t, k, v)$ for such a design. (S comes from Steiner system)

Example. (1) $(V, \binom{V}{k})$ *is a* $S_\lambda(t, k, v)$ *for* $v = |V|$ *and all* $t \leq k$ *and* $\lambda_t = \binom{v-t}{k-t}$*. These are called trivial designs.*

(2) *Fano plane:* $v = 7, k = 3, t = 2, \lambda = 1$



(3) $\mathcal{P} = \mathbb{F}_2^n, v = 2^n$

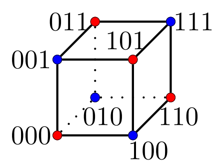
Blocks are of the form $\{x, y, z, x+y+z\} \forall x, y, z$ *different in* $\mathbb{F}_2^n \Rightarrow |B| = 4 \forall B \in \mathcal{B}$.

Every block is uniquely determined by any subset of 3 *elements. Therefore* $(\mathcal{P}, \mathcal{B})$ *is a* $S_1(3, 4, 2^n)$

(4) *Let* $(\mathcal{P}, \mathcal{B})$ *be the* $S_1(3, 4, 8)$ *from the previous example. Let* $p \in \mathcal{P}$ *and consider* $(\mathcal{P} \setminus p, \mathcal{B}_p$ *with* $\mathcal{B}_p = \{B \setminus p : p \in B \in \mathcal{B}\}$ *This is a* $S_1(3, 4, 7)$ *and this is the Fano plane*

Remark. $\lambda = 1$ *is often omitted:* $S(t, k, v) = S_1(t, k, v)$

Example. *The* $S(3, 4, 8)$ *in geometric view*



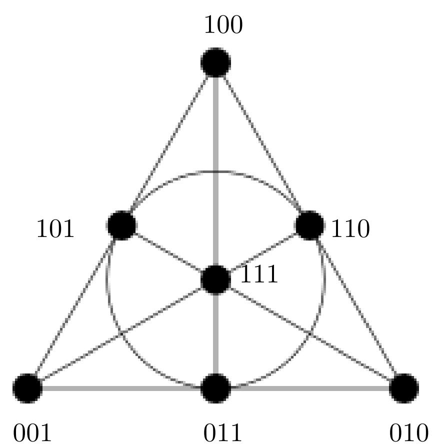
The blocks are (all of size 4*):*

- 6 *sides of the cube*
- 6 *complementary pairs of edges (opposite vertices sum to* 111*)*
- 2 *independent sets of vertices (marked in blue and red)*

These are all blocks as $|\mathcal{B}| \binom{k}{t} = \binom{v}{t} \lambda$ *(double counting)*

$\Rightarrow 4\,|\mathcal{B}| = \frac{8 \cdot 7 \cdot \cancel{6}}{\cancel{3 \cdot 2} \cdot 1} \Rightarrow |\mathcal{B}| = 14$

Example (Vectors for Fano plane)**.** $p = (000)$



Questions in design theory, given (λ, t, k, v):

- Is there a $S_\lambda(t, k, v)$?
- Structure of designs with these parameters?
- How many designs with these parameters are there (with respect to equivalencies)?

Remark. *Questions of existence can be translated into an integer programming problem: Consider a matrix* A *with*

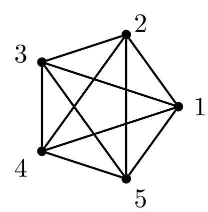
- $\binom{v}{t}$ *rows correspoding to t-subsets of V*
- $\binom{v}{k}$ *columns corresponding to k-subsets of V*
- $A_{T,K} = \begin{cases} 1 & T \subseteq K \\ 0 & \textit{otherwise} \end{cases}$
- $Ax = \lambda \mathbb{1}$

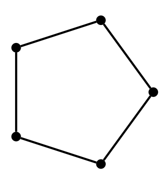
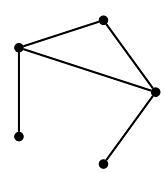
A solution x is a fractional design. If the entries of x are in $\mathbb{N}$ we get a design.

One approach to reduce the size of the IP (integer program) is to look at orbits of some group action.

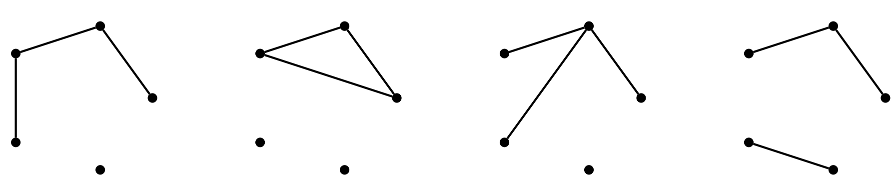
12.1 Construction of an $S_6(3,5,10)$

V = edges of the complete graph on 5 vertices, $|V| = 10$

For the blocks we take all instances of 

 and 

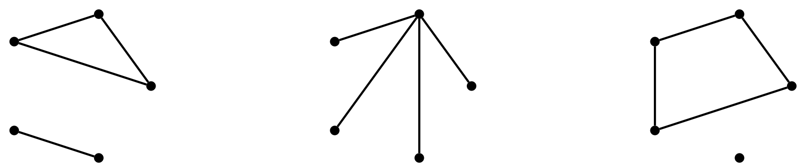
As rotations and reflections are equivalent there are $\frac{120}{10} = 12$ blocks of the first type.
For the second type there are $\binom{5}{3} \cdot 3 \cdot 2 = 60$ blocks.
The possible 3-sets are (these are all)



For every 3-set shown above there are 6 possibilities to complete them to one of the two blocks.

12.2 Construction of $S(3,4,10)$

Again we use the orbits of the action of S_5 on the edges of the complete graph with 5 vertices. For the blocks (of length 4) we take the orbits of



The first block occurs $\binom{5}{3} = 10$ times, the second one appears 5 times and for the last one there are $5 \cdot \frac{4!}{8} = 15$ possibilities.

How often do the 3-sets occur in these blocks? There is always exactly one possibility to construct one of the blocks by adding an edge.

These are all possibilities because we know the number of blocks:

$$\begin{aligned} b\binom{k}{t} &= \binom{v}{t} \cdot \lambda \\ b \cdot 4 &= \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} \cdot 1 \\ \Rightarrow b &= 30 \end{aligned}$$

12.3 Necessary conditions for the existence of designs

If a $S_\lambda(t, k, v)$ exists than it has

$$b = \lambda \cdot \frac{\binom{v}{t}}{\binom{k}{t}}$$

blocks. Therefore b must be an integer.

Example. *For an $S(2, k, v)$ to exist we must have*

$$\begin{aligned} \frac{\binom{v}{2}}{\binom{k}{2}} &\in \mathbb{N} \\ \frac{\binom{v}{2}}{\binom{k}{2}} &= \frac{v(v-1)}{k(k-1)} \in \mathbb{N} \\ \Rightarrow v(v-1) &= 0 \ mod \ k(k-1) \end{aligned}$$

If $k = 3$ then for $v = 5$ mod 6 there is no $S(2, 3, v)$.

Consider $J \subseteq V, |J| = j < t$. Let $\lambda(J) = \#\{B : J \subseteq B, B \in \mathbb{B}\}$

Double counting pairs (T, B) with $J \subseteq T \subseteq B$, B block, $|T| = t$:

$$\begin{aligned} \sum_{T:J\subseteq T} \sum_{B:T\subseteq B} 1 &= \sum_{B:J\subseteq B} \sum_{T:J\subseteq T\subseteq B} 1 \\ \Rightarrow \sum_{T:J\subseteq T} \lambda &= \sum_{B:J\subseteq B} \binom{k-j}{t-j} \\ \Rightarrow \binom{v-j}{t-j} \cdot \lambda &= \lambda(J)\binom{k-j}{t-j} \end{aligned}$$

Remark. *$\lambda(J)$ is counting something therefore it has to be a natural number. It also is independent of J, it only depends on $|J|$*

Proposition 12.3.1. *If a $S_\lambda(t,k,v)$ exists then for $0 \le j \le t$ we know that*

$$\lambda_j = \lambda \cdot \frac{\binom{v-j}{t-j}}{\binom{k-j}{t-j}} \in \mathbb{N}$$

Corollary. *A $S_\lambda(t,k,v)$ is a $S_{\lambda_j}(j,k,v)$ for $j < t$.*

Example (Conditions for $S(2,3,v)$)**.** *With $j = 1$ we get*

$$\frac{\binom{v-1}{1}}{\binom{3-1}{1}} = \frac{v-1}{2} \in \mathbb{N}$$

Therefore v must be odd. As we have seen before $v = 5$ mod 6 is impossible.

$$\Rightarrow v = 1, 3 \ mod \ 6$$

12.4 Steiner triple systems

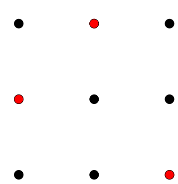
Definition. *A $STS(v)$ is a $S_1(2,3,v)$.*

Problem (Kirkman's problem, 1850)**.** 15 *school girls walk each day in* 5 *groups of* 3*. Arrange the girls walk for a week such that each pair of girls walk together in a group just once.*

Groups are blocks of size $3, t = 2, \lambda = 1, v = 15$.
Kirkman is asking for a $S_1(2,3,15), b = \frac{15 \cdot 14}{6} = 35$ with an extra condition on partition of the block sets.

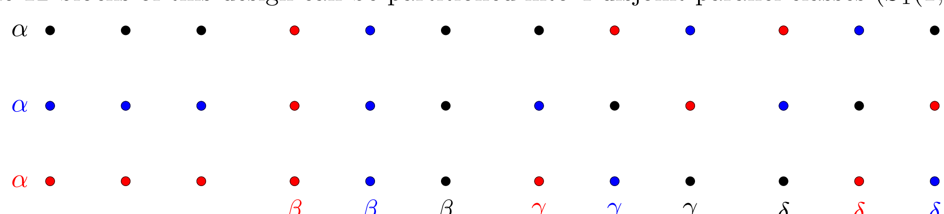
Example (Construction of a $STS(9)$)**.** *The following* 3*-sets of points are blocks:*

- *horizontal and vertical lines $\Rightarrow$ 6 blocks*
- *permutations $\pi \in S_3$ (The block $B_\pi, \pi = (213)$ is marked red)*



Claim 12.4.1. *They form a $S(2,3,9)$.*

Proof. Every pair is covered by a block. As $b = \frac{9 \cdot 8}{3 \cdot 2} = 12$ every pair is covered at most once.
The 12 blocks of this design can be partitioned into 4 disjoint parallel classes ($S_1(1,k,v)$):



The design is resolvable. ∎

Theorem 12.4.2. *The resolvable $STS(9)$ can be used to construct a $S(2,4,13)$.*

Proof. Let $\alpha, \beta, \gamma, \delta$ be additional points. Add one of these to each block of each of the 4 parallel classes. Add the block $\{\alpha, \beta, \gamma, \delta\}$. This yields the result. ∎

Kirkman asks for a resolvable $STS(15)$ (days correspond to the parallel classes of the resolution)

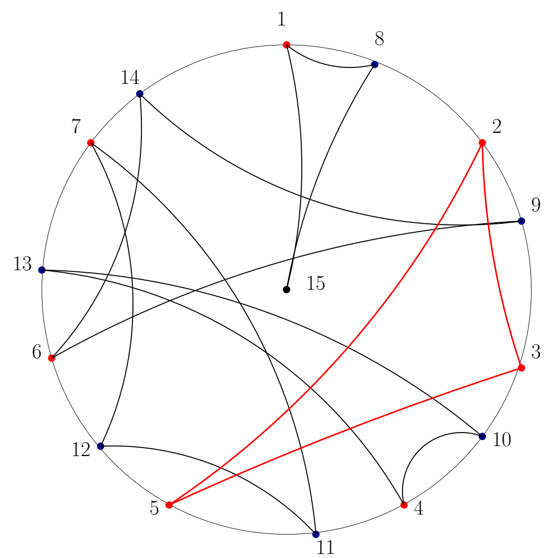
12.5 Construction of $STS(15)$

In $\mathbb{F}_2^n$ there is a $S(3,4,2^n)$. For $n = 4$ we get a $S(3,4,16)$: take the point derived design for any point. This is a $S(2,3,15)$.

$$\mathcal{B}_p = \{B \setminus p : p \in B \in \mathcal{B}\}$$

The blocks are $\{x, y, z, x+y+z\}, x, y, z$ different in $\mathbb{F}_2^4 \setminus \{(0,0,0,0)\}$. Is it resolvable?

A second construction: One of the classes is shown in the picture below.



There are 7 rotations that map red on red, these yield the classes. The red block and its rotations form a subsystem $S(2,3,7)$ on the red points.

Problem. *Show that the two constructions yield equivalent $S(2,3,15)$ systems.*

Theorem 12.5.1. *For all $v = 3$ mod 6 there is a $STS(v)$.*

Proof. $v = 6s+3 = 3(2s+1)$. Let $m = 2s+1, V = \{a_i, b_i, c_i : i \in \mathbb{Z}_m\}$. The blocks are then of the following form (4 block classes):

$$\begin{aligned} &I(a_i, b_i, c_i), i \in \mathbb{Z}_m \\ &II(a_i, a_j, b_k), i+j = 2k \text{ mod } m \\ &III(b_i, b_j, c_k), i+j = 2k \text{ mod } m \\ &IV(c_i, c_j, a_k), i+j = 2k \text{ mod } m \end{aligned}$$

Claim 12.5.2. *These blocks form the system we want to have.*

Every pair (a_i, b_i) is in block class I. All the other pairs $(a_i, a_j), (a_i, b_k), (a_k, c_i)$ appear because of the following lemma:

Lemma 12.5.3. *In the equation $i + j = 2k$ any two of i, j, k determine the third one uniquely.*

If i, k or j, k are given we can compute the third by subtraction.

If i, j are given we need that $i + j$ can be written as $2k$ mod m. $x \mapsto 2x$ is a bijection on $\mathbb{Z}_m$. For this it is enough to show injectivity. Suppose $2s = 2y$ mod m, $\quad x, y \in \mathbb{Z}_m$.

Without loss of generality (Wlog) $2x + m = 2y$ but m is odd so this is a contradiction.

Therefore all pairs appear in a unique block. $\square$

Remark. *It is known that $STS(v)$ for $v = 1$ mod 6 also exist.*

12.6 Projective planes

Definition (projective plane)**.** *A design $S(2, n+1, n^2+n+1)$ is called a projective plane.*

The number of blocks of a projective plane $S(2, n+1, n^2+n+1)$ is

$$\frac{(n^2+n+1)(n^2+n)}{(n+1)n} = n^2+n+1$$

We will count triples (B, B', p) with $B \cap B' = p$. For fixed q there are n^2+n points $\neq q$ in a block with q. Each block contributes exactly n points. Therefore p is contained in $n+1$ blocks. This yields

$$\sum_{P:p\in P} \binom{n+1}{2} = \frac{(n^2+n+1)(n^2+n)}{2} = \binom{n^2+n+1}{2}$$

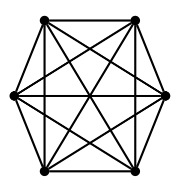
and this equals the number of pairs of blocks. This implies that any two blocks intersect. We have thus shown that a $S(2, n+1, n^2+n+1)$ indeed respects the axioms of projective geometry.

12.6.1 Constructing projective planes

Let q be a prime power. Then there exists a field $\mathbb{F}_q$. Take the 3-dimensional space $\mathbb{F}_q^3$ and consider 1-dimensional subspaces as points and 2-dimensional subspaces as lines.

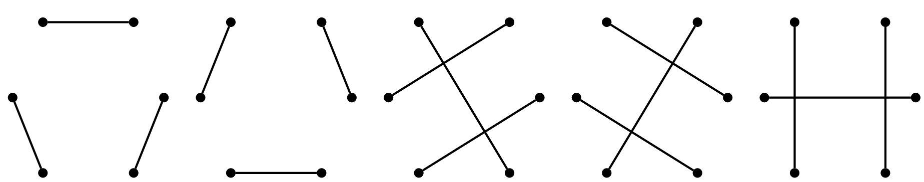
$$\begin{aligned} \#\text{ points } &= \frac{q^3-1}{q-1} = q^2+q+1 \\ \#\text{ lines} &= \begin{bmatrix} 3 \\ 2 \end{bmatrix} \end{aligned}$$

Example (A projective plane of order 4)**.** *By K_6 we denote the complete graph on 6 vertices. We construct a $S(2, 5, 21)$ K_6 has 15 edges.*



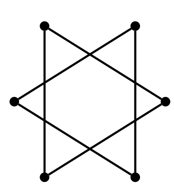
A P-class (or 1-factor) is a set of 3 disjoint edges. A Q-class (or 1-factorization) is a family of 5 disjoint P-classes.

The following 5 P-classes form a Q-class:



There are 15 P-classes in total as we have 5 choices to fix the first edge, 3 for the next one and the last one is then already fixed by the first two edges.

Two disjoint P-classes always form a cycle of length 6 and therefore determine a unique Q-class. A class P that is disjoint from P_1 and P_2 contains exactly one diagonal (relative to P_1, P_2) as there is no perfect matching on the graph without the outer cycle and diagonals.



The incidence structure of the blocks and classes can be written down in some kind of incidence matrix given here.

	blocks	
	edges of K_6	Q*-classes*
vertices of K_6	$\in$	$\emptyset$
P*-classes*	$\ni$	$\in$

The size of the blocks of this incidence structure is given by

$$|B_e| = 2 + 3 = 5 = 0 + 5 = |B_Q|$$

The number of points is $6 + 15 = 21$. *Two points appear in exactly one block:*

2 **vertices of** K_6 *they form an edge*

2 P**-classes** *if they are disjoint there exists a unique* Q*-class containing the two. Otherwise there exists a unique edge in the intersection.*

A vertex v *of* K_6 *and a* P*-class define the unique edge of the* P*-class containing* v. *By this construction we get a* $S(2, 5, 21)$

Remark. *On the basis of this construction it is possible to build the small Witt design* $S(5, 6, 12)$. *The large Witt design is* $S(5, 8, 24)$.

12.7 Fisher's inequality

Theorem 12.7.1 (Fisher's inequality)**.** *A* $S_\lambda(t, k, v)$ *with* $v \geq k + 1$ *can only exist if* $b \geq v$ *where* $b = \#$ *blocks* $= \frac{\binom{v}{t}}{\binom{k}{t}}$

Remark. *As an application we recognize that* $S(2, 6, 16)$ *and* $S_3(2, 10, 25)$ *do not exist.*

The proof of Fisher's theorem 12.7.1 is based on a rank inequality for the incidence matrix.

Chapter 13

Inversions And Involutions

13.1 Möbius inversion

Definition (Incidence algebra of a poset)**.** *Let $P = (X, \leq)$ be a poset, $F : X \times X \to \mathbb{F}$ a map with $F(x, y) = 0$ if $x \not\leq y$. The incidence algebra of P is the ring of maps F with this property.*

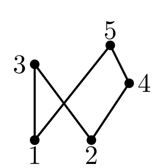
Proposition 13.1.1. *These maps form a ring with pointwise addition and the convolution as multiplication:*

$$(F \circ G)(x, y) = \sum_{x \leq z \leq y} F(x, z)G(z, y)$$

For the proof fix a linear extension L of P ($L = x_1, \ldots, x_n$) and look at

$$F \longrightarrow M_F = (F(x_i, x_j))_{i,j \in [n]}$$

Example. $L = 1, 2, 3, 4, 5$



$$M_F = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & * & 0 & * & 0 & * \\ 2 & 0 & * & * & * & * \\ 3 & 0 & 0 & * & 0 & 0 \\ 4 & 0 & 0 & 0 & * & * \\ 5 & 0 & 0 & 0 & 0 & * \end{array}$$

From the shape of M_F we see:

- multiplication is the matrix multiplication
- F has an inverse $\Leftrightarrow F(x, x) \neq 0 \forall x \in X$

Proof. $\Rightarrow$ $(F \circ G)(x, x) = F(x, x)G(x, x)$

$\Leftarrow$ The entries of F^{-1} can be computed by increasing distance from diagonal. (Left and right inverses are the same)

□

13.2 The Zeta-function

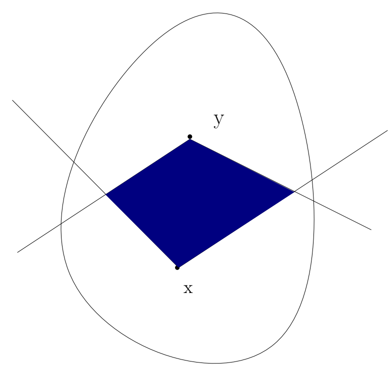
Definition (Zeta-function)**.**

$$Z(x,y)=\begin{cases}1 & x\le y\\ 0 & \textit{else}\end{cases}$$

is called the Zeta-function.

The powers of Z have a special interpretation:

$$\begin{aligned}Z^2(x,y) &= \sum_{x\le z\le y} Z(x,z)Z(z,y)\\ &= \sum_{x\le z\le y} 1\\ &= \#\text{ elements in the interval } [x,y] \text{ of } P\\ &= \#\,(\text{downset}(y)\cap\text{upset}(x))\end{aligned}$$



$$\begin{aligned}Z^k(x,z) &= \sum_{x\le y_1\le\ldots\le y_k=z} 1\\ &= \#\text{ multichains with } k+1 \text{ elements from } x \text{ to } z\end{aligned}$$

Example.

$$Z^3(2,5)=\#\ \textit{multichains with } 4 \textit{ elements from } 2 \textit{ to } 5$$

chain	*multichain*
$2\le 4\le 5\le 5$	$1,2,34$
$2\le 2\le 5\le 5$	$12,,34$
$2\le 4\le 4\le 5$	$1,23,4$
$2\le 2\le 4\le 5$	$12,3,4$
$2\le 5\le 5\le 5$	$1,,234$
$2\le 2\le 2\le 5$	$123,,4$

$$(Z-1)(x,z)=\begin{cases}1 & x<z\\ 0 & \text{else}\end{cases}$$

$$\begin{aligned}(Z-1)^k(x,z) &= \sum_{x=y_0<y_1<\ldots<y_k=z} 1\\ &= \#\text{ chains with } k+1 \text{ elements from } x \text{ to } z\end{aligned}$$

$$(2-Z)(x,z)=\begin{cases}1 & x=z\\ -1 & x<z\\ 0 & \text{else}\end{cases}\quad\text{(this function has an inverse)}$$

Proposition 13.2.1.

$$(2-Z)^{-1}(x,z) = \textit{ \# chains from } x \textit{ to } z$$

Proof.

$$\begin{aligned}
(2-Z)^{-1} &= (1-(Z-1))^{-1} \\
&= \frac{1}{1-(Z-1)} \\
&\overset{\text{geometric series}}{=} \sum_{k\geq 0} (Z-1)^k \\
&= \sum_{k\geq 0} \text{\# chains with } k+1 \text{ elements from } x \text{ to } z \\
&= \text{\# chains from } x \text{ to } z
\end{aligned}$$

□

Verify:

$$\begin{aligned}
(2-Z)\sum_{k\geq 0}(Z-1)^k &= (1-(Z-1))\sum_{k\geq 0}(Z-1)^k \\
&= \sum_{k\geq 0}(Z-1)^k - (Z-1)^{k+1} \\
&\overset{\text{telescope sum}}{=} (Z-1)^0 \\
&= 1
\end{aligned}$$

Remark.

$$H(x,y) = \begin{cases} 1 & x < y \\ 0 & \textit{else} \end{cases} \Rightarrow (1-H)^{-1}(x,y) = \textit{\# maximal chains from } x \textit{ to } y$$

This denotes the cover relation, i. e. x, y is an edge in the diagram.

13.3 The Möbius function of P

Z is invertible. Let $M = (\mu(x,z))$ be the inverse of Z, i. e.

$$\sum_{x\leq z\leq y} \mu(x,z)Z(z,y) = \delta_{x=y}$$

Therefore

M1 $\mu(x,x) = 1 \forall x \in X$

M2 $\mu(x,y) = -\sum_{x\leq z\leq y}\mu(x,z)$ if $x < y$

M3 $\mu(x,y) = 0$ if $x \not\leq y$

Theorem 13.3.1 (Möbius inversion)**.** *If $P = (X, \leq)$ is a poset, $f_=, f_\leq : X \to F$ then*

$$\left[f_\leq(y) = \sum_{x\leq y} f_=(x) \quad \Leftrightarrow \quad f_=(y) = \sum_{x\leq y} f(x)\mu(x,y) \right] \forall y$$

Proof. Look at it in terms of vectors and matrices:

$$f_{\leq} = f_{=}Z \Rightarrow f_{\leq}M = f_{=}$$

Similarly we have

$$\begin{aligned}
&Zf_{=} = f_{\geq} \\
\Leftrightarrow &f_{=} = Mf_{\geq} \\
\Leftrightarrow &f_{=}(y) = \sum_{y\leq z} \mu(y,z)f_{\geq}(z) \\
\Leftrightarrow &f_{\geq}(y) = \sum_{y\leq z} f_{=}(z)
\end{aligned}$$

□

We still do not know how to compute Möbius functions. To analyse this we will first have a look at chains. Then we will analyse products of posets and Boolean lattices.

P is a chain

$$\begin{aligned}
\mu(a,a) = 1 &\quad \forall a \\
\mu(a,a) + \mu(a,a+1) = 0 \Rightarrow &\mu(a,a+1) = -1 \\
\mu(a,a) + \mu(a,a+1) + \mu(a,a+2) = 0 \Rightarrow &\mu(a,a+2) = 0 \\
\Rightarrow \mu(a,a+k) = 0 \forall k > 1&
\end{aligned}$$

$P = \mathbb{N}, P$ infinite is ok as long as all intervals are finite. For functions $f, g : \mathbb{N} \to \mathbb{F}$ we have:

$$g(n) = \sum_{k\leq n} f(k) \forall n \Leftrightarrow f(n) = g(n) - g(n-1)$$

This is a special instance of Möbius inversion.

products $P \times Q$ Elements $(x_1, x_2), x_1 \in X_P, x_2 \in X_Q$. Relations:

$$(x_1, x_2) \leq_X (y_1, y_2) \Leftrightarrow x_1 \leq_P y_1, x_2 \leq_Q y_2$$

Proposition 13.3.2.

$$\mu_{P\times Q}\left((x_1,x_2),(y_1,y_2)\right) = \mu_P(x_1,y_1)\mu_Q(x_2,y_2)$$

Proof. $[M1]$ and $[M3]$ are immediate. To prove $[M2]$

$$\begin{aligned}
\sum_{(x_1,x_2)\leq(z_1,z_2)\leq(y_1,y_2)} \mu\left((x_1,x_2),(y_1,y_2)\right) &= \sum_{x_1\leq z_1\leq y_1, x_2\leq z_2\leq y_2} \mu(x_1,z_1)\mu(x_2,z_2) \\
&= \left(\sum_{x_1\leq z_1\leq y_1} \mu(x_1,z_1)\right)\left(\sum_{x_2\leq z_2\leq y_2} \mu(x_2,z_2)\right) \\
&= 0
\end{aligned}$$

□

$P = \mathcal{B}_n = (C_2)^n \ \ (C_2)^n = n$-fold product of 2-elements.

$$\begin{aligned}
\mu_{C_2}(0,0) &= \mu_{C_2}(1,1) = 1 \\
\mu_{C_2}(0,1) &= -1 \\
\mu_{C_2}(1,0) &= 0 \\
\Rightarrow \mu_{\mathcal{B}_n}(A,B) &= \prod_{i=1}^{n} \mu_{C_2}(A_i, B_i) \\
&= \begin{cases} (-1)^{|B-A|} & \text{if } A \subseteq B \\ 0 & \text{else} \end{cases}
\end{aligned}$$

Corollary.

$$N_{\geq}(A) = \sum_{B \supseteq A} N(B)$$

$$\Rightarrow N(A) = \sum_{B \supseteq A} (-1)^{|B-A|} N_{\geq}(B)$$

In particular we get the principle of inclusion and exclusion:

$$N(\emptyset) = \sum_{B} (-1)^{|B|} N_{\geq}(B)$$

Applications:

1. Derangements $\pi \in S_n$:

$$\begin{aligned}
fix(\pi) &= \{i : \pi(i) = i\} \\
N(A) &= \#(\pi \in S_n : fix(\pi) = A)
\end{aligned}$$

   We are interested in $N(\emptyset)$

$$\begin{aligned}
N_{\geq}(A) = \sum_{B \supseteq A} N(B) \\
&= \ \# \ \pi \in S_n \text{ with } fix(\pi) \supseteq A \\
&= (n - |A|)!
\end{aligned}$$

   From the principle of inclusion and exclusion we get

$$\begin{aligned}
N(\emptyset) &= \sum_{B} (-1)^{|B|} (n - |B|)! \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n-k)! \\
&= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \\
&= der(n)
\end{aligned}$$

2. $f_{=}(A) = a_k \forall A$ with $|A| = k$

$$f_{\leq}(B) = \sum_{A \subseteq B} f_{=}(A)$$

$$\begin{aligned}
\Rightarrow \quad b_n &= \sum_{k=0}^{n} \binom{n}{k} a_k \\
a_n &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k
\end{aligned}$$

3. Möbius inversion in number theory: posets of divisors are products of chains.

$$\mu\left(\frac{r}{s}\right) = \mu(r, s)$$

13.4 Involutions

Involutions can be used for bijective proof of identities. For example the proof of

$$\sum_{A \subseteq [n]} (-1)^{|A|} = 0$$

in the first lecture was using involution.

The general setting is a set $S = S^+ \dot\cup S^-$ and a bijection $\varphi : S \to S$.

Definition. *φ is a sign reversing involution iff.*

- $\varphi^2 = id_S$ *(involution property)*
- $\forall x \ni Fix(\varphi) : sgn(x) \neq sgn(\varphi(x))$

$$sgn(y) = \begin{cases} + & y \in S^+ \\ - & y \in S^- \end{cases}$$

Lemma 13.4.1. *If there exists a sign reversing involution φ on $S^+ \dot\cup S^-$ then*

$$\left|S^+\right| - \left|S^-\right| = \left|Fix(\varphi) \cap S^+\right| - \left|Fix(\varphi) \cap S^-\right|$$

Remark. *Useful situations mostly have $Fix(\varphi) \cap S^- = \emptyset$*

Definition (Vandermonde matrix)**.**

$$V = \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{pmatrix}$$

is called Vandermonde matrix.

Theorem 13.4.2.

$$\det(V) = \prod_{i<j} (x_j - x_i)$$

Remark. *On the right hand side there are $2^{\binom{n}{2}}$ terms. From the Leibniz rule we then know that this is a polynomial of degree $\binom{n}{2}$.*

On the other hand we know thet $x_i = x_j \Rightarrow \det(V) = 0 \Rightarrow (x_j - x_i)$ is a divisor of the polynomial.

These two arguments imply $\det(V) = c \prod_{i<j} (x_j - x_i)$. Comparing coefficients yields $c = 1$.

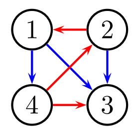
We now turn to a more combinatorial proof that uses involution.

The aim is to design an involution that kills the excess-terms on the right hand side. To do this we look at tournaments.

Definition (tournaments)**.** *A tournament on $\{1, \dots, n\}$ is an orientation of the complete graph K_n on $\{1, \dots, n\}$*

There are $2^{\binom{n}{2}}$ tournaments and there exists a bijection between tournaments and the summands of the expanded product $\prod(x_j - x_i)$

Example. *We look at a tournament on K_4*



$$\begin{aligned}
(x_4 - x_3)(x_4 - x_2)(x_4 - x_1)& \\
(x_3 - x_2)(x_3 - x_1)& \\
(x_2 - x_1)&
\end{aligned}$$

The picture above shows the tournament T on the right hand side corresponding to the term $-(x_1 x_2 x_3^3 x_4)$.

Definition (weight and sign of tournament)**.** *The weight of an edge $e = (i, j)$ is $w(e) = x_j$. The sign of an edge $e = (i, j)$ is $sgn(e) = \begin{cases} -1 & j < i \\ +1 & j > i \end{cases}$*

$$w(T) = \prod_{e \in T} w(e) \qquad sgn(T) = \prod sgn(e)$$

Proposition 13.4.3.

$$\sum_{T\, tourn.\ on\ 1,\ldots,n} sgn(T)w(T) = \prod_{i<j} x_j - x_i$$

Lemma 13.4.4. *T is transitive iff. $i \to j \to k \Rightarrow i \to k$*

$$\begin{aligned}
&\Leftrightarrow T \text{ is acyclic} \\
&\Leftrightarrow \text{no two vertices have the same outdegree in } T
\end{aligned}$$

Proof. Assume transitivity. Then by ordering the elements from left to right we get that the graph is acyclic.

Transitive tournaments are in a bijection to $\pi \in S_n$

$$\begin{aligned}
T &\sim \pi_1 \pi_2 \ldots \pi_n \\
w(T) &= x_{\pi_2} x_{\pi_3}^2 \ldots x_{\pi_n}^{n-1} \\
sgn(T) &= (-1)^{\# i<j : \pi_j < \pi_i} = sgn(\pi)
\end{aligned}$$

$\square$

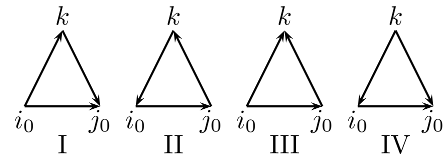
Proposition 13.4.5.

$$\det(V) = \sum_{T\, trans.\ tourn.\ on\ 1,\ldots,n} sgn(T)w(T)$$

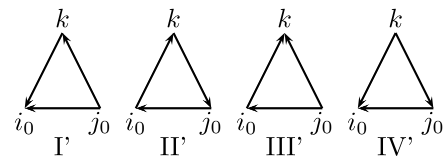
To prove the Vandermonde identity we need a sign reversing involution on tournaments that keeps the transitive tournaments fixed. The involution:

$$T \text{ is not transitive} \to \exists i, j \text{ with } outdeg(i) = outdeg(j)$$

Let (i_0, j_0) be a lexicographic minimal pair with this property (assume $i_0 \to j_0$ from $lexmin(i_0 < j_0)$) Look at triples i_0, j_0, k and classify. For the edge $i_0 \to j_0$ there are 4 possibilities:



The involution reverts (i_0, j_0) and all edges involved in Type I and II. We then obtain a new tournament T'.



Claim 13.4.6. *For all x we have $outdeg_T(x) = outdeg_{T'}(x)$.*

This is clearly true for $x \neq i_0, j_0$

$$\begin{aligned} outdeg_{T'}(j_0) &= a_I + a_{III} + 1 \\ &= outdeg_T(i_0) \\ &= outdeg_T(j_0) \\ &= a_{II} + a_{III} \\ &= outdeg_{T'}(i_0) \end{aligned}$$

Since $w(T) = \prod x_i^{indeg_T(i)}$ and $indeg_T(i) = (n-1) - outdeg_T(i)$ we obtain $w(T) = w(T')$

Claim 13.4.7. *This map is an involution, i. e. $T \to T' \to T'' \Rightarrow T'' = T$*

Proof. The candidates (i, j) with $outdeg(i) = outdeg(j)$ are the same. Therefore (i_0, j_0) are the same. □

Claim 13.4.8. *If T is not transitive then $sgn(T) \neq sgn(T')$*

Proof. We show $sgn(T)sgn(T') = -1$

$$\begin{aligned} sgn(T)sgn(T') &= \prod_{e \in T, e \notin T'} -1 \\ &= (-1)^{2a_I + 2a_{II} + 1} \\ &= -1 \\ \prod (x_j - x_i) &= \sum_{T \text{ tourn.}} w(T)sgn(T) \\ &\overset{involution}{=} \sum_{T \text{ trans. tourn.}} w(T)sgn(T) \\ &\overset{Leibniz}{=} \sum_{\pi \in S_n} sgn(\pi) x_{\pi_2} x_{\pi_3}^2 \dots x_{\pi_n}^{n-1} \\ &\overset{Leibniz}{=} \det(V) \end{aligned}$$

□

13.5 Lemma of Lindström Gessel-Viennot

Let $G = (V, E)$ be a directed acyclic graph, $w : E \to \mathbb{R}$ weights, $A = \{a_1, \dots, a_n\}, B = \{b_1, \dots, b_n\}$ subsets of V.

Definition (path matrix)**.** *$\mathcal{P}(i,j)$ denotes the family of paths from a_i to b_j. The path matrix M is defined as $M = (m_{ij})_{i,j\in[n]}$ with*

$$m_{ij} = w\left(\mathcal{P}(i,j)\right) = \sum_{P \text{ path from } a_i \text{ to } b_j} w(P) = \sum_{P \text{ path } a_i \to b_j} \prod_{e\in P} w(e)$$

Definition (path system)**.** *Let $P^\pi(A,B)$ be the set of connecting path systems $p_1 : a_1 \to b_{\pi_1}, \ldots .p_n : a_n \to b_{\pi_n}$, $\pi \in S_n$.*

The weight of a path system $(p_1,\ldots,p_n)$ is $\prod_{i=1}^n w(p_i)$. The set of vertex disjoint path systems with $p_i \cap p_j = \emptyset$ is called $P_{VD}(A,B)$.

Lemma 13.5.1 (Lindström Gessel-Viennot)**.**

$$\det(M) = \sum_{P=(p_1,\ldots,p_n)\in P_{VD}(A,B)} sgn(P)w(P)$$

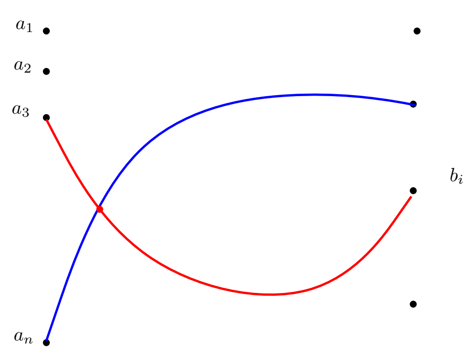
Remark. *The sign of a path system $p_1,\ldots,p_n$ is the sign of the permutation realized by $p_1,\ldots,p_n$, i. e. $p_i : a_i \to b_{\pi_i}$.*

Proof. A typical contribution of the Leibniz expansion of M:

$$\begin{aligned} sgn(\pi)m_{1\pi_1}m_{2\pi_2}\ldots m_{n\pi_n} &= sgn(\pi)\left(\left(\sum_{p:a_1\to b_{\pi_1}} w(p)\right)\cdots\left(\sum_{p:a_n\to b_{\pi_n}} w(p)\right)\right) \\ &= sgn(\pi)\sum_{P=(p_1,\ldots,p_n)\in\mathcal{P}^\pi(A,B)} w(P) \\ \Rightarrow \det(M) &= \sum_{\pi\in S_n}\sum_{P=(p_1,\ldots,p_n)\in\mathcal{P}^\pi(A,B)} sgn(P)w(P) \end{aligned}$$

□

An involution allows to get rid of the path systems that are not vertex disjoint. Sketch of the involution:

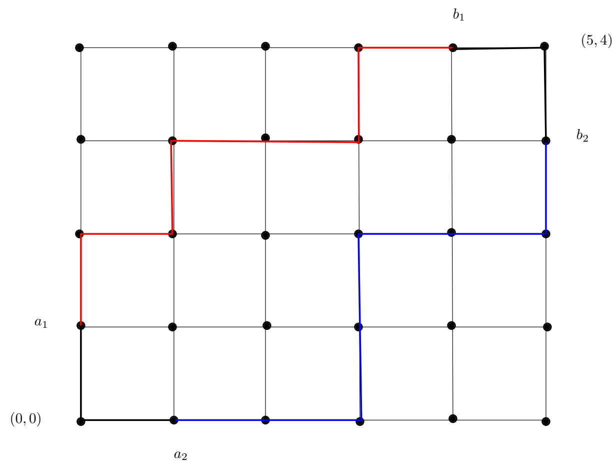


In the picture two paths intersect in at least one vertex. Choose one of these vertices carefully and exchange the tails of the paths. Then the weight remains the same and the sign changes.

Applications:

- an easy proof of the determinant multiplication formula
- combinatorial counting: a special increasing polyomino yields a disjoint path system.

Example. *Increasing polyomino between* $(0,0)$ *and* $(5,4)$



The number of these polyominos is

$$\det \begin{pmatrix} \binom{7}{3} & \binom{7}{2} \\ \\ \binom{7}{4} & \binom{7}{3} \end{pmatrix} = \det \begin{pmatrix} 35 & 21 \\ 35 & 35 \end{pmatrix} = 490$$

Chapter 14

Catalan-Numbers

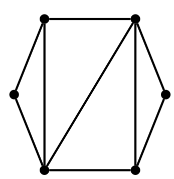
We will analyze Catalan numbers in three steps[1]:

- Catalan families
- Get the numbers
- Further aspects

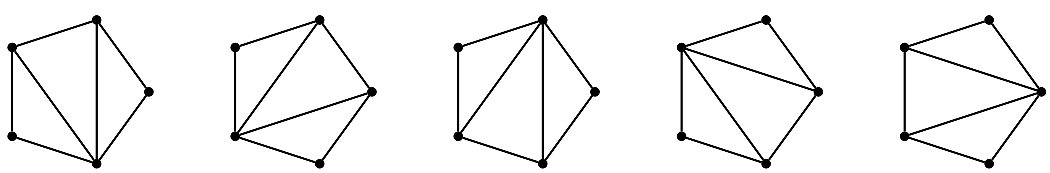
The first few families[2] are:

(a) triangulations of an $n+2$-gon

Example ($n = 4$)**.** *There are several triangulations. One of them is shown below:*

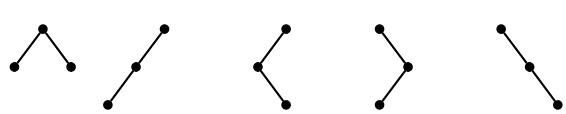


Example ($n = 3$)**.** *There are five triangulations:*



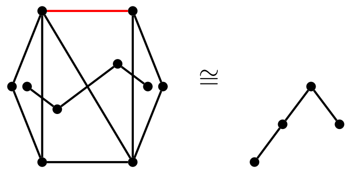
(c) Binary trees with n vertices.

Example ($n = 3$)**.** *The trees with three vertices are*

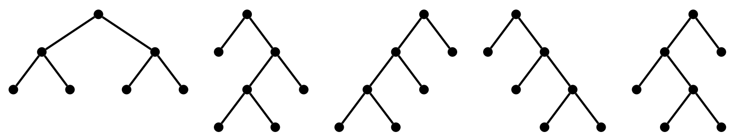


[1] see `http://www-math.mit.edu/~rstan/~ec/catalan.pdf` or `/catadd.pdf`
[2] from Stanley, "Enumerative Combinatorics", Vol. 2, Excercise 15

Bijection between (a) and (c): Distinguish one outer edge and draw the dual graph inside. The root then comes from the distinguished edge:

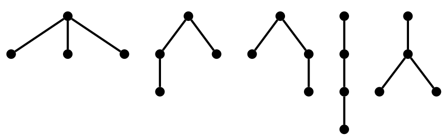


(d) Full binary trees with with $n+1$ edges:

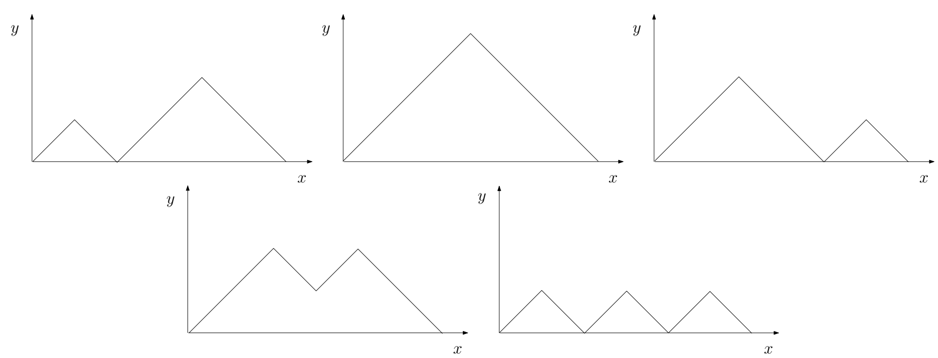


Bijection between (d) and (c): delete or add leafs.

(e) plane trees with $n+1$ vertices

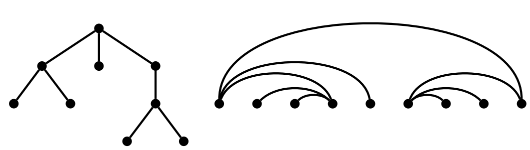


(i) Dyck path: path with diagonal steps to the up and down right from $(0,0)$ to $(2n,0)$, staying above the x-axis.



Bijection between (e) and (i): Start with a Dyck path, stick glue on the inside, stick it together and pull it out again. For the other direction record a bug walking around the tree. He either walks up or down. This yields the corresponding Dyck path.

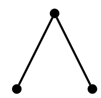
A more interesting bijection can be seen between (e) and (d). For this draw the plane tree as an alternating tree.



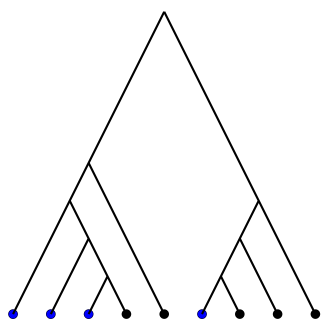
Definition (Alternating tree)**.** *An alternating tree has the following properties:*

- *all vertices are on the x-axis*
- *edges are non-crossing and above the x-axis*
- *the root is on the left*
- *each vertex is either the left end or the right end of all its incident edges*

The replacement

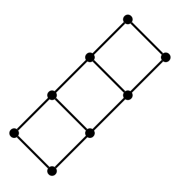
 $\leftrightarrow$ 

yields a bijection with binary trees. The figure shows the binary tree corresponding to the alternating tree of the previous example.



14.1 5 more families

(aaa) Linear extension of the poset $2 \times n$



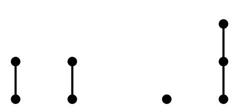
Bijection to Dyck paths: picking an element in the left n-chain corresponds to going down whereas any element from the right n-chain corresponds to a step up.

The initial part of a linear extension is a down set. This implies that at least as many elements from the lower chain as from the upper chain. Therefore we always need as many up as down steps and the path then stays above the x-axis.

It is easy to see that every Dyck path forms a linear extension.

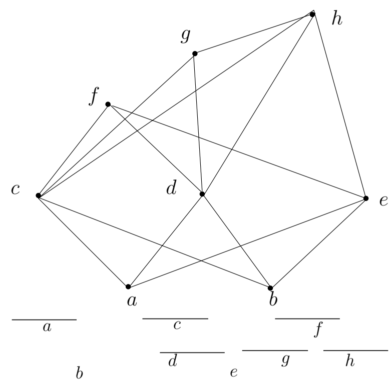
(ddd) Semi-orders with n elements.

Definition (Semi-order)**.** *A semi-order is a poset without induced* $2+2$ *and* $1+3$.



Semi-orders correspond to unit-interval-orders, i. e. every element has an interval of length 1. If $a < b$ then the interval of a is left of the interval of b.

Example. *A $2 + 2$ is not unit-interval-order skipped. The same is with $1 + 3$.*



Bijection with Dyck paths: Colour the left interval end points blue and the right ones red. If blue means "up" and red means "down" then this yields a Dyck path.

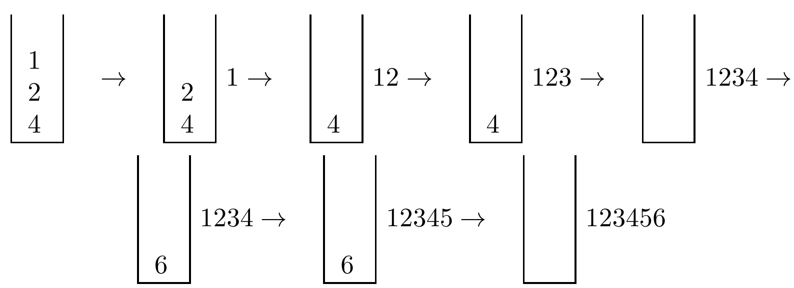
The way back leads to a proper interval order (i. e. no interval is contained in another). This can be made into an unit interval order since it does not contain a $1 + 3$.

(ii) Stack sortable permutations. There are two operations:

push element from the stack → out

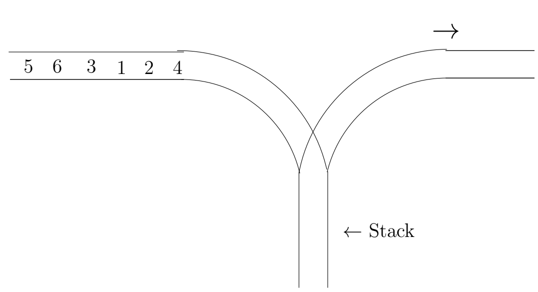
pop element from in → stack

Example. $\pi = (421365)$



The operations push and pop act on the next element of π, the aim is to produce the identity.

Another picture is the one with trains and waggons:



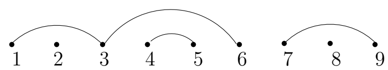
Remark. *Not every permutation is stack sortable, $\pi = (231)$ is not.*

(231) is a forbidden pattern in all permutations. Permutations with $\ldots i \ldots j \ldots k \ldots$, $k < i < j$ are not stack sortable, all others are. Prove this as an exercise.

A bijection to Dyck paths is given by the interpretation "push=down" and "pop=up".

(pp) Non-crossing partitions of $[n]$

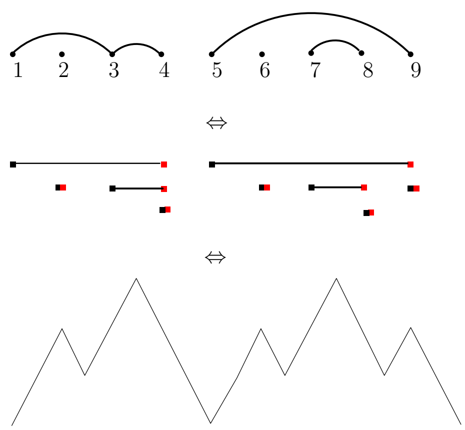
A non-crossing partition can be written in terms of non-crossing arcs:



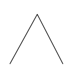
$\Leftrightarrow$ $(136)(2)(4,5)(7,9)(8)$

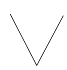
Bijection to Dyck paths: for all vertices create an interval, starting at the vertex itself and ending at block end. Left interval end points then correspond to steps up and right end points mrk steps down. Draw the path from the intervals.

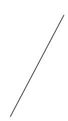
For the other direction first draw the intervals from the Dyck path and then the partition.

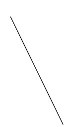


Second bijection:

Every single element produces a 

Every internal element produces a 

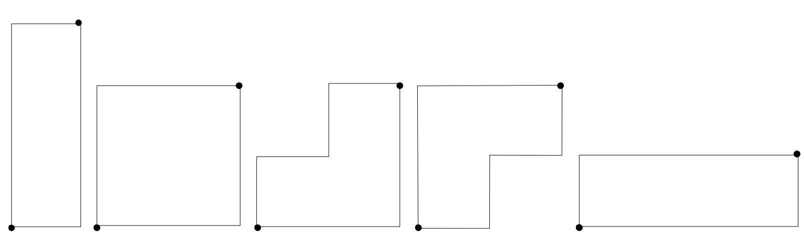
Every block start produces a 

Every block end produces a 

These again yields a bijection to Dyck paths but the paths are different.

(l) Pairs of lattice paths of length $n+1$ that only meet at the two end points

Example $(n = 3)$**.** *There are* 5 *such path pairs:*



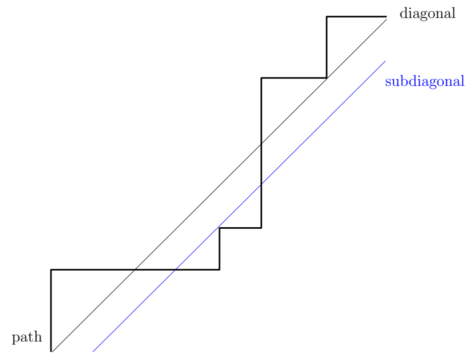
The actual Catalan numbers are

$$1, 2, 5, 14, 42, 132, \ldots$$
$$c_n = \frac{1}{n+1}\binom{2n}{n}$$

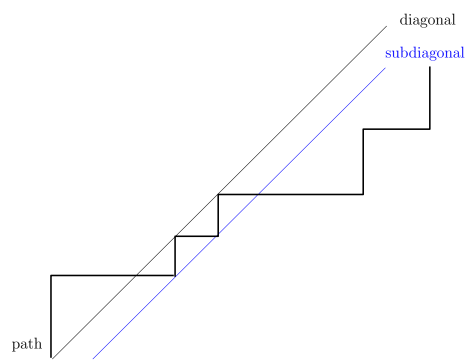
14.2 The reflection principle

Look at lattice paths from $(0,0)$ to (n,n). There are $\binom{2n}{n}$ such paths. A path is called "good" if it stays above the diagonal (i. e. it is a Dyck path, rotated by 45°, "bad" otherwise.

$$\binom{2n}{n} = \underbrace{|D_n|}_{good} + \underbrace{|B_n|}_{bad}$$
$$|D_n| = C_n$$



Every bad path touches the subdiagonal. Reflect the rest of the path at the first touching point. The path now goes from $(0,0)$ to $(n+1, n-1)$



Every path from $(0,0)$ to $(n+1, n-1)$ touches the subdiagonal and can be reflected back.

$$B_n = \#\text{ paths from } (0,0) \text{ to } (n+1, n-1)$$
$$\Rightarrow |B_n| = \binom{2n}{n-1}$$
$$\Rightarrow C_n = \frac{1}{n+1}\binom{2n}{n}$$

14.3 The cycle lemma

There are $\binom{2n+1}{n}$ words in $-1, +1$ with $n+1$ entries -1 and n entries $+1$. These words correspond to paths with steps up and down. They contain Dyck paths extended by an up-step at the start.

Let a cyclic group act on these words (rotations)

$$a_1 \dots a_{2n+1} \sim a_{2n+1} a_1 a_2 \dots a_{2n}$$

Observation 1 Every orbit (equivalence class) contains $2n+1$ words. If not then $a = bb \dots b$ exists, $length(a) = 2n+1$, $length(b) = k | 2n+1$ but

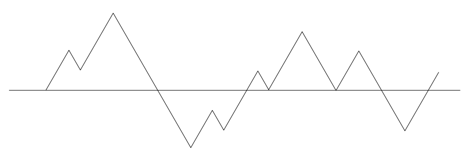
$$\sum a = +1 = \underbrace{\frac{2n+1}{k}}_{\in \mathbb{N}} \underbrace{\sum b}_{\in \mathbb{N}}$$

$$\Rightarrow k = 2n+1$$

$$\Rightarrow \text{ there is no solution except } a = b$$

Observation 2 Every equivalence class contains a unique word with all prefix sums > 0.

Rotate s.t. start and end are at the rightmost global minimum. This is a $+1 +$ Dyck path.

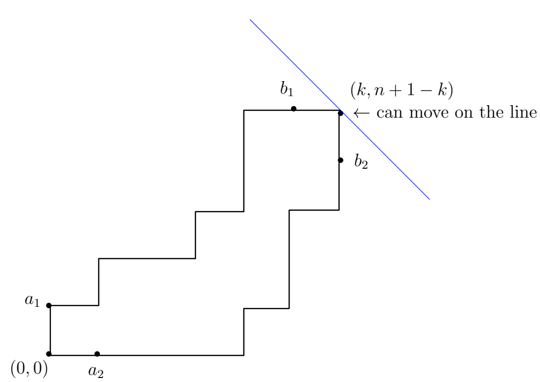


$$\begin{aligned} \# \text{ Dyck paths of length } 2n = {} & \# \text{ equivalence classes} \\ = {} & \frac{\binom{2n+1}{n}}{2n+1} \\ = {} & \frac{1}{n+1}\binom{2n}{n} \end{aligned}$$

14.4 A bijection for family (l) and a count via 13.5.1

(l) contains pairs of paths of length $n+1$ that only intersect at the end points. end points are always of the form $(k, n+1-k)$. Let $c_n^k = \#$ pairs of paths in family (l) ending in $(k, n+1-k)$

$$c_n = \sum_{k=1}^{n} c_n^k$$



We then cut the first and the last step since they are fixed anyway.

$$\begin{aligned}\Rightarrow c_n^k = \#\text{ of disjoint pairs } (p,q) \text{ of paths}\\ p : (0,1) \to (k-1, n+1-k)\\ q : (1,0) \to (k, n-k)\end{aligned}$$

This can be evaluated with the Lindström - Gessel - Viennot lemma 13.5.1.

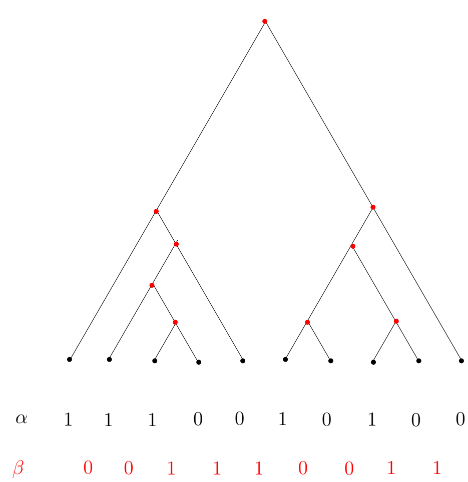
$$\begin{aligned}c_n^k &= \left|\begin{pmatrix}\#\text{ paths: } a_1 \to b_1 & \#\text{ paths: } b_1 \to b_2\\ \#\text{ paths: } a_2 \to b_1 & \#\text{ paths: } a_2 \to b_2\end{pmatrix}\right|\\ &= \left|\begin{pmatrix}\binom{n-1}{k-1} & \binom{n-1}{k}\\ \binom{n-1}{k-2} & \binom{n-1}{k-1}\end{pmatrix}\right|\\ &= \binom{n-1}{k-1}^2 - \binom{n-1}{k}\binom{n-1}{k-2}\\ &= \frac{1}{n}\binom{n}{k}\binom{n}{k-1}\\ \Rightarrow c_n &= \sum_k c_n^k\\ &= \frac{1}{n}\sum_k \binom{n}{k}\binom{n}{k-1}\end{aligned}$$

This yields the Narayana numbers c_n^k, $c_4 = 1 + 6 + 6 + 1$.

Remark. *One can look for a partition of other Catalan families into Narayana families.*

14.5 Bijection between Narayana numbers

In this section we will refine one of the the previous bijections. First look at a full binary tree with $n + 1$ leaves.



$\alpha(T)$ is the finger print of T. Left leaf induces a 1, right leaf a 0.

$\beta(T)$ is called body print of T. It depends on the red inner vertices between leaves. If it is a right child then it induces a 1, a left child induces a 0. The root is referred to as a 1.

$\alpha(T)$ always is of the form

$$\alpha(T) = 1\widehat{\alpha}(T)0$$

as the the first leaf always is a left one and the last leaf is always a right one.

$$\beta(T) = \widehat{\beta}(T)1$$

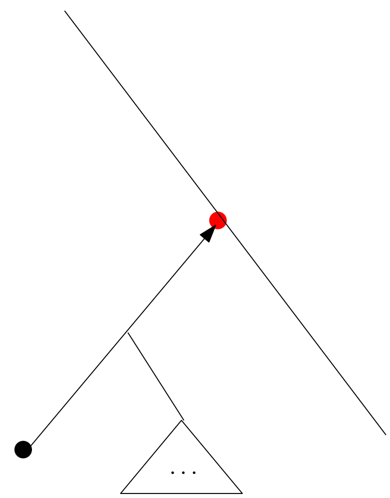
$\widehat{\alpha}(T)$ and $\widehat{\beta}(T)$ are reduced finger and body prints, both of length $n-1$.

Lemma 14.5.1. *Let T be a binary tree with $n+1$ leaves, k of them left.*

$$\Rightarrow \sum_{i=1}^{n-1} \widehat{\alpha}_i = \sum_{i=1}^{n-1} \widehat{\beta}_i = k-1$$
$$\text{and } \sum_{i=1}^{j} \widehat{\alpha}_i = \sum_{i=1}^{j} \widehat{\beta}_i \quad \forall j \leq n-1$$

Proof. We use a bijection between left leaves and inner vertices that are right children + root.

The proof of the lemma is given by the following picture:



A left leaf occurs in $\widehat{\alpha}$ not after correspoding inner right child which is found in $\widehat{\beta}$. □

Lemma 14.5.2. *Full binary trees with $n+1$ leaves and k left ones are in bijection with pairs $\widehat{\alpha}, \widehat{\beta}$ of length $n-1$ such that*

$$\sum \widehat{\alpha} = \sum \widehat{\beta} = k-1 \text{ and } \sum_{i=1}^{j} \widehat{\alpha}_i \geq \sum_{i=1}^{j} \widehat{\beta}_i \forall j$$

are in bijection to pairs (p, q) of lattice paths $p, q : (0, 0) \to (k, n+1-k)$, p is above q and they only meet at end points.

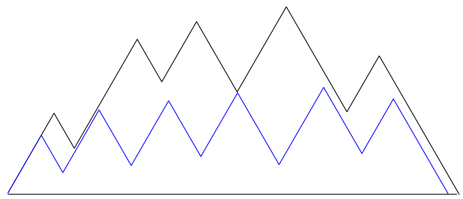
A consequences of this lemma is that these trees are counted by Narayana numbers.

Example. *Strings $\to$ Trees skipped*

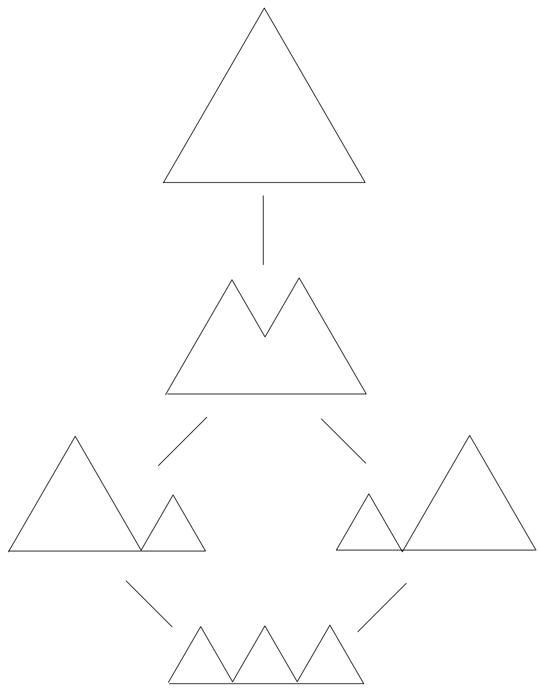
14.6 Structure on Catalan families

There is a poset structure on the family of Dyck paths by regions with inclusion.

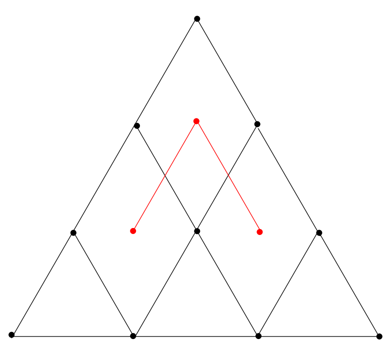
Example. *In the picture there are shown two paths. As the blue one is completely under the black one it is smaller in the inclusion order of the poset.*



Example ($n = 3$)**.** *We show the complete picture of the Dyck path poset for $n = 3$.*

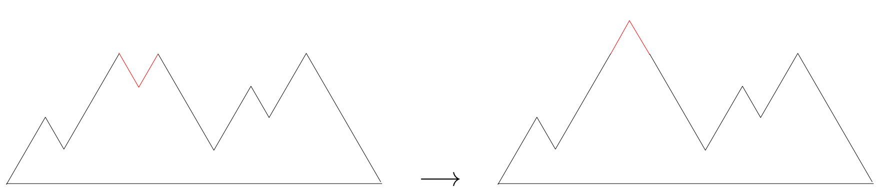


These posets are the down set posets of the cells



There is a rank function on the paths depending on the area of the region.

The diagram edges can be modified to increase the area of a region by 1. For this invert "peaks":

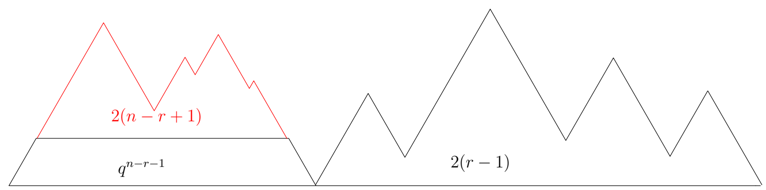


We can define $F_n^k = \#$ paths on rank k. This is another refinement of the Catalan numbers.

This yields some q-Catalan number $C_n(q) = \sum_{k=1}^{n} F_n^k q^k$. There is a recursive formula for the q-Catalan numbers.

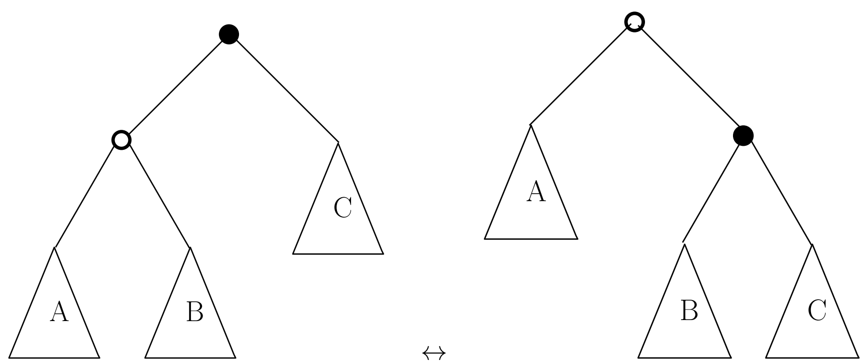
$$C_n(q) = \sum_{r=1}^{n-1} C_{r-1}(q) C_{n-r}(q) q^{n-r+1}$$

The idea is to split up every path into two regions of size $n-r+1$ and $r-1$:



14.7 Tamari lattice

In this section we will have a look on rotation on binary trees which is often used in computer science for search trees.



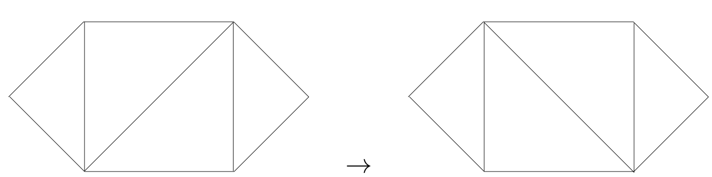
Some useful facts about these rotations are listed below.

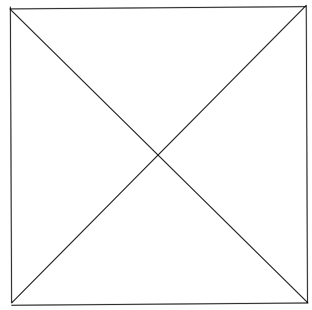
- You can always apply left rotation until reaching a chain
- Every two trees can be transformed into each other
- Interpretating right rotations as steps up this yields a poset which is know as a Tamari lattice

Theorem 14.7.1 (Thurston, Slater, Tarjan)**.** *Let S, T be binary trees on n nodes. Then $2n-4$ rotations suffice to move from S to T and this is best possible.*

Consider trees as trangulations of an n-gon. Then a rotation corresponds to an edge flip between two adjacent triangles. Blowing up means inserting the flipped edge in addition to the first one and yields a tetrahedron.

edge flip:



blow up: 

Appendix A

Practice Sheets

1. Practice sheet for the lecture: **Felsner/ Heldt**
Combinatorics (DS I) 7. April

Delivery date: 20. April
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI11.html`

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(1) A spider has a sock and a shoe for each of his eight feet. In how many different ways can he put on his shoes and socks, assuming that on each foot he has to put on the sock first?

(2) Consider a chess tournament of n players, each playing against every other participant. Show that at each point of time during the tournament there exist at least two players, having finished the same number of games.

(3) In the german lottery system "6 aus 49", six pairwise different numbers $a_1 < a_2 < \ldots < a_6$ are drawn from the set $[49] := \{1, 2, \ldots, 49\}$. What is the chance that no two adjacent numbers are picked, i.e. there is no $i \in \{1, 2, \ldots, 5\}$, with $a_i + 1 = a_{i+1}$?

(4) A sequence of numbers $a_1, a_2, \ldots, a_n$ is called *unimodal,* if there exists an $m \in [n]$, such that $a_i \leq a_{i+1}$ for all $i < m$ and $a_i \geq a_{i+1}$ for all $i \geq m$. Give three different proofs of the unimodality of the sequence $\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \ldots, \binom{n}{n}$ for all $n \in \mathbb{N}$, based on the three given hints:

(a) Use the definition
$$\binom{n}{k} := \frac{n!}{k! \cdot (n-k)!}.$$

(b) Consider the recursive definition
$$\binom{n}{k} := \binom{n-1}{k-1} + \binom{n-1}{k} \text{ and } \binom{n}{0} = \binom{n}{n} = 1,$$
based on Pascale's triangle.

(c) Use the bijection between $\binom{n}{k}$ and the number of subsets of $[n]$, having k elements.

(5) There are nine caves in a forest, each being the home of one animal. Between any two caves there is a path, but they do not intersect. Since it is election time, the candidating animals travel through the forest to advertise themselves. Each candidate visits every cave, starting from their own cave and arriving there at the end of the tour again. Also each path is only used once in total, since nobody wants to look at the embarrassing posters (not even their own), put there on the first traverse. How many candidates are there at most?

2. Practice sheet for the lecture: Combinatorics (DS I) — **Felsner/ Heldt, Knauer**

18. April

Delivery date: 25. –29.. April

`http://www.math.tu-berlin.de/~felsner/Lehre/dsI11.html`

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(1) Let $d(n)$ be the number of derangements of S_n. Prove $d(n) = n \cdot d(n-1) + (-1)^n$ for all $n \in \mathbb{N}$.

(2)

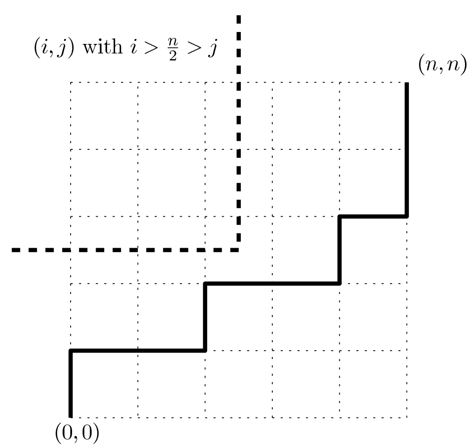
(a) Prove $f_n + f_{n-1} + \sum_{k=0}^{n-2} 2^{n-k-2} f_k = 2^n$ for the Fibonacci numbers f_n, using a bijection.

(b) Decompose the generating function of the Fibonacci numbers, such that

$$\sum_{k=0}^{\infty} f_k z^k = \frac{1}{1-z-z^2} = \frac{A}{1-\alpha z} + \frac{B}{1-\beta z}$$

with partial fractial decomposition (Partialbruchzerlegung) and find an explizit formula for its coefficients based on geometric progression (Geometrische Reihe).

(3) Let $n \in \mathbb{N}$ be odd. Count the number of lattice paths from $(0,0)$ to (n,n), moving only up and right, which avoid all lattice points (i,j), such that $i > \frac{n}{2} > j$ (the figure below shows one of theses paths for $n = 5$).



(4) Please hand your solution of this exercise in a written form in:

(a) Let $x^{\underline{n}} := (x)_n$ denote the falling factorials and $x^{\overline{n}} := x \cdot (x+1) \cdots (x+n-1)$ the *raising factorials.* Deduce the following equation from Vandermonde's identity:

$$(x+y)^{\overline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\overline{k}} y^{\overline{n-k}} \quad \text{(Hint: } (-x)^{\overline{n}} = (-1)^n x^{\underline{n}}\text{)}$$

(b) The Stirling numbers of first kind $s(n,k)$ count the number of permutations in S_n consisting of k disjoint cycles. Compute all values of $s(n,k)$ for $n \leq 4$ and give a (bijective) proof of the equation $s(n,k) = (n-1)s(n-1,k) + s(n-1,k-1)$.

(c) Show that the identities

$$x^{\overline{n}} = \sum_{k=0}^{n} s(n,k)x^k \text{ and } x^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k} s(n,k)x^k$$

hold (Hint: prove the first one inductively and deduce the second from the first).

(5)

$$\text{Let } Q_n := \sum_{k=0}^{k=2^n} \binom{2^n - k}{k} (-1)^k. \text{ What is } Q_{100000}?$$

3. Practice sheet for the lecture: Combinatorics (DS I) | **Felsner/ Heldt, Knauer**

21. April

Delivery date: 2. – 6. May

`http://www.math.tu-berlin.de/~felsner/Lehre/dsI11.html`

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(1) Let $F(n)$ be the number of walks in $\mathbb{Z}^2$, moving either one unit up, down, left or right, which start in $(0,0)$ and return to $(0,0)$ aftern n steps. Give a closed form for $F(n)$. There are many correct solutions; give a nice one!

(2) Complete the lecture's proof of the binomial theorem. To do so, consider polynomials $g(x,y), h(x,y) \in \mathbb{C}[x,y]$ with $\deg(g) \le \deg(h) \le m$. Here the degree is $\deg(x^i y^j) := i + j$ and $\deg(\sum_{i,k} c_{i,k} x^i y^k) := \max_{i,k:c_{ik} \neq 0} \deg(x^i y^k)$. Show, if there are $x_1, \ldots, x_{m+1} \in \mathbb{C}$, such that $x_i \neq x_j$ for $i \neq j$ and $g(x_i, y) = h(x_i, y) \in \mathbb{C}[y]$ we have $g(x,y) = h(x,y)$.
Can you even weaken the requirements?

(3)

(a) What is the expected number E_ℓ of cycles of length ℓ of permutations $\sigma \in S_n$? Use the equation

$$E_\ell = \frac{1}{n!} \sum_{\sigma \in S_n} \#(\ell\text{–cycles of } \sigma).$$

(b) Another (more complicated) way to establish E_ℓ is, to start with the probability $P(k,\ell)$, that a permutation $\sigma \in S_n$ has exactly k cycles of length ℓ and write E_ℓ as

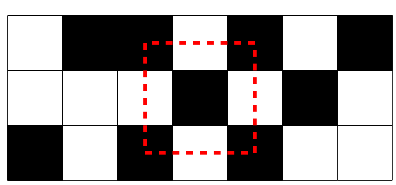
$$E_\ell = \sum_{k-1}^{n} k \cdot P(k,\ell).$$

Compute

$$P(k,\ell) = \frac{|\{\sigma \in S_n | \sigma \text{ has exactly } k \text{ cycles of length } \ell\}|}{n!}$$

for all k and $\frac{1}{2}n < \ell \le n$ as well as $\frac{1}{3}n < \ell \le \frac{1}{2}n$.

(4) Show, that any coloring of the fields of an 3×7 chess–board with colors black and white contain a rectangle, such that all four corners have the same color. Is the same claim true for a 4×6 board?



one coloring of a 3×7 board with a (highlighted) black rectangle

(5) *Please hand your solution of this exercise in a written form in:* In the parliament of some country there are $2n + 1$ seats filled by 3 parties. How many possible distributions (i, j, k), (i.e. party 1 has i, party 2 has j, and party 3 has k seats) are there, such that no party has an absolute majority?
(Hint: look at small numbers and make a good guess.)

4. Practice sheet for the lecture: **Felsner/Heldt, Knauer**
Combinatorics (DS I) 29. April

Delivery date: 9. – 13. May
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI11.html`

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(1)

(a) How many subsets of the set $[n]$ contain at least one odd integer?

(b) How many sequences $(T_1, T_2, \dots, T_k)$ with $T_1 \subset T_2 \subset \dots \subset T_k \subseteq [n]$ are there?

(2) *Please hand your solution of this exercise in a written form in:*
A composition of n is an ordered set of numbers $(a_1, \dots, a_k)$ with $a_i \in \mathbb{N}$ and $0 < a_i < n$ such that $a_1 + \dots + a_k = n$. For example, 4 has the compositions $1+1+1+1$, $1+1+2$, $1+2+1$, $2+1+1$, $2+2$, $3+1$, $1+3$ and 4.

(a) Prove that n has $\binom{n-1}{k-1}$ compositions into exactly k parts/numbers.

(b) Use (a) to show, that n has in total 2^{n-1} compositions.

(c) Let $c(n)$ be the number of compositions of n into an even number of even parts. For $n = 8$ the compositions (with above's additional constraints) are $2+2+2+2$, $4+4$, $6+2$ and $2+6$. Give a closed formula for $c(n)$.

(3)

(a) The *Durfee square* of a partition P is the largest square fitting in the top left corner of P's Ferrers shape. How can you determine the square's size directly from the partition without considering the Ferrers diagram?

(b) Show that the number of partitions of n into at most k parts equals the number of partitions of n, such that each part is $\leq k$.

(c) Show that the number of partitions of n into at most k parts is as big as the number of partitions of $n+k$ into exactly k parts.

(d) Show that the number of partitions of n into exactly k different odd parts equals the number of self-conjugated partitions of n into k parts.

(e) Show, that each partition of n has either at least $\sqrt{n}$ parts or the biggest part is $\geq \sqrt{n}$.

(4) How many functions $f : [n] \to [n]$ exist, such that at most two elements are mapped to the same image, i.e. $|f^{-1}(a)| \leq 2$ for all $a \in [n]$? (Hint: Do not expect to find a nice closed form. A sum is fine)

(5)

(a) The Bell number $B(n) = \sum_{k=1}^{n} S(n,k)$ is the number of all partitions of $[n]$. Give a combinatorial proof for the recursion $B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k)$.

(b) Give a combinatorical argument, which proves that the number of partitions of $[n]$, such that no two consecutive numbers appear in the same block, is $B(n-1)$.

5. Practice sheet for the lecture: Combinatorics (DS I) **Felsner/ Heldt, Knauer**

9. Mai

Delivery date: 16. – 20. May

`http://www.math.tu-berlin.de/~felsner/Lehre/dsI11.html`

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(1) In how many ways can you pay n Dollar with 1\$, 5\$ and 10\$ notes? Find a generating function and compute the number of ways to pay 50 Dollar.
(Hint: find three distinct generating functions, each for one type of notes only, and put them together in the right way)

(2) *Please hand in your solution of this exercise in written form:* Let a_k be the number of words of length k over the alphabet $\{n, w, e\}$ with no w next to an e (i.e. no substring of the form we or ew). These words can be interpreted as lattice paths of length k which go north, west, or east and never intersect themselves. Find

(a) a recurrence equation of fixed depth for the numbers a_k.

(b) a generating function for them, computed from the linear recursion.

(c) a closed form for a_k.

(3)

(a) Count words of length k over the alphabet $\{n, w, e\}$ with no we or ew with respect to their last step, i.e. let n_k be the number of these words, ending with n, w_k the number of words ending with w and e_k the number of remaining words. So we have $a_k = n_k + w_k + e_k$ (with a_k from exercise 2). Find a matrix A such that

$$\begin{pmatrix} n_k \\ w_k \\ e_k \end{pmatrix} = A \cdot \begin{pmatrix} n_{k-1} \\ w_{k-1} \\ e_{k-1} \end{pmatrix}$$

and use the characteristic polynomial of A to derive a linear recursion for A^n and therefore one for a_k (Hint: Look at the derivation of the explicit form for Fibonacci numbers in the lecture; the same techniques are applied there). Can you also directly derive a closed form for a_k from A?

(b) Let $A(z) = \sum_{k=0}^{\infty} a_k z^k$ be the generating function for the words. Give a combinatoric proof for the equation

$$A(z) = \left(1 + 2\sum_{k=1}^{\infty} z^k\right) \cdot (z \cdot A(z) + 1)$$

and directly deduce a closed form for $A(z)$ without using exercise (1).

(4) Use the recursion for the Bell numbers $B(n+1) = \sum\limits_{k=0}^{n} \binom{n}{k} B(n-k)$ to find a exponential generating function $F(z) = \sum \frac{B(n)}{n!} z^n$ for the bell numbers (Hint: Look at $\frac{d}{dz} B(z)$).

(5) Let $k \in \mathbb{N}$ be fixed. Prove, that for each $n \in \mathbb{N}$ there are unique $a_k > a_{k-1} > \ldots > a_t \geq t \geq 1$ with $a_i \in \mathbb{N}$, such that

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_t}{t}.$$

6. Practice sheet for the lecture: **Felsner/ Heldt, Knauer**
Combinatorics (DS I) 12. Mai

Delivery date: 23. – 27. May
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI11.html`

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(1) The q–binomials fullfill the equation

$$\begin{bmatrix} n \\ m \end{bmatrix}\begin{bmatrix} m \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} n-k \\ m-k \end{bmatrix}$$

for all $n \geq m \geq k \geq 0$. Prove this via the model on $\mathbb{F}_q$ subspaces for q–binomials. Can you give other proofs?

(2) The q–binomials fullfill the equation

$$\sum_{i=0}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \cdot q^{(k+1)(n-i)} = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}$$

for all $n \geq m \geq k \geq 0$. Prove this via the lattice path model for q–binomials. Can you give other proofs?

(3) *Please hand in your solution of this exercise:* A permutation $\pi \in S_n$ is *alternating* if $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \ldots$ holds. Let $\mathrm{Alt}_n \subseteq S_n$ be the set of alternating permutations. A permutation σ is *reverse alternating* if $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \ldots$ holds. Let $\mathrm{RAlt}_n \subseteq S_n$ be the set of reverse alternating permutations.

(a) Prove $|\mathrm{Alt}_n| = |\mathrm{RAlt}_n|$.

(b) Let $E_n := |\mathrm{Alt}_n|$ and prove $2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}$ for all $n \geq 1$ (Hint: Apply (a)).

(c) Let $E_n(q) := \sum_{\pi \in \mathrm{RAlt}_n} q^{inv(\pi)}$ and $E_n^\star(q) := \sum_{\pi \in \mathrm{Alt}_n} q^{inv(\pi)}$. Prove

$$E_n^\star(q) = q^{\binom{n}{2}} E_n\left(\frac{1}{q}\right).$$

(4) Let $\mathrm{des}(\sigma) := |\{i \in [n-1] \mid \sigma_i > \sigma_{i+1}\}|$ be the number of *descents* of $\sigma \in S_n$ and

$$A_n(x) := \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)+1} = \sum_{k=1}^{n} a_{n,k} x^k.$$

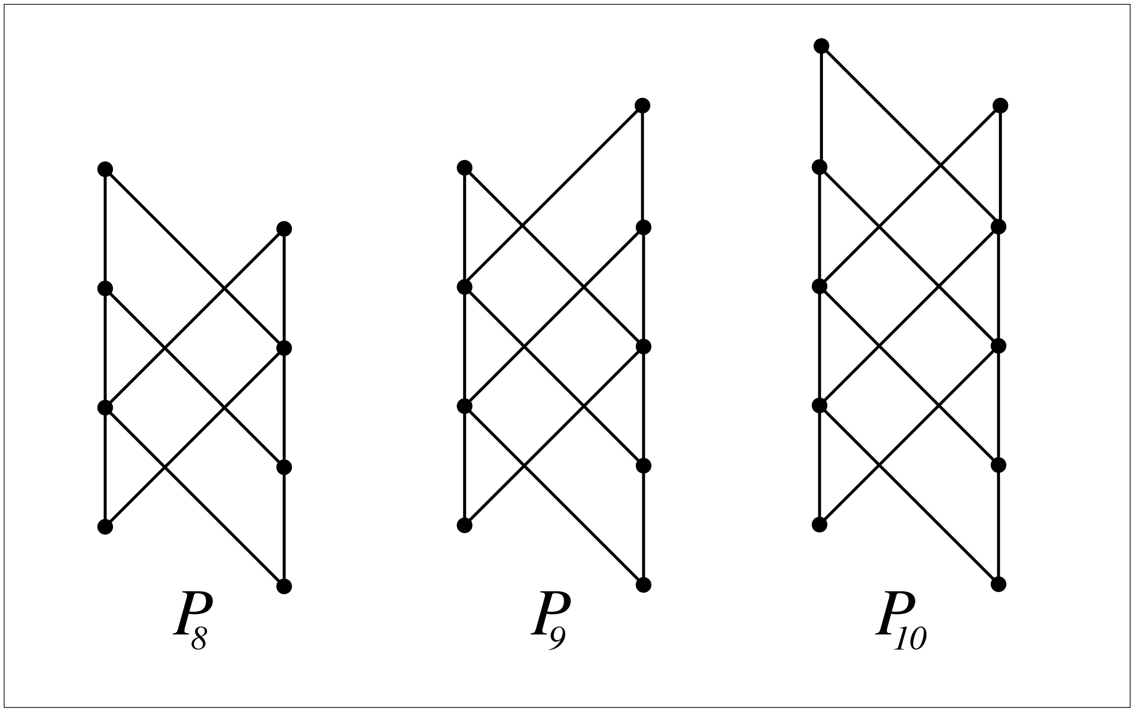
(a) Compute $A_1(x), \ldots, A_4(x)$.

(b) Prove $a_{n,k} = a_{n,n-k}$.

(c) Prove the linear recursion $a_{n,k+1} = (k+1)a_{n-1,k+1} + (n-k)a_{n-1,k}$.

(d) Use (c) to deduce an (differential) equation for $A_n(x)$.

(5) A *linear extension* of a poset $(\{m_1, \ldots, m_k\}, \leq)$ is a total ordering $m_1 <' m_2 <' \cdots <' m_k$ such that there is no $i > j$ with $m_i \leq m_j$, i.e. the total order $\leq'$ respects the poset's partial order $\leq$. Now consider the poset P_n on the set $\left\{a_1, \ldots, a_{\lceil \frac{n}{2} \rceil}, b_1, \ldots, b_{\lfloor \frac{n}{2} \rfloor}\right\}$ with the cover relations $a_i < a_{i+1}$, $b_i < b_{i+1}$, and $b_i > a_{i-1}$ as well as $a_i > b_{i-2}$ for all i. Count the linear extensions of P_n (Hint: There are pictures of the Hasse diagrams of P_8, P_9 and P_{10} on the homepage of this course available at
`http://www.math.tu-berlin.de/~felsner/Lehre/DSI11/ladder-n.pdf`).


P_8
P_9
P_{10}

7. Practice sheet for the lecture: Combinatorics (DS I) — **Felsner/ Heldt, Knauer**

20. May

Delivery date: 30. Mai – 3. June

`http://www.math.tu-berlin.de/~felsner/Lehre/dsI11.html`

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(1)

(a) Let $(P, \leq)$ be a poset, consisting of n disjoint chains of length $a_1, a_2, \ldots, a_n$. How many linear extensions does P have?

(b) Solve again exercise 1 of sheet 1. Use part (a) of this exercise to do so. The corresponding exercise was: *A spider has a sock and a shoe for each of his eight feet. In how many different ways can he put on his shoes and socks, assuming that on each foot he has to put on the sock first?*

(c) Suppose the largest chain in a finite poset P contains m elements. Show that P can be partitioned into m antichains.

(d) Let $(P, \leq)$ be a poset and $\max((P, \leq)) := \{x \in P \mid x \leq y \Rightarrow y = x\}$ be the set of its maxima. Let $e((P, \leq))$ be the number of linear extensions of $(P, \leq)$. Prove

$$e((P, \leq)) = \sum_{x \in \max(P)} e((P \setminus \{x\}, \leq')),$$

where $\leq'$ is the restriction of $\leq$ to $P \setminus \{x\}$, i.e. $\leq' := \leq \cap (P \setminus \{x\}) \times (P \setminus \{x\}) \subseteq P \times P$.

(2) A family of k-sets is *compressed* if it is an initial segment in the reverse lexicographic order of k-sets.

(a) Prove that the shadow of a compressed family is compressed.

(b) Is any compressed set the shadow of a compressed set?

(3) Let $(P, \leq)$ be a poset. An *up set* is a set $U \subseteq P$, such that for all $x \in U$ and $y \in P$ with $x \leq y$ we have $y \in U$. A *down set* is a set $D \subseteq P$, such that for all $x \in D$ and $y \in P$ with $y \leq x$ we have $y \in D$. Let A, B be up sets and C, D down sets of the boolean lattice $\mathcal{B}_n$.

(a) Prove $|A| \cdot |B| \leq 2^n \cdot |A \cap B|$ (Hint: Induction on n).

(b) Prove $|C| \cdot |D| \leq 2^n \cdot |C \cap D|$ (Hint: Apply (a)).

(c) Prove $|A| \cdot |C| \geq 2^n \cdot |A \cap C|$ (Hint: Apply (a)).

(This is known as Kleitman's Lemma)

(4) Let $(P, \leq)$ be a *tree–shaped poset,* i.e. a poset such that for each $x \in P \setminus \min(P)$ there is a parent $y \in P$ with $z \leq y < x$ for all $z \in P$ with $z < x$. Consider a recursive procedure to count the linear extensions of $(P, \leq)$. Use this procedure to derive an explicit formula involving the hook $h(x) := \#\{z \in P : x \leq z\}$ of the elements of $(P, \leq)$.

(5) *Please hand in your solution of this exercise:* Let $k \leq \frac{n}{2}$. Describe a bijection

$$f : \binom{[n]}{k} \longrightarrow \binom{[n]}{n-k}$$

with the property $A \subseteq f(A)$ for all $A \in \binom{[n]}{k}$.

8. Practice sheet for the lecture: Combinatorics (DS I) — **Felsner/Heldt, Knauer**

30. May

Delivery date: 6. – 10. June

`http://www.math.tu-berlin.de/~felsner/Lehre/dsI11.html`

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(1) *Please hand in your solution of this exercise:* Take a standard deck of 52 playing cards. Split them into 13 piles S_i, each containing 4 cards. Show that for any such splitting you can choose one card $a_i \in S_i$ from each set, such that the set $\{a_1, \ldots, a_{13}\}$ contains one card of each of the ranks $\{2, 3, 4, \ldots, 10, \text{jack}, \text{queen}, \text{king}, \text{ace}\}$.

(2)

(a) A graph $G = (V, E)$ is bipartite iff V can be partitioned into X and Y, such that all edges have one endpoint in X and one in Y. Prove that G is bipartite iff there is no odd cycle in G. An odd cycle is a sequence of vertices and edges $v_1 e_1 v_2 e_2 \ldots e_{2k} v_n e_{2k+1}$, such that $e_i = (v_{i-1}, v_i)$ and $e_{2k+1} = (v_{2k+1}, v_1)$ with $e_i \in E$ and $v_i \in V$.

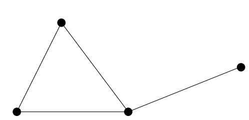
(b) Show that every regular, bipartite graph has a perfect matching. A bipartite graph is called *regular* if every vertex has degree d and a matching is a *perfect matching* if every vertex is incident to a matching edge. Does every bipartite graph have a perfect matching? Give a lower bound for the number of perfect matchings of a regular bipartite graph.

(c) Show that a regular bipartite graph can be covered with perfect matchings, i.e. that the set of edges can be partitioned into perfect matchings. Give a lower bound for the number of covers with perfect matchings.

(3) Let G be a graph and M be a matching of G. Color all edges e of G blue if $e \in M$ and red otherwise. The vertex v is *exposed* if all adjacent edges are red (i.e. do not belong to the matching). Furthermore a path between two vertices is *alternating colored,* if the path's edges are alternating red and blue. Show, that a matching is maximum (i.e. there is no matching, containing more edges) if and only if for all pairs of exposed vertices v, w there is no alternating path.

(4) The following is an incorrect proof of Dilworth's Theorem. Find the mistake:
Induction on $n := |P|$; $n = 1$ is obvious. For the induction step $n \to (n+1)$ let us assume the theorem holds for posets of n elements. Let $m \in P$ be a maxima of P. Apply the hypothesis to $P \setminus \{m\}$ and gain a decomposition into chains $C_1, \ldots, C_w$ of $P \setminus \{m\}$, with $w = \text{width}(P \setminus \{m\})$. If $w < \text{width}(P)$ then add $C_{w+1} = \{m\}$ as additional chain and we have a chain decomposition of P. If $w = \text{width}(P)$ the set $\{\max(C_i) \mid i = 1, \ldots, w\} \cup \{m\}$ can not be an antichain. Therefore $m \geq \max(C_i)$ for some $i = 1, \ldots, w$. Now $C_i \cup \{m\}$ is a chain, so $C_1, \ldots C_{i-1}, C_i \cup \{m\}, C_{i+1}, \ldots C_w$ is a chain decomposition of P. Thus in any case we have a chain decomposition of P with width(P) chains.

(5) Consider the graph G in the picture. Let $P := \left\{x \in \mathbb{R}^4_{\geq 0} \mid A \cdot x \leq \mathbb{1}\right\} \subseteq \mathbb{R}^4$, where A is the incidence matrix of G. Let $Q \subseteq [0,1]^4$ be the convex hull of the characteristic vectors of matchings of G. Visualize and compare P and Q.



The graph G.

What can you say about the relation between these polytopes for general graphs?

9. Practice sheet for the lecture: Combinatorics (DS I) **Felsner/Heldt, Knauer**

7. June

Delivery date: 13. –17. June

`http://www.math.tu-berlin.de/~felsner/Lehre/dsI11.html`

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(1) *Please hand in your solution of this exercise:* How many necklaces of length 12 with beads of 3 colors are there? Count them modulo all symmetries, i.e. rotations and flipping the necklace over.

(2) What is the Cayley graph of a group? Do some literature research to learn about this graph.

(3)

(a) Prove that every group $(G, \circ)$ with an even number of elements contains at least one element $g \neq id$ of order 2, i.e. $g \circ g = id$.

(b) Let $p \in \mathbb{N}$ be a prime number and G a group with p elements. Prove that G is a cyclic group, i.e. there is an element $a \in G$ such that all elements of G are of the form a^k for some $k \in \mathbb{N}$.

(4) Count the number of functions $f : \{0,1\}^3 \to \{0,1\}$ modulo permutations of the variables, i.e. g is equivalent to f if there is $\pi \in S_3$ such that $g(x_1, x_2, x_3) = f(\pi(x_1, x_2, x_3))$.

(5) A graph $G = (V, E)$ is *isomorphic* to a graph $H = (V', E')$, if a re-labeling of the vertices of G equals H, i.e. if there is a bijection $\Phi : V \to V'$ such that the mapping $\Psi((v, w)) := (\Phi(v), \Phi(w))$ is a bijection from E to E'. Furthermore, *loops* are edges, starting and ending in the same vertex.

(a) Count the number of non–isomorphic graphs (without loops) with four vertices.

(b) Count the number of non–isomorphic graphs with four vertices and when loops are allowed.

10. Practice sheet for the lecture: Combinatorics (DS I) **Felsner/Heldt, Knauer**

10. June

Delivery date: 20. – 24. June

`http://www.math.tu-berlin.de/~felsner/Lehre/dsI11.html`

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(1)

(a) Compute the cycle index of the symmetry group of the solid dodecahedron, acting on the dodecahedron's faces. (Hint: Use flags to identify group elements; solid means,that there are no mirror symmetries)

(b) Let D_n be the Dihedral group, i.e. the symmetry group of a regular n-gon. Compute the cycle index of D_n.

(2)

(a) Let S_n act on $\binom{[n]}{k}$ by $\pi(\{c_1, \ldots, c_k\}) := \{\pi(c_1), \ldots, \pi(c_k)\}$ with $\pi \in S_n$ and $\{c_1, \ldots, c_k\} \in \binom{[n]}{k}$. For $\sigma \in S_n$ let a_i be the number of cycles of σ of length i and b_j be the number of cycles of σ acting as a permutation on $\binom{[n]}{k}$ of length j. Express b_j in terms of $a_i, i = 1, \ldots, n$.

(b) Let $g_{n,k}$ be the number of non-isomorphic graphs on n vertices with k edges. Let G be the symmetric group on the vertices, which acts on $\binom{[n]}{2}$ by $\pi(\{i,j\}) := \{\pi(i), \pi(j)\}$. Prove

$$\sum_{k=0}^{\binom{n}{2}} g_{n,k} x^k = P_G(1+x, 1+x^2, \ldots 1+x^{\binom{n}{2}}).$$

(3) *Please hand in your solution of this exercise:* Let G be a group acting on the set M and H be a group acting on the set N with $N \cap M = \emptyset$. Let P_G be the cycle index of G with respect to M and P_H be the cycle index of H with respect to N. Consider $G \cdot H := \{g \cdot h \mid g \in G \text{ and } h \in H\}$, where

$$(g \cdot h)(a) := \begin{cases} g(a) & \text{if} \quad a \in M \\ h(a) & \text{if} \quad a \in N \end{cases}$$

Prove that $G \cdot H$ is a group acting on $M \cup N$ and that $P_{G \cdot H} = P_G \cdot P_H$, where $P_{G \cdot H}$ is the cycle index of $G \cdot H$ with respect to $M \cup N$.

(4) Let G be a group, acting on the set D. Let P_G be the corresponding cycle index. Then $F(t) = \sum_{k=1}^{|D|} f_k t^k := P_G(1+t, 1+t^2, \ldots, 1+t^{|D|})$ is a polynomial in t.

(a) Give an interpretation for f_k. (Hint: Consider the action of G on $\binom{D}{k}$ and the CFB Lemma 10.2.1)

(b) Interpret $F(t)$ in terms of Polya's fundamental Theorem (weighted case).

(5) Consider an $n \times n$ chess–board for even $n \in \mathbb{N}$. How many configurations (up to the symmetries of D_4) of n rooks (Türme) on the board are there, such that no rook can capture another one? (Hint: Use the CFB Lemma) How many distinct configurations exist, if you are only considering the symmetries of D_4, which map black squares to black squares?

11. Practice sheet for the lecture: Combinatorics (DS I) — **Felsner/Heldt, Knauer**

21. June

Delivery date: 27. June – 1. July

http://www.math.tu-berlin.de/~felsner/Lehre/dsI11.html

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(1) *Please hand in your solution of this exercise:* Let $P := (M, \leq)$ be a finite poset with $|M| = n$. A relation matrix $(a_{ij}) = A \in \{0,1\}^{n\times n}$ is a matrix, such that $a_{ij} = 1 \Leftrightarrow m_i \leq m_j$ for some order $M = (m_1, m_2, \ldots, m_n)$.

- (a) Which conditions on $(m_1, \ldots, m_n)$ suffice to ensure, that A is an upper triangle matrix?
- (b) Show that $\#\{(i,j)|a_{ij} = 1\} = \frac{n^2+n}{2} \Leftrightarrow P$ is a total order .
- (c) Which conditions on a $\{0,1\}$-matrix B imply that B represents a partial order relation?
- (d) Let k be the length of P's biggest chain. Show that A has the minimal polynomial $\mu_A(x) = (x-1)^k$.

(2) Consider the following algorithm:

Input: Poset $(P, \leq)$, linear exension L of $(P, \leq)$.
$\tilde{L} := []$.
while $P \neq \emptyset$ do:
  $x := \max_L(\mathrm{Min}_{\leq}(P))$.
  $\tilde{L} := \tilde{L} + x$ and $P = P - x$.
Output: $\tilde{L}$

Let $(P, \leq)$ be a poset, L a linear extension of $(P, \leq)$ and $\tilde{L}$ the output of the algorithm above w.r.t. $(P, \leq)$ and L.

- (a) Show that $\tilde{L}$ is a linear extension of $(P, \leq)$.
- (b) Show that $L = \tilde{L}$ if and only if $\dim(P) = 1$.
- (c) Let L be a non-separating linear extension of P. Show that $\{L, \tilde{L}\}$ is a realizer of $(P, \leq)$.
- (d) Give an example of a 2-dimensional poset $(P, \leq)$ and a linear extension L such that $\{L, \tilde{L}\}$ is not a realizer of $(P, \leq)$.

(3) Let $(P, \leq)$ be a *tree–shaped poset,* i.e. a poset such that for each $x \in P \setminus \min(P)$ there is a parent $y \in P$ with $z \leq y < x$ for all $z \in P$ with $z < x$. What is the dimension of P?

(4) For $n \in \mathbb{N}$, the *divisor–poset* P_n is the set of all divisors of n, ordered by divisibility, i.e. $P_n := \big\{\{x \in \mathbb{N} : x \mid n\}, \leq\big\}$, such that $x \leq y \Leftrightarrow x \mid y$.

- (a) Sketch the Hasse diagrams of P_{60} and P_{1001} (Hint: $1001 = 7 \cdot 11 \cdot 13$).
- (b) What is the dimension of P_n (Hint: Use the dimension of $B_{n'}$ for a well-chosen n')?

(5)

- (a) Let $(P, \leq)$ be a poset, c_k the maximal size of a k-chain of $(P, \leq)$ and $\mathcal{A}$ an anti-chain decomposition of $(P, \leq)$. Prove

$$c_k \leq \sum_{A\in\mathcal{A}} \min\{|A|, k\}.$$

- (b) Let $\lambda(B_n)$ be the Ferrer's diagram of the partition of 2^n corresponding to B_n by the Greene-Kleitman theorem. Find the shape of $\lambda(B_n)$.

12. Practice sheet for the lecture: Combinatorics (DS I) **Felsner/Heldt, Knauer**

28. June

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http://www.math.tu-berlin.de/~felsner/Lehre/dsI11.html

(1) For which of the parameter sets does a design exist? Either show that there is no design or present one.

(a) $t = 4$, $k = 7$, $v = 13$ and $\lambda = 2$.

(b) $t = 2$, $k = 7$, $v = 36$ and $\lambda = 1$.

(c) $t = 2$, $k = 5$, $v = 125$ and $\lambda = 1$.

(d) $t = 1$, $k = 4$, $v = 124$ and $\lambda = 1$.

(2) Let $(V, \mathcal{B})$ be a $S_\lambda(t, k, v)$ design. Let $p \in V$ and $\mathcal{B}^p := \{B : p \notin B \in \mathcal{B}\}$ be the set of blocks, which do not contain p. Show that $(V \setminus \{p\}, \mathcal{B}^p)$ is a design. What are its parameters?

(3) Let $(V, \mathcal{B}) = S(2, n + 1, n^2 + n + 1)$ be a projective plane and fix $B \in \mathcal{B}$. Show that $(V \setminus B, \{C \setminus B \mid C \in (\mathcal{B} \setminus \{B\})\})$ is a $S(2, n, n^2)$ design.

(4) Prove Fisher's proposition, which states that every $S_\lambda(t, k, v)$ design $(V, \mathcal{B})$ with $t \geq 2$ fullfills $|V| \leq |\mathcal{B}|$ (Hint: Use the adjacency matrix $A \in \mathbb{R}^{|V| \times |B|}$ with $a_{v,B} = 1$ if $v \in B$ and $a_{v,B} = 0$ otherwise and consider the rank of $A \cdot A^T$).

(5) Let $(V, \mathcal{B})$ be a design, $I, J \subseteq V$ with $I \cap J = \emptyset$ and $|I| = i, |J| = j$ such that $i + j \leq t$ and $k < v$. Let $\lambda_{I,J} = \#\{B \in \mathcal{B} \mid I \subseteq B \text{ and } J \cap B = \emptyset\}$.

(a) Show that $\lambda_{I,J}$ does only depend on i and j and not on I and J, i.e. $\lambda_{i,j} := \lambda_{I,J}$ is well defined.

(b) Compute all λ_{ij} for the $S_6(3, 5, 10)$ design from the lecture.

(c) Prove $\lambda_{i,j} = \lambda_{i+1,j} + \lambda_{i,j+1}$ for $i + j < t$.

(d) Prove $\lambda_{i,j} = \sum_{r=0}^{j} (-1)^r \binom{j}{r} \lambda_{i+r,0}$.