9. Practice sheet for the lecture: Combinatorics (DS I) **Felsner/ Tiwary, Heldt**

23. June

Delivery date: 1. July

`http://www.math.tu-berlin.de/~felsner/Lehre/dsI09.html`

---

(1)

(a) List all lattices consisting of five vertices. Which of them are distributive?

(b) For all distributive lattices $L = (X, \leq)$ with $|X| = 5$ give a poset P, such that L is the down set lattice of P.

(c) Let $(L, \vee, \wedge)$ be a lattice such that for all $x, y, z \in L$ we have $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. Show that this already implies $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

(2) Let $P = (X, \leq)$ be a poset and $\leq'$ a linear extension of P. Further let S be the set of all anti–chains in P. For $A, B \in S$ we say $A \prec B$ if and only if $\max_{\leq'}(A \triangle B) \in B$.

(a) Show, that $(S, \preceq)$ is a well defined poset and $\dim((S, \preceq)) = 1$.

(b) Let C and D be down sets of P. We say $C \preceq' D$ if $\max(C) \preceq \max(D)$. Show that $\preceq'$ forms a linear extension of P's down set lattice.

(3) Let $A \subseteq \mathbb{R}^n_{\geq 0}$ be a finite set and $K = \{x \in \mathbb{R}^n_{\geq 0} \mid \langle x, a \rangle \leq 1 \quad \forall\ a \in A\}$ a polytope. The polotype $K^* := \{y \in \mathbb{R}^n_{\geq 0} \mid \langle y, x \rangle \leq 1 \quad \forall\ x \in K\}$ is the *anti–blocker* of K.

(a) Show $(K^*)^* = K$ for all polytopes K.

(b) Let G be a graph and P its matching polytope. Characterize the vertices of P^*.

(4)

(a) Sketch the Hasse diagramms of the incidence posets of

$$G := \begin{array}{ccccc} \circ & - & \circ & - & \circ \\ & & | & & \\ & & \circ & & \end{array} \quad \text{and } H := \begin{array}{ccccc} \circ & - & \circ & - & \circ \\ | & & & & | \\ \circ & - & \circ & - & \circ \end{array}$$

and compute the graph's dimensions.

(b) Compute the *graph dimension* of H and G, which is the minimal size of a realizer of permutations, such that for all edges (a, b) and vertices c there is a permutation $\pi = (\ldots, a, \ldots, b, \ldots, c, \ldots)$ and a permutation $\pi' = (\ldots, c, \ldots, a, \ldots, b, \ldots)$ in the realizer.