6. Practice sheet for the lecture: Combinatorics (DS I) — **Felsner/ Tiwary, Heldt**

20. Mai

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`http://www.math.tu-berlin.de/~felsner/Lehre/dsI09.html`

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(1) For a Poset $(\{m_1,\dots,m_k\},\leq)$ a *linear extension* is total ordering $m_1 <' m_2 <' \cdots <' m_k$ such that there is no $i > j$ with $m_i \leq m_j$, i.e. the total order $<'$ respects the poset's partial order $\leq$.
Now consider the poset P_{k+j} on the set $\{a_1,\dots,a_k,b_1,\dots,b_j \mid j \in \{k-1,k\}\}$ with the relations $a_i < a_{i+1}$, $b_i < b_{i+1}$, and $b_i > a_{i-1}$ as well as $a_i > b_{i-2}$ for all i.
Count the number linear extensions of P_k.

(2) Let $k \in \mathbb{N}$ be fixed. Prove, that for each $n \in \mathbb{N}$ there are unique $a_k > a_{k-1} > \ldots > a_t \geq t \geq 1$ with $a_i \in \mathbb{N}$, such that

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_t}{t}.$$

(3) Suppose the largest chain in a finite poset P contains m elements. Show that P can be partitioned into m antichains.

(4) Let $P := (M,\leq)$ be a finite poset with $|M| = n$. A relation matrix $(a_{ij}) = A \in \{0,1\}^{n\times n}$ is matrix, such that

$$a_{ij} = 1 \Leftrightarrow m_i \leq m_j$$

for some order $M = (m_1, m_2, \dots, m_n)$.

(a) Which conditions on $(m_1,\dots,m_n)$ suffice to ensure, that A is an upper triangle matrix?

(b) Show that

$$\#\{(i,j)|a_{ij} = 1\} = \frac{n^2+n}{2} \Leftrightarrow P \text{ is a total order .}$$

(c) Which conditions on a $\{0,1\}$–matrix B imply that B represents a partial order relation?

(d) Let k be the size of P's biggest chain. Show that A has the minimal polynomial $\mu_A(x) = (x-1)^k$.

(5) Let $k \leq \frac{n}{2}$. Describe a bijection

$$f : \binom{[n]}{k} \longrightarrow \binom{[n]}{n-k}$$

with the property $A \subseteq f(A)$ for all $A \in \binom{[n]}{k}$.