5. Practice sheet for the lecture: Combinatorics (DS I)

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`http://www.math.tu-berlin.de/~felsner/Lehre/dsI09.html`

Solutions have to be handed in!

(1) For $n \in \mathbb{N}$, the *divisor–poset* P_n of n is the set of all divisors, ordered by divisibility, i.e.

$$P_n := \big\{\{x \in \mathbb{N} : x \mid n\}, \{(x,y) \in \mathbb{N}^2 : x \mid y \text{ and } y \mid n\}\big\}.$$

Proof that P_n has a rank–function. Sketch the Hasse–Diagramm of P_{60} and compute its rank–numbers.

(2) How many pairs X, Y of distinct subsets of $[n]$ with $X \subset Y \subseteq [n]$ are there? Give three different approaches to determine the number of pairs based on the three given hints:

(a) Count the pairs inductively.

(b) Give a generating function of the form $\frac{Q(z)}{P(z)}$ with polynomials P and Q.

(c) Use a direct bijection to a set of your choice.

(3) Let $\mathcal{A}$ be an *up–set* of $[n]$, i.e a set of subsets such that for every $A \in \mathcal{A}$ and every B with $A \subseteq B \subseteq [n]$ we have $B \in \mathcal{A}$. Show, that the average size of the sets in $\mathcal{A}$ is at least $\frac{n}{2}$.

(4) Let M be a finite set and let $\mathbb{P}(M) := \{X \mid X \subseteq M\}$ be the power set of M. A subset $\mathcal{A} \subseteq \mathbb{P}(M)$ is called *intersecting,* if for all $A, A' \in \mathcal{A}$ we have $A \cap A' \neq \emptyset$. Show that for each intersecting $\mathcal{A} \subseteq \mathbb{P}([n])$ there is an intersecting set of subsets $\mathcal{B} \subseteq \mathbb{P}([n])$ such that $\mathcal{A} \subseteq \mathcal{B}$ and $|\mathcal{B}| = 2^{n-1}$.

(5) Consider $x_1, x_2, \ldots, x_n \in \mathbb{R}$ such that $|x_i| \geq 1$ for $i = 1, 2, \ldots, n$ and let $I \subseteq \mathbb{R}$ be an open interval of length 1. Show that there are at most $\binom{n}{\lfloor n/2 \rfloor}$ vectors $(v_1, \ldots, v_n) \in \{0,1\}^n$ such that

$$\sum_{i=1}^{n} v_i x_i \in I.$$