3. Practice sheet for the lecture: Combinatorics (DS I)

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`http://www.math.tu-berlin.de/~felsner/Lehre/dsI09.html`

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(1) Take a walk in $\mathbb{Z}^2$, i.e. you start at $(0,0)$ and in each step you either go one unit up, down, left or right. So for example from $(2,-3)$ you can reach $(2,-2),(2,4),(1,-3)$ and $(3,-3)$ with one step. How many different tours can you take, such that you reach again $(0,0)$ after n steps? There are many correct solutions; give a nice one!

(2) Complete the lecture's proof of the binomial theorem. To do so, consider polynomials $g(x,y), h(x,y) \in \mathbb{C}[x,y]$ with $\deg(g) \leq \deg(h) \leq m$. Here the degree is $\deg(x^i y^j) := i+j$ and $\deg(\sum_{i,k} c_{i,k} x^i y^k) := \max_{i,k:c_{ik}\neq 0} \deg(x^i y^k)$. Show, if there are $x_1, \ldots, x_{m+1} \in \mathbb{C}$, such that $x_i \neq x_j$ for $i \neq j$ and $g(x_i, y) = h(x_i, y) \in \mathbb{C}[y]$ we have $g(x,y) = h(x,y)$.
Can you even weaken the requirements?

(3)

(a) The *Durfee square* of a partition P is the largest square fitting in the top left corner of P's Ferrers shape. How can you determine the square's size directly from the partition without considering the Ferrers diagram?

(b) Give a proof, stating that the number of partitions of n into at most k parts is as big as the number of partitions of $n+k$ into exactly k parts.

(c) Proof, that the number of partitions of n into different and odd parts equals the number of self-conjugated partitions of n.

(d) Show, that each partition of n has either at least $\sqrt{n}$ parts or the biggest part is $\geq \sqrt{n}$.

(4) Starting with $n! = \sum_{k=0}^{n} \binom{n}{k} d(k)$, find a equation of two exponential generating functions to derive a generating function for the number of derangements. Then deduce an explicit way to compute $d(n)$ by comparing coefficients.

(5)

(a) Let $(a)_n$ be a sequence and $b_n = \sum_{i=0}^{n} a_i$. Express $\sum_{i=0}^{\infty} b_i z^i$ in terms of $\sum_{j=0}^{\infty} a_j z^j$.

(b) A child walks up a staircase. With each step she moves up one or two stairs. Give a generating function for the number of ways to reach the n-th stair. How does the function change, if the child is also able to move up three stairs in one step?