2. Practice sheet for the lecture: Combinatorics (DS I) — **Felsner/ Tiwary, Heldt**

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`http://www.math.tu-berlin.de/~felsner/Lehre/dsI09.html`

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(1) The student lockers at a high school are numbered consecutively beginning with locker number 1. The plastic digits used to to number the lockers cost two cents a piece. Thus, it costs two cents to label locker number 9 and four cents to label locker number 10. If it costs 137 Dollar and 94 Cents to label all the lockers, how many lockers are there at this school?

(2) The mensa's cook has to set up a menu for each week. He can decide between fish–, chicken, beef–, and lamb–dishes. How many different menus can he set up,

- (a) if there are no further restrictions?
- (b) if he must cook fish on fridays?
- (c) if he can make one beef–dish at most?
- (d) if he can afford two lamb–dishes at most?
- (e) if he has to serve different dishes on consecutive days?
- (f) if each day's dish has to differ from the two precending ones?

(3)

- (a) Prove Vandermonde's identity

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}.$$

Give two different proofs, one based on the binomeal theorem and the second using the inductive definition of $\binom{n}{k}$. Which of your proofs admits what kind of values for m, n and r?

- (b) Let $x^{\underline{n}} := (x)_n$ denote the falling factorials and $x^{\overline{n}} := x \cdot (x+1) \cdots (x+n-1)$ the *raising factorials.* Deduce

$$(x+y)^{\overline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\overline{k}} y^{\overline{n-k}}$$

from Vandermonde's identity.

(4)

- (a) Let $s(n,k)$ be the Stirling number of first kind, i.e. the number of permutations in S_n consisting of k disjoint cycles. Compute all values of $s(n,k)$ for $n \leq 4$ and prove the equation

$$s(n,k) = (n-1)s(n-1,k) + s(n-1,k-1).$$

- (b) Show that the identities

$$x^{\overline{n}} = \sum_{k=0}^{n} s(n,k)x^k \text{ and } x^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k} s(n,k) x^k$$

hold (Hint: proof the first one inductively and deduce the second from the first).

(5) Let

$$Q_n := \sum_{k=0}^{k=2^n} \binom{2^n-k}{k} (-1)^k.$$

What is Q_{100000}?