10. Practice sheet for the lecture: Combinatorics (DS I) | **Felsner/ Tiwary, Heldt** | 01. July

Delivery date: 8. July

`http://www.math.tu-berlin.de/~felsner/Lehre/dsI09.html`

Solutions have to be handed in until July the 8th!

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(1)

(a) Proof that for every graph G, where all vertices have degree at most k, $\chi(G) \leq k+1$ holds.

(b) Show that for any graph $G = (V, E)$, with $\chi(G) > 1$ we can find a partition of the vertices $V = V_1 \cup V_2$ (with $V_1 \cap V_2 = \emptyset$ and $V_1 \neq \emptyset \neq V_2$) of G leading to graphs $G_1 = (V_1, E \cap (V_1 \times V_1))$ and $G_2 = (V_2, E \cap (V_2 \times V_2))$, such that

$$\chi(G_2) + \chi(G_1) = \chi(G).$$

(c) Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ be graphs. Show

$$\max\{\chi(G_1), \chi(G_2)\} \leq \chi(G_1 \star G_2) \leq \chi(G_1) \cdot \chi(G_2),$$

where $G_1 \star G_2$ is the *strong product* of G_1 and G_2. The vertices of the strong product are $V_1 \times V_2$ and two vertices $(x, y), (x', y')$ are connected if $x = x'$ and$(y, y') \in E_2$ or $y = y'$ and $(x, x') \in E_1$ or $(x, x') \in E_1$ and $(y, y') \in E_2$.
Alternatively you can add all loops (i.e. vertices (x, x)) to G_1 and G_2, take the normal product and delete all loops in the new graph to describe $G_1 \star G_2$.

(d) Which of the claims hold for $\chi_b(G)$ and/or $\chi_f(G)$?

(2) There are n children and n toys in a room. Each child wants to play with r specific toys, and for each toy, there are r children who want to play with that toy. Prove that we can organize r playing rounds so that in each of them, each child plays with the toy he wanted to, and no child plays with the same toys twice.

(3) Prove or disprove: Every b–clique is the sum of b 1–cliques and vice versa.

(4) Let G be a graph and $\chi_G(\lambda, b)$ the number of b–colourings of G with λ colours. Compute $\chi_G(\lambda, b)$ for all trees G with n vertices and deduce an upper bound of $\chi_{G'}(\lambda, b)$ for all graphs G'.

(5) Let $P = (\mathbb{N}, \leq)$ be the divisor poset of all natural numbers, i.e. $a \leq b$ if and only if $a \mid b$. Let μ be the Möbius–function of P and $r, s \in \mathbb{N}$. Compute $\mu(r, s)$ (hint: Möbius functions will be introduced in the lectures on Monday, July the 6th). You can conclude that for $\frac{s}{r} = a \in \mathbb{N}$ you can define $\mu(a) := \mu(r, s)$ which is the Möbius function from number theory.