Übungsaufgaben

Algebraische und Probabilistische Methoden in der Diskreten Mathematik

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Last change **20. Mai 2022**

(1) Show that the coefficients of the chromatic polynomial $p_G(x)$ alternate in sign.

(2) Let $p_G(x) = \sum_{k=\ell}^{n} a_k x^k$ interpret ℓ and n as parameters of G and give an interpretation of a_{n-1}.

(3) Determine the chromatic polynomial of

- cycles,
- maximal outerplanar graphs,
- the Petersen graph.

(4) Show that the chromatic polynomial of chordal graphs can be computed in polynomial time.

(5) Determine the chromatic polynomial and the number of acyclic orientations of $K_{n,m}$.

(6) Show that for 4-regular graphs

$$\#(3\text{–flows}) \quad = \quad \#(\text{2-in 2-out orientations})$$

(7) Give a proof for the delete-contract relation for the flow polynomial $F_G(y)$.

(8) An orientation of G is *strong* if whenever (u, v) belong to the same component of G there is a directed $u \to v$ path in G. Find a delete-contract relation for strong orientations. Can the number of strong orientations be expressed as $p_G(x)$ or $F_G(y)$ for some x or some y?

(9) Show that the Petersen graph admits no 4-flow but it has a 5-flow.

(10) Show that there are pairs of distinct graphs of high connectivity with identical chromatic polynomials.

(11) Show that for a connected graph G

- $x\, T_G(1+x,1) = \sum_{A \subseteq E \text{ forest}} x^{k(A)}$
- $y^{n-1}\, T_G(1,1+y) = \sum_{A \subseteq E,\ k(A)=1} y^{|A|}$

(12) Prove the uniqueness (Wohldefiniertheit) of $T_G(x,y)$ by showing that the edge orderings $e_1, \ldots, e_i, e_{i+1}, \ldots, e_m$ and $e_1, \ldots, e_{i+1}, e_i, \ldots, e_m$ yield the same polynomial.

(13) Sei $\mathbb{F}$ ein Körper und $p(x)$ ein Polynom vom Grad $\leq k$ mit $> k$ Nullstellen in $\mathbb{F}[x]$. Zeige, dass $p \equiv 0$.

(14) *Kombinatorischer Nullstellensatz II:* Sei $\mathbb{F}$ ein Körper und $P = P(x_1, \ldots, x_n)$ ein Polynom in $\mathbb{F}[x_1, \ldots, x_n]$. Angenommen es existieren $r_1, \ldots, r_n \in \mathbb{N}$, sodass $\text{Grad}(P) = \sum_{i=1}^n r_i$ ist und der Koeffizient von $\Pi x_i^{r_i} \neq 0$. Seien $S_i \subseteq \mathbb{F}$ mit $|S_i| > r_i\ \forall i$. Dann existieren $(t_1, \ldots, t_n) \in S_1 \times \cdots \times S_n$ mit $P(t_1, \ldots, t_n) \neq 0$. Beweise den Kombinatorischen Nullstellensatz II mit Hilfe des Kombinatorischen Nullstellensatzes aus der Vorlesung.

(15) Sei $H_1, H_2, \ldots H_m$ eine Familie von Hyperebenen in $\mathbb{R}^n$, die alle Ecken des Einheitswürfels bis auf den Ursprung überdeckt.

(a) Finde kardinalitätsminimale Familien für $n \leq 3$. Was passiert wenn man auf die Unüberdecktheit des Ursprungs verzichtet?

(b) Zeige, dass n Hyperebenen ausreichen.

(c) Zeige, dass $m \geq n$.
Hinweis: Die Hyperebene H_i lässt sich durch $\langle a_i, x \rangle = b_i$ beschreiben. Es gilt $b_i \neq 0\ \forall i$, da der Ursprung nicht überdeckt wird. Nimm an, dass die Aussage falsch ist, d.h. $m < n$, und betrachte das folgende Polynom:

$$P(x) = (-1)^{n+m+1} \prod_{j=1}^{m} b_j \prod_{i=1}^{n} (x_i - 1) + \prod_{i=1}^{m} (\langle a_i, x \rangle - b_i)$$

(16) The theorem of Erdős-Ginzburg-Ziv reads as follows.

- *If m is a positive integer and $a_1, \ldots, a_{2m-1}$ is a sequence of elements from the cyclic group $\mathbb{Z}_m$, then there exists a set $I \subset [2m-1]$ of cardinality m such that $\sum_{i \in I} a_i \equiv 0 \mod m$.*

Assume that the statement of the theorem is true for primes and show that this implies the statement for all $m \in \mathbb{Z}$.

Given two subsets, A, B, of a ring R, their *sumset* is defined as

$$A + B = \{a + b : a \in A, b \in B\}.$$

The following was proved in 1813 by Cauchy

- If p is prime and A, B are non empty subsets of $\mathbb{Z}_p$ then

$$|A + B| \geq \min\{p, |A| + |B| - 1\}.$$

(17) Prove Cauchy's Theorem along the following line:

- For $|A| + |B| > p$ show that $A + B = \mathbb{Z}_p$.
- If $|A| + |B| \leq p$ consider $C \supseteq A + B$ with $|C| = |A| + |B| - 2$ and the polynomial $Q(x, y) = \prod_{c \in C}(x+y-c)$. Use the CNSS to show that there exists and $a \in A$ and $b \in B$ such that $Q(a, b) \neq 0$.

A *zero p-flow* is an map $f : E \to \mathbb{Z}_p \setminus \{0\}$ such that for all $v \in V$ the sum of $f(e)$ over all e incident to v is zero. (Note that e behaves the same at both ends.)

(18) Let p be a prime. Show that G has a zero p-flow if and only if the polynomial

$$g = \prod_{v \in V} \Big(\big(\sum_{e : v \in e} x_e \big)^{p-1} - 1 \Big)$$

does not belong to the ideal generated by the polynomials $x_e^{p-1} - 1$ for $e \in E$.

- Reformulate the existence of a zero p-flow in terms of evaluations of g.
- Let g' be a polynomial of minimal degree in $g + \langle x_e^{p-1} - 1 : e \in E \rangle$ and argue that evaluations of g' and g on zero-free vectors agree.
- Use CNSS to show that G has a zero p-flow iff $g' \neq 0$.