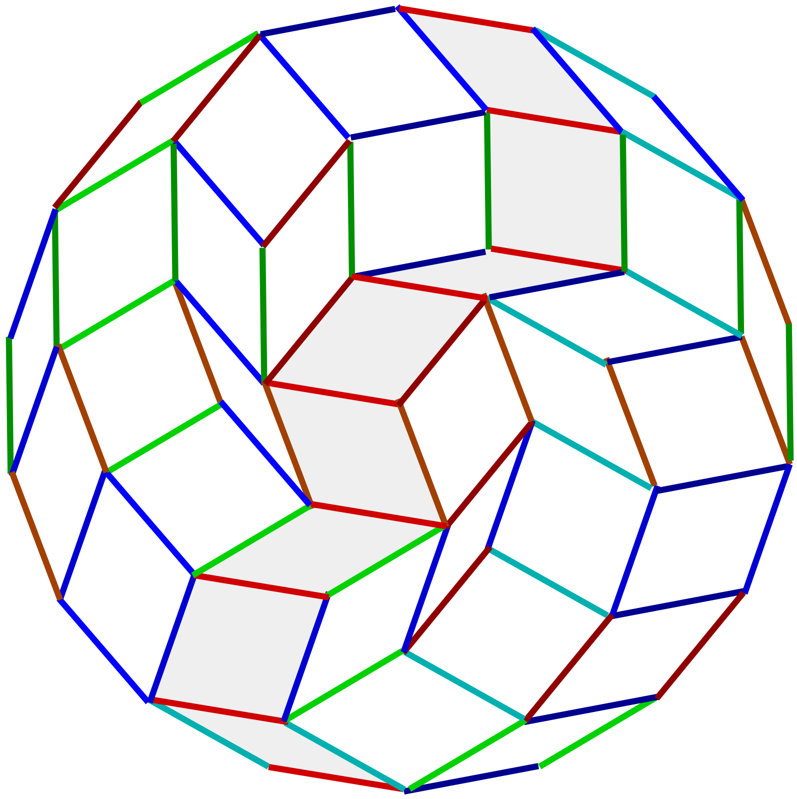
Geometric Graphs and Arrangements

Stefan Felsner

Technische Universität Berlin,
Institut für Mathematik,
Straße des 17. Juni 135,
10623 Berlin, Germany
felsner@math.tu-berlin.de



16. Dezember 2003

Preface

The body of this text is written. It remains to find some words to explain what to expect in this book. A first attempt of characterizing the content could be:

MSC 2000: 05-01, 05C10, 05C62, 52-01, 52C10, 52C30, 52C42.

In words: The questions posed and partly answered in this book are from the intersection of graph theory and discrete geometry. The reader will meet some graph theory with a geometric flavor and some combinatorial geometry of the plane. Though, the investigations always start in the geometry of the plane it is sometimes appropriate to pass on to higher dimensions to get a more global understanding of the structures under investigation. This is the in Chapter 7, for example, when the study of triangulations of a point configuration leads to the definition of secondary polytopes.

David Hilbert said: *Im großen Garten der Geometrie kann sich jeder nach seinem Geschmack einen Strauß pflücken.* I like to think of this book as a collection which makes up a kind of bouquet. A bouquet of problems, ideas and results, each of a special character and beauty, put together with the intention that they supplement each other to form an interesting and appealing whole.

The main mathematical part of the text contains only few citations and references to related material. These additional bits of information are provided in the last section of each chapter, 'Notes and References'. On average the bibliography of a chapter contains about thirty items. This is far from being a complete list of the relevant literature. The intention is to just indicate the most valuable literature so that these sections can serve as entry points for further studies. The text is supplemented by many figures to make the material more attractive and help the reader get a sensual impression of the objects. In some cases, I have confined the presentation to results which fall behind today's state of the art. I wanted to emphasize the main ideas and stop before technical complexity starts taking over. This strategy should make the mathematics accessibility to a relatively broad audience including students of computer science, students of mathematics, instructors and researchers.

The book can serve different purposes. It may be used as textbook for a course or as a collection of material for a seminar. It should also be helpful to people who want to learn something about specific themes. They may concentrate on single chapters because all the chapters are self-contained and can be read as stand alone surveys.

Topics

Chapter 1. We introduce basic notion graph theory and explain what geometric and topological graphs are. Planar graphs and some important theorems about them are reviewed. The main results of this chapter are bounds for some extremal problems for geometric graphs.

Chapter 2. We show that a 3-connected planar graph with f faces admits a convex drawing on the $(f-1) \times (f-1)$ grid. The result is based on Schnyder woods, a special cover of the edges of a 3-connected planar graph with three trees. Schnyder woods bring along connections to geodesic embeddings of planar graphs and to the order dimension of planar graphs and 3-polytopes.

Chapter 3. This is about non-planar graphs. How many crossing pairs of edges do we need in any drawing of a given graph in the plane? The Crossing Lemma provides a bound and has beautiful applications to deep extremal problems. We explain some of them.

Chapter 4. Let $\mathcal{P}$ be a configuration of n points in the plane. A k-set of $\mathcal{P}$ is a subset $\mathcal{S}$ of k points of $\mathcal{P}$ which can be separated from the complement $\bar{\mathcal{S}} = \mathcal{P} \setminus \mathcal{S}$ by a line. The notorious k-set problem of discrete geometry asks for asymptotic bounds of this number as a function of n. We present bounds, Welzl's generalization of the Lovász Lemma to higher dimensions and close with the surprisingly related problem of bounding the rectilinear crossing number of complete graphs from below.

Chapter 5. This chapter contains selected results from the extremal theory for configurations of points and arrangements of lines. The main results are bounds for the number of ordinary lines of a point configuration and for the number of triangles of an arrangement.

Chapter 6. Compared to arrangements of lines, arrangements of pseudolines have the advantage that they can be nicely encoded by combinatorial data. We introduce several combinatorial representations and prove relations between them. For each representation we give an applications which makes use of specific properties. The encoding by triangle orientations has a natural generalization which leads to higher Bruhat orders.

Chapter 7. In this chapter we study triangulations of a point configuration. The flip operation allows to move between different triangulations. The Delaunay triangulation is investigated as a special element in the graph of triangulations. This graph is shown to be related to the skeleton graph of the secondary polytope. In the special case of a point configuration in convex position they coincide. In this case, we make use of hyperbolic geometry to get a lower bound for the diameter of the graph of triangulations.

Chapter 8. Rigidity allows a different view to geometric graphs. We introduce rigidity theory and prove three characterizations of minimal generically rigid graphs in the plane. Pseudotriangulations are shown to be the planar minimal generically rigid graphs in the plane. The set of pseudotriangulations with vertices embedded in a fixed point configuration $\mathcal{P}$ has a nice structure. There is a notion of flip that allows to move between different pseudotriangulations. The flip-graph is a connected graph and it turns out that it is the skeleton graph of a polytope. The beautiful theory finds a surprising application in the Carpenter's Rule Problem.

The selection of topics is clearly governed by my personal taste. This is a drawback, because your taste is likely to differ from mine, at least in some details. The advantage is that though there are several books on related topics each of these books is clearly distinguishable by its style and content. In the spirit of 'Customers who bought this book also bought' I recommend the following four books:

- J. Matoušek
  Lectures on Discrete Geometry
  Graduate Texts in Math. 212
  Springer-Verlag, 2002.
- J. Pach and P. K. Agarwal
  Combinatorial Geometry
  John Wiley & Sons, 1995.
- G. M. Ziegler
  Lectures on Polytopes
  Graduate Texts in Math. 152
  Springer-Verlag, 1994.

- H. EDELSBRUNNER *Geometry and Topology for Mesh Generation*, Cambridge University Press, 2001.

Feedback

There will be something wrong. You may find errors of different nature. Inadvertently, I may not have given proper credit for certain contribution. You may know of relevant work that I have overlooked or you may have additional comments. In all these cases: Please let me know.

- `felsner@math.tu-berlin.de`

You can find a list of errata and a collection of comments and pointers related to the book at the following web-location:

- `http://www.math.tu-berlin.de/~felsner/gga-book.html`

Acknowledgments

There are *sine qua non* conditions for the existence of this book: The work of a vast number of mathematicians who laid out the ground and produced what I am retelling. Friends and family of mine, in particular Diana and my parents. without their persistent trust -blind or not- I would not have been able to finish this task. I am truly grateful to all of them.

In Berlin we have a wonderful environment for discrete mathematics. While working on this book, I had the privilege of working in teams of discrete mathematicians at the Freie Universität and at the Technische Unversität. I want to thank the nice people in these groups for providing me with such a friendly and supportive ‘ambiente’.

I have been involved in the European graduate college ‘Combinatorics, Geometry and Computing’ and in the European network COMBSTRU and I am glad to acknowledge support from their side.

I'm grateful to Martin Aigner, he furthered this project with advice and the periodically renewed question: “What about your book Stefan?”. For valuable discussions and for help by way of comments and corrections I want to thank Patrick Baier, Peter Braß, Frank Hoffmann, Klaus Kriegel, Mareike Massow, Ezra Miller, Walter Morris, János Pach, Günter Rote, Sarah Renkl, Volker Schulz, Ileana Streinu, Tom Trotter, Pavel Valtr, Uli Wagner, Helmut Weil, Emo Welzl, Florian Zickfeld and Günter Ziegler.

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1 Geometric Graphs: Turán Problems

The first part of this chapter collects rather elementary and well known material: We start with definitions of geometric and topological graphs, rush through basic notions from graph theory and report on facts about planar graphs. Beginning with Section 1.4 we discuss problems from the extremal theory for geometric graphs. That is, we deal with questions of Turán type: How many edges can a geometric graph avoiding a specified configuration of edges have?

1.1 What is a Geometric Graph?

As usual in texts dealing with graph theory, we shall begin by defining a *graph* G to be a pair (V, E) where V is the set of *vertices*, E is the set of *edges* $\{v, w\}$ each joining two vertices $v, w \in V$. To illustrate the definition, let us look at some pictures.

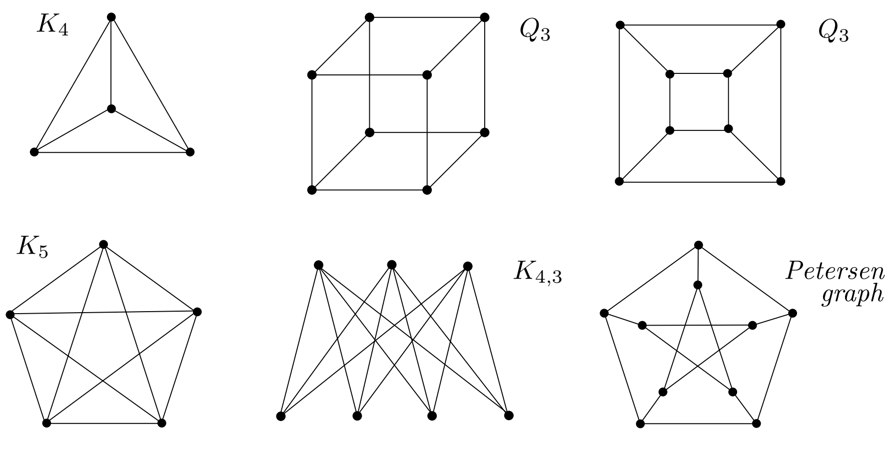


Figure 1.1 A little gallery of graphs, the cube graph Q_3 is shown with two drawings.

A *geometric graph* is a graph $G = (V, E)$ drawn in the plane with straight edges. That is, V is a set of points in the plane and E is a set of line segments with endpoints in V. For convenience, we often assume that the points of V are in general position, i.e., no three points of V are collinear. Pictures of graphs, as Figure 1.1, display geometric graphs. Hence, the study of geometric graphs can be viewed as the study of straight line drawings of a graph.

A *topological graph* is a graph drawn in the plane such that edges are represented by Jordan curves with the property that any two of these curves share at most one point. Obviously, geometric graphs are a subclass of topological graphs

In general, a drawing of a graph G will display *crossings* of edges that are disjoint as edges of G. Many interesting questions about geometric and topological graphs are concerned with crossing patterns in one way or the other. Among the simplest and most natural questions of this type is the quest of characterizing those graphs admitting a drawing without crossing edges. These *planar graphs* have been intensely studied for more than hundred years. Before initiating our studies on geometric graphs with a section on planar graphs, we continue with a brief overview of basic concepts of graph theory.

1.2 Fundamental Concepts in Graph Theory

Two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if there is a bijection $V \to V'$ such that two vertices are joined by an edge in G iff the corresponding vertices are joined by an edge in G'. When discussing the structure of a graph G the concrete set of vertices is, usually, of no relevance and G is used as an arbitrary representative of its isomorphism class. If this is not the case, then we emphasize the fact that the vertex set of G is distinguished by calling G a *labeled graph*. Two vertices joined by an edge are called *adjacent*. If vertex v is a vertex of edge e, then v and e are said to be *incident*. The *neighborhood* $N(v)$ of vertex v is the set of adjacent vertices. The size of the neighborhood is the degree $d(v) = |N(v)|$. A *complete graph* is a graph such that every pair of its vertices is an edge. With K_n we denote a complete graph (the isomorphism class of complete graphs) on n vertices.

A *path* in G is a sequence $v_1, v_2, \ldots, v_k$ of vertices of G such that $\{v_i, v_{i+1}\}$ is an edge for $1 \le i < k$. We call $v_1, v_2, \ldots, v_k$ *a path from* v_1 *to* v_k and vertices v_1 and v_k the *endpoints* of the path. A path is a *simple path* if all its vertices are distinct. A *cycle* is a path with identical endpoints, i.e., $v_1 = v_k$. A graph without cycles is a *forest*.

A graph G is *connected* if any two vertices can be joined by a path, otherwise G is *disconnected*. The maximal connected pieces of a graph G are the *components* of G. A connected forest is a *tree*. Trees have many characterizations.

Proposition 1.1 *For a graph G with n vertices the following are equivalent:*

(a) *G is connected and has no cycles (i.e., G is a tree).*

(b) *G is connected and has $n - 1$ edges.*

(c) *G has no cycles and $n - 1$ edges.*

A *leaf* is a vertex of degree 1, every tree with at least two vertices has at least two leaves. If $G = (V, E)$ is connected and W is a set of vertices, such that removing W and all edges incident to vertices in W from E leaves a disconnected graph, then we call W a *separating set* of G. For $k \ge 2$ we say that a graph G is *k-connected* if every separating set of G has cardinality at least k. For complete graphs we make the exceptional agreement that K_{k+1} is k-connected.

A *subgraph* of $G = (V, E)$ is a graph $H = (W, F)$ with $W \subseteq V$ and $F \subseteq E$. The subgraph H of G is *spanning* if H has the full vertex set of G, i.e., $W = V$. A particularly important class of spanning subgraphs are the *spanning trees*. An *induced subgraph* of G is a subgraph H containing all edges of G joining two vertices in W. The subgraph of G induced by W is denoted $G[W]$. A *clique* in $G = (V, E)$ is a set W of vertices such that $G[W]$ is a complete graph. If $G[W]$ has no edges, then we call W an *independent set*. A graph $G = (V, E)$ is a *bipartite* graph if V can be partitioned into two independent sets V_1, V_2 such that $E \subseteq V_1 \times V_2$. The graphs Q_3 and $K_{4,3}$ from Figure 1.1 are examples of bipartite graphs. Similarly, G is *k-partite* if there is a partition $V_1, V_2, \ldots, V_k$ of V such that each V_i is independent. With $K_{n_1,n_2,\ldots,n_k}$ we denote a *complete k-partite* graph: It has a k-partition $V_1, V_2, \ldots, V_k$ such that V_i contains n_i vertices and the edges are all pairs of vertices from distinct classes.

Many problems in *extremal graph theory* can be stated in the form: How many edges can a graph on n vertices satisfying a certain property P have? An important special case of this question is when P is the property of avoiding a subgraph isomorphic to some fixed graph H, graphs avoiding such a subgraph are called *H-free.* Turán initiated extremal graph theory by solving a problem of this type.

Theorem 1.2 (Turán 1941)
The number of edges of a K_{k+1}-free graph on n vertices is at most $(1 - \frac{1}{k})\frac{n^2}{2}$.

Turán also provided a complete characterization of the extremal examples. There is -up to isomorphism- a unique K_{k+1}-free graph with a maximum number of edges, this graph, the *Turán graph* $T_{k+1}(n)$ is the complete k-partite graph with $|V_i| = \lceil n/k \rceil$ or $|V_i| = \lfloor n/k \rfloor$ for all $i = 1, \ldots, k$.

Before studying extremal problems for geometric graphs, we have a section on planar graphs.

1.3 Planar Graphs

A graph is *planar* if it can be drawn in the plane without crossing edges. A *plane graph* is a planar graph together with such a drawing. In this context we do not insist that the edges are straight, that is, a plane drawing of a planar graph shows a topological graph but not necessarily a geometric graph. The existence of a straight line drawing for every planar graph is a nontrivial fact (the next chapter contains a proof). A planar drawing decomposes the plane into connected regions, one of them is the unbounded (outer) region, these regions are referred to as *faces*. To circumvent the special role taken by the outer face it is sometimes convenient to think of a planar drawing as a drawing on the sphere. A graph has a non-crossing drawing in the plane iff it has such a drawing on the sphere. This can be shown by stereographic projections: Place a sphere S on the plane and identify a point x in the plane with the point x' on the sphere where the ray from x to the north-pole N intersects S, see Figure 1.3.

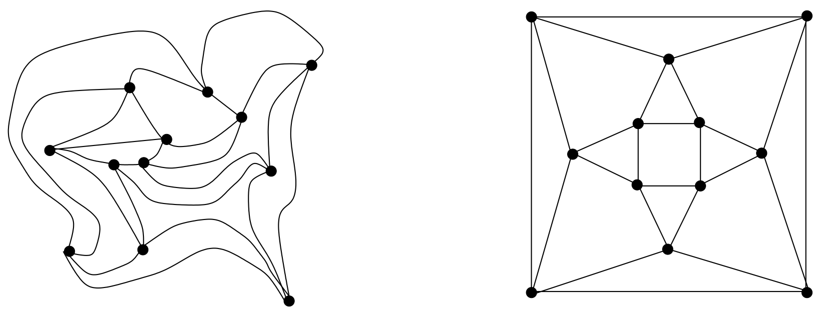


Figure 1.2 A topological and a straight plane drawing of the same graph.

One of the most fundamental and useful facts about planar graphs is Euler's Formula.

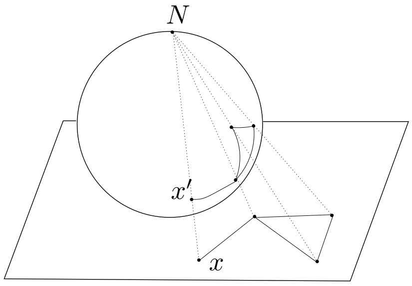


Figure 1.3 A stereographic projection.

Theorem 1.3 (Euler's Formula)
If G is a connected plane graph with v vertices, e edges and f faces, then

$$v - e + f = 2.$$

Proof. Remove the edges of G in any order from the drawing. Upon removal of any given edge a there are two possibilities:

(1) Edge a separates two different faces. Removing a reduces the number of faces by one and leaves the number of components invariant.

(2) Edge a is incident to the same face on both sides. Removing a increases the number of components by one and leaves the number of faces invariant*.

In the original graph we have one component and f faces. The final graph consisting of isolated vertices has v components and one face. Therefore, the number of edge removals of type (1) and (2) was $f - 1$ and $v - 1$ respectively. The total number of edge removals was e, hence, $e = (f - 1) + (v - 1)$. □

Note that the formula even holds if G has multiple edges connecting the same pair of vertices or loops, i.e, edges with a double endpoint. For the next result, however, we have to stay in the realm of (simple) graphs. This application of Euler's Formula gives the solution of an extremal problem.

Theorem 1.4 *A planar graph G with n vertices has at most $3n - 6$ edges. Moreover, if G is planar and contains no 3-cycle then it has at most $2n - 4$ edges.*

Proof. Assume that G is connected, otherwise some edges could be added such that the graph remains planar. Let f_k be the number of faces with k bounding edges, clearly, $\sum f_k = f$. Counting the bounding edges of all the faces we count every edge exactly twice, i.e., $2e = \sum k f_k$. Since every face has at least three edges on its boundary, this yields $2e \geq 3f$. Inserting this into Euler's Formula we obtain $3n - 6 = 3e - 3f \geq e$.

If G contains no 3-cycle every face has at least four edges on its boundary and $2e \geq 4f$. In this case we obtain $2n - 4 = 2e - 2f \geq e$. □

The proof shows that $e = 3n - 6$ and $f = f_3$ are equivalent. Hence, the extremal graphs are exactly those having only triangular faces, they are known with the name *planar triangulation*.

* This innocent statement is based on the Jordan Curve Theorem.

The count of edges alone allows the conclusion that K_5 and $K_{3,3}$ are not planar. In fact, K_5 has ten edges, one to much for a planar graph with five vertices. For $K_{3,3}$ it can be used that this graph contains no 3-cycle and $e = 9 > 8 = 2n - 4$.

Since every subgraph of a planar graph is planar we may ask for a characterization of planarity in terms of forbidden subgraphs. In a classical theorem Kuratowski has shown that the two graphs K_5 and $K_{3,3}$ are the only minimal obstructions against planarity. Let a *subdivision* of a graph G be a graph obtained from G by inserting new vertices of degree two into the edges of G. Planarity is invariant under taking subdivisions, in particular, subdivisions of K_5 or $K_{3,3}$ are not planar.

Theorem 1.5 (Kuratowski 1930)
A graph G is planar if and only if G contains no subgraph which is a subdivision of K_5 or $K_{3,3}$.

A drawing of a planar graph is called *convex* if every face boundary (including the unbounded face) is a convex polygon. Note that the 2-connected planar graph $K_{2,4}$ has no convex drawing.

Theorem 1.6 (Tutte 1960)
Every 3-connected planar graph admits a convex drawing.

Tutte's theorem can also be derived from a much older result from geometry, the theorem of Steinitz.

Theorem 1.7 (Steinitz 1922)
Every 3-connected planar graph is the skeleton graph, i.e., the graph of 0-dimensional faces (vertices) and 1-dimensional faces (edges), of a 3-dimensional polytope.

Given a 3-connected planar graph G, let P be a 3-dimensional polytope with skeleton G. Place P on the plane and choose a point a below the face of contact of P and the plane. Project each point x from P to the point x' in the plane where the ray from x to a intersects the plane. If a was chosen close enough to the plane this yields a convex drawing of G.

Planar graphs have many special properties. One of the most fundamental is the existence of a dual graph. A *dual* G^* of a plane graph G is a plane graph having a vertex in each face of G. Every edge e of G has a corresponding dual edge e^* in G^*: If F and F' are the faces on the two sides of e then e^* connects the vertices of G^* in F and F'.

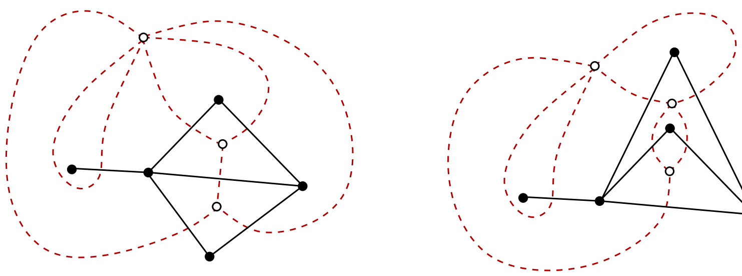


Figure 1.4 A planar graph with two non-isomorphic drawings and the corresponding duals.

Figure 1.4 illustrates that a dual may have some features which make it necessary to extend our concept of graphs. The dual G^* may have loops and multiple edges even though G is a simple graph. Having extended the definition of a graph accordingly it is almost evident from the drawings that $(G^*)^* = G$ for every connected plane graph G.

The figure also shows that a planar graph may have different drawings which have non-isomorphic duals. Whitney has characterized the well-behaving planar graphs.

Theorem 1.8 (Whitney 1933)
Let G be a planar graph. The graphs G and G^* uniquely determine each other iff G is 3-connected. Moreover, the dual is a simple graph in this case.

Let G be a plane graph and C be the edge set of a cycle in G. This cycle C splits the plane into at least two connected regions and each of these regions contains at least one face of G. If v^* and w^* are dual vertices from different regions, then every path connecting v^* and w^* has to cross some edge of C and, hence, contain an edge from the dual set C^*. Therefore, C^* is a *cut*, i.e., a disconnecting set of edges for G^*. In fact there is a bijection between simple cycles of G and minimal cuts of G^* and vice versa.

1.4 Outerplanar Graphs and Convex Geometric Graphs

A planar graph is *outerplanar* if it has a plane drawing such that all vertices lie on the boundary of the exterior face. Equivalently, G is outerplanar if the graph obtained by adding a new vertex x joined to all vertices of G is still planar.

Using Kuratowski's Theorem and the second definition of outerplanar graphs we conclude that a graph is outerplanar if and only if it contains no subgraph which is a subdivision of K_4 or $K_{2,3}$. The maximum number of edges of an n vertex outerplanar graph is $2n-3$. The extremal graphs can be drawn as convex n-gons with a triangulation of the interior by chords.

A *convex geometric graph* is a geometric graph with the property that the vertices are the vertices of a convex polygon.

A convex geometric graph without crossings is just a maximal outerplanar, therefore, a convex geometric graph with n vertices and without crossings has at most $2n-3$ edges. The next theorem deals with a less trivial extremal problem for convex geometric graphs. A pair of edges of a geometric graph is called *disjoint* if they do not cross in the geometric representation and share no vertex.

Theorem 1.9 *Let $dc_k(n)$ be the maximum number of edges that a convex geometric graph can have without containing $k+1$ pairwise disjoint edges. If $n \geq 2k+1$ then*

$$dc_k(n) = kn.$$

Proof. Let $x_0, x_1, \ldots, x_{n-1}$ be the cyclic order of the vertices of a convex geometric graph. Consider the set of all interior segments $x_i x_j$. Two segments $x_i x_j$ and $x_k x_l$, with i, j, k, l distinct are called *parallel*, if there is a t such that $k = i + t$ and $l = j - t$, where indices are taken modulo n. This equivalence relation partitions the segments into n classes of pairwise parallel segments. If G has no $k+1$ pairwise disjoint edges, then G can have at most k edges from each class. This implies $dc_k(n) \leq kn$.

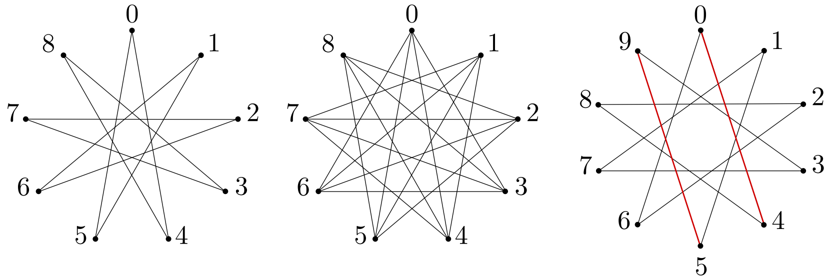


Figure 1.5 The graphs $G_1(9)$, $G_2(9)$ and $G_1(10)$.

To see that the bound can be attained we need appropriate graphs. The idea is to build a graph whose edges are the long segments: Let $G_k(n)$ be the graph whose edges are the segments $x_i x_{i+\lfloor \frac{n}{2} \rfloor + j}$ for $0 \le i \le n-1$ and $1 \le j \le k$. Figure 1.5 shows three examples.

Suppose that the vertices $x_0, x_1, \ldots, x_{n-1}$ are in clockwise order. Think of the edge $x_i x_{i+\lfloor \frac{n}{2} \rfloor + j}$ as being oriented $x_i \to x_{i+\lfloor \frac{n}{2} \rfloor + j}$. The oriented edge has $\lceil \frac{n}{2} \rceil - j - 1$ vertices on its right side and $\lfloor \frac{n}{2} \rfloor + j - 1$ vertices on its left side. Hence, the right side is always smaller than the left side. Therefore, all the edges are different and $G_k(n)$ has nk edges.

Let P be a set of pairwise disjoint edges of $G_k(n)$, we want to show $|P| \le k$. Consider the plane graph G_P consisting of the cycle $x_0, x_1, \ldots, x_{n-1}, x_0$ together with the edges of P, these edges are disjoint chords in the cycle. Therefore, G_P is an outerplanar graph. Consider the dual graph of G_P and remove the vertex corresponding to the outer face of G_P. The remaining truncated dual is a tree, hence, has at least two leaves. They correspond to faces F_1 and F_2 of G_P which have only one edge e_1 resp. e_2 of P on the boundary. The vertices of F_i, $i = 1, 2$, are the two vertices of e_i together with all vertices on one side of e_i. From the above we know that each side of each edge of $G_k(n)$ contains at least $\lceil \frac{n}{2} \rceil - k - 1$ vertices. Vertices of F_i which are not on e_i cannot be incident to edges of P. We conclude, that there are at most $n - 2(\lceil \frac{n}{2} \rceil - k - 1)$ vertices which are incident to edges of P. In general this number is $\le 2k + 2$ and if n is odd it is $\le 2k + 1$. Edges of P are disjoint, therefore, the same bound holds for the number of incidences between edges of P and vertices. This later number, however, is twice the number of edges of P. Hence $|P| \le k$ for n odd.

For n odd and $n \ge 2k + 1$ we have shown that the graph $G_k(n)$ proves $dc_k(n) \ge kn$. This is also true if n and k are both even and $n \ge 2k$. For n even and k odd however, $G_k(n)$ contains sets of $k + 1$ pairwise disjoint edges. For an example look at the graph $G_1(10)$ from Figure 1.5. It is possible to modify $G_k(n)$ to show that $dc_k(n) \ge kn$ also holds for the pairs n even and k odd. □

1.5 Geometric Graphs without $(k+1)$-Pairwise Disjoint Edges

We consider the question: How many edges can a geometric graph on n points have without having $k+1$ pairwise disjoint edges? This is an old problem with a long history, see the notes at the end of the chapter. We begin with the easiest special case.

Theorem 1.10 *A geometric graph with no two disjoint edges can have at most n edges.*

Proof. Every vertex may mark an edge incident to it. For vertices of degree one there is no choice. For all other vertices mark the right edge at the largest angle at the vertex. If there remains an unmarked edge $e = uv$ we find a situation as shown in Figure 1.6. The

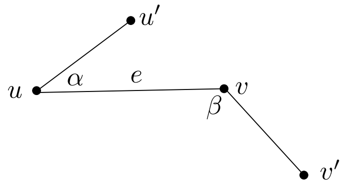


Figure 1.6 An unmarked edge e and its two disjoint adjacent edges.

angles α and β cannot be the largest angles at u and v. Hence, they are both less than π and the edges uu' and vv' must be disjoint because they reach into different halfplanes of the line supporting e. We conclude: In the absence of disjoint edges there is no unmarked edge and the number of edges is at most as large as the number of vertices. □

More complex marking arguments have been used to show that $3n+1$ edges force three disjoint edges and $10n + 1$ edges force four disjoint edges. The following theorem deals with the general case.

Theorem 1.11 *A geometric graph with no $k + 1$ disjoint edges cannot have more then $256\, k^2 n$ edges.*

Proof. Let G be a geometric graph on n vertices containing no $k + 1$ disjoint edges. Let $x(v)$ and $y(v)$ denote the x- and y-coordinate of point v. If there are two vertices with the same x-coordinate we slightly rotate the plane to make all x-coordinates different. Edge e is said to *lie above* edge e' if every vertical line intersecting both intersects e above e'. Define four relations $\prec_1, \prec_2, \prec_3$ and $\prec_4$ on disjoint pairs of edges. Let $e = vw$ and $e' = v'w'$ be two edges such that e is above e' and $x(v) < x(w)$ and $x(v') < x(w')$, we define:

$$\begin{aligned} e' \prec_1 e \quad &\text{iff} \quad x(v) < x(v') \text{ and } x(w) < x(w'), \\ e' \prec_2 e \quad &\text{iff} \quad x(v) > x(v') \text{ and } x(w) > x(w'), \\ e' \prec_3 e \quad &\text{iff} \quad x(v) < x(v') \text{ and } x(w) > x(w'), \\ e' \prec_4 e \quad &\text{iff} \quad x(v) > x(v') \text{ and } x(w) < x(w'). \end{aligned}$$

The definitions are illustrated in Figure 1.7. It is important to note that each of these relations is transitive, i.e., an order relation.

For any vertex v_i we partition the edges incident to v_i into *left edges*, those edges v_i, v_j with $x(v_j) < x(v_i)$, and *right edges*, those edges v_i, v_j with $x(v_i) < x(v_j)$. The *left degree* l_i is the number of left edges at v_i and the *right degree* r_i is the number of right edges at vertex v_i.

For every vertex v define two linear orders on the right edges of v. $R_s(v)$ is the order by decreasing slope and $R_x(v)$ is the order by increasing x-coordinate. The intersection $R_s(v) \cap R_x(v)$ is a partial order, actually a two-dimensional order.

The following theorem is a corollary from the Greene-Kleitman duality theory:

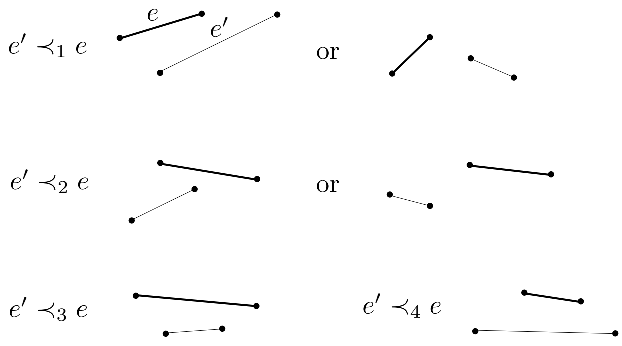


Figure 1.7 The four orders defined on disjoint edges.

Theorem 1.12 *Let $P = (X, <)$ be a partial order on n elements, then there is a family $\mathcal{C}$ of at most $\sqrt{n}$ chains and a family $\mathcal{A}$ of at most $\sqrt{n}$ antichains such that the chains and antichains in $\mathcal{C} \cup \mathcal{A}$ cover all elements of P.*

Hence, the right edges of a vertex v can be covered with at most $\sqrt{r_v}$ chains and $\sqrt{r_v}$ antichains. Let $C^r(v)$ be the set of edges covered by chains and $A^r(v)$ be the set of edges covered by antichains.

One of $\bigcup_v C^r(v)$ and $\bigcup_v A^r(v)$ contains at least one half of all edges of G. Let G' be the graph restricted to the edges of the larger class, clearly $e(G') > e(G)/2$. The *right block* of an edge of G' is the set of edges of the chain (or antichain) it belongs to. The order of edges of a right block is the order by decreasing slope, in this order the edges of a right block form an x-increasing (chain) or x-decreasing (antichain) sequence.

Let l'_v be the left degree of vertex v in G', clearly $l'_v \leq l_v$. We now uniformize the left edges of every vertex v in G'. Let $L_s(v)$ be the order by increasing slope and $L_x(v)$ be the order by increasing x-coordinate. As before, the intersection order $L_s(v) \cap L_x(v)$ can be covered by at most $\sqrt{l'_v}$ chains and $\sqrt{l'_v}$ antichains. Let $C^l(v)$ be the set of edges covered by chains and $A^l(v)$ be the set of edges covered by antichains. Let G'' be the graph restricted to the edges of the larger class of $\bigcup_v C^l(v)$ and $\bigcup_v A^l(v)$, clearly $e(G'') > e(G)/4$. The *left block* of an edge of G'' is the set of edges of the chain (or antichain) it belongs to. The order of edges of a left block is the order by increasing slope, in this order the edges of a left block form an x-increasing (chain) or x-decreasing (antichain) sequence.

The restriction of a right block of G' to the edges of G'' is a right block of G''. For two edges with a common endpoint, $e = vu$ and $e' = vw$, we say that (e, e') is a *right-zag*, if e' follows immediately after e in the same right block at vertex v. Analogously, for two edges, $e = uv$ and $e' = wv$, we say that (e, e') is a *left-zag*, if e' follows immediately after e in the same left block at v.

A path $e_1, e_2, \ldots, e_m$ of G'' is a *zig-zag* if in every pair e_i, e_{i+1} of consecutive edges is either a right-zag or a left-zag with left and right alternating along the path.

Claim 1. Every zig-zag of G'' contains at most $2k$ edges.

There are four cases, the right/left blocks can be increasing/decreasing, these cases correspond to the four order relations $\prec_i$. We detail one case, the other cases can be handled analogously.

Assume that all right blocks of G' (and hence of G'') are x-increasing and all left blocks of G'' are x-decreasing. Let $e_1, e_2, \ldots, e_{2k+1}$ be a zig-zag of length $2k+1$. We claim that $e_{i+2} \prec_3 e_i$ for all $1 \le i \le 2k-1$. Consider e_i, e_{i+1}, e_{i+2} and let a, b, c, d be the sequence of vertices of this zig-zag. The pair (e_i, e_{i+1}) is a right-zag or a left-zag at vertex b. We distinguish these two cases, see Figure 1.8.

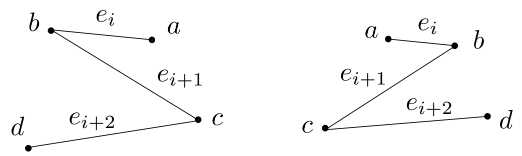


Figure 1.8 Edges e_i, e_{i+1} form a right-zag or a left-zag at b.

(1) If (e_i, e_{i+1}) is a right-zag then $x(a) < x(c)$. Since (e_{i+1}, e_{i+2}) is a left-zag at c it follows that $x(b) > x(d)$. Clearly, e_{i+2} is below e_i, so $e_{i+2} \prec_3 e_i$.

(2) If (e_i, e_{i+1}) is a left-zag then $x(a) > x(c)$. Since (e_{i+1}, e_{i+2}) is a right-zag at c it follows that $x(b) < x(d)$. Again, e_{i+2} is below e_i and $e_{i+2} \prec_3 e_i$.

Consequently, a zig-zag of length $2k+1$ contains a chain $e_{2k+1} \prec_3 e_{2k-1} \prec_3 \ldots \prec_3 e_1$ of length $k+1$. This is a contradiction, since the edges of a chain are pairwise disjoint. $\triangle$

Claim 2. If Z is the number of maximal zig-zags in G'', then $Z \le 2\sqrt{e(G)n}$.

Let $e_1, e_2, \ldots, e_m$ be a maximal zig-zag if v is the vertex of e_1 which is not incident to e_2 then e_1 is the first element of its block at v. Therefore, the number of maximal zig-zags starting at a vertex v is at most the number of blocks of edges at v. By construction v has at most $\sqrt{r_v}$ right blocks and at most $\sqrt{l'_v} \le \sqrt{l_v}$ left blocks. This can be turned into an estimate for Z. Apply a variant of the Cauchy-Schwarz inequality, namely, $(\sum_{i=1}^n a_i)^2 \le n(\sum_{i=1}^n a_i^2)$ twice and use the obvious equation $\sum_{i=1}^n (r_i + l_i) = 2e(G)$, to obtain:

$$Z \le \sum_{i=1}^n (\sqrt{r_i} + \sqrt{l_i}) \le \sqrt{n \sum_{i=1}^n (\sqrt{r_i} + \sqrt{l_i})^2} \le \sqrt{n \sum_{i=1}^n 2(r_i + l_i)} = \sqrt{4ne(G)}.$$

$\triangle$

Each edge of G'' is covered by at least one maximal zig-zag and each zig-zag contains at most $2k$ edges, hence $e(G'') \le 2kZ$. By construction $e(G'') \ge e(G)/4$, together with Claim 2 this yields $e(G) \le 16\, k\sqrt{e(G)n}$. Squaring the inequality and dividing by $e(G)$ we obtain $e(G) \le 256\, k^2 n$. $\square$

1.6 Geometric Graphs without Parallel Edges

Assuming general position for the endpoints of two disjoint segments in the plane we can distinguish two cases. Either the convex hull of their four endpoints is a triangle or a quadrangle, in the second case we say that the two *edges are parallel*, see Figure 1.9.

A geometric graph on n points with no $k+1$ pairwise parallel edges can have more edges than a geometric graph with no $k+1$ pairwise disjoint edges. In the case of parallel edges, already the solution to the extremal problem for $k = 1$ requires the use of interesting techniques.

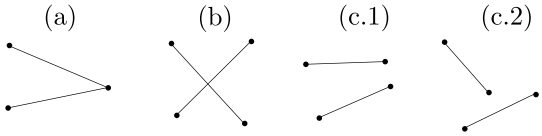


Figure 1.9 The four possible positions of two edges in a geometric graph, (a) adjacent, (b) crossing, (c.1) parallel, (c.2) stabbing.

Kupitz gave examples showing that in the absence of parallel edges a geometric graph can have as much as $2n-2$ edges, see Figure 1.10, he conjectured that this is the maximum.

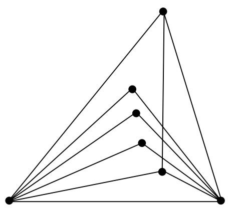


Figure 1.10 The construction of Kupitz for $n = 7$.

Theorem 1.13 *A geometric graph on n vertices with no two parallel edges can have at most $2n-2$ edges.*

Let G be a geometric graph on n vertices and with $e(G)$ edges. For an edge e of G let l_e be the line supporting e. Choose a circle C such that all vertices $v_1, v_2, \dots, v_n$ of G are in the interior of C. For every edge $e = v_i v_j$ consider the two points of intersection of line l_e with circle C and label them with the index of the closer vertex, i.e., traversing l_e we find point i, vertex v_i, vertex v_j and point j in this order. Let $S(G)$ be the circular sequence obtained from reading the labels of the intersection points along C in clockwise order. The length $|S(G)|$ of $S(G)$ is $2e(G)$.

From $S(G)$ we construct a reduced circular sequence $RS(G)$ by erasing all the labels which are equal to their predecessor labels. The reduced sequence for the graph in Figure 1.11 is $RS(G) = (4, 2, 3, 5, 3, 1, 3, 2, ($.

Lemma 1.14 *If G is a geometric graph without parallel edges, then $|RS(G)| \geq e(G)$.*

Proof. Consider an edge $e = v_i v_j$ and the labels $i = i_e$ and $j = j_e$ in $S(G)$ that come from e. Assuming that the successor of i_e in $S(G)$ is i and the successor of j_e is j we find parallel edges e_i and e_j as indicated in Figure 1.12. This shows that if G has no parallel edges, then every edge has at least one remaining label in $RS(G)$. In other words $|RS(G)| \geq e(G)$. □

Lemma 1.15 *If G is a geometric graph without parallel edges, then there is no circular subsequence i, j, i, j in $RS(G)$.*

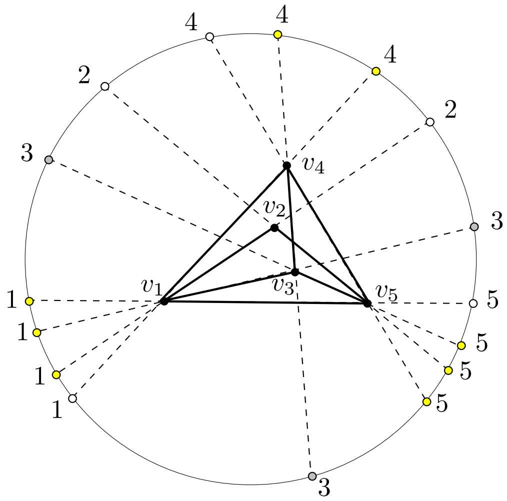


Figure 1.11 A graph G with circular sequence $S(G) = (4, 4, 4, 2, 3, 5, 5, 5, 5, 3, 1, 1, 1, 1, 3, 2($.

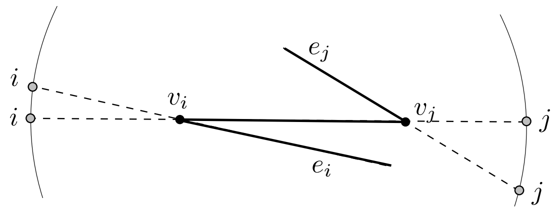


Figure 1.12 An edge with two erased successor labels implies a pair of parallel edges.

Proof. Suppose that there is a subsequence i, j, i, j in $RS(G)$. Let ℓ be the line through v_i and v_j. A rotation makes ℓ vertical, such that v_i is above v_j. One of the following two events is unavoidable: Reading clockwise there is a ji subsequence right of ℓ or a ij subsequence left of ℓ. In the second case exchange the labels i and j and rotate. Now we have the situation illustrated in Figure 1.13. The crossing right of ℓ of the supporting lines

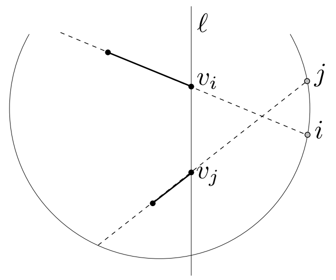


Figure 1.13 A i, j, i, j subsequence implies a pair of parallel edges.

of two edges situated left of ℓ shows that these two edges are parallel, a contradiction. □

The next lemma implies that $|RS(G)| \leq 2n - 2$. Together with Lemma 1.14 this completes the proof of Theorem 1.13.

Lemma 1.16 *If R is a circular sequence with entries from a set of $n > 1$ symbols such that no two adjacent entries are identical and R contains no circular subsequence of type abab, then the length of R is at most $2n - 2$.*

Proof. Let R be a circular sequence on n symbols such that no two adjacent entries are identical and R contains no circular subsequence of type *abab*. In the literature such a sequence is called a circular Davenport-Schinzel sequence of order 2. Let a be a symbol with $k \geq 2$ occurrences in R and decompose the sequence using this symbol, $R = aS_1aS_2a \dots aS_k$. The forbidden pattern enforces that for $i \neq j$ the subsequences S_i and S_j have no symbol in common. Let λ_i be the number of symbols of S_i. If we consider S_i as a circular sequence it contains no *abab* but the first and the last element may be equal. After removal of such a duplication the length of the sequence is at most $2\lambda_i - 2$, by induction. Therefore, $|S_i| \leq 2\lambda_i - 1$ and

$$|R| \leq \sum_{i=1}^{k} (2\lambda_i - 1) + k = 2\sum_{i=1}^{k} \lambda_i = 2(n-1).$$

□

1.7 Notes and References

A good introduction to many aspects of graph theory is the book of West [214], with its more then 500 references this book can serve as a starting point for deeper studies. Biggs et al. [26] give a nice account to the early history of graph theory. Extremal graph theory is the topic of a survey by Bollobás [29]. Four different proofs of Turán's theorem can be found in The Book by Aigner and Ziegler [8]. The existence of a straight line drawing for every planar graph is a theorem independently obtained by Wagner [208], Fáry [79] and Stein [187]. As part of his "Geometry Junkyard" Eppstein has collected fifteen proofs of Euler's Formula [72]. A chapter in The Book [8] is devoted to applications of the formula. The classification of regular polytopes is another application, see e.g. [214]. Extensions of Euler's formula in the theory of polytopes, in particular the Euler-Poincaré formula, are presented in the polytope book of Ziegler [219]. West [214] reproduces ideas of Thomassen to give a joint proof for the theorems of Kuratowski and Tutte. The theorem of Tutte is one of the main results in the next chapter of this book. The hard direction of Steinitz's theorem, to produce a polytope with a prescribed skeleton, can be proven in two ways. An inductive approach based on ΔY transformations is detailed by Ziegler [219]. Richter-Gebert [162] extends the approach used by Tutte to prove his theorem, this leads to a special plane drawing of the graph which has the property that the vertices can be lifted so that they form the desired polytope. More detailed comments about this approach can be found in the notes section of Chapter 2.

The book [142] of Mohar and Thomassen contains a rich chapter on planar graphs with many references. In particular they are careful about the use of the Jordan Curve Theorem in this theory.

Theorem 1.8 is a byproduct of Whitney's characterization of all plane embeddings of a 2-connected planar graph. He shows [217] that all these embeddings can be obtained from a given one by a series of *switchings* of two connected components. The construction of the dual graph introduced here is geometric. A multi-graph G^* is an *algebraic dual* of G if there is a bijection $E(G^*) \leftrightarrow E(G)$ between the edge-sets which maps simple cycles of G to minimal cuts of G^* and vice versa. Whitney characterized planar graphs as those graphs admitting an algebraic dual. The study of cuts and duals leads to the definition of a *matroid* and the rich theory thereof, West [214] gives an introduction and further references.

Outerplanar graphs and convex geometric graphs.

According to Pach [148] Theorem 1.9 was first obtained by Kupitz 1982. To show that the bound of the theorem is best possible the family $G_k(n)$ is described in [148]. The "dual" extremal problem was solved by Capoyleas and Pach [45]: The maximum number of edges that a convex geometric graph can have without containing $k+1$ pairwise crossing edges is $k(2n-2k-1)$ for all $n \geq 2k+1$.

Kupitz and Perles [127] study the maximum number of edges that a convex geometric graph can have without containing $k+1$ disjoint edges in convex position, that is all the edges are edges of the convex hull of their vertices. If $n \geq 2k+1$ then this number is $t_k(n)+n-k$, where $t_k(n)$ is the Turán number of Theorem 1.2.

Geometric graphs with no $k+1$ pairwise disjoint edges.

Besides the study of planar graphs, the first investigations on geometric graphs were in the context of repeated distances. For example the problem of determining the maximum number of diametral pairs of a set of n points in the plane is closely related to the maximum number of edges of a geometric graph with no disjoint edges. A related problem was posed by Hopf and Pannwitz in 1934. They ask for a proof that only for odd n it is possible to choose n points such that all the pairs x_i, x_{i+1}, $i = 1, .., n$ are diametral. Sutherland [191] and Fenchel [90] proposed solutions. Erdős [74] proved the bound of Theorem 1.10 in the context of distance problems.

The question about the maximum number $d_k(n)$ of edges of a geometric graph with no $k+1$ pairwise disjoint edges was raised by Kupitz and Perles [125] and again stated by Akiyama and Alon [10]. Alon and Erdős proved, $d_2(n) \leq 6n$. Goddard et al. [99] improved to $d_2(n) \leq 3n+1$, they also proved $d_3(n) \leq 10n+1$ and gave the first general upper bound $d_k(n) \in O(m(\log n)^{k-3})$. Pach and Törőcsik [154] introduced the order relations $\prec_i$, $i = 1,2,3,4$, on disjoint edges. As an application of Dilworth's Theorem about chain decompositions of orders they could show that $d_k(n)$ is linear in n, more precisely $d_k(n) \leq k^4 n$. Tóth and Valtr [199] added the concept of a zig-zag and improved to $d_k(n) \leq k^3(n+1)$. They also construct examples to show the lower bound $d_k(n) > 1.5(k-1)n - 2k^2$. Tóth [197] obtained the bound on $d_k(n)$ of order ck^2n (Tóth says $c = 2^9$ but his argument only yields $c = 2^{13}$). Tóth is using a greedy approach based on the Erdős-Szekeres Theorem to define the blocks of edges at each vertex. Replacing this by an application of the more sophisticated Theorem 1.12 made the slightly improved constant in Theorem 1.11 possible.

Theorem 1.12 is an offspring of the Greene-Kleitman theory about chain and antichain families in orders. A nice proof based on min-cost flows is given by Frank [92]. The Greene-Kleitman Theory was surveyed by West [213] and more recently by Britz and Fomin [40].

It is widely believed that $d_k(n) \sim ckn$ for some very moderate c. It would already be very nice to know this for the restricted class of those geometric graphs admitting a line which intersects all the edges. Let $dl_k(n)$ be the maximum number of edges such a geometric graph can have if it has no $k+1$ pairwise disjoint edges. If $dl_k(n) \leq ckn$ the bound on $d_k(n)$ would drop to $d_k(n) \leq c'(k \log k)n$.

Conway defines a *thrackle* as a graph drawn in the plane such that any two distinct edges either have a common vertex or meet at exactly one point where they cross, i.e., a thrackle is a topological graph without disjoint edges. Every cycle is a thrackle (exercise).

Conway conjectured that a thrackle with n vertices has at most n edges, i.e., that the bound of Theorem 1.10 remains valid in the more general context of topological graphs.

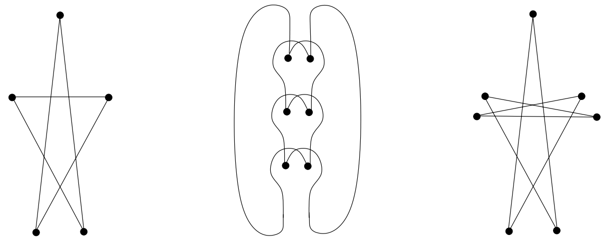


Figure 1.14 Thrackle-representations of the 5-, 6- and 7-cycle.

For about 40 years the best known upper bound on the number of edges of a thrackle was $O(n^{3/2})$. This follows from the fact that a subgraph of a thrackle is a thrackle and the four-cycle is not a thrackle. In a very nice paper Lovász, Pach and Szegedy [134] show that the number of edges of a thrackle on n vertices cannot exceed $2n-3$. Cairns and Nikolayevsky [43] improved the bound from $2n-3$ to $(3/2)(n-1)$.

Graphs with no $k+1$ pairwise parallel edges.
Kupitz [126] was the first who considered geometric graphs with no pair of parallel edges. He constructed examples of such graphs with n vertices and $2n-2$ edges and conjectured that this is the maximum. Katchalski and Last [119] proved an upper bound of $2n-1$. The argument was sharpened by Valtr [206] to yield the conjectured bound. The proof given here is a simplification of the original argument, it first appeared in Valtr [207]. Valtr also considers geometric graphs with no $k+1$ pairwise parallel edges, using Dilworth's theorem and generalized Davenport-Schinzel sequences he proves that such graphs have $O(n)$ edges.

Graphs with no $k+1$ pairwise crossing edges.
For $k = 1$ these graphs are just the planar graphs. Graphs with no three pairwise crossing edges have been named quasi-planar. Agarwal et al. [2] have shown that quasi-planar graphs have a linear number of edges. A nice question stated in that paper is: Can the edges of a quasi-planar graph G be colored with a constant number of colors, such that the edges in each color class form a planar subgraph of G? For general k the currently best bound for the number of edges of a graph with no $k+1$ pairwise crossing edges is $O(n \log n)$, due to Valtr [206]. He makes use of the bound for the extremal problem for pairwise parallel edges. The key idea is to use the mapping $T : (x, y) \to (1/x, y/x)$, this sends a pair of crossing edges, both with one endpoint in the halfplane $x < 0$ and one in the halfplane $x > 0$, to a pair of parallel edges.

In the case of convex geometric graphs the problem has a complete solution. Capoyleas and Pach [45] have shown that for $n \geq 2k+1$ the maximum number of edges of a convex geometric graph without $k+1$ pairwise crossing edges is exactly $(2n-2k-1)k$.

Several authors have considered more complex extremal problems for convex geometric graphs. A convex geometric graph H with vertices $w_1, \ldots, w_r$ in cyclic order is a *convex subgraph* of a convex geometric graph G if there is a circular subsequence of vertices $v_1, \ldots, v_r$ of G, such that $w_i w_j \in E(H)$ implies $v_i v_j \in E(G)$. Kupitz and Perles [127] solve the extremal problem for the graph H on $2k$ vertices with k edges in convex position, i.e., $E(H) = \{w_1 w_2, w_3 w_4, \ldots, w_{2k-1} w_{2k}\}$.

Gritzmann et al. [106] study the following situation: H is a 2-connected outerplanar graph on $k + 1$ vertices represented as a convex $(k + 1)$-gon with some non-crossing chords. They show that a convex geometric graph not containing H as convex subgraph can have at most $t_k(n)$ edges, where $t_k(n)$ is the Turán number. In a recent paper [34] Braß, Károlyi and Valtr review the known results and work towards an extremal theory for convex geometric graphs.

2 Schnyder Woods or How to Draw a Planar Graph?

One of the most fundamental problems around a planar graph is the question: How should the graph be drawn? This, of course, is less a mathematical question and more a matter of taste. In the graph drawing literature many answers are offered. In this chapter we present some results about drawings and other representations of (3-connected) planar graphs. The results are based on the structure of Schnyder woods.

The main drawing result shown here is Tutte's Theorem (Theorem 1.6) it says that every 3-connected planar graph admits a convex drawing. This implies that every planar graph can be drawn with straight line segments. Early proofs for convex or straight line embeddings, like Tutte's proof, would produce drawings of little practical use, because, under realistic assumptions, they could not be perceived by a human eye. More formally, the ratio between the largest and the smallest distance of vertices would be unreasonably large. This motivated the problem of computing a straight line embedding placing the verices on a grid of small size. Schnyder proved the existence of an embedding on the $(n-2) \times (n-2)$ grid. Here we use extensions of Schnyder's ideas to produce embeddings of 3-connected planar graphs with f faces on the $(f-1) \times (f-1)$ grid.

In Section 2.3 we relate planar graphs and orthogonal surfaces in $\mathbb{R}^3$. Geodesic embeddings of planar graphs on orthogonal surfaces naturally lead to a notion of a dual Schnyder wood on an appropriately defined dual graph.

Section 2.5 deals with a concept of dimension for graphs and polytopes closely related to order dimension. Theorem 2.17 reproduces Schnyder's first application of Schnyder woods: A characterization of planar graphs as those graphs whose incidence order is of order dimension at most three. The Brightwell Trotter Theorem (Theorem 2.19) is a generalization of Schnyder's Theorem to polytopes: The dimension of a 3-polytope is four but removing any face makes the dimension drop to three.

2.1 Schnyder Labelings and Woods

A *planar map* M is a simple planar graph G together with a fixed planar embedding of G in the plane. A *suspension* M^σ of M is obtained by selecting three different vertices a_1, a_2, a_3 in clockwise order from the outer face of M and adding a half-edge that reaches into the outer face to each of these special vertices.

Let M^σ be the suspension of a 3-connected planar map. A *Schnyder labeling* with respect to a_1, a_2, a_3 is a labeling of the angles of M^σ with the labels $1, 2, 3$ (alternatively: red, green, blue) satisfying three rules*:

(A1) The two angles at the half-edge of the special vertex a_i have labels $i+1$ and $i-1$ in clockwise order.

(A2) *Rule of vertices:* The labels of the angles at each vertex form, in clockwise order, nonempty intervals of 1's 2's and 3's.

* We assume a cyclic structure on the labels so that $i+1$ and $i-1$ is always defined.

(A3) *Rule of faces:* The labels of the angles at each face form, in clockwise order, a nonempty interval of 1's, a nonempty interval of 2's and a nonempty interval of 3's. At the outer face the same is true in counterclockwise order.

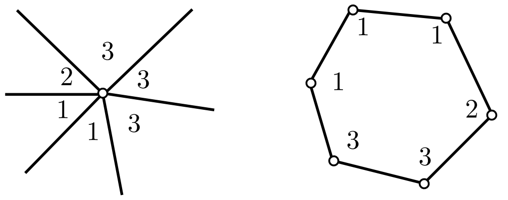


Figure 2.1 Rule of vertices and rule of faces

Let M^σ be the suspension of a 3-connected planar map. A *Schnyder wood* rooted at a_1, a_2, a_3 is an orientation and labeling of the edges of M^σ with the labels $1, 2, 3$ satisfying the following rules.

(W1) Every edge e is oriented by one or two opposite directions. The directions of edges are labeled such that if e is bioriented the two directions have distinct labels.

(W2) The half-edge at a_i is directed outwards and labeled i.

(W3) Every vertex v has outdegree one in each label. The edges e_1, e_2, e_3 leaving v in labels 1,2,3 occur in clockwise order. Each edge entering v in label i enters v in the clockwise sector from e_{i+1} to e_{i-1}.

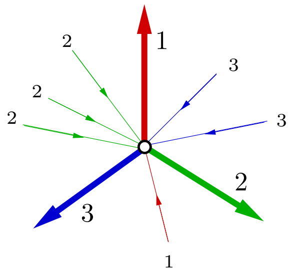


Figure 2.2 Edge orientations and edge labels at a vertex.

(W4) There is no interior face whose boundary is a directed cycle in one label.

The following lemma is central to the fact that there is a one-to-one correspondence between Schnyder labelings and Schnyder woods.

Lemma 2.1 *Let M^σ be a suspended planar map with a Schnyder labeling, then the four angles of each edge contain all three labels 1,2,3. Thus every edge has one of the two types shown in Figure 2.3.*

Proof. The proof is based on double counting and Euler's formula. Define the degree $d(v)$ of a vertex v as the number of edges incident with v whose angles at v have distinct labels. By the rule of vertices $d(v) = 3$ for every vertex v. Similarly, the degree $d(F)$ of a face F is the number of boundary edges of F whose angles in F have distinct labels. By

Figure 2.3 The two types of labeling for an edge.

the rules of faces $d(F) = 3$ for every face, in the case of the outer face the half-edges are counted as edges with distinct labels. The sum S of degrees of vertices and faces is:

$$S = \sum_v d(v) + \sum_F d(F) = 3n + 3f = 3|E| + 6.$$

The same number S can be obtained by counting the changes of label around the edges. Each of the half-edges contributes two. Hence, the average contribution of full-edges is 3. Consider the four angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of an edge in counterclockwise order. Define $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ so that $\alpha_2 = \alpha_1 + \epsilon_1$, $\alpha_3 = \alpha_2 + \epsilon_2$, $\alpha_4 = \alpha_3 + \epsilon_3$ and $\alpha_1 = \alpha_4 + \epsilon_4$. From the rules of vertices and faces $\epsilon_j \in \{0, 1\}$, for all j. The cyclic nature of the linear system implies that $\sum_{j=1}^{4} \epsilon_j = 0 \mod 3$, hence, either $\sum_j \epsilon_j = 0$ or $\sum_j \epsilon_j = 3$. Since the contribution of an edge e to the degree sum S is $\sum_j \epsilon_j(e)$ we must have $\sum_j \epsilon_j(e) = 3$ for every full-edge e. Up to rotational symmetry this only leaves the two cases shown in Figure 2.3. □

Note that from the exterior labels at special vertices (A1) and the rule of faces for the outer face we know all the edge labels at outer angles. Every outer angle on the clockwise outer path from a_i to a_{i+1} has label $i - 1$. With Lemma 2.1 we can deduce two labels i at angles incident to a_i. Together with rule (A2) applied to vertex a_i this implies:

Corollary 2.2 *In a Schnyder labeling all interior angles at the special vertex a_i are labeled i.*

Let a Schnyder labeling of the angles of a planar map be given, using the following rule the labeling induces a Schnyder wood.

($\star$) If edge $\{u, v\}$ has different angular labels i and j at vertex u, then we direct the edge from u to v in the third label k.

This rule can be reversed, so that a Schnyder wood induces a Schnyder labeling. See Figure 2.4.

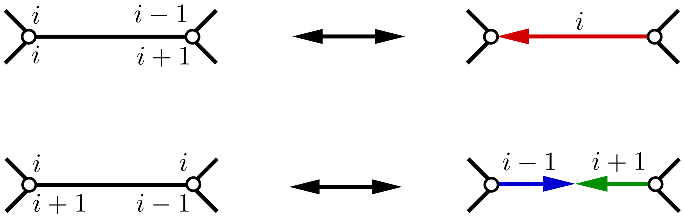


Figure 2.4 The correspondence between angle labels at an edge and the coloring and orientation of the edge.

Theorem 2.3 *Let M^σ be the suspension of a 3-connected planar map. The above correspondence is a bijection between the Schnyder labelings (axioms A1,A2,A3) and Schnyder woods (axioms W1,W2,W3,W4) of M^σ.*

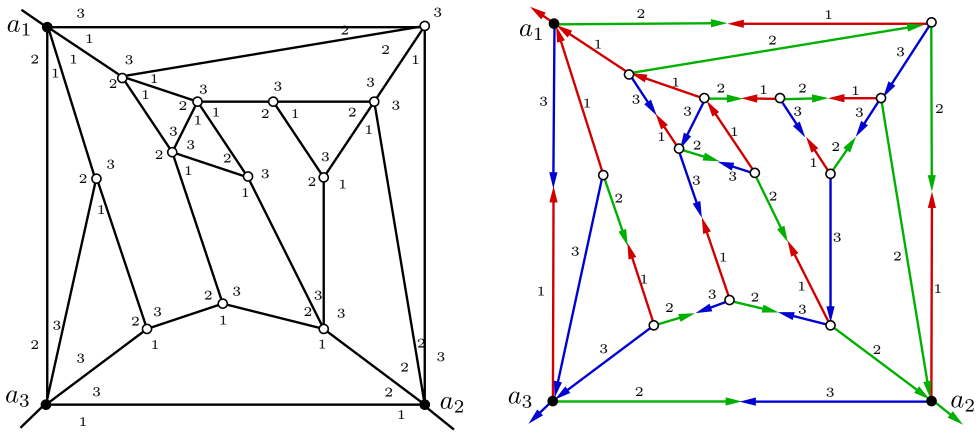


Figure 2.5 A suspended planar map with a Schnyder labeling (left) and the corresponding Schnyder wood (right).

Proof. Let a Schnyder angle labeling be given and use rule $\star$ to define orientations and labelings for the edges of the map. Lemma 2.1 shows that this orientation and labeling obeys (W1). There is an immediate correspondence of (A1) and (W2). Also the two rules of vertices (A2) and (W3) correspond to each other. If there is an interior face whose boundary is directed in one label, then we infer that all interior angles of this face have the same label. Hence, the rule of faces (A3) forces (W4). Together this shows that the construction yields a Schnyder wood.

Conversely, let a Schnyder wood be given. If the direction (u, v) of an edge is colored i then we color the angle at u to the left of edge uv with $i+1$ and the angle to the right of uv with $i-1$. If uv is unidirectional we also color the two adjacent angles at v with color i. From (W3) it follows that the two colors assigned to an angle from its two adjacent edges coincide, i.e., the coloring of angles is well-defined.

Again, the correspondence of (A1) and (W2) and of the two rules of vertices (A2) and (W3) is trivial. To show that (A3), i.e., the rule of faces, is valid is more subtle: As in the proof of Lemma 2.1 we count color changes of the angles at vertices, faces and edges. (W1) implies that at a full-edge e there are three changes, $d(e) = 3$, see Figure 2.4. For a half-edge e we let $d(e) = 2$. The contribution of vertices is $\sum_v d(v) = 3n$. The degree $d(F)$ of a bounded face F is the number of color changes at the angles when we cycle around F. If we cycle clockwise then an angle colored i is always followed by an angle colored i or $i+1$. Consequently, $d(F)$ must be a multiple of 3. Rule (W4) enforces $d(F) \neq 0$. For the unbounded face we count one color change at each half-edge, hence, $d(F) \geq 3$.

$$\sum_v d(v) + \sum_F d(F) = \sum_e d(e) \qquad \Longrightarrow \qquad 3n + \sum_F d(F) = 3|E| + 6.$$

With Euler's Formula $\sum_F d(F) = 3f$ which is only possible if $d(F) = 3$ for every face F.

We have already seen that the colors go clockwise at bounded faces and counterclockwise at the outer face. This proves the rule of faces (A3). □

Henceforth, when we have a given Schnyder wood or a Schnyder labeling we may be sloppy and refer to properties of the corresponding other structure.

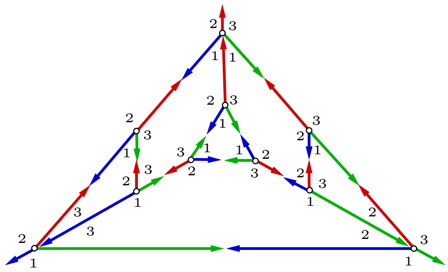


Figure 2.6 The shown orientation and coloring of edges obeys (W1), (W2) and (W3) but not (W4). The induced angle labeling is not a Schnyder labeling.

Let M be a planar map with a Schnyder wood. Let T_i denote the digraph induced by the directed edges of label i. Every inner vertex has outdegree one in T_i, therefore, every v is the starting vertex of a unique i-path $P_i(v)$ in T_i. The next lemma shows that each of the digraphs T_i is acyclic, actually we prove a bit more.

Lemma 2.4 *Let M be a planar map with a Schnyder wood (T_1, T_2, T_3). Let T_i^{-1} be obtained by reverting all edges from T_i. The digraph $D_i = T_i \cup T_{i-1}^{-1} \cup T_{i+1}^{-1}$ is acyclic for $i = 1, 2, 3$.*

Proof. Edges in $T_{i-1} \cap T_{i+1}$ remain bidirected in D_i. We do not consider bidirected edges and paths as directed cycles, hence, a directed cycle in D_i will enclose a non-empty set of faces. Let Z be a directed cycle such that the number of faces enclosed by Z is minimum. Let F be the interior region of Z, we first show that F consists of a single face.

Suppose F contains a vertex x. Start at x and always use the outgoing edge of color i to leave a vertex. This defines the i-path $P_i(x)$ of vertex x. By minimality of Z there is a simple initial part $P_i'(x)$ of $P_i(x)$ connecting x to Z. Let $P_{i-1}'(x)$ be defined analogously. By the minimality of Z the paths $P_i'(x)$ and $P_{i-1}'(x)$ have no common vertex other than x. Together with one of the two segments they determine on Z these two paths form a directed cycle in $T_i \cup T_{i-1}^{-1} \cup T_{i+1}^{-1}$ which encloses fewer faces than Z. This contradiction shows that Z contains no vertex. An edge lying in F and joining two non-consecutive vertices of Z would similarly determine a cycle enclosing fewer faces than Z.

Therefore, F is a face and Z its boundary cycle. If the traversal of Z is clockwise no angle of F has label $i+1$ and if this traversal is counterclockwise no angle has label $i-1$. Both cases are excluded by the rule of faces. □

By the rule of vertices (W3) every vertex has out-degree one in T_i. Disregard the half-edges at special vertices. This makes the vertex a_i a sink of T_i. Since T_i is acyclic and has $n-1$ edges we readily obtain:

Corollary 2.5 *T_i is a directed tree rooted at a_i, for $i = 1, 2, 3$.*

The i-path $P_i(v)$ is the unique path in T_i from v to the root a_i. Lemma 2.4 implies that for $i \neq j$ the paths $P_i(v)$ and $P_j(v)$ have v as the only common vertex. Therefore, $P_1(v), P_2(v), P_3(v)$ divide M into three regions $R_1(v)$, $R_2(v)$ and $R_3(v)$, where $R_i(v)$ denotes the region bounded by and including the two paths $P_{i-1}(v)$ and $P_{i+1}(v)$, see Fig. 2.7. The open interior of region $R_i(v)$, denoted $R_i^o(v)$, is $R_i(v) \setminus (P_{i-1}(v) \cup P_{i+1}(v))$.

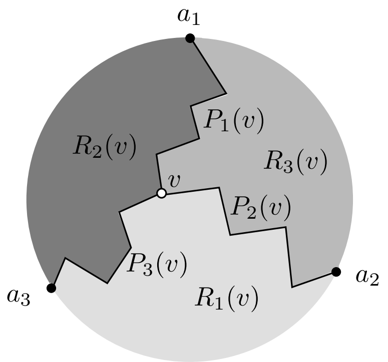


Figure 2.7 The three regions of a vertex

Lemma 2.6 *If u and v are vertices of a labeled graph with $u \in R_i(v)$, then $R_i(u) \subseteq R_i(v)$. If $u \in R_i^o(v)$, then the inclusion is proper: $R_i(u) \subset R_i(v)$.*

Proof. By symmetry it suffices to consider the case $i = 1$. Suppose $u \in R_1^o(v)$ and let x be the first vertex of $P_2(u)$ that belongs to $P_2(v) \cup P_3(v)$. From the edge orientations at x (Figure 2.2) it follows that $x \notin P_3(v)$. By the same reason $x \neq v$, hence, $x \in P_2(v)$. Similarly the first vertex y of $P_3(u)$ that belongs to $P_2(v) \cup P_3(v)$ is on $P_3(v)$ and $y \neq v$. Hence, $R_1(u) \subseteq R_1(v)$, see Figure 2.8, the inclusion is proper as $v \notin R_1(u)$.

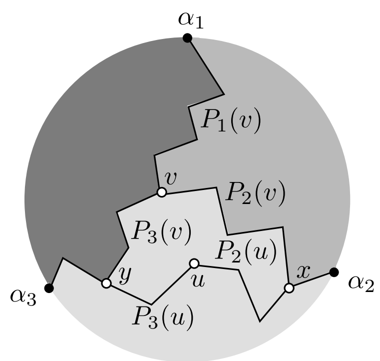


Figure 2.8 If $u \in R_1^o(v)$ then $R_1(u)$ is a proper subset of $R_1(v)$.

Now let $u \in R_1(v) \setminus R_1^o(v)$, by symmetry we only consider the case $u \in P_3(v)$. If at u the outgoing edge in label 2 is different from the incoming edge on $P_3(v)$ then a reasoning as in the previous case shows that the inclusion is proper, $R_1(u) \subset R_1(v)$. Otherwise, if u' is the other vertex of the bidirected edge leaving u in label 2 and entering in label 3, then $R_1(u') = R_1(u)$. However $R_1(u') \subseteq R_1(v)$ by induction on the number of vertices between u and v on $P_3(v)$. □

Let vertices u, v be neighbors such that the edge $e = (u, v)$ is directed from u to v in label i, see Figure 2.9. Since $v \in P_i(u)$ vertex v is contained in $R_{i-1}(u)$ and $R_{i+1}(u)$. The orientations of edges at v imply $u \in R_i(v)$. Therefore the following inclusions of regions hold:

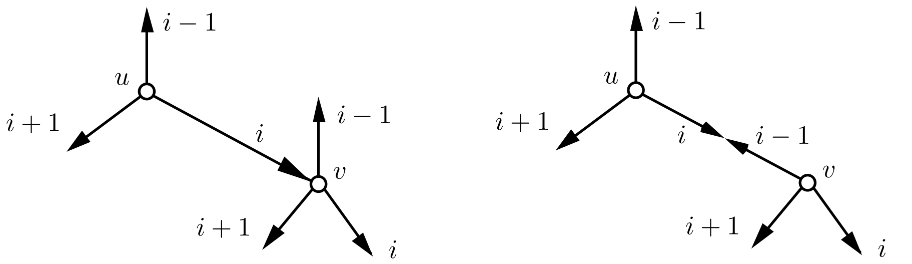


Figure 2.9 Two cases for an edge (u, v) with label i from u to v.

(E_1) If $e = (u, v)$ is an unidirectional edge with label i from u to v then $R_i(u) \subset R_i(v)$ and $R_{i-1}(u) \supset R_{i-1}(v)$ and $R_{i+1}(u) \supset R_{i+1}(v)$.

(E_2) If $e = (u, v)$ is bidirectional with label i from u to v and label $i-1$ from v to u, then $R_{i+1}(u) = R_{i+1}(v)$ and $R_i(u) \subset R_i(v)$ and $R_{i-1}(u) \supset R_{i-1}(v)$.

2.2 Regions and Coordinates

Let M^σ be a planar map with f faces and a Schnyder wood. With every vertex v of M we associate a *region vector* (v_1, v_2, v_3):

$$v_i = \#\ \text{ faces of } M \text{ in the region } R_i(v).$$

Note that the special vertices a_1, a_2, a_3 have region vectors $(f-1, 0, 0)$, $(0, f-1, 0)$ and $(0, 0, f-1)$. Translating our knowledge about inclusion of regions, in particular (E_1) and (E_2), to the region vectors we obtain:

(1) $v_1 + v_2 + v_3 = f - 1$ for all vertices v.

(2) If $u \in R_i(v)$ then $u_i \leq v_i$ and if $u \in R_i^o(v)$ then $u_i < v_i$.

(3) If an edge of M is directed from u to v in label i then $u_i < v_i$, $u_{i+1} \geq v_{i+1}$ and $u_{i-1} \geq v_{i-1}$.

(4) For every edge (u, v) of a labeled graph there are indices i, j such that $u_i < v_i$ and $u_j > v_j$.

Given three non-collinear points α_1, α_2 and α_3 in the plane. These points and the region vectors of the vertices of M can be used to define an embedding of M in the plane. A vertex of M is mapped to the point

$$\mu : v \to \frac{1}{f-1}(v_1\alpha_1 + v_2\alpha_2 + v_3\alpha_3),$$

an edge (u, v) is represented by the line segment $(\mu(u), \mu(v))$. Note that any two drawings based on points α_1, α_2 and α_3 and β_1, β_2 and β_3 can be mapped onto each other by an affine map. Geometrically μ is a linear map from the affine plane $A_f \subset \mathbb{R}^3$ defined by $x_1 + x_2 + x_3 = f - 1$ to the standard plane $\mathbb{R}^2$.

Theorem 2.7 *If M is a planar map and the coordinate v_i of vertex v counts the number of faces in $R_i(v)$ with respect to a Schnyder wood of M, then the drawing $\mu(M)$ is a convex drawing of M.*

The proof of the theorem is postponed until page 28.

Modulo the existence of Schnyder woods for 3-connected planar graphs this theorem is Tutte's Theorem. The existence of Schnyder woods is shown in Section 2.6. With the special choice $\alpha_1 = (0, f-1)$, $\alpha_2 = (f-1, 0)$ and $\alpha_3 = (0,0)$ every vertex v of M is mapped to an integral point in the $(f-1) \times (f-1)$ grid. This yields the announced version of Tutte's Theorem.

Corollary 2.8 *If M is a 3-connected planar map with f faces, then there is a convex drawing of M on the $(f-1) \times (f-1)$ grid.*

Let v be a vertex of M with coordinates (v_1, v_2, v_3). The μ-images of the three lines in A_f given by $x_1 = v_1$, $x_2 = v_2$ and $x_3 = v_3$ cross in $\mu(v)$ and partition the triangle with vertices α_1, α_2 and α_3 into six regions, see Figure 2.10. Each of the three closed shaded parallelograms contains exactly one neighbor of v. This is because if (u, v) is directed towards v then by property (3) vertex u is contained in one of the three white triangles.

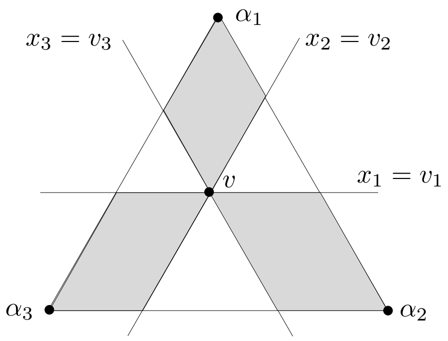


Figure 2.10 Each of the three shaded parallelograms contains exactly one neighbor of v, these are the outgoing edges at v of the three Schnyder trees.

In the proof of Theorem 2.7 we will use properties which are best understood in the context geodesic embeddings. These embeddings of planar maps are the topic of the next section.

2.3 Geodesic Embeddings of Planar Graphs

Consider $\mathbb{Z}^3$ as subsets of $\mathbb{R}^3$ and the *dominance order* on these sets, i.e, $(u_1, u_2, u_3) \leq (v_1, v_2, v_3)$ iff $u_i \leq v_i$ for $i = 1, 2, 3$. Use $u \vee v$ and $u \wedge v$ to denote the *join* (component-wise maximum) and *meet* (component-wise minimum) of $u, v \in \mathbb{R}^3$. Let $\mathcal{V} \subset \mathbb{Z}^3 \subset \mathbb{R}^3$ be an antichain, i.e., a set of pairwise incomparable elements. The *filter* generated by $\mathcal{V}$ in $\mathbb{R}^3$ is the set

$$\langle \mathcal{V} \rangle = \{\alpha \in \mathbb{R}^3 \mid \alpha \geq v \text{ for some } v \in \mathcal{V}\}.$$

The boundary $\mathcal{S}_\mathcal{V}$ of $\langle \mathcal{V} \rangle$ is the *orthogonal surface* generated by $\mathcal{V}$. Orthogonal projection onto the plane $x+y+z=0$ yields a picture of $\mathcal{S}_\mathcal{V}$ in the plane. When the integral level curves are emphasized in the picture this results in a rhombic tiling of the plane, see Figure 2.11.

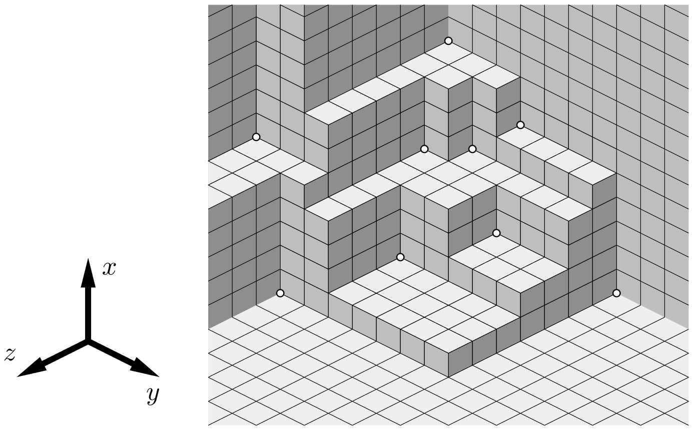


Figure 2.11 The orthogonal surface $\mathcal{S}_\mathcal{V}$ generated by $\mathcal{V} = \{(0,0,7), (0,7,0), (1,2,4), (2,4,2), (4,1,2), (4,2,1), (5,-2,6), (5,3,0), (7,0,0)\}$.

If $u, v \in \mathcal{V}$ and $u \vee v \in \mathcal{S}_\mathcal{V}$ then $\mathcal{S}_\mathcal{V}$ contains the union of the two line segments joining u and v to $u \vee v$; we refer to such arcs as *elbow geodesics* in $\mathcal{S}_\mathcal{V}$. The *orthogonal arc* of $v \in \mathcal{V}$ in direction of the standard basis vector e_i is the intersection of the ray $v + \lambda e_i$, $\lambda \geq 0$, with $\mathcal{S}_\mathcal{V}$. Clearly every vector $v \in \mathcal{V}$ has exactly three orthogonal arcs, one parallel to each coordinate axis. Some orthogonal arcs are unbounded while others are bounded. Observe that $u \vee v$ must share two coordinates with at least one (and perhaps both) of u and v, so every elbow geodesic contains at least one bounded orthogonal arc.

Let M be a planar map, a drawing $M \hookrightarrow \mathcal{S}_\mathcal{V}$ is a *geodesic embedding* of M in $\mathcal{S}_\mathcal{V}$, if the following axioms are satisfied:

(G1) *Vertex axiom.* There is a bijection between the vertices of M and $\mathcal{V}$.

(G2) *Elbow geodesic axiom.* Every edge of M is an elbow geodesic in $\mathcal{S}_\mathcal{V}$, and every bounded orthogonal arc in $\mathcal{S}_\mathcal{V}$ is part of an edge of M.

(G3) There are no crossing edges in the embedding of M on $\mathcal{S}_\mathcal{V}$.

An antichain $\mathcal{V}$ in $\mathbb{Z}^3$ is called *axial* if it contains exactly three unbounded orthogonal arcs. The example from Figure 2.11 is not axial, however, removing the point $(5,-2,6)$ from the set $\mathcal{V}$ leads to an axial antichain, see Figure 2.12.

Theorem 2.9 *Let $\mathcal{V}$ be axial and $M \hookrightarrow \mathcal{S}_\mathcal{V}$ be a geodesic embedding, then the embedding induces a Schnyder wood of M^σ. Conversely, given a Schnyder wood of a planar graph M^σ define $\mathcal{V}$ as set of region vectors of vertices of M^σ. This yields a geodesic embedding of $M \hookrightarrow \mathcal{S}_\mathcal{V}$ with an axial $\mathcal{V}$.*

Proof. Let $M \hookrightarrow \mathcal{S}_\mathcal{V}$ be an axial geodesic embedding. The edges of M are colored with the direction of the orthogonal arc contained in the edge: Arcs parallel to the x_i-axis are colored i. The orientation of an edge is chosen in accordance with the axis used to color the edge, Figure 2.12 shows an example. The claim is that this coloring is a Schnyder wood for M. Since every elbow geodesic contains one or two orthogonal arcs, axioms (W1) and (W2) are obvious. A vertex $v \in \mathcal{V}$ has exactly three outgoing orthogonal arcs. In the standard projection of $\mathcal{S}_{\mathcal{V}}$ this yields a clockwise sequence of outgoing edges in colors 1,2,3. Consider an edge $\{u, v\}$ represented by an elbow geodesic that arrives at v through the sector $S_3(v)$ between or on the outgoing edges in colors 1 and 2. The sector $S_3(v)$ is contained in the plane $x_3 = v_3$. As an elbow geodesic the edge has to pass the join $u \vee v$ of its two vertices. The join has coordinates $u \vee v = (u_1, u_2, v_3)$. Therefore the geodesic representing $\{u, v\}$ contains the orthogonal arc leaving u in direction of the x_3-axis, whence $\{u, v\}$ is oriented as (u, v) in color 3. This together with symmetric arguments for the other sectors shows (W3). A path of edges colored i gives a sequence of vertices with increasing ith coordinate. Therefore, the directed graph T_i of i colored edges is acyclic, this implies (W4).

Note that the corresponding Schnyder labeling of the angles of M is the labeling by the three different shades in the tiling figure.

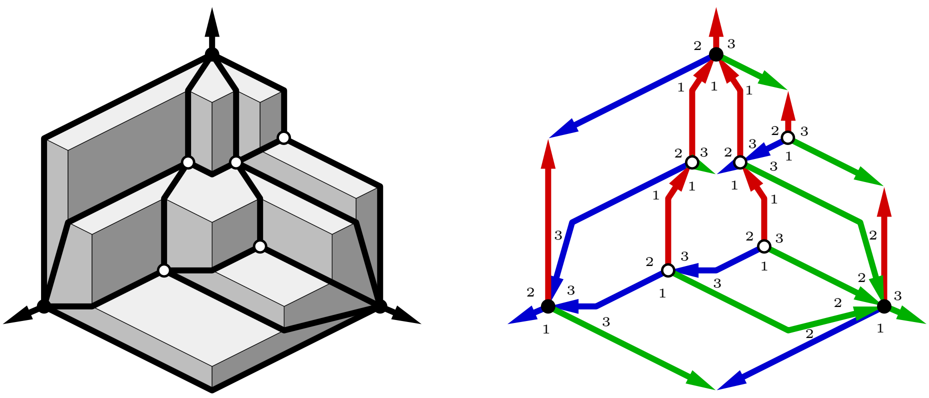


Figure 2.12 Schnyder labeling and wood induced by a geodesic embedding.

Given a Schnyder wood of M^σ embed every vertex v at its region vector $(v_1, v_2, v_3) \in \mathbb{N}^3 \subset \mathbb{Z}^3$, i.e., $\mathcal{V} = \{(v_1, v_2, v_3) : v \text{ is a vertex of } M\}$. Let f be the number of faces of M, then $v_1 + v_2 + v_3 = f - 1$ is independent of v. Hence, $\mathcal{V}$ is an antichain in $\mathbb{Z}^3$. Since the mapping is injective we have (G1). If $e = \{u, v\}$ is an edge of M and $x \notin e$ a vertex, then for some i edge e is contained in region $R_i(x)$. This implies $R_i(u) \subseteq R_i(x)$ and $R_i(v) \subseteq R_i(x)$ hence, $u_i \le x_i$ and $v_i \le x_i$. This shows that with $e = \{u, v\}$ the join $u \vee v$ and hence the elbow geodesic $[u, v]$ is on the surface $\mathcal{S}_{\mathcal{V}}$. If edge $e = (v, w)$ is directed in color i from v to w then $v_i < w_i$, $v_{i+1} \ge w_{i+1}$ and $v_{i-1} \ge w_{i-1}$ (property (3) on page 23). Therefore, the orthogonal arc of v in direction e_i is used by this edge. This yields (G2).

It remains to prove the non-crossing condition (G3). Every edge is represented by an elbow geodesic consisting of two straight legs. At least one of the legs is an orthogonal arc. An elbow geodesic cannot intersect another orthogonal arc. Suppose there is a pair $\{u, v\}$ and $\{y, z\}$ of crossing edges. The elbow geodesics representing these edges cross with their legs on a plane orthogonal to one of the coordinate axes. Up to symmetry the situation is as illustrated in Figure 2.13. We may thus assume that names of vertices and orientation are as in the figure, in particular $u_1 = y_1$, $u_3 > y_3$ and $z_3 > v_3$.

Between u and y there is a path consisting of orthogonal arcs only. With (G2) this

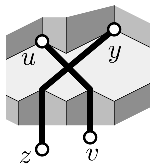


Figure 2.13 A pair of crossing elbow geodesics.

implies that in the Schnyder wood of M^σ there is a bidirected path P^* in colors 2 and 3 between u and y. P^* is directed from u to y in color 2. The crossing edges have color 1, they are unidirected as (v, u) and (z, y). Let s be the first common vertex of the color 2 paths $P_2(u)$ and $P_2(v)$. If $s \in P^*$ then there is a cycle in $T_1 \cup T_2^{-1} \cup T_3^{-1}$, hence $s \in P_2(y)$ and $s \neq y$. This proves $y \in R_3^o(v)$ and since $\{y, z\}$ is an edge also $z \in R_3(v)$. Expressed in terms of coordinates this yields $v_3 \geq z_3$, a contradiction. □

Let M be a planar map with a Schnyder wood. The corresponding surface $\mathcal{S}_\mathcal{V}$ is above the plane X defined by $x_1 + x_2 + x_3 = f - 1$. The elements of $\mathcal{V}$ which are the minima of $\mathcal{S}_\mathcal{V}$ are in this plane X. With a point $p \in \mathbb{R}^3$ above X consider the points of X dominated by p. This set is a triangle ∇_p. The border of ∇_p consists of those elements of X having a common coordinate with p.

Lemma 2.10 *Let $e = \{u, v\}$ be an edge of M and $\nabla_e = \nabla_{u \vee v}$. The triangle ∇_e has u and v on its border and the interior of ∇_e contains no vertex of M (see Figure 2.14).*

Proof. In the proof of property (G2) we have shown that $u \vee v \in \mathcal{S}_\mathcal{V}$. □

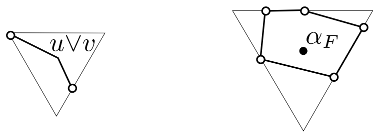


Figure 2.14 Triangles ∇_e of an edge and ∇_{α_F} of a face in the plane X.

Lemma 2.11 *Let F be a bounded face of M and $\alpha_F = \bigvee_{w \in F} w$ be the join of the vertices of F, then $\alpha_F \in \mathcal{S}_\mathcal{V}$, moreover, α_F is a maximum of $\mathcal{S}_\mathcal{V}$. All $v \in F$ are on the border of ∇_{α_F} and the interior of ∇_{α_F} contains no vertex of M (see Figure 2.14).*

Proof. Let w be any vertex of M and suppose that F is contained in region $R_i(w)$. For $v \in F$ let v^* be the last vertex of path $P_i(v)$ in region $R_i(w)$. From $v^* \in P_{i-1}(w) \cup P_{i+1}(w)$ it follows that $w_i \geq v_i^* \geq v_i$. Hence, for every w there is a coordinate i such that $w_i \geq [\alpha_F]_i$. This proves that α_F is on the surface $\mathcal{S}_\mathcal{V}$.

For $v \in F$ we have $(v_1, v_2, v_3) \leq \alpha_F$ by definition and $v_i \geq [\alpha_F]_i$ if i is such that $F \subset R_i(v)$. Therefore, $v \in F$ is on the border of ∇_{α_F}.

Let v be a vertex with $v_i = [\alpha_F]_i$. For the clockwise neighbor u of v at F we find $R_{i+1}(u) \supset R_{i+1}(v)$, hence, $[\alpha_F]_{i+1} \geq u_{i+1} > v_{i+1}$. Considering the counterclockwise neighbor it follows that $[\alpha_F]_{i-1} > v_{i-1}$. Since there is a vertex $v \in F$ with $v_i = [\alpha_F]_i$ for $i = 1, 2, 3$ it can be concluded that α_F is a maximum of $\mathcal{S}_\mathcal{V}$. □

Proof of Theorem 2.7. Projecting the geodesic embedding of M into the plane X gives a planar drawing of M. In this drawing every edge is composed by two straight segments. The claim is that the straight drawing, obtained by replacing each bend edge by the single segment connecting the vertices, is a convex drawing.

Claim 1: The straight drawing is crossing free. Suppose two edges e and e' cross in the straight drawing. These edges do not cross in the geodesic embedding. Therefore, one vertex of $e = \{u, v\}$ is contained in the triangle formed by the two representations of $e' = \{y, z\}$, or the other way round. This shows that a vertex is embedded in the interior of $\nabla_{e'}$ or of ∇_e. In either case a contradiction to Lemma 2.10.

Claim 2: The straight drawing is convex. Let F be a bounded face of M. By Lemma 2.11 all vertices of F are embedded on the border of the triangle ∇_{α_F}. The resulting shape of F in the straight drawing is a triangle with some truncated corners. In particular F is convex. The shape of the outer face is the triangle spanned by α_1, α_2 and α_3. $\square$

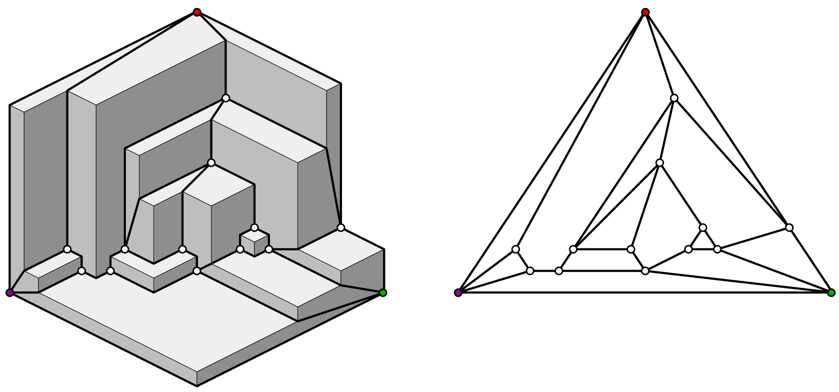


Figure 2.15 Geodesic and convex embedding for the graph of Figure 2.5

2.4 Dual Schnyder Woods

Let M be a planar map with a suspension M^σ and dual M^*. The *truncation* $M^{*\tau}$ of the dual of M is obtained by deleting the vertex corresponding to the unbounded face of M but leaving the edges incident to this vertex as half-edges. Furthermore these half-edges are partitioned into blocks B_1, B_2, B_3, where B_i contains the duals of the edges of M on the exterior path between the special vertices a_j and a_k, $\{i, j, k\} = \{1, 2, 3\}$.

Suppose a Schnyder wood of M^σ is given and $M^\sigma \hookrightarrow \mathcal{S}_\mathcal{V}$ is the corresponding geodesic embedding. With each bounded face F there is a maximum $\alpha_F \in \mathcal{S}_\mathcal{V}$ (Lemma 2.11). Actually, the maxima of $\mathcal{S}_\mathcal{V}$ are in bijection with bounded faces of M. With two faces F and F' sharing an edge $e = \{u, v\}$ we have $\alpha_F \wedge \alpha_{F'} = u \vee v \in \mathcal{S}_\mathcal{V}$. Let a dual elbow geodesic in a surface $\mathcal{S}_\mathcal{V}$ be the union of two line segments in $\mathcal{S}_\mathcal{V}$ connecting maxima α and α' to $\alpha \wedge \alpha'$. The dual of an edge $\{u, v\}$ separating the unbounded face from a bounded face F is a half-edge. It consists of the orthogonal arc reaching from α_F to $u \vee v$. A *dual geodesic embedding* is a set of dual elbow geodesics without crossings that uses all orthogonal arcs incident to maximal points.

Proposition 2.12 *With a geodesic embedding $M^\sigma \hookrightarrow \mathcal{S}_\mathcal{V}$ there is a dual geodesic embedding of the truncation $M^{*\tau}$ of the dual of M.*

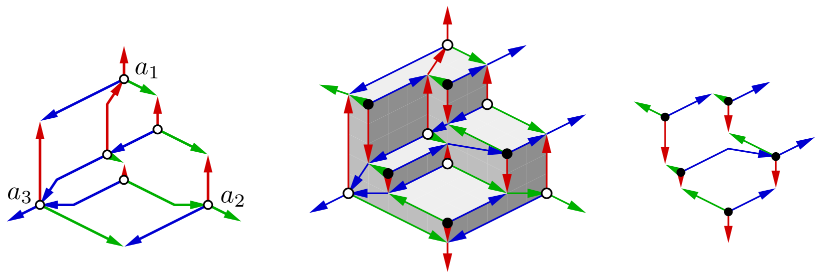


Figure 2.16 A suspended graph M^σ with a Schnyder wood, a corresponding embedding and the edge coloring and orientation of the truncation $M^{*\tau}$.

The dual geodesic embedding can be used to induce a coloring and orientation on the edges of $M^{*\tau}$ which is almost a Schnyder wood. The axioms (W1), (W3) and (W4) are fulfilled, instead of (W2) we have: all half-edges in B_i are colored i and directed outward. Based on $M^{*\tau}$ we construct the *suspension dual* M^{σ^*} of M^σ by connecting half edges in B_i to a new vertex b_i and adding the triangle edges $\{b_1, b_2\}, \{b_2, b_3\}, \{b_1, b_3\}$ and a half-edge for each b_i. Coloring and orientation of edges of $M^{*\tau}$ with the above properties can be extended in a unique way to a Schnyder wood of M^{σ^*}.

Proposition 2.13 *There is a bijection between the Schnyder woods of M^σ and the Schnyder woods of the suspension dual M^{σ^*}.*

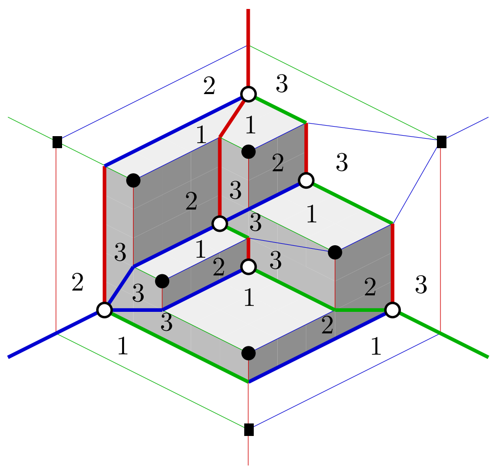


Figure 2.17 Bold edges show a suspended graph M^σ, light edges correspond to M^{σ^*}. The Schnyder angle labeling shown is valid for both graphs.

Proof. The proof becomes particularly simple in the terminology of Schnyder angle labelings. There is an obvious one-to-one correspondence between the angles of M^σ and the inner angles of M^{σ^*}. This correspondence yields an exchange between the rule of vertices

(A2) and the rule of faces (A3). Therefore, any Schnyder labeling of M^σ is a Schnyder labeling of M^{σ^*} and vice versa. This is exemplified in Figure 2.17. $\square$

We now define the completion of a planar suspension M^σ and its dual M^{σ^*}. Superimpose M^σ and M^{σ^*} so that exactly the primal dual pairs of edges cross (the half edge at a_i has a crossing with the dual edge $\{b_j, b_k\}$, for $\{i, j, k\} = \{1, 2, 3\}$). The common subdivision of each crossing pair of edges by a new edge-vertex gives the *completion* $\widetilde{M^\sigma}$. The completion $\widetilde{M^\sigma}$ is planar and has six half-edges reaching into the unbounded face.

Proposition 2.14 *The Schnyder woods of a planar suspension M^σ are in bijection with orientations of $\widetilde{M^\sigma}$ such that*

- $\mathsf{outdegree}(v) = 3$ *for all primal- and dual-vertices v,*
- $\mathsf{outdegree}(v_e) = 1$ *for all edge-vertices v_e*

and all half-edges are oriented away from their incident edge-vertex.

This proposition leads to an easy technique for modifying Schnyder woods of M^σ. Given a Schnyder wood consider the corresponding orientation of $\widetilde{M^\sigma}$. If this orientation contains an oriented cycle C revert the orientation of all edges of C. This construction yields another orientation with the same outdegrees, hence, another Schnyder wood of M^σ. This observation is the starting point for the proof of the following theorem. It describes a global structure on all Schnyder woods of M^σ.

Theorem 2.15 *The set of Schnyder woods of a planar suspension M^σ form a distributive lattice.*

We abstain from proving the theorem and pass on to another application of Schnyder woods.

2.5 Order Dimension of 3-Polytopes

Let $G = (V, E)$ be a finite simple graph. A nonempty family $\mathcal{R}$ of linear orders on the vertex set V of graph G is called a *realizer* of G provided:

(∗) For every edge $e \in E$ and every vertex $x \in V \setminus e$, there is some $L \in \mathcal{R}$ so that $x > y$ in L for every $y \in e$.

The *dimension* of G, denoted $\dim(G)$, is then defined as the least positive integer t for which G has a realizer of cardinality t.

An intuitive formulation for condition (∗) is as follows: For every vertex v and edge e with $v \notin e$ the vertex has to get over the edge in at least one of the orders of a realizer. From the defining condition it is obvious that dimension is monotone under

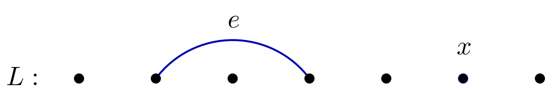


Figure 2.18 Vertex x is over edge e in L.

taking subgraphs, i.e.,

- if H is a subgraph of G then $\dim(H) \le \dim(G)$.

For readers who are new to the concept of dimension for graphs, we first prove an elementary proposition.

Proposition 2.16 *If a graph G contains a cycle, i.e., if G is not a tree, then* $\dim(G) \ge 3$.

Proof. By the monotonicity of dimension we only have to show that $\dim(C_n) \ge 3$, where C_n is the cycle on $n \ge 3$ vertices. Assume that C_n has a realizer L_1, L_2. Suppose that two vertices u and v are in the same order $u < v$ in L_1 and L_2. Let $v' \ne u$ be a neighbor of v. One of the orders of a realizer has to bring u over (v, v'); this contradicts $u < v$ in L_1 and L_2. Therefore, L_2 has to be the reverse of L_1. In this case the two vertices of every edge have to be adjacent in L_1, otherwise, if (v, v') is an edge and $v < u < v'$ in L_1 then u does not get over (v, v') in L_1 and not in L_2. The cycle C_n has n edges but L_1 only has $n - 1$ adjacent pairs, this shows that C_n has no realizer consisting of only two linear extensions. □

It is easy to construct a realizer consisting of 3 linear orders for C_n, $n \ge 3$. The dimension of the complete graph K_5 is 4, but the removal of any edge reduces the dimension to 3. Similarly, the dimension of the complete bipartite graph $K_{3,3}$ is 4 and again the removal of any edge reduces the dimension to 3. These examples are instances of the classical theorem of Schnyder.

Theorem 2.17 (Schnyder)
A graph G is planar if and only if its dimension is at most 3.

Proof. Let G be a non-planar graph and suppose $\dim(G) \le 3$. Let $\{L_1, L_2, L_3\}$ be a realizer of G. For a vertex v of G let v_i be the position of v in L_i. Define an embedding ϕ of G in $\mathbb{R}^3$ by $v \to \phi(v) = (s_1 v_1, s_2 v_2, s_3 v_3)$, where the s_i are scalars which will be fixed later. For an edge $e = \{u, v\}$ let $e_i = \max(u_i, v_i)$ and embed e by $e \to \phi(e) = (s_1(e_1 + \frac{1}{2}), s_2(e_2 + \frac{1}{2}), s_3(e_3 + \frac{1}{2}))$. Note that by the definition of a realizer we have

($\star$) $\phi(v)_i < \phi(e)_i$ for $i = 1, 2, 3$ if and only if $v \in e$.

Adjust the s_i such that under the orthogonal projection π to the plane $x_1 + x_2 + x_3 = 0$ all points in $\phi(V \cup E)$ project to distinct points and these points are in general position.

Now G is drawn in the plane by joining $\pi(\phi(v))$ and $\pi(\phi(e))$ with a straight line segment whenever $v \in e$. Assuming that G has no planar representation there are[†] crossing segments $[\pi(\phi(u)), \pi(\phi(e))]$ and $[\pi(\phi(v)), \pi(\phi(f))]$ with $u \notin f$ and $v \notin e$. Let p be the crossing point and suppose that the ray starting in p and leaving the plane orthogonally meets the segment $[\phi(u), \phi(e)]$ in $\mathbb{R}^3$ at x no later than it meets $[\phi(v), \phi(f)]$ at y. Now consider the path formed by straight segments from $\phi(u)$ to x to y and $\phi(f)$. This path is increasing in each coordinate, hence $u \in f$ by property $\star$. The contradiction shows that G is planar.

It remains to show that every planar graph G admits a realizer $\{L_1, L_2, L_3\}$. By monotonicity we may assume that G is a maximal planar graph, i.e., a triangulation. Consider the trees of a Schnyder wood of G. Since each of the trees has $n - 1$ edges and the graph has $3n - 6$ edges the only bidirected edges are the three edges of the exterior triangle. Therefore, $R_i(u) \subset R_i(v)$ whenever $u \in R_i(v)$. For $i = 1, 2, 3$ let the inclusion order on the regions induce the order Q_i on the vertices, i.e., $u < v$ in Q_i iff $R_i(u) \subset R_i(v)$.

[†] This is a nice exercise: If G is drawn such that all crossing pairs of edges share a vertex, then G is a planar graph.

For any edge (u, v) and vertex $w \neq u, v$ the edge is in one of the regions $R_i(w)$ of w, hence, $u < w$ and $v < w$ in Q_i. This shows that any choice of linear extensions L_i of Q_i, $i = 1, 2, 3$, will produce a realizer for G. □

In complete analogy to the definition of the dimension of a graph the dimension of a hypergraph can be defined. A particularly interesting instance is related to 3-dimensional polytopes and hence, by Steinitz's theorem also to planar graphs.

Let P be a polytope with vertex set $\mathcal{V}(P)$ and facets $\mathcal{F}(P)$. Given a subset $\mathcal{G}$ of $\mathcal{F}(P)$ a *realizer* for $(P, \mathcal{G})$ is a nonempty family $\mathcal{R}$ of linear orders on $\mathcal{V}(P)$ provided

(∗∗) For every facet $F \in \mathcal{G}$ and every vertex $x \in \mathcal{V}(P) \setminus \mathcal{V}(F)$, there is some $L \in \mathcal{R}$ so that $x > y$ in L for every $y \in \mathcal{V}(F)$.

The *dimension* of $(P, \mathcal{G})$, denoted $\dim(P, \mathcal{G})$, is then defined as the least positive integer t for which $(P, \mathcal{G})$ has a realizer of cardinality t. In the case $\mathcal{G} = \mathcal{F}(P)$ we simply write $\dim(P)$ and call this the *dimension of the polytope P*.

Theorem 2.18 *If P is a d-polytope with $d \geq 2$, i.e., a polytope whose affine hull is d-dimensional, then* $\dim(P) \geq d + 1$.

Proof. The proof is by induction on d. If $d = 2$ then the vertices and facets of P have the structure of the cycle C_n for $n = |\mathcal{V}(P)|$. It follows from Proposition 2.16 that $\dim(P) \geq 3$.

Let P be a d-polytope embedded in $\mathbb{R}^d$ for some $d > 2$ with realizer $L_1, L_2, \ldots, L_t$. Let v be the highest vertex in L_t and consider a hyperplane H which separates v from all the other vertices of P. The intersection $P \cap H$ is a $(d-1)$-polytope P/v, the so called

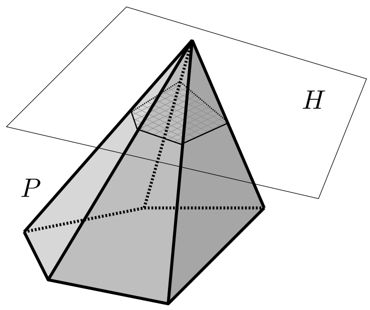


Figure 2.19 The *vertex figure* of the tip vertex of P.

vertex figure of P at v. The $(k-1)$-dimensional faces of P/v are in bijection with the k-dimensional faces of P that contain v. In particular an edge (u, v) of P corresponds to a vertex $u' = (u, v) \cap H$ of P/v and for every facet $\{u'_1, \ldots, u'_r\}$ of P/v there is a facet $\{v, u_1, \ldots, u_r, w_1, \ldots, w_s\}$ of P. Let $\mathcal{F}_v$ be the set of facets of P containing v. The correspondence $\mathcal{F}(P/v) \leftrightarrow \mathcal{F}_v$ shows that $\dim(P/v) \leq \dim(P, \mathcal{F}_v)$. Since P/v is $(d-1)$-dimensional $\dim(P/v) \geq d$ by induction. Now let $F \in \mathcal{F}_v$ and $w \notin \mathcal{V}(F)$, by the choice of v the order L_t cannot bring w over F. Therefore, $L_1, L_2, \ldots, L_{t-1}$ is a realizer for $(P, \mathcal{F}_v)$, i.e., $\dim(P, \mathcal{F}_v) \leq t - 1$. Combine the inequalities to deduce $t \geq d + 1$. □

It is known that for $d \geq 4$ a polytope in d-space can have arbitrarily high dimension. For $d = 3$, however, the situation is different. By Steinitz's theorem polytopes and 3-connected planar graphs are essentially the same. Making use of our knowledge about Schnyder woods we prove:

Theorem 2.19 (Brightwell–Trotter)
If P is a 3-polytope, then $\dim(P) = 4$. *Moreover, if $I \in \mathcal{F}(P)$ and $\mathcal{F}_I = \mathcal{F}(P) \setminus \{I\}$, then* $\dim(P, \mathcal{F}_I) = 3$.

Proof. Let $\mathcal{R}$ be a realizer of $(P, \mathcal{F}_I)$. To obtain a realizer for P we only have to add a single linear order with $v < w$ for all $v \in \mathcal{V}(I)$ and $w \in \mathcal{V}(P) \setminus \mathcal{V}(I)$ to $\mathcal{R}$. Combined with the lower bound from Theorem 2.18 this yields $4 \leq \dim(P) \leq \dim(P, \mathcal{F}_I) + 1$. To prove the theorem it remains to show $\dim(P, \mathcal{F}_I) \leq 3$.

Let G be the graph of P. The graph G is planar and 3-connected by Steinitz Theorem, Theorem 1.7. Choose a planar embedding of the graph G of P with I as the exterior face. Specify vertices a_1, a_2, a_3 in clockwise order around I. As 3-connected planar graph G has a Schnyder woood for every choice of three special vertices at the exterior face, see Section 2.6. Consider a Schnyder wood of G with special vertices a_1, a_2, a_3. As in the proof of Theorem 2.17 we define linear extensions L_i of the inclusion order Q_i of regions $i = 1, 2, 3$, i.e., $u < v$ in Q_i iff $R_i(u) \subset R_i(v)$. To bring every vertex y over every face $F \in \mathcal{F}_I$ with $y \notin F$, however, more care in the choice of L_i is required.

Define Q_i^* such that $u < v$ in Q_i^* if either

(a) $u < v$ in Q_i or

(b) $u || v$ in Q_i and $u < v$ in Q_{i+1}.

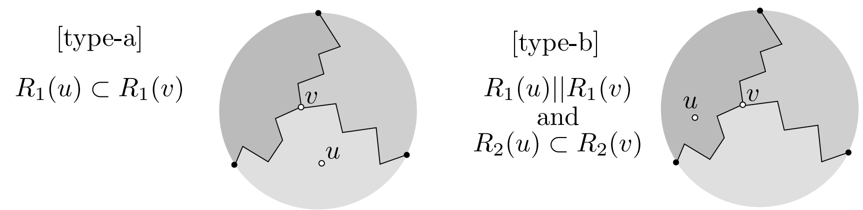


Figure 2.20 The two types of comparabilities $u < v$ in Q_1^*.

Lemma 2.20 Q_i^* *is acyclic for* $i = 1, 2, 3$.

Proof. Call (u, v) a type-a pair if $u < v$ in Q_i^* by part (a) of the definition and call it a type-b pair if $u < v$ in Q_i^* by (b). A cycle in Q_i^* has to contain both a type-a pair and a type-b pair. We claim that if $u < v$ is a type-a pair and $v < w$ is a type-b pair then $u < w$ is also in Q_i^*. Since $u < v$ and $v < u$ cannot be both in Q_i^* the claim yields a contradiction to the assumption that Q_i^* contains a cycle.

Claim. If $u < v$ is a type-a pair and $v < w$ is a type-b pair then $u < w$ is also in Q_i^*.

By symmetry we may assume that $i = 1$. If $R_1(v) = R_1(w)$ then with (u, v) the pair (u, w) also is type-a. Therefore, we assume $R_1(v) \not\subseteq R_1(w)$, since (v, w) is type-b this implies $w \notin R_1(v)$ and $w \notin R_2(v)$. Therefore, $w \in R_3^o(v)$ and $R_3(w) \subset R_3(v)$. Since $u \in R_1(v)$ we either find u in $R_1(w)$ or in $R_2(w)$. If u in $R_1(w)$ then $R_1(u) \subseteq R_1(w)$ but equality is impossible since $w \notin R_1(u)$, i.e., (u, w) is type-a pair in this case. Otherwise $u \in R_2^o(w)$, i.e., $R_2(u) \subset R_2(w)$, and the 1–regions of u and w are incomparable. This shows that (u, w) is a type-b pair in this case. □

Let L_i be a linear extension[‡]of Q_i^*. We have to show that L_1, L_2, L_3 is a realizer for the incidence hypergraph of vertices and bounded faces of G, i.e., a realizer for $(P, \mathcal{F}_I)$. Consider a pair (F, y), where F is a face and y is a vertex not on F. Face F is contained in one of the regions of y, by symmetry we may assume that $F \in R_1(y)$. Hence, $R_1(x) \subseteq R_1(y)$ for all $x \in F$. If $R_1(x) \subset R_1(y)$ for all $x \in F$ then F is below y in L_1. Assume that there is an $x \in F$ with $R_1(x) = R_1(y)$.

Note that it is impossible that F contains vertices x and x' with $R_1(x) = R_1(y) = R_1(x')$ and $x \in P_3(y)$ while $x' \in P_2(y)$. This would lead to the placement of y on some edge bounding F in the drawing $\mu(G)$. This is a contradiction since $\mu(G)$ is a convex drawing.

Suppose that for all $x \in F$ either $R_1(x) \subset R_1(y)$ or $R_1(x) = R_1(y)$ and $x \in P_3(y)$. From (E_2), page 23, it follows that $R_2(x) \subset R_2(y)$ for all $x \in F$ with $R_1(x) = R_1(y)$. By the definition of Q_1^* this shows that F is below y in L_1.

Finally, consider the situation that for all $x \in F$ either $R_1(x) \subset R_1(y)$ or $R_1(x) = R_1(y)$ and $x \in P_2(y)$. We claim that F is below y in L_3 in this case. All x in F with $R_1(x) = R_1(y)$ have $R_3(x) \subset R_3(y)$ by (E_2), hence, they are below y in L_3. If $x||y$ in Q_3 for all $x \in F$ with $R_1(x) \subset R_1(y)$ these vertices also go below y in L_3 and we are done. This is shown to be true in the lemma below which completes the proof of the theorem.

Lemma 2.21 *If $R_1(x) = R_1(y)$, $x \in P_2(y)$ and F is a face in $R_1(x)$ with $x \in F$ and $y \notin F$ then $R_3(y) \not\subseteq R_3(v)$ for all $v \in F$.*

Proof. Consider the triangle ∇_F enclosing F in the convex drawing $\mu(G)$ (Lemma 2.11). Vertex y is placed on the horizontal line ℓ_1 bounding ∇ and y is left of all vertices of F on ℓ_1, Figure 2.21 shows the situation.

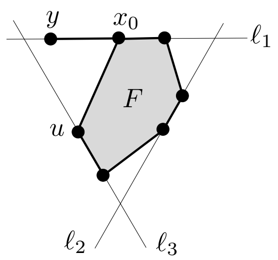


Figure 2.21 Crucial for this case is the orientation of u, x_0.

Let x_0 be the leftmost vertex of F on ℓ_1 and u be the uppermost vertex of F on ℓ_3. Let x_1 be the other neighbor of x_0 at F, i.e., $u \neq x_1$. Even so x_1 need not be on ℓ_1 the edge (x_0, x_1) is the outgoing edge of x_0 in label 2, cf. Figure 2.10. Also (x_0, y) is the outgoing edge of x_0 in label 3. The edge orientations at vertex x_0 imply that (u, x_0) is oriented from u to x_0 in label 1. This shows that x_0 is on $P_1(v)$ for all $v \in F \cap \ell_3$. The paths $P_1(v)$ for $v \in F \cap \ell_2$ clearly cross ℓ_1 to the right of x_0. This shows that $y \notin R_3(v)$ for all $v \in F$, hence $R_3(y) \not\subseteq R_3(v)$. □

[‡] Actually Q_i^* is already a total order.

2.6 Existence of Schnyder Labelings

A route plan for the construction of a Schnyder labeling for a 3-connected planar graph could be the following:

(1) Choose an edge e of G and let G/e be the graph obtained by contraction of e.

(2) Recursively construct a Schnyder labeling of G/e.

(3) Expand the labeling of G/e to a Schnyder labeling of G.

A detail that deserves some cautiousness is the choice of edge e. For the induction it is required that G/e is again 3-connected. We call an edge e of a 3-connected graph G such that G/e is again 3-connected a *contractible edge*. The existence of a contractible edge is warranted by the following lemma of Thomassen [196]:

Lemma 2.22 *A 3-connected graph G with at least five vertices contains an edge e whose contraction leaves a 3-connected graph G/e.*

If we let e be an arbitrary contractible edge, however, the proof that the expansion of the labeling can be carried out may involve excessive case distinctions. To reduce the case analysis it would be desirable to have a contractible edge of a special form. But the existence of such an edge will likewise not come for free. Schnyder has taken this approach in his work about planar triangulations. Here we take a different inductive approach.

Let G be a 3-connected planar graph with three special vertices a_1, a_2, a_3 in clockwise order on the boundary cycle C of the outer face. Let $x \notin C$ be a neighbor of a_1.

Suppose that $e = (a_1, x)$ is contractible and let $a_1, x_1, x_2, \ldots x_k$ be the neighbors of x in clockwise order. Since e is contractible only x_1 and x_k may be neighbors of both a_1 and x. Figure 2.22 shows a generic contraction of the edge (a_1, x) into a_1.

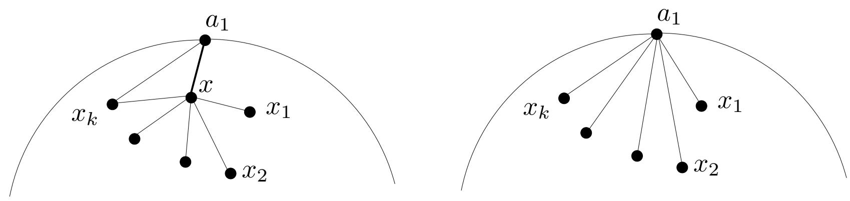


Figure 2.22 Contraction of edge (a_1, x) into a_1.

By the rule for the labels at special vertices all inner angles at a_1 are labeled 1. The angles of edge (a_1, x_i) at x_i have to be labeled 2 and 3 as shown in the left part of Figure 2.23. The right part of Figure 2.23 shows that the labeling of G/e can be expanded to a Schnyder labeling of G. Note that the expansion leaves the labels in all faces that do not have (a_1, x) as boundary edge unchanged.

Next suppose that $e = (a_1, x)$ is not contractible, i.e., G/e is only 2-connected. Clearly, every cutset of size two in G/e has to contain a_1, let y be the second vertex of such a cutset. The set $S = \{a_1, x, y\}$ is a cutset of G, denote the components of $G \setminus S$ by H and K. Let H' be $G \setminus K$ and $K' = G \setminus H$. The idea is to take Schnyder labelings of the two smaller graphs H' and K' and to show that they can be pasted together.

The first problem is that H' and K' need not be 3-connected, resolve this by augmenting both graphs with the edges (a_1, y) and (x, y), provided these edges are not existent, this yields H'' and K''.

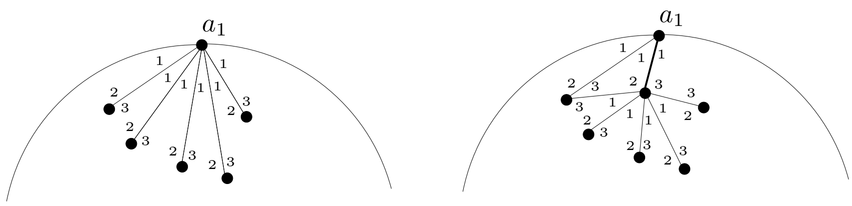


Figure 2.23 Extending the Schnyder labeling of G/e to G.

We have to consider two cases. First suppose that one of the graphs, say H'', contains all special vertices a_1, a_2 and a_3. A Schnyder labeling of H'' contains all three labels in the triangle $T = (a_1, x, y)$. The label at a_1 is 1, let the second special vertex for K'' be the vertex with label 2 in T and the third special vertex be the vertex with label 3 in T. Construct a Schnyder labeling for K'' with this assignment of special vertices. If all three edges of T have been present in G, then the pasting of the labelings makes no problem: the rules of vertices and faces can be verified in the labelings of H'' and K''.

It remains to consider a face that was cut by a new edge. We treat this case with the edge (x, y) with the assumption that the angle of x in T has label 2 in the labeling of H''. Let F be the face of G containing x and y and let F_H and F_K be the parts of this face after insertion of the edge (x, y) such that F_H belongs to H'' and F_K to K''. Figure 2.24 shows the situation.

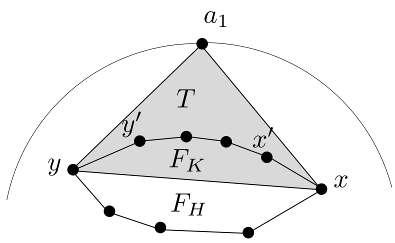


Figure 2.24

Since in the labeling of H'' the angles of x and y at T are 2 and 3 the label of x at F_H is 1 or 2 and the label of y at F_H is 1 or 3. The claim is that we can use the same labels in G. Now consider the labeling of K''. Both labels of x at the edge (x, x') are 2 and both labels of y at the edge (y, y') are 3 by the rule for special vertices. Therefore, the labels of x' and y' at F_K are both 1, see Figure 2.3. All vertices between x' and y' in K'' also have label 1 at F_K by the rule for the face. This proves that using the labels of H'' for the angles at x and y in G gives a consistent labeling.

It remains to consider the case where the two special vertices a_2 and a_3 are separated by $\{a_1, x, y\}$. Assume $a_3 \in H''$ and $a_2 \in K''$ vertex y has to play the role of the missing special vertex in both graphs, i.e., the role of a_2 in H'' and the role of a_3 in K''. Figure 2.25 shows some of the labels in the Schnyder labelings of H'' and K'', we have to prove that they can be pasted together to yield a Schnyder labeling of G.

The edge a_1, y was not present in G, so we remove it from both graphs and identify the two copies of a_1, x and y. Since the edges (a_1, x) and (x, y) from the two graphs are also identified the labels in the triangles formed by a_1, x, y in H'' and K'' vanish. However, if we assign label 1 to the outer angle at y the rule of vertices is satisfied at x and y. If

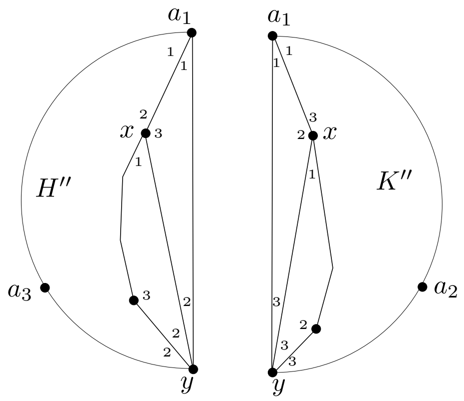


Figure 2.25

the edge (x, y) is in G then the rules of all the other vertices and faces can be verified in the labelings of H'' and K''. If (x, y) has to be removed assign label 1 to the angle at x and one of the labels 2 or 3 to the angle at y. Again all the conditions for a Schnyder labeling are easily verified.

2.7 Notes and References

The existence of straight line drawings for planar graphs was independently proven by Wagner [208], Fáry [79] and Stein [187]. The question whether every planar graph has a straight line embedding on a grid of polynomial size was raised by Rosenstiehl and Tarjan [165]. Unaware of the problem Schnyder [172] constructs a barycentric representation which immediately translates to an embedding on the $(2n-6)\times(2n-6)$ grid. The first explicit answer to the question was given by de Fraysseix, Pach and Pollack [56, 57]. They construct straight line embeddings on an $(2n-4)\times(n-2)$ grid and show that the embedding can be computed in $O(n \log n)$. De Fraysseix et al. also observed a lower bound of $(\frac{2}{3}n-1)\times(\frac{2}{3}n-1)$ for grid embeddings of the n vertex graph containing a nested sequence of $n/3$ triangles. Xin He [113] mentions the conjecture that every planar graph can be embedded on the $(\frac{2}{3}n-1)\times(\frac{2}{3}n-1)$ grid. In [173] Schnyder improved on his first result and shows the existence of an embedding on the $(n-2)\times(n-2)$ grid which can be computed in $O(n)$ time. The difference between the two algorithms of Schnyder is that in the first case coordinate v_i of vertex v is obtained by counting the faces in region $R_i(v)$. In the second algorithm v_i is computed using the vertices in this region. More compact representations can be found for 4-connected planar graphs. Xin He [113] shows that every such graph embeds on a $W \times H$ grid with $W + H \leq n$.

Tutte [202, 203] shows that every 3-connected planar graph G admits a strictly convex drawing. The idea for Tutte's proof is to nail down at least three vertices of the outer face of G and consider edges as springs. The equilibrium state of the self-stress of this framework is a convex drawing of G and can be computed by solving a system of linear equations. This basic idea of spring-embeddings has been modified and developed in many directions, confer the graph drawing book [61] for further references. Schnyder and Trotter [174] have worked on convex grid embeddings. Felsner [81] elaborated the idea of using Schnyder woods for convex drawings. Kant [118] has extended the approach of de Fraysseix et al. to construct convex drawings on the $(2n-4) \times (n-2)$ grid, the grid size was further reduced to $(n-2) \times (n-2)$ by Chrobak and Kant [49]. Straight line grid drawings with area $O(n^2)$ are not strictly convex, in fact, every strictly convex embedding of the n-cycle requires an area of $\Omega(n^3)$.

Every graph admits a straight line embedding in $\mathbb{R}^3$ without crossing edges. This can be achieved, e.g., by placing the vertices on different points of the moment curve $x \to (x, x^2, x^3)$. Three-dimensional grid drawings have been studied, among others, by Pach, Thiele and Tóth [153]. They prove that every r-colorable graph has a three-dimensional grid drawing in a box of volume cr^2n^2.

The self-stress approach of Tutte can be extended to prove Steinitz's Theorem. Details for the following outline are elaborated by Hopcroft and Kahn [115] and Richter-Gebert [162]. Let G be a planar graph with $V(G) = \{1, \ldots, n\}$. A *stress* on G is a symmetric $n \times n$ matrix $W = (w_{ij})$ with $w_{ij} = 0$ for all non-edges ij of G. An embedding $i \to p_i$ of the vertices of G to $\mathbb{R}^2$ is said to be in equilibrium at i if $\sum_{j=1}^{n} w_{ij}(p_i - p_j) = 0$. Let W be a positive stress, i.e., $w_{ij} > 0$ for every edge ij of G and let G be a 3-connected planar graph. Suppose that a set C of at least three vertices of G is nailed down, i.e., has been assigned unchangeable positions in the plane. Then:

(1) There exists unique positions p_i for the vertices of $V \setminus C$ such that all these vertices are in equilibrium.

(2) If the set C is a face of G and C is nailed down as a convex polygon in the plane, then the equilibrium is a strictly convex drawing of G.

(3) Such a strictly convex equilibrium drawing of a 3-connected graph can be lifted to a 3-polytope P, see Figure 2.26.

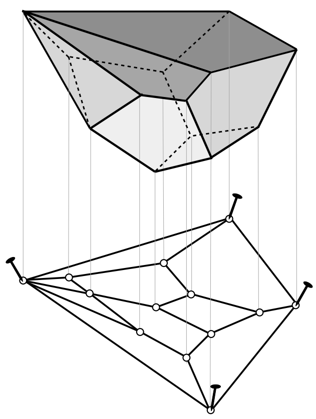


Figure 2.26 A stressed drawing lifted to space

A different and very beautiful approach to both, convex drawings and Steinitz's Theorem, is based on the Koebe Circle Packing Theorem.

Theorem 2.23 *Every planar graph G can be represented by a set of non-overlapping circles in the plane, a circle C_v for every vertex v, so that two vertices u, v are adjacent in G if and only if circles C_u and C_v touch in a point $t_{u,v}$, i.e., the two circles are tangent to each other.*

The result originated from complex analysis and was rediscovered at several times and places. Sachs [168] and Ziegler [219] comment on the history of the theorem. A decade ago the theorem and some generalizations where popularized in the graph theory community. Particularly appealing is the following generalization.

Theorem 2.24 *Let G be a 3-connected planar graph with dual G^* and let o be the vertex of G^* corresponding to the outer face of G. Then there are circle representations of G and $G^* \setminus o$ and a circle C_o so that:*

- *Circle C_o contains all face circles and touches C_f, in a point $t_{f,o}$, iff (f, o) is an edge of G^*.*
- *For every edge (u, v) of G and its dual edge (f, g) the touching points coincide, i.e., $t_{u,v} = t_{f,g}$, and the tangents of C_u, C_v in $t_{u,v}$ and of C_f, C_g in $t_{f,g}$ cross perpendicular.*

Furthermore, the representation is unique up to linear fractional transformations of the plane.

A proof of Theorem 2.23 can be found in Pach and Agarwal [148]. Theorem 2.24 is proved by Brightwell and Scheinerman [39]. We sketch the steps for a proof of the stronger of the two theorems.

From 3-connectivity it follows that either G or G^* contains a vertex of degree three, so we may assume that the outer face o of G is a triangle with vertices a, b, c. To each vertex $v \neq a, b, c$ and to each face $f \neq o$, assign a variable r_v, respectively r_f which is thought of as the radius of the corresponding circle. The radii of the circles corresponding to a, b, c are fixed at 1. With an assignment of radii there is a right angled triangle with side length r_v, r_f and $\sqrt{r_v^2 + r_f^2}$ corresponding to every incident pair v vertex, f face. Two such triangles are combined to form the kite k_{vf}, see Figure 2.27.

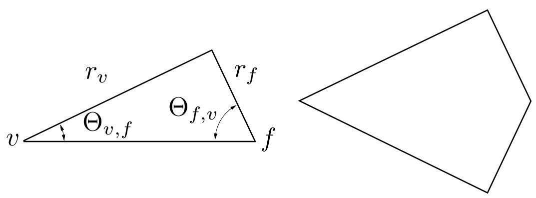


Figure 2.27 The triangle and the kite of v and f.

It has to be shown that some assignment of radii allows to lay out the kites nicely side by side as in Figure 2.28. A necessary local condition for this is that $\sum_{f:vIf} \Theta_{v,f} = \pi$ holds for every vertex v and of course a similar condition for faces. It can be shown that under the 3-connectivity assumption this set of conditions has a unique solution. The second main step in the proof is to show that the triangles obtained that way indeed fit together globally. This is shown using a discrete homotopy argument.

Lifting a primal-dual circle representation to the sphere and using the planes supported by the face cycles yields a polytope with skeleton graph G. This polytope can, in an essentially unique way, be adapted to have all edges tangent to the sphere, Schramm [175]. Higher-dimensional sphere representations and connections to the Colin de Verdière number of a graph are discussed by Kotlov, Lovász and Vampala [124]. Another application

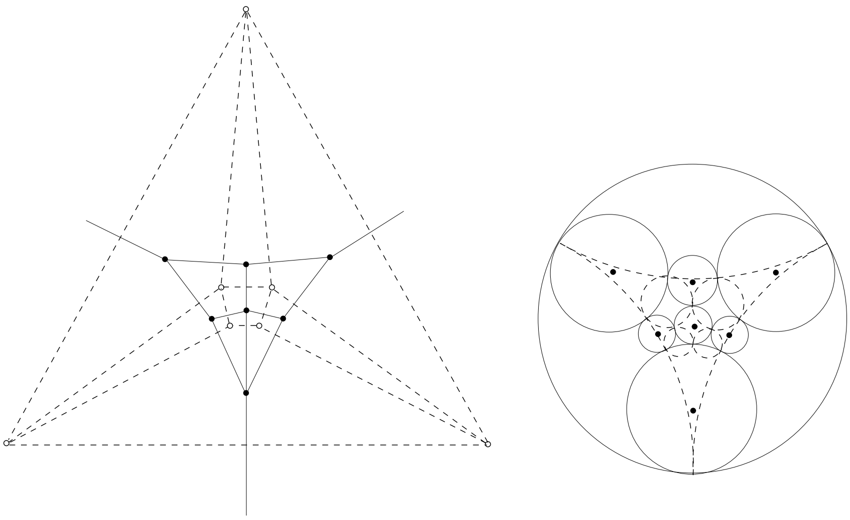


Figure 2.28 A planar graph layed out with kites (left) and with circles (right).

of circle representations is a new proof of the Lipton-Tarjan separator theorem for planar graphs, cf. the book of Pach and Agarwal [148].

Schnyder woods and applications.

In his two fundamental papers [172, 173] Schnyder developed a theory of Schnyder labelings and Schnyder woods for planar triangulations.

In the first publication [172] Schnyder gave the characterization of planar graphs. He stated Theorem 2.17, however, he used the following slightly different concept of dimension. With a finite graph $G = (V, E)$, associate a height two order P_G whose ground set is $V \cup E$. The order relation is defined by setting $x < e$ in P_G if $x \in V$, $e \in E$ and $x \in e$. P_G is called the *incidence order* of G.

When $P = (X, <)$ is an order, and $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ is a family of linear orders on X, we call $\mathcal{R}$ a realizer of P if $P = \cap\mathcal{R}$, i.e., $x < y$ in P if and only if $x < y$ in L_i for all $i = 1, 2, \ldots, t$. The *dimension* of an order is then defined as the minimum cardinality of a realizer.

With this notation at hand, here is the original form of Schnyder's Theorem [172].

Theorem 2.25 *A graph is planar if and only if the dimension of its incidence order is at most 3.*

In the same paper Schnyder also shows that the dimension of the face lattice of a simplicial 3-polytope, i.e., of a polytope with only triangular faces, is 4 and drops to 3 upon removal of a face.

If G is a graph with minimum degree at least 2, then the dimension of G and the dimension of its incidence order P_G agree. Also the dimension of a polytope as defined in this chapter and the order dimension of the face lattice of the polytope agree. The two order theoretic facts that sit in the background of the phenomenon are: All critical pairs of face lattices are min-max pairs, and secondly, if all critical pairs of an order are min-max pairs then its interval dimension equals its order dimension. For additional information on the order theoretical background we recommend Trotter's monograph [200].

A particularly interesting problem about dimension of graphs is the dimension of the complete graph. The order theoretic counterpart to this problem is to determine the order dimension of the first two levels of the Boolean lattice. Spencer and Trotter determined the asymptotic growth of this function as:

$$\dim(K_n) \sim \log_2 \log_2 n + \big(\frac{1}{2} + o(1)\big) \log_2 \log_2 \log_2 n.$$

Hoşten and Morris [114] found a surprising connection to the number of intersecting antichains in the Boolean lattice. Building upon the Hoşten-Morris result Felsner and Trotter [86] found a variant of dimension for graphs such that this dimension of the complete graph relates to the Dedekind numbers, i.e., numbers of all antichains in Boolean lattices. Based on Schnyder's ideas there is also a dimension theoretic characterization of outerplanar graphs, this and further results and references about the dimension of graphs can be found in [86]. Studies of the order dimension of polytopes were carried out by Reuter [160], he proves the lower bound in lattice theoretic terms. Brightwell and Trotter [37] prove Theorem 2.19, however, their definition of the three linear orders for the realizer is more involved. The proof given here is adapted from [81]. Brightwell and Trotter [38] extend the approach to show that the inclusion order of vertices, edges and faces of any planar multi-graph is of dimension at most 4. The dimension of higher-dimensional polytopes does not behave as nicely as the dimension of 3-polytopes. Four-dimensional cyclic polytopes have a complete graph as skeleton graph, since the dimension of these graphs is unbounded the dimension of 4-polytopes is unbounded as well.

A connection between orthogonal surfaces and planar graphs came up in a series of papers by Sturmfels and others, e.g. [21, 140]. These authors were interested in the structure of minimal resolutions of monomial ideals in three variables. Miller [139] began studying geodesic embeddings from this perspective. He observes and exploits a connection with Schnyder labelings. Miller defines a *rigid geodesic* on a surface $\mathcal{S}_\mathcal{V}$ generated by $\mathcal{V}$ as an elbow geodesic connecting u and v via $u \vee v$ with the additional property that u and v are the only elements of $\mathcal{V}$ which are dominated by $u \vee v$. A rigid surface is an orthogonal surface such that all elbow geodesics on the surface are rigid. The main result in Miller's paper is that every 3-connected planar graph is induced by a rigid orthogonal surface. The Brightwell-Trotter Theorem (Theorem 2.19) is a corollary to the existence of rigid geodesic embeddings. Confirming a conjecture from Miller [139], Felsner [84] shows a bijection between planar graphs with a Schnyder wood and combinatorially different rigid orthogonal surfaces.

The set of Schnyder woods of a planar triangulation has the structure of a distributive lattice, this was shown by de Mendez [58] and Brehm [36]. The first step in the proof consists in showing that Schnyder woods of a suspended triangulation M^σ are in bijections with 3-orientations, i.e, orientations such that every vertex has outdegree 3. Given a 3-orientation of M^σ and a directed triangle we obtain another 3-orientation by reverting the orientation of the triangle. Let a flip be the operation from the counterclockwise orientation of the triangle to the clockwise, see Figure 2.29. The transitive hull of the flip operation is an order relation on the set of all 3-orientations and, hence, on the set of all Schnyder woods, of M^σ. This order is a distributive lattice.

As shown by Felsner [83] and de Mendez [58] a much more general result is true: If G is a plane map and $\alpha : V \to \mathbb{N}$ assigns a non-negative integer to every vertex of G, then the set of all orientations of G with $\mathsf{outdegree}(v) = \alpha(v)$ for all $v \in V$ has the

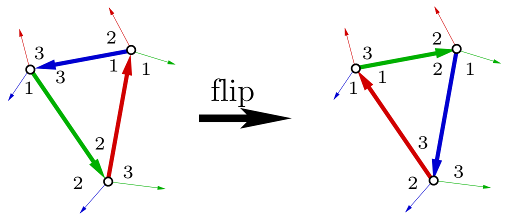


Figure 2.29 Flip of a triangle. These flips generate a distributive lattice on Schnyder woods of a planar triangulation.

structure of a distributive lattice. With this at hand Theorem 2.15 is a direct corollary of Proposition 2.14. The general theorem about α-orientations has several interesting applications (see [83]). To mention just one: The set of rooted spanning trees of a planar graph has the structure of a distributive lattice.

Besides the triangle-flip another type of local flip on Schnyder woods on planar triangulations was recently introduced by Bonichon and others, see [32]. The generic instance of this flip is shown in Figure 2.30. This type of flip makes a change in the underlying graph. Actually, the first application of this operation is a very nice proof of a theorem

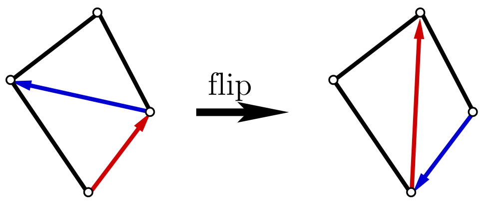


Figure 2.30 Flip of diagonals in a quadrangle.

of Wagner. This theorem says that any two planar triangulations on n vertices can be transformed into each other by a sequence of diagonal flips in quadrangles. The technical hard part of the usual proofs for this theorem is where it comes to show that multiple edges can be avoided in the course of the flipping. This assertion becomes very easy when transforming a given Schnyder wood of one triangulation into a Schnyder wood of the other triangulation via the flip of Figure 2.30. Other interesting applications of this flip lie in the area of counting various kinds of planar map, see [30, 31].

3 Topological Graphs: Crossing Lemma and Applications

Intuitively a handy drawing of a non-planar graph will be a drawing with few crossings. The *crossing number* of a graph G is the least possible number of pairs of crossing edges in a drawing of G. This measure for the non-planarity of a graph has been studied for more than thirty years now. The main result is the Crossing Lemma (Theorem 3.3) it provides a lower bound for the crossing number in terms of the numbers of vertices and edges of a graph. In Section 3.3 the constant in the Crossing Lemma is improved. This improvement is an application of bounds for the number of edges of topological graphs with the property that every edge participates at at most one or two crossings.

With the simple probabilistic proof the Crossing Lemma seems to be an innocent result. However, as first observed by Székely, it is the key to simplified proofs for some deep and important questions of Erdős type. Examples of this phenomenon are the subject of Section 3.4.

3.1 Crossing Numbers

A drawing of a graph is an embedding of the graph in the plane with vertices represented by points and edges represented by Jordan curves. Topological graphs are drawings with the following three additional restrictions:

- No edge has a crossing with itself.
- Two edges with a common endpoint do not cross.
- Two edges cross at most once.

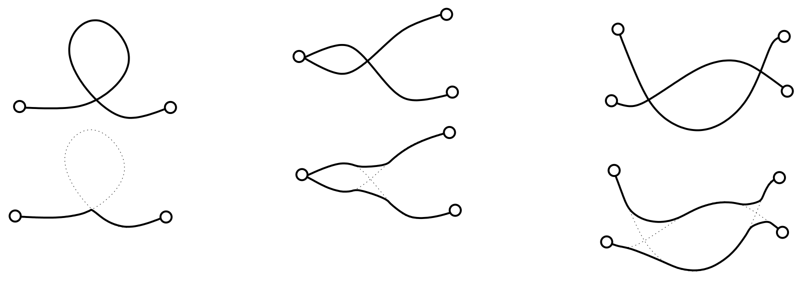


Figure 3.1 Simplifying and crossing reducing modifications of a topological graph.

Figure 3.1 indicates how to modify a drawing in order to obtain a topological drawing with fewer crossings. So, a drawing which achieves the minimal number $\mathrm{cr}(G)$ of crossings of G is a topological drawing. For a topological drawing it can also be assumed that there is no single point where three or more edges cross. In this chapter the drawings under discussion are topological drawings.

The attentive reader may already suspect that the reason for allowing curved edges in drawings is that they may help reduce the number of crossings. This is indeed true, $\mathrm{cr}(K_8) = 18$, but a straight-line drawing of K_8 contains at least 19 crossings. The least number of crossings in a straight-line drawing of a graph G is the *rectilinear crossing number* $\overline{\mathrm{cr}}(G)$. The ratio of $\overline{\mathrm{cr}}(G)$ and $\mathrm{cr}(G)$ can get arbitrarily large. For every m there is a graph with $\mathrm{cr}(G) = 4$ and $\overline{\mathrm{cr}}(G) \geq m$. The construction is based on the graph shown in Figure 3.2.

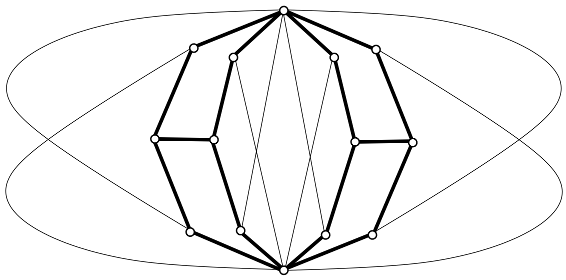


Figure 3.2 A graph constructed to separate $\mathrm{cr}(G)$ and $\overline{\mathrm{cr}}(G)$.

The key for the proof is the following: A straight line drawing of this graph either has a crossing between bold edges or between a bold and a skinny edge. Having shown this, replace each bold edge by a bundle of m edges and subdivide these edges by vertices of degree two. The resulting graph still has crossing number 4 but the rectilinear crossing number will be at least m.

3.2 Bounds for the Crossing Number

A planar graph with n vertices has at most $3n - 6$ edges (Theorem 1.4). This fact is the basis for estimates of the crossing number in terms of the number of vertices and edges of a graph.

Proposition 3.1 *A drawing of a graph G with n vertices and m edges has at least $m - 3n + 6$ crossings.*

Proof. Let H be a maximal planar subgraph of G. Every edge which is not in H has a crossing with some edge in H. Since H has at most $3n - 6$ edges the bound follows. □

With a slightly more subtle counting argument we can raise the lower bound on the crossing number to the order of $m^2/6n$.

Proposition 3.2 *Any drawing of a graph G with n vertices and m edges has at least $\sum_{i=1}^{\lfloor m/r \rfloor - 1} i \cdot r$ crossings, where $r \leq 3n - 6$ is the number of edges of a maximal planar subgraph of G.*

Proof. Let $G_0 = G$ and H_0 be a maximal planar subgraph of G_0. Inductively, define $G_{i+1} = G_i \setminus H_i$ and let H_{i+1} be a maximal planar subgraph of G_{i+1}. Every edge of H_i has at least i crossings, one with each subgraph H_j, $j < i$. The observation $\sum_{i=1}^{\lfloor m/r \rfloor - 1} i \cdot r \leq \sum_{i \geq 1} i \cdot |E(H_i)|$ completes the proof. □

Erdős and Guy conjectured that the true order of magnitude of the crossing number is cm^3/n^2 for some constant c. The positive answer is known as the *Crossing Lemma*, it deserves the name lemma because it has very nice applications as we will see in subsequent sections.

Theorem 3.3 (Crossing Lemma)
If G is a graph with n vertices and $m \geq 4n$ edges, then

$$\mathrm{cr}(G) \geq \frac{1}{64}\frac{m^3}{n^2}.$$

Proof. Fix a drawing of G with a minimal number of crossings and construct a random subgraph H of G as follows. Take a p-biased coin, i.e., a coin showing head with probability p. The graph H is constructed by flipping the coin for every vertex v of G, if the coin shows head, then v is accepted for H, otherwise v is refused. The graph H is the graph induced by the accepted vertices.

Let $n(H)$ and $m(H)$ be the number of vertices and edges of H. Denote the number of crossings in the induced drawing of H by $\hat{\mathrm{cr}}(H)$, clearly $\hat{\mathrm{cr}}(H) \geq \mathrm{cr}(H)$. Since H is a random graph these quantities are random variables. The expected values are easily computed

$$\begin{aligned}
\mathrm{Ex}(n(H)) &= p \cdot n(G)\\
\mathrm{Ex}(m(H)) &= p^2 \cdot m(G)\\
\mathrm{Ex}(\hat{\mathrm{cr}}(H)) &= p^4 \cdot \mathrm{cr}(G).
\end{aligned}$$

The exponents of p on the right-hand side just tell how many coin-flips have to show head to make a single vertex, edge or crossing of G appear in H.

Proposition 3.1 implies $\hat{\mathrm{cr}}(H) - m(H) + 3n(H) \geq 0$. Taking expectations and making use of linearity of expectations:

$$\begin{aligned}
\mathrm{Ex}(\hat{\mathrm{cr}}(H) - m(H) + 3n(H)) &\geq 0,\\
\mathrm{Ex}(\hat{\mathrm{cr}}(H)) &\geq \mathrm{Ex}(m(H)) - \mathrm{Ex}(3n(H)),\\
p^4 \cdot \mathrm{cr}(G) &\geq p^2 \cdot m(G) - 3p \cdot n(G),\\
\mathrm{cr}(G) &\geq \frac{m(G)}{p^2} - \frac{3n(G)}{p^3}.
\end{aligned}$$

Set $p = 4n/m$, here we need the assumption that $m \geq 4n$, because as a probability $p \leq 1$. With this p we obtain:

$$\mathrm{cr}(G) \geq \frac{m^3}{16n^2} - \frac{3m^3}{64n^2} = \frac{1}{64}\frac{m^3}{n^2}.$$

□

The bound given by the Crossing Lemma is tight up to the constant: Consider a convex n-gon with vertices $x_0, x_1, \ldots, x_{n-1}$. Let G_k be the geometric graph on this set whose edges are the pairs $x_i x_j$ (indices modulo n) with $j \leq i + k$, i.e., edges are just the 'short' segments. This graph is regular of degree $2k$ and has $m = nk$ edges. An edge $x_i x_j$ with $j = i + (l + 1)$ is involved in $l \cdot 2k - l(l + 1)$ crossings. Summation over all edges yields:

$$\mathrm{cr}(G_k) = \frac{1}{3}nk^3 + O(nk^2) \approx \frac{1}{3}\frac{m^3}{n^2}.$$

3.3 Improving the Crossing Constant

The constant 1/64 resulting from the probabilistic proof of the crossing lemma is surprisingly good. However, with more effort an improvement by a factor of almost two is possible. The improvement is obtained by applying the probabilistic proof technique to an inequality of the form

$$\mathrm{cr}(G) \geq tm - (\binom{t}{2} + 3t)n. \tag{3.1}$$

This inequality is known to be valid for $t \leq 5$. The crossing constant $1/33.75$ is obtained from the probabilistic argument using the above equation with $t = 5$ and $p = 7.5\,n/m$. Here we prove (3.1) only for $t = 3$ and obtain a slightly weaker constant. In the computation we use the probability $p = 6n/m$.

Theorem 3.4 *If G is a graph with n vertices and $m > 6n$ edges, then*

$$\mathrm{cr}(G) \geq \frac{1}{36}\frac{m^3}{n^2}.$$

The proof of (3.1) is based on another nice extremal problem. A drawing is said to be *k-restricted* if every edge is crossed by at most k other edges. The question is: How many edges can a graph G with n vertices have, if G admits a k-restricted drawing?

Theorem 3.5 *A graph with n vertices admitting a k-restricted drawing, $k = 0, 1$ or 2 has at most $(k+3)(n-2)$ edges.*

Before starting into the proof we show how to derive (3.1) for $t = 3$ from the theorem. Let $G_3 = G$ and for $i = 2, 1, 0$ define a subgraph G_i of G_{i+1} by removing edges which have $i+1$ crossings until there are no such edges left. The construction implies that G_i is i-restricted and hence, $|E(G_i)| \leq (i+3)(n-2)$. Now, the crossings of G can be estimated as follows: $\mathrm{cr}(G) \geq 3(|E(G_3)| - |E(G_2)|) + 2(|E(G_2)| - |E(G_1)|) + (|E(G_1)| - |E(G_0)|) \geq 3m - \sum_{k=0}^{2}(k+3)(n-2) > 3m - 12n$.

Proof. $\mathbf{k = 0}$: Since 0-restricted graphs are exactly the planar graphs they have at most $3n - 6 = 3(n-2)$ edges.

$\mathbf{k = 1}$: Consider a 1-restricted drawing of a graph G with n vertices maximizing the number of edges of such a graph. Let H be a maximal planar subgraph of G. Let e be an edge in $E(G) \setminus E(H)$. Since $e \notin E(H)$ there is an edge $xy \in E(H)$ crossed by e. Let v be a vertex of e and note that it is possible to include edges vx and vy in the drawing of G, so that they are drawn crossing free. The maximality of G implies that the edges vx and vy are in $E(G)$. Since they are crossing free they already belong to $E(H)$.

Consequently, every edge $e \notin E(H)$ traverses a triangular face of H at each of its vertices. Call such a triangular face a *final triangle* for e. Since the drawing of G is 1-restricted a triangular face of H can be the final triangle for only one edge $e \notin E(H)$. Since H has at most $2n - 4$ triangular faces and every $e \in E(G) \setminus E(H)$ requires two final triangles there are at most $n - 2$ edges in $E(G) \setminus E(H)$. Since H has at most $3n - 6$ edges we obtain $|E(G)| \leq 4(n-2)$.

$\mathbf{k = 2}$: Consider a 2-restricted drawing of a graph G with n vertices maximizing the number of edges of such a graph. Let $vw \in E$ be an edge with a crossing. Start traversing vw at vertex v and let xy be the first edge crossed by vw. Since xy has at most two crossings one of the endpoints of xy can be reached from v without crossing an edge, say, this vertex is x. By maximality of G the edge vx is in G and moreover, this edge is a crossing-free edge, see Figure 3.3.

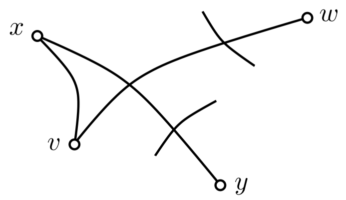


Figure 3.3 A crossing-free edge vx associated to vw.

Let H be the subgraph of G consisting of all edges of G which are crossing free in the drawing of G. We summarize the above argument as follows:

> If $e \in E(G) \setminus E(H)$ and v is a vertex of e, then one of the neighboring edges of e in the cyclic order of edges at v is in H. We say that this edge of H *takes care* of e at v.

An edge of H can take care of at most four crossed edges, i.e., edges of $E(G) \setminus E(H)$. Every crossed edge is taken care of at least twice. Hence, $2|E(G) \setminus E(H)| \leq 4|E(H)|$, or more simply $|E(G)| \leq 3|E(H)|$. To improve on this consider the faces of the plane graph H.

- If there is a four face F in H, then there are (at most) two edges interior to F and from the eight angles of edges of H in F only four are, actually, needed to take care of the interior edges, a loss of four.
- If there is a triangular face F in H, then from the six angles of edges of H in F non is needed to take care of interior edges, a loss of six.

Let f_4, f_3 be the number of 4-faces and 3-faces of H and let $m = |E(H)|$. We have shown $2|E(G) \setminus E(H)| \leq 4|E(H)| - 4f_4 - 6f_3$, or more simply $|E(G)| \leq 3m - 2f_4 - 3f_3$.

Let f be the number of faces of H and $f^+ = f - f_4 - f_3$. Since H may be disconnected the Euler relation turns into an inequality: $n - m + (f^+ + f_4 + f_3) \geq 2$. Double counting the edge-face incidences we further obtain $2m \geq 5f^+ + 4f_4 + 3f_3$. Subject to these two inequalities together with $f_3, f_4 \geq 0$ the objective $3m - 2f_4 - 3f_3$ is maximized when $f_3 = f_4 = 0$, $f^+ = \frac{2}{3}(n-2)$ and $m = \frac{5}{3}(n-2)$. This completes the proof of $|E(G)| \leq 5(n-2)$. □

Remark. The bounds given in the theorem for the number of edges of k-restricted drawings, $k = 1, 2$, are best possible. For $k = 1$ let Q be a planar graph such that all faces of Q are 4-gons. Q has $n-2$ faces and $2n-4$ edges. Adding the two diagonals to each face of Q gives a 1-restricted graph with $4(n-2)$ edges. Candidates for Q are the cube-graph and graphs obtained by gluing cubes together, face by face, see Figure 3.4.

For $k = 2$ let D be a planar graph such that all faces of D are 5-gons. D has $\frac{2}{3}(n-2)$ faces and $\frac{5}{3}(n-2)$ edges. Adding the five diagonals to each face of D gives a 2-restricted graph with $5(n-2)$ edges. Candidates for D are the dodecahedron-graph and graphs obtained by gluing dodecahedra together, face by face, see Figure 3.4.

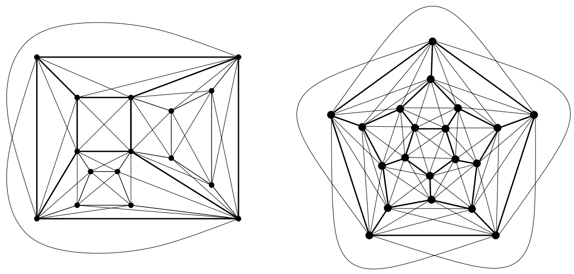


Figure 3.4 Examples of 1- and 2-restricted geometric graphs with maximal number of edges.

3.4 Crossing Numbers and Incidence Problems

Points, lines and incidences between them give raise to elementary and appealing problems. The celebrated Szemerédi–Trotter theorem gives a bound on the number of incidences in terms of the numbers of points and lines. The original proof of the theorem was a remarkable tour de force and would result in enormous constants. It came as a big surprise when László Székely observed that the crossing lemma could be used in an extremely short and elegant proof.

If $\mathcal{P}$ is a set of points and $\mathcal{L}$ is a set of lines, let $I(\mathcal{P}, \mathcal{L})$ be the number of incidences between points of $\mathcal{P}$ and lines of $\mathcal{L}$.

Theorem 3.6 (Szemerédi–Trotter)
Let $I(n, m)$ denote the maximum of $I(\mathcal{P}, \mathcal{L})$ taken over all sets $\mathcal{P}$ of n points and $\mathcal{L}$ of m lines, then

$$I(n, m) \leq 2.7\, n^{2/3} m^{2/3} + 6n + 2m.$$

Proof. Given $\mathcal{P}$ and $\mathcal{L}$ with I incidences we consider the arrangement as a geometric graph G. The vertices of G are the points in $\mathcal{P}$, i.e., $n(G) = n$. The edges of G are segments between consecutive points on a line. A line $\ell \in \mathcal{L}$ containing k points of $\mathcal{P}$ contributes $k - 1$ edges, hence, $m(G) = I - m$.

Since two lines have at most one intersection we have a bound on the crossing number: $\mathrm{cr}(G) \leq \binom{m}{2}$. With the Crossing Lemma (Theorem 3.4) we obtain

$$\frac{m^2}{2} > \binom{m}{2} \geq \mathrm{cr}(G) \geq \frac{1}{36} \frac{(I - m)^3}{n^2}.$$

This converts to $(18\, m^2 n^2)^{1/3} > I - m$ and further to $I < 2.7\, n^{2/3} m^{2/3} + m$. But recall that $I - m > 6n$ was a precondition for the application of the Crossing Lemma. If this precondition fails $I \leq 6n + m$. Adding the two bounds on I yields the theorem. □

The following is a surprising application of the incidence bound to a problem from number theory. Let $A = \{a_1, \ldots, a_n\}$ be a set of distinct numbers. Let $\Sigma = |A + A|$ and $\Pi = |A \cdot A|$, i.e., Σ and Π are the number of different sums and products of pairs of elements of A. Clearly, $\Sigma, \Pi \leq n^2/2$. Considering only sums and products involving $\min(A)$ and $\max(A)$ it follows that $\Sigma, \Pi \geq 2n - 1$. Erdős and Szemerédi asked whether one of the two quantities is always large. Elekes made big progress at this problem by applying the Szemerédi–Trotter Theorem.

Let $\mathcal{P}_A$ be the set of all points with coordinates of the form $(a_i + a_j\,,\; a_k \cdot a_j)$. The number of points in this set is upper bounded by $\Sigma \cdot \Pi$. Consider the set $\mathcal{L}_A$ of the n^2 lines with equations $y = a_k(x - a_i)$. Since point $(a_i + a_j\,,\; a_k \cdot a_j)$ is on line $y = a_k(x - a_i)$ each line contributes at least n incidences making a total of at least n^3 incidences. From the Szemerédi–Trotter Theorem

$$n^3 \leq C \cdot (|\mathcal{P}_A|^{\frac{2}{3}} |\mathcal{L}_A|^{\frac{2}{3}} + |\mathcal{P}_A| + |\mathcal{L}_A|).$$

With $|\mathcal{L}_A| = n^2$ this yields $|\mathcal{P}_A| \geq cn^{5/2}$. Consequently, one of $A + A$ and $A \cdot A$ has to be of size $c^{1/2} n^{5/4}$.

Back to plane geometry. Let $\mathcal{P}$ be a set of n points and $k \in \mathbb{N}$. A line ℓ is called a *big line* if it contains at least k points from $\mathcal{P}$. How big can the number of big lines be?

Theorem 3.7 *If B_k denotes the maximal number of big lines of a set $\mathcal{P}$ of n points and $2 \leq k \leq \sqrt{n}$, then*

$$B_k \leq c \frac{n^2}{k^3}.$$

Proof. Let $\mathcal{L}$ be a maximal set of big lines for $\mathcal{P}$. As in the previous proof we have a geometric graph G with vertex set $\mathcal{P}$. Since all lines are big $m(G) \geq B_k(k-1)$. Assume $m(G) \geq 6n$ and apply the Crossing Lemma to obtain

$$B_k^2 \geq \operatorname{cr}(G) \geq \frac{1}{36} \frac{(B_k(k-1))^3}{n^2}.$$

Since $k/(k-1) \leq 2$ we can estimate the constant as $c \leq 144$. Now suppose that the graph has too few edges for the Crossing Lemma. From $B_k(k-1) < 6n$, i.e., $B_k < 12n/k$, the bound stated in the theorem follows from $n/k^2 \geq 1$ which is true since $k \leq \sqrt{n}$. $\square$

The next question is about distances. Let $U(\mathcal{P})$ denote the number of unit distances among points in $\mathcal{P}$.

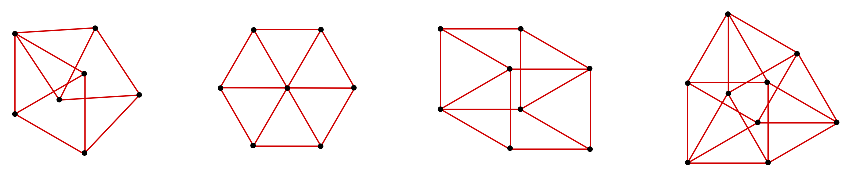


Figure 3.5 Four unit distance graphs, i.e., geometric graphs with edges of length one.

Theorem 3.8 *If $U(n)$ denotes the maximal number of unit distances among n points in the plane, then*

$$U(n) \leq 3.3 n^{4/3}.$$

Proof. Draw a circle of radius one around each of the n points in a maximizing configuration of points $\mathcal{P}$. Clearly each of the n circles contains at least two points of $\mathcal{P}$. Consider the arcs between consecutive points on the circles as edges of a topological graph G. The circle centered at point p is subdivided into as many arcs as there are points at unit distance from p. Therefore, the number of arcs of G is $2U(\mathcal{P})$.

Some pairs of points may be connected by two different arcs of G, discard one of every such pair of arcs. For the number of edges of the resulting graph G' we have $m(G') \geq U(\mathcal{P})$. If $m(G') \leq 6n$, then $U(n)$ is only linear in n. Otherwise, note that any two circles cross at most twice and apply the Crossing Lemma

$$n^2 \geq 2\binom{n}{2} \geq \operatorname{cr}(G') \geq \frac{1}{36}\frac{m(G')^3}{n^2} \geq \frac{1}{36}\frac{U(\mathcal{P})^3}{n^2}.$$

□

3.5 Notes and References

The crossing number problems for complete graphs and complete bipartite graphs have a long history. The problem for complete bipartite graphs is known as Turán's brick-factory problem. Turán was prisoner in a German labor camp in 1944 when the problem occurred to him. In [201] he tells the following story:

We worked near Budapest, in a brick factory. There were some kilns where the bricks were made and some open storage yards where the bricks were stored. All the kilns were connected by rail with all the storage yards. The bricks were carried on small wheeled trucks to the storage yards. All we had to do was to put the bricks on the trucks at the kilns, push the trucks to the storage yards, and unload them there. We had a reasonable piece rate for the trucks, and the work itself was not difficult; the trouble was only at the crossings. The trucks generally jumped the rails there, and the bricks fell out of them, in short this caused a lot of trouble and loss of time which was precious to all of us. We were all sweating and cursing at such occasions, I too; but nolens volens the idea occurred to me that this loss of time could have been minimized if the number of crossings of the rails had been minimized. But what is the minimum number of crossings? I realized after several days that the actual situation could have been improved, but the exact solution of the general problem with m kilns and n storage yards seemed to be very difficult ... the problem occurred to me again ... at my first visit to Poland where I met Zarankiewicz. I mentioned to him my "brick-factory"-problem ... and Zarankiewicz thought to have solved (it). But Ringel found a gap in his published proof, which nobody has been able to fill so far – in spite of much effort. (Turán 1977)

The conjecture of Zarankiewicz asserts:

$$\operatorname{cr}(K_{m,n}) = \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor.$$

This has been verified in some cases but the general problem remains unresolved. There is an easy upper bound construction matching this bound: Let $\lceil m \rceil = 2s$ and $\lceil n \rceil = 2t$ and define $X = \{(-a,0),(+a,0) : 0 < a \leq s\}$ and $Y = \{(0,-b),(0,+b) : 0 < b \leq t\}$ if m or n is odd delete a point from X or Y. Finally, connect each point in X by a straight segment with each point of Y. Due to this construction a proof of Zarankiewicz's conjecture would also imply $\operatorname{cr}(K_{m,n}) = \overline{\operatorname{cr}}(K_{m,n})$. Very readable accounts to the state of crossing number problems in the early 1970's are given by Guy [110] and by Erdős and Guy [75]. Guy [110] proved that rectilinear crossing number and crossing number can differ, he shows $\operatorname{cr}(K_8) = 18$ and $\overline{\operatorname{cr}}(K_8) = 19$. The example of Figure 3.2, which shows that $\overline{\operatorname{cr}}(G)/\operatorname{cr}(G)$ is unbounded, is due to Bienstock and Dean [25]. Garey and

Johnson [96] prove that computing the crossing number of a graph is a computationally hard problem.

The Crossing Lemma (Theorem 3.3) was conjectured by Erdős and Guy [75]. The first proofs were given by Ajtai, Chvátal, Newborn and Szemerédi [9] and independently Leighton [131]. The probabilistic proof presented in the first section was communicated by Emo Welzl who attributed it to discussions with Chazelle and Sharir. Pach and Tóth [155] improved the constant from $1/64 \approx 0.015$ to 0.029. They prove the statement of Theorem 3.5 for k up to 4 and conclude inequality (3.1) for $t = 5$. The latest improvement of the crossing constant is due to Pach, Radoicic, Tardos and Tóth [150], the new constant is 0.31. The result is based on an improved bound for the number of edges in a 3-restricted drawing. They show that a graph admitting a 3-restricted drawing has at most $5.5(n-2)$ edges.

Pach and Tóth [155] also discuss upper bound constructions. The geometric graph consisting of all short edges (edges of length $\leq (2m/\pi n)^{1/2}$) in a slightly perturbed $n^{1/2} \times n^{1/2}$ grid is shown to have $\leq 0.06(m^3/n^2)$ crossings. Variants of crossing numbers and of the Crossing Lemma have been studied by Pach and Tóth [156] and Pach, Spencer and Tóth [151]. One of the variants is the *odd-crossing number*, this is the minimum number of pairs of edges which have an odd number of crossings. Planar graphs can be characterized as those graphs which have odd-crossing number zero. An easy proof for this remarkable result of Tutte [204] is still missing.

The geometric applications of the crossing lemma shown in Section 3.4 are taken from Székely's stupendous paper [193]. That paper also contains an improved bound on the number of different distances determined by some point from a set of n points based on the same technique. The number theoretic application of the Szemerédi–Trotter Theorem is from Elekes [69]. There are many more applications of the Crossing Lemma and the incidences bound than shown in this chapter. For further reading we recommend the books of Matoušek [138] and of Pach and Agarwal [148] and the survey of Erdős and Purdy [77].

The Szemerédi–Trotter theorem (Theorem 3.6) is one of the strong tools in combinatorial geometry. The original proof of the theorem [195] was extremely complicated and resulted in enormous constants. A proof using cuttings was found by Clarkson et al. [50], a simplified version of this proof is due to Aronov and Sharir [16]. Székely [193] gave the extremely short and elegant proof reproduced here.

The Szemerédi–Trotter paper [195] contains a first collection of geometric extremal problems which could be attacked using their result. One of these first applications is the big-line theorem (Theorem 3.7). At about the same time Beck [22] proved a slightly weaker result and used this for improvement in several other geometric extremal problems, among them Dirac's problem (cf. the notes section of Chapter 5).

Erdős [74] initiated the study of the distribution of distances in planar configurations of points. In particular he asked for two quantities: The minimum number $D(n)$ of distinct distances among n points in the plane and the maximum number $U(n)$ of unit distances among n points in the plane. Erdős remarks that n points arranged on a square-grid of suitable step-size realize $n^{(1+c/\log\log n)}$ unit distances, the computation verifying this is detailed in the book [138] of Matoušek. As upper bound Erdős proved $U(n) \leq n^{3/2}$. The bound given in Theorem 3.8 was first obtained by Spencer, Szemerédi and Trotter [182]. It is conjectured that the actual growth of $U(n)$ is close to the lower bound. Erdős had offered \$500 for a proof or disproof of this conjecture.

Another fascinating problem related to unit distances is the *chromatic number of the* *plane*: What is the minimum number of colors needed to color the points of $\mathbb{R}^2$ such that any two points at unit distance receive different colors, this number is denoted $\chi(\mathbb{R}^2)$. The first of the small unit distance graphs of Figure 3.5 shows that $\chi(\mathbb{R}^2) \geq 4$. The upper bound $\chi(\mathbb{R}^2) \leq 7$ can be shown by an appropriate coloring of the cells of a regular honeycomb tiling with side-length slightly less than 1/2, see Figure 3.6. The

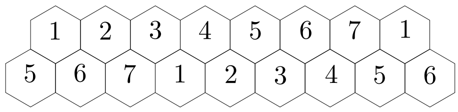


Figure 3.6 Part of a periodic 7 coloring of the plane.

bounds $4 \leq \chi(\mathbb{R}^2) \leq 7$ have stood for over 50 years now. A recent survey was given by Székely [194].

4 k-Sets and k-Facets

Let $\mathcal{P}$ be a set of n points in $\mathbb{R}^d$ in general position, this assumption means that no subset of $j \leq d+1$ points is affinely dependent.

A *k-set* of $\mathcal{P}$ is a k element subset $\mathcal{S}$ of $\mathcal{P}$ such that $\mathcal{S}$ and $\bar{\mathcal{S}} = \mathcal{P} \setminus \mathcal{S}$ can be separated by a hyperplane H. The problem of finding good upper and lower bounds for the number of k-sets is central to combinatorial geometry.

In this chapter we present some of the results that have been obtained. In Section 4.1 we deal with the k-set problem in the plane. We collect classical observations and bounds and continue with the new $O(n\,k^{1/3})$ upper bound (Theorem 4.5). One of the key ingredients is again the Crossing Lemma (Theorem 3.3).

With Section 4.2 we leave the plane. Welzl's Theorem (Theorem 4.7) gives a higher-dimensional analog of the Lovász Lemma (Lemma 4.3) in the plane. The section ends with a bound for the 3-dimensional problem. Here the Crossing Lemma again plays a crucial role.

Section 4.3 shows a surprising application of a lower bound for the total number of t-sets with $t \leq k$ ($\leq$$k$-sets for short). The lower bound is used to prove that every straight line drawing of the complete graph K_n has at least $\frac{3}{8}\binom{n}{4}$ crossings. This result of Lovász, Vesztergombi, Wagner and Welzl is a very recent breakthrough at the old and notorious problem of rectilinear crossing numbers.

4.1 k-Sets in the Plane

Let $\mathcal{P}$ be a set of n points in the plane. A k-edge of $\mathcal{P}$ is a directed edge with endpoints in $\mathcal{P}$ such that exactly k points of $\mathcal{P}$ are on the positive (left) side of the line supporting the edge.

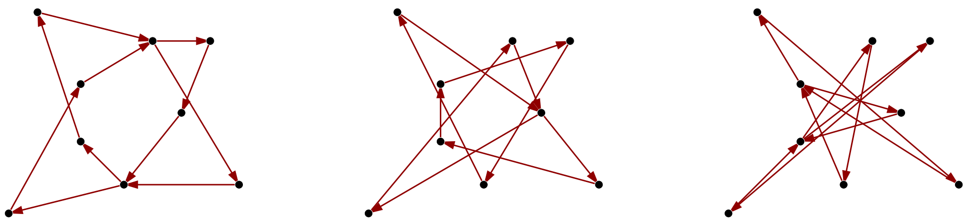


Figure 4.1 A set of 9 points with its k-edges for $k = 1, 2, 3$.

Most contributions in the area of k-sets of planar point sets actually work with k-edges. This is legitimized by the following simple observation.

Observation 1. Let $\mathcal{P}$ be a set of n points in the plane and $1 \leq k < n$. The number of k-sets of $\mathcal{P}$ equals the number of $(k-1)$-edges of $\mathcal{P}$.

Proof. Let $\mathcal{S}$ be a k-set of $\mathcal{P}$. There are exactly two lines which are tangent to $\mathcal{S}$ and $\bar{\mathcal{S}}$ and separate them. Note that being tangent these lines contain exactly one point from each of $\mathcal{S}$ and $\bar{\mathcal{S}}$. Orient these bi-tangents such that $\mathcal{S}$ is left and $\bar{\mathcal{S}}$ right. On exactly one of the lines the direction is from the point in $\mathcal{S}$ to the point in $\bar{\mathcal{S}}$, call this line $g(\mathcal{S})$. The segment between the two points on $g(\mathcal{S})$ is a $(k-1)$-edge $e(\mathcal{S})$ of $\mathcal{P}$. The mapping $\mathcal{S} \to e(\mathcal{S})$ is a bijection between k-sets and $(k-1)$-edges. △

The following lemma has an elementary proof but a wide range of applications (it really deserves to be called a lemma).

Lemma 4.1 (Alternation Lemma)
Let p be a fixed point from $\mathcal{P}$ and ℓ be a directed line which is split by p into a front-half and a back-half. Rotate ℓ with center p in clockwise direction and consider the out-events *where the front-half covers a k-edge which is outgoing at p and the* in-events *where the back-half covers a k-edge which is incoming at p. Out-events and in-events alternate during the rotation.*

Proof. Let $\mu(\ell)$ be the number of points on the positive (left) side of the rotating line ℓ. If the front-half of the line scans a point, then $\mu(\ell)$ is increased by one, $\mu(\ell) \to \mu(\ell)+1$. Symmetrically, $\mu(\ell) \to \mu(\ell) - 1$ corresponds to the event that the back-half of the line scans a point. At an out-event $\mu(\ell)$ changes from k to $k+1$. Between two out-events the value of $\mu(\ell)$ must get back to k. This happens first when the value changes from $k+1$ to k and this corresponds to an in-event. Similarly, there is an out-event between any two in-events. Together this enforces the claimed alternation of out- and in-events. □

One immediate consequence of the lemma is that every point has the same number of incoming and outgoing k-edges. In particular, the number of k-edges incident to a point is even. The following lemma pinpoints another important fact.

Lemma 4.2 *Let ℓ be a line containing a unique point p from $\mathcal{P}$. If from the positive side of ℓ there are a incoming k-edges at p, then there are either $a-1$ or a or $a+1$ outgoing k-edges reaching from p into the negative side. If $k < \frac{n-1}{2}$ the precise statement is:*

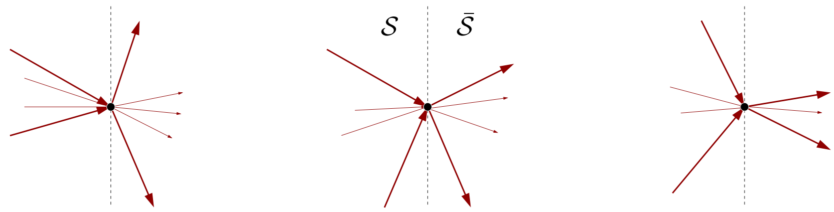
- *If the positive side of ℓ contains at most k points, then the number of outgoing k-edges reaching from p into the negative side is $a+1$.*
- *If both sides of ℓ contain more than k points, then the number of outgoing k-edges reaching from p into the negative side is a.*
- *If the negative side of ℓ contains at most k points then the number of outgoing k-edges reaching from p into the negative side is $a-1$.*

Proof. As in the proof of the Alternation Lemma we consider rotations of a line centered at p. Starting in the position of ℓ consider a clockwise rotation. If left of ℓ there are at most k points, then the line first arrives at an out-event and if there are more than k points left of ℓ, then the line first arrives at an in-event.

Next we have to determine whether in counterclockwise direction the line first reaches an incoming k-edge from the left of ℓ or an outgoing k-edge to the right. Note that reverting the orientation of an k-edge, yields an $(n-k-2)$-edge and conversely. Therefore we consider $(n-k-2)$-edges, those which are outgoing left of ℓ and those which are incoming right of ℓ and adopt the terminology of out- and in-events for them. Rotating counterclockwise an out-event corresponds to a decrease from $n-k-1$ to $n-k-2$

points on the left side of the line. An in-event corresponds to an increase from $n-k-2$ to $n-k-1$ points on the left side. Therefore, starting with $n-k-1$ or more points on the left the line first arrives at an out-event while starting with $n-k-2$ or fewer points on the left side the line first arrives at an in-event.

The three cases of the lemma are shown schematically in Figure 4.2. The Alternation



$|\mathcal{S}| \leq k < n-k-1$ $k < |\mathcal{S}| < n-k-1$ $k < n-k-1 \leq |\mathcal{S}|$

Figure 4.2 A small, medium and large set $\mathcal{S}$ left of the vertical through p.

Lemma allows to draw the claimed conclusions. □

An important special case is when the number n of points is even and $k = \frac{n-2}{2}$. In this case the negative and the positive side of a line supporting a k-edge contain the same number of points, hence, the reorientation of a k-edge is again a k-edge, i.e., k-edges come in bidirected pairs. Consider such pairs of k-edges as single undirected edges and call them *halving edges*. The lines supporting the halving edges are the *halving lines*. Note that the previous lemma implies that every point of an even point set $\mathcal{P}$ is incident to an odd number of halving edges.

Lemma 4.3 (Lovász Lemma)
Let ℓ be a line disjoint from $\mathcal{P}$ which separates $\mathcal{P}$ into a left set $\mathcal{S}$ and a right set $\bar{\mathcal{S}}$. If $|\mathcal{S}| = s$, then for $0 \leq k \leq \frac{n-2}{2}$, the number $e_k(\ell)$ of k-edges (p,q) which cross ℓ from left to right i.e., with $p \in \mathcal{S}$ and $q \in \bar{\mathcal{S}}$, is exactly $\min(k+1, s, n-s)$.

Proof. For simplicity we assume that ℓ is vertical and that no two points of $\mathcal{P}$ have the same x-coordinate. Consider a vertical line ℓ_0 which has $\mathcal{P}$ on its right side. Let ℓ_t be a continuously moving vertical line starting at ℓ_0 and moving across the plane such that ℓ_t, $t \in [0,1]$ are the intermediate positions of the line. Let $e_k(\ell_t)$ be the number of k-edges crossing ℓ_t from left to right. Starting with $e_k(\ell_0) = 0$ the value of $e_k(\ell_t)$ can only change when ℓ_t moves over a point of $\mathcal{P}$. At the first $k+1$ points met by the line the set of points left of ℓ_t has cardinality at most k. Therefore, the first case of Lemma 4.2 tells us that at each of these points the value of $e_k(\ell_t)$ is increasing by one. While the set of points left of ℓ_t has cardinality between $k+1$ and $n-k-2$ there is no change in the value $e_k(\ell)$ as shown by the second case of Lemma 4.2. Hence, $e_k(\ell_t) = k+1$ in this range. At each of the last $k+1$ points the value of $e_k(\ell_t)$ is decreased by one, this is implied by the thirds case of Lemma 4.2.

Since $\ell_t = \ell$ for some t the value of $e_k(\ell)$ only depends on the number of points left of ℓ and can be put into the closed form $e_k(\ell) = \min(k+1, s, n-s)$. □

The classical upper bound for the number of k-sets follows quite easily from the Lovász Lemma: With $n-1$ vertical lines we partition the plane into n stripes, each containing a single point of $\mathcal{P}$. Each vertical line is intersected by $2(k+1)$ or fewer k-edges. This gives a total of $\approx 2nk$ intersections between a k-edge and one of the vertical lines. It follows that the number of k-edges which cross $\sqrt{k}$ or more of the vertical lines is at most $2n\sqrt{k}$. A point of $\mathcal{P}$ is incident to at most $2\sqrt{k}$ of the remaining short edges. Therefore, the total number of short k-edges is again bounded by $2n\sqrt{k}$. Hence, the total number of k-edges is at most $4n\sqrt{k}$ and we have proven the following proposition.

Proposition 4.4 *A set of n points in the plane has at most $O(n\sqrt{k})$ k-edges.*

Proof. Here is another nice proof which makes use of the Crossing Lemma (Theorem 3.3): Lemma 4.3 implies that any given k-edge is crossed by at most $2k$ other k-edges. Consider the geometric graph G_k of all k-edges and let m_k be the number of edges. We have $\mathrm{cr}(G_k) \leq km_k$. In combination with the Crossing Lemma this yields, $km_k \geq c\frac{m_k^3}{n^2}$, i.e., $m_k \leq c'n\sqrt{k}$. □

Actually, the Crossing Lemma was the essential ingredient that led to the following improved bound for the number of k-edges of planar point sets. We define the following quantities which depend on a point set $\mathcal{P}$ which is usually suppressed in the notation. With $e_j = e_j(\mathcal{P})$ we denote the number of j-edges and $E_k = E_k(\mathcal{P})$ is the total number of j-edges for $0 \leq j \leq k$, i.e., $E_k = \sum_{j=0}^{k} e_j$.

Theorem 4.5 *A set of n points in the plane has at most $O(nk^{1/3})$ k-edges.*

Proof. The work in proving the theorem is to prove a sensible bound on the crossing number $\mathrm{cr}(G_k)$ of the graph of k-edges. Here this is done via the identity from Proposition 4.6, which implies $\mathrm{cr}(G_k) \leq E_{k-1}$. It is known* that $E_{k-1} \leq kn$, with the Crossing Lemma this implies $kn \geq c\frac{m_k^3}{n^2}$ and hence, $m_k \leq c'n\sqrt[3]{k}$. □

Proposition 4.6 *Let $\mathcal{P}$ be a set of n points in general position and $k \leq \frac{n-3}{2}$. If $\deg_k^+(p)$ denotes the out-degree of a point p in the graph G_k of k-edges of $\mathcal{P}$, then:*

$$\mathrm{cr}(G_k) + \sum_{p \in \mathcal{P}} \binom{\deg_k^+(p)}{2} = E_{k-1}.$$

Proof. The proof is based on a *continuous motion argument*. The idea is as follows: First, the identity is verified for a specific set $\mathcal{P}_o$. Then we consider a motion of the points starting with $\mathcal{P}_o$ and, eventually, stopping when the points are in the designated positions, i.e., when the point set is $\mathcal{P}$. The key observation is that changes in the graph of k-edges, which may affect the identity, correspond to situations where three points move through a collinearity. These situations, called *mutations*, are analyzed and it is shown that in each case the increase or decrease on both sides of the formula is the same so that the identity remains valid.

Convex position. Let $\mathcal{P}_o$ be a set of $n \geq 2k+3$ points in convex position. Each point $p \in \mathcal{P}_o$ has precisely one outgoing k-edge, $\deg_k^+(p) = 1$, hence, all binomial coefficients in the sum are of type $\binom{1}{2} = 0$. The outgoing k-edge at p is crossed from left to right by those k-edges which are outgoing at one of the k points left of the edge. In total this makes nk crossings and nk is the value of the left hand side. The number of j-edges is n for each $j = 0, \ldots, n-2$, therefore, the value of E_{k-1} is kn and the identity holds for $\mathcal{P}_o$.

* A proof for the slightly weaker bound $E_{k-1} \leq 2kn$ follows from Proposition 6.15 via duality.

Moving points. For the movement of the points from $\mathcal{P}_0$ to $\mathcal{P}$ we make the following assumptions:

- The moving set is in general position, except for finitely many moments when collinearities of more than two points occur.
- The collinearities never involve more than three points and at any moment there is at most one collinearity.

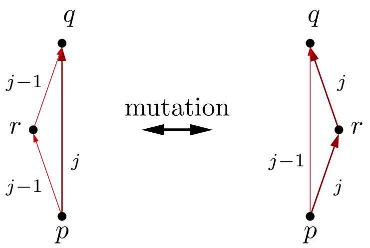


Figure 4.3 Point r crossing from the left through a j-edge or from the right through a $(j-1)$-edge.

Lets take a closer look at a mutation on three points $\{p, q, r\}$. We may assume that the mutation is such that in the moment of collinearity r is between p and q. We can imagine such a mutation as a movement of point r across the line spanned by p and q. Let (p, q) be a j-edge and let r cross the edge from left to right. It follows that immediately before the mutation the edges (p, r) and (r, q) both are $(j-1)$-edges. After the mutation (p, q) has lost the point r on the positive side, so (p, q) has become a $(j-1)$-edge and (p, r), (r, q) have become j-edges, see Figure 4.3. The reverse of the picture describes a mutation with r is crossing through a $(j-1)$-edge (p, q) from right to left.

From this analysis we conclude that a mutation involving j- and $(j-1)$-edges, keeps the value of both sides of the claimed identity unaffected unless $j = k$ or $j - 1 = k$.

Mutations involving a k-edge. We separately analyze the mutations involving k-edges and $(k-1)$-edges (type M_k^-) and those mutations involving k-edges and $(k+1)$-edges (type M_k^+), see Figure 4.4 for both types.

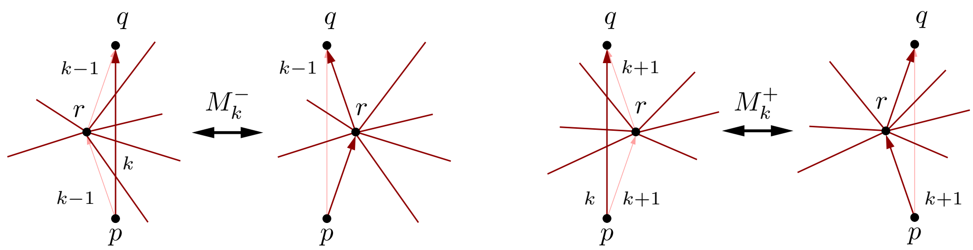


Figure 4.4 The two types M_k^- and M_k^+ of mutations involving k-edges.

Mutation M_k^-: Suppose M_k^- transforms the left configuration in the figure to the right configuration. The left configuration contains one $(k-1)$-edge more than the right one, therefore, E_{k-1} is decreased by one. Let $s = \deg_k^+(r)$ be the out-degree of r in the left configuration. The contribution of r in the sum changes from $\binom{s}{2}$ to $\binom{s+1}{2}$ which amounts to an increase by s.

It remains to consider the number b of crossings lost under the mutation. Let ℓ be a line through r parallel to (p, q). Note that b is just the number of k-edges incident to r and reaching into the right of ℓ. Let a count the number of k-edges incident to r and reaching into the left of ℓ. The sum $a + b$ is the degree of r in G_k which is $2\deg_k^+(r) = 2s$, (see Lemma 4.1). Refine the counts of a and b by separately counting incoming and outgoing edges as a^-, b^- and a^+, b^+. Since line ℓ has $k-1$ points on the left, Lemma 4.2 implies that $a^- = b^+ - 1$. Consider ℓ in the reverse direction, such that the line has $(n-k)$ points on the positive side. Again with Lemma 4.2 we obtain $b^- = a^+ + 1$. Together with the trivial equations $a = a^- + a^+$ and $b = b^- + b^+$ this yields $b = a + 2$ and, hence, $b = s + 1$.

Summarizing: A mutation of type M_k^- (from left to right) implies the following changes on the two rides of the identity:

- E_{k-1}: decrease by 1.
- $\mathrm{cr}(G_k) + \binom{\deg_k^+(r)}{2}$: (decrease by $\deg_k^+(r) + 1$) + (increase by $\deg_k^+(r)$).

Therefore, both sides of the identity are decreased by one if a mutation M_k^- is performed from left to right. If M_k^- is performed from right to left, then both sides of the identity are increased by one.

Mutation M_k^+. Again we assume that the mutation is from the left to the right configuration in Figure 4.4. No j-edges with $j < k$ are involved in the mutation, hence, the value E_{k-1} remains unaltered. Let $s = \deg_k^+(r)$ be the out-degree of r in the left configuration. The contribution of r in the sum changes from $\binom{s}{2}$ to $\binom{s+1}{2}$ which amounts to an increase by s.

It remains to consider the number b of crossings lost under the mutation. Let ℓ be a line through r parallel to (p, q) and let a and b be the number of k-edges incident to r from the left and right of ℓ. Again $a + b = 2\deg_k^+(r) = 2s$ as well as $a = a^- + a^+$ and $b = b^- + b^+$ where a^-, b^- count incoming and a^+, b^+ outgoing edges. Since line ℓ has $k+2$ points on the left Lemma 4.2 implies[†] that $a^- = b^+$. Consider ℓ in the reverse direction, such that the line has $n - (k + 3)$ points on the positive side. Again with Lemma 4.2 we obtain $b^- = a^+$. Hence, $b = s$ and the value of the left hand side of the identity is also unaffected by a mutation of type M_k^+. □

4.2 Beyond the Plane

The Lovász Lemma (Lemma 4.3) is a key tool for bounding the number of k-edges of point sets in the plane. A similarly prominent role is taken by the analog of the lemma in higher dimensions. In this section we prove Welzl's Theorem (Theorem 4.7) which is a precise version.

Let $\mathcal{P}$ be a set of n points in general position in $\mathbb{R}^d$. A *k-facet* of $\mathcal{P}$ is an oriented $(d-1)$-simplex spanned by d points of $\mathcal{P}$, such that exactly k points of $\mathcal{P}$ are on the (strictly) positive side of the hyperplane affinely generated (spanned) by the d points of the simplex. The terminology is motivated by the fact that 0-facets are exactly the facets of the convex hull of $\mathcal{P}$, oriented such that the positive side contains no points from $\mathcal{P}$.

We say that *a directed line ℓ enters a k-facet F* if the line intersects F in a single point where it changes from the positive to the negative side of F.

[†] If $n = 2k + 3$ this is not quite true: In this case replace ℓ by a slightly tilted line through r which separates p and q. With the resulting $(k + 1, n - (k - 2))$ partition the argument works.

If F is a $(d-1)$-simplex and ℓ a directed line intersecting F, then we assign an orientation to F such that the facet F_ℓ corresponding to this orientation of F is entered by ℓ.

Let ℓ be a directed line which is disjoint from all convex hulls of $d-1$ points in $\mathcal{P}$ and define $\bar{h}_j = \bar{h}_j(\ell, \mathcal{P})$ as the number of j-facets entered by line ℓ.

The aim is to find upper bounds on the $\bar{h}_j$'s. The 0-facets are facets of the convex hull of $\mathcal{P}$. Since a line can enter the convex hull at most once, we have:

Fact 0. $\bar{h}_0 \leq 1$.

Define $s^0 = \sum_j \bar{h}_j$. Every $(d-1)$-simplex based on points of $\mathcal{P}$ that is intersected by ℓ is a k-facet for some k. Therefore, s^0 is just the total number of $(d-1)$-simplices intersected by ℓ.

Define $s^1 = \sum_j j\bar{h}_j$. This counts pairs (F, p), where F is a $(d-1)$-simplex and p is a point on the positive side of F_ℓ. Together F and p span a d-simplex intersected by ℓ and, indeed, every d-simplex Σ intersected by ℓ gives a unique pair (F, p): facet F of Σ is specified by the property that ℓ leaves Σ through F. The bijection shows that s^1 is the number of d-simplices intersected by ℓ.

More generally define $s^k = \sum_j \binom{j}{k}\bar{h}_j$. This counts pairs (F, B), where F is a $(d-1)$-simplex and B is a set of k points from the positive side of F_ℓ. A bijection as in the previous case shows that s^k counts the number of $(k+d)$-element subsets of $\mathcal{P}$ whose convex hull is intersected by ℓ.

Fact 1. The vector $(\bar{h}_0, \bar{h}_1, \ldots, \bar{h}_{n-d})$ and the vector $(s^0, s^1, \ldots, s^{n-d})$ completely determine each other.

Proof. By definition $s^k = \sum_j \binom{j}{k}\bar{h}_j$ is expressed in terms of the $\bar{h}_j$. The explicit representation of $\bar{h}_j$ in terms of the s^k is $\bar{h}_j = \sum_k (-1)^{j+k}\binom{k}{j}s^k$. This inversion can be verified by elementary manipulations of binomial coefficients. △

Let ℓ' be the line parallel to ℓ through p. The interpretation of s^k directly implies: Replacing a point p by another point p' on ℓ' leaves the vector $(s^0, s^1, \ldots, s^{n-d})$ unaltered. Combined with Fact 1 this implies the same invariance for the $\bar{h}$-vector:

Fact 2. The vector $(\bar{h}_0, \bar{h}_1, \ldots, \bar{h}_{n-d})$ does not change when we replace a point p by another point p' on ℓ_p.

Let H be a hyperplane perpendicular to ℓ, such that all points of $\mathcal{P}$ are on one side of H. Let $p \to \hat{p}$ be the reflection at H, i.e., p and $\hat{p}$ determine a line parallel to ℓ and have the same distance from H. Let $\hat{\mathcal{P}}$ denote the reflected set and note that F is a j-facet of $\mathcal{P}$ iff $\hat{F}$ is a $(n-d-j)$–facet of $\hat{\mathcal{P}}$. With Fact 2 this implies the symmetry of the $\bar{h}$-vector:

Fact 3. $\bar{h}_j = \bar{h}_{n-d-j}$ for $0 \leq j \leq n-d$.

Fact 4. $\sum_p \bar{h}_j(\ell, \mathcal{P} \setminus \{p\}) = (n-d-j)\bar{h}_j + (j+1)\bar{h}_{j+1}$

Proof. The left hand side counts all pairs (p, F) where p is a point of $\mathcal{P}$ and F is a j-facet of $\mathcal{P} \setminus \{p\}$ entered by ℓ. These pairs are of two types. Either F is a j-facet of $\mathcal{P}$ and p a point on the negative side of F. Each j-facet entered by ℓ comes with $(n-d-j)$ points on the negative side. This gives a total of $\bar{h}_j(n-d-j)$ such pairs. The other possibility is that F is a $(j+1)$-facet of $\mathcal{P}$ and p is a point on the positive side of F. There are $\bar{h}_{j+1}(j+1)$ such pairs. △

Fact 5. $\bar{h}_j \geq \bar{h}_j(\ell, \mathcal{P} \setminus \{p\})$ for every $p \in \mathcal{P}$.

Proof. Fix $p \in \mathcal{P}$, above p in the direction of ℓ there is a point p' which is on the negative side of all $(j+1)$-facets generated by points in $\mathcal{P} \setminus \{p\}$. Replace p in $\mathcal{P}$ by p' to obtain $\mathcal{P}'$. Fact 2 implies $\bar{h}_j(\ell, \mathcal{P}) = \bar{h}_j(\ell, \mathcal{P}')$.

Let F be a j-facet of $\mathcal{P}' \setminus \{p'\}$. By construction p' is on the negative side of F. Therefore, F is also a j-facet of $\mathcal{P}'$. This shows $\bar{h}_j(\ell, \mathcal{P}' \setminus \{p'\}) \leq \bar{h}_j(\ell, \mathcal{P}')$. With $\mathcal{P}' \setminus \{p'\} = \mathcal{P} \setminus \{p\}$ and the identity $\bar{h}_j(\ell, \mathcal{P}) = \bar{h}_j(\ell, \mathcal{P}')$ we obtain $\bar{h}_j(\ell, \mathcal{P} \setminus \{p\}) \leq \bar{h}_j(\ell, \mathcal{P})$. △

Combining Fact 4 and 5 we obtain $n\bar{h}_j \geq (n - d - j)\bar{h}_j + (j+1)\bar{h}_{j+1}$ which can be rewritten as:

$$\bar{h}_{j+1} \leq \frac{j+d}{j+1}\bar{h}_j.$$

This allows to prove the next theorem with induction on j. The initial condition $\bar{h}_0 \leq 1$ for the induction is provided by Fact 0.

Theorem 4.7 (Welzl)
Let $\mathcal{P}$ be a set of n points in general position and ℓ be a directed line disjoint from all convex hulls of sets of $d-1$ points of $\mathcal{P}$. The number $\bar{h}_j$ of j-facets of $\mathcal{P}$ entered by ℓ satisfies

$$\bar{h}_j \leq \binom{j-1+d}{j} = \binom{j+d-1}{d-1}.$$

k-Facets in Three Dimensions

We apply previous findings to give an upper bound for the number of k-facets in three dimensions. The bound in the next theorem is weaker than the best known, still, the proof carries some ideas that are also used for the stronger results.

Theorem 4.8 *The number of k-facets of a set of n points in $\mathbb{R}^3$ is $O(n^2k^{2/3})$.*

Proof. In $\mathbb{R}^3$ a k-facet is an oriented triangle. Let T be the set of k-facets of a point set $\mathcal{P}$ of n points in general position. A pair of triangles Δ_1, Δ_2 is a *crossing pair* iff they share one vertex p and the edge of Δ_1 opposite of p intersects the interior of Δ_2, see Figure 4.5.

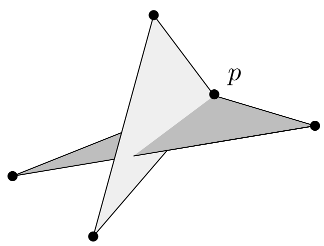


Figure 4.5 A pair of crossing triangles.

We give two estimates for the number X of crossing pairs. The first estimate is based on Theorem 4.7: A line spanned by two points x, y of $\mathcal{P}$ can enter at most $\binom{k+2}{2}$ triangles from T. Hence, a line can cross at most $2\binom{k+2}{2}$ triangles from T. Each crossed triangle Δ provides three corners p, such that $(\{p, x, y\}, \Delta)$ can be a crossing pair. Therefore $X \leq 3\binom{n}{2}2\binom{k+2}{2} \in O(n^2k^2)$.

For a second estimate of X we associate a geometric graph G_p with each $p \in \mathcal{P}$: Take a sphere S_p with center p and project all triangles of T which have p as a corner radially onto S_p. If there are t_p triangles with corner p we obtain a geometric graph G_p with t_p edges. Crossing pairs of edges in G_p correspond to pairs of crossing triangles with common vertex p. If x_p is the number of crossings of G_p, then $X = \sum_p x_p$. From the Crossing Lemma[‡] $x_p \geq c\frac{t_p^3}{n^2}$. Using the inequality $\sum_p t_p^3 \geq n\left(\frac{1}{n}\sum_p t_p\right)^3$ this implies a lower bound on X:

$$X = \sum_p x_p \geq \sum_p c\frac{t_p^3}{n^2} = \frac{c}{n^2}\sum_p t_p^3 \geq \frac{c}{n^2}\frac{(\sum_p t_p)^3}{n^2} \geq \frac{c'|T|^3}{n^4}.$$

Combining the upper and the lower bound for X we obtain $|T|^3 \leq c''n^6k^2$, as claimed. □

4.3 The Rectilinear Crossing Number of K_n

We close the chapter with a problem about crossing numbers. Precisely, we ask for the rectilinear crossing number of the complete graph.

We begin with a simple observation: The number of crossings of a straight-line drawing of K_n with the vertices embedded in an n element point set $\mathcal{P}$ in general position is exactly the number $\square(\mathcal{P})$ of 4-element subsets of $\mathcal{P}$ which are in convex position, i.e., whose convex hull is a quadrilateral.

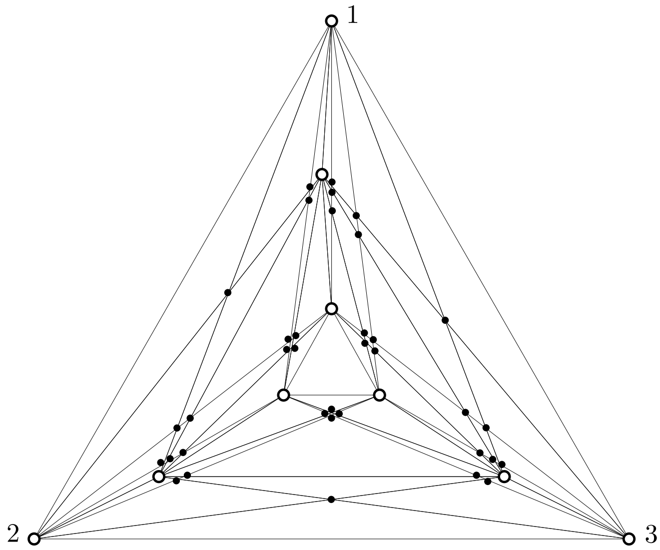


Figure 4.6 A drawing of K_9 with realizing the rectilinear crossing number $\overline{\mathrm{cr}}(K_9) = 36$. Removing the vertices labeled 1, 2, 3 in this order yields optimal drawings realizing $\overline{\mathrm{cr}}(K_8) = 19$, $\overline{\mathrm{cr}}(K_7) = 9$, $\overline{\mathrm{cr}}(K_6) = 3$.

Proposition 4.9 *A set of five points in general position in the plane always determines at least one convex quadrilateral.*

[‡] Points with $t_p < 4n$ would help strengthening the bound.

Proof. If the convex hull of the five points contains four or five of the points we are done. Suppose the convex hull is a triangle. One edge of the triangle is disjoint from the line which is spanned by the two points interior to the triangle. The two endpoints of this edge and the two points in the triangle form a quadrilateral. $\square$

This gives a nice direct proof for the non-planarity of K_5 (with straight edges). The proposition implies $\square(\mathcal{P})(n-4) \geq \binom{n}{5}$ and, hence, $\square(\mathcal{P}) \geq \frac{1}{5}\binom{n}{4}$ for every set $\mathcal{P}$ of n points. The trivial upper bound is $\square(\mathcal{P}) \leq \binom{n}{4}$. The remaining question is to determine good bounds for the constant $C_\square$ in front of the binomial coefficient.

Very recently, the lower bound was strongly improved leading to a decrease in the gap between the bounds for $C_\square$ from ≈ 0.05 to ≈ 0.005.

Theorem 4.10 *For every point set $\mathcal{P}$ of n points in general position*

$$\square(\mathcal{P}) \geq \frac{3}{8}\binom{n}{4}.$$

The proof is based on a surprising connection between the number of convex quadrilaterals of a point set $\mathcal{P}$ and the numbers of k-edges. The connecting formula (Lemma 4.11) translates lower bounds for numbers of k-edges to lower bounds for the number of convex quadrilaterals.

From now on we suppress references to the underlying set of n points in general position on which all counting variables of the play are based. With $\square$ and $\triangle$ we denote the numbers of 4-element subsets which are in convex position and in non-convex position, in particular $\square + \triangle = \binom{n}{4}$. The number of k-edges of the point set is e_k and the number of $\leq k$-edges is $E_k = e_0 + \ldots + e_k$.

Lemma 4.11

$$\square = \sum_{k < \frac{n-2}{2}} e_k \left(\frac{n-2}{2} - k\right)^2 - \frac{3}{4}\binom{n}{3}.$$

Proof. We count the number Z of ordered 4-tuples $(u|v,w|x)$ such that the directed line v,w has u on its left and x on its right side. The contribution of a set $\{u,v,w,x\}$ in convex position to Z is four. The contribution of a set $\{u,v,w,x\}$ in non-convex position to Z is six. That is, $Z = 4\square + 6\triangle$ and after eliminating $\triangle$:

$$Z = 6\binom{n}{4} - 2\square \tag{4.1}$$

Now, consider a k-edge (v,w). This pair is in the central pair of $k(n-2-k)$ of the 4-tuples counted by Z. This gives another count for Z as:

$$Z = \sum_{k=0}^{n-2} k(n-2-k)e_k \tag{4.2}$$

Starting from equation 4.1 and using the obvious identity $\sum_k e_k = n(n-1)$ we get:

$$Z + 2\square + \frac{3}{2}\binom{n}{3} = 6\binom{n}{4} + \frac{3}{2}\binom{n}{3} = \left(\frac{n-2}{2}\right)^2 (n-1)n = \sum_{k=0}^{n-2}\left(\frac{n-2}{2}\right)^2 e_k \tag{4.3}$$

Below, we start from 4.3 and first replace Z with the expression from 4.2, then we use the symmetry of the e_j, namely $e_j = e_{n-j-2}$ together with the fact that in the even case the coefficient of $e_{\frac{n-2}{2}}$ vanishes:

$$\begin{aligned} 2\Box &= \sum_{k=0}^{n-2} \Big(\frac{n-2}{2}\Big)^2 e_k - \sum_{k=0}^{n-2} k(n-k-2)e_k - \frac{3}{2}\binom{n}{3} \\ &= \sum_{k<\frac{n-2}{2}} 2\left[\Big(\frac{n-2}{2}\Big)^2 - k(n-k-2)\right] e_k - \frac{3}{2}\binom{n}{3} \\ &= 2\sum_{k<\frac{n-2}{2}} \Big(\frac{n-2}{2} - k\Big)^2 e_k - \frac{3}{2}\binom{n}{3} \end{aligned}$$

$\Box$

Corollary 4.12

$$\Box = \sum_{k<\frac{n-2}{2}} (n-2k-3)E_k + O(n^3).$$

Proof. In Lemma 4.11 substitute $e_k = E_k - E_{k-1}$ for all k. Simplify the resulting formula with $\big(\frac{n-2}{2} - k\big)^2 - \big(\frac{n-2}{2} - (k+1)\big)^2 = n - 2k - 3$. Errors made at the boundary of the summation and by omitting the term $\frac{3}{2}\binom{n}{3}$ are subsumed in the big-O. $\Box$

In Theorem 4.16 we prove the lower bound $E_k \geq 3\binom{k+2}{2}$ for $k < n/2$. Substitute this into Corollary 4.12 and concentrate on the leading coefficient:

$$\Box \geq \frac{3}{2}\Big(\,\frac{n}{3}\big(\frac{n}{2}\big)^3 - 2\frac{1}{4}\big(\frac{n}{2}\big)^4\Big) + O(n^3) = \frac{1}{64}n^4 + O(n^3).$$

To conclude the bound of Theorem 4.10 it is enough to show that the contribution of the $O(n^3)$ is non-negative, we omit the details.

A Lower Bound for $\leq k$-Edges

The lower bound on E_k is proved in the combinatorial disguise of allowable sequences. These objects will be central in Chapter 6 (pages 91ff). We briefly introduce the setting:

A sequence $\Sigma = \pi_0, \ldots, \pi_z$ of permutations of an n-element set is an *allowable sequence* if π_0 and π_z are reverse of each other, i.e., $\pi_0(t) = \pi_z(n-t+1)$ for all t, and each consecutive pair π_{i-1}, π_i differs by an adjacent transposition.

Let $\mathcal{P}$ be a set of n points. We require general position for $\mathcal{P}$ and assume in addition that no two lines spanned by the points are parallel. With $\mathcal{P}$ we associate an allowable sequence as follows: Let ℓ be a directed line such that the orthogonal projections of the points are all different. Let π_0 be the order of the projections of the points to ℓ. Start rotating ℓ keeping track of the order of the orthogonal projections of the points. This order changes whenever ℓ gets orthogonal to a line spanned by two points p_i, p_j of $\mathcal{P}$. The two orders before and after differ by the transposition exchanging these two adjacent points. The sequence of the different orderings, as permutations of labels, arising while

ℓ rotates 180° is the allowable sequence associated with $\mathcal{P}$ (and ℓ). The following lemma emphasizes an easy observation, essential in our context.

Lemma 4.13 *Let Σ be an allowable sequence associated to a set $\mathcal{P}$ of n points by the above construction. The j-edges of $\mathcal{P}$ correspond bijectively to pairs π_{i-1}, π_i of permutations in Σ which are related by one of the two transpositions $(j{+}1, j{+}2)$ or $(n{-}j{-}1, n{-}j)$.*

To bound E_{k-1} we will prove a lower bound on the number of occurrences of transpositions from the set

$\big\{(1,2),(2,3),\ldots,(k,k+1)\big\} \cup \big\{(n-k,n-k+1),(n-k+1,n-k+2),\ldots,(n-1,n)\big\}$

these are the *critical transpositions.* To simplify the exposition let σ_i be the transposition $(k-i+1, k-i+2)$ and σ^i be the transposition $(n-k+i-1, n-k+i)$ so that the set of critical transpositions is $\{\sigma_i, \sigma^i : i = 1,\ldots,k\}$. We let $m = n-2k$ and assume that the first permutation of Σ is:

$$\pi_0 = (a_k, a_{k-1}, \ldots, a_1, b_1, b_2, \ldots, b_m, c_1, c_2, \ldots, c_k)$$

and the last is:

$$\pi_z = \overline{\pi_0} = (c_k, c_{k-1}, \ldots, c_1, b_m, \ldots, b_2, b_1, a_1, a_2, \ldots, a_k)$$

To get from π_0 to $\pi_z = \overline{\pi_0}$ the element a_j has to be moved to the right by critical transpositions of types $\sigma_j, \sigma_{j-1}, \ldots, \sigma_1$ and $\sigma^1, \sigma^2, \ldots, \sigma^j$. Symmetrically, element c_j is moved left by critical transpositions of types $\sigma^j, \sigma^{j-1}, \ldots, \sigma^1$ and $\sigma_1, \sigma_2, \ldots, \sigma_j$. Counting these critical transpositions used by the a_j's and c_j's we get a total of $2\sum_{i\le k} 2i = 4\binom{k+1}{2}$. The problem is that a critical transposition can simultaneously move an a right and a c left. Further analysis will allow us to bound the number of critical transpositions counted twice.

Let us think of the sequence Σ as a sequence $\tau_1, \tau_2, \ldots, \tau_{z-1}$ of transpositions and of the order of the transpositions as time. We say that an element a_j (respectively, c_j) is *confined* until it is moved by a transposition of type σ_1 (respectively, σ^1), after that it becomes *free.* An element b_j is always free.

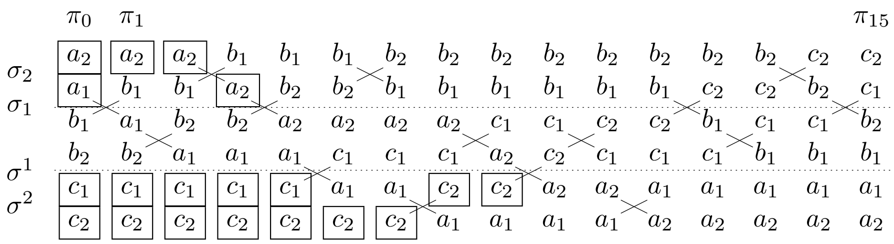


Figure 4.7 An example with $n = 6$ and $k = 2$, confined elements are boxed.

Lemma 4.14 *A sequence Σ' can be replaced by a sequence Σ which uses the same number of critical transpositions and without transpositions which move two confined elements.*

Proof. Suppose Σ' contains a transposition moving two confined elements, a *confined transposition.* Let τ_s be the first confined transposition. The elements moved by τ_s are either two a's or two c's, we assume they are a's. From the choice of τ_s it follows that the two elements exchanged by τ_s are adjacent in the starting permutation π_0. Otherwise, one of the elements exchanged by τ_s would have been involved in another confined transposition which precedes τ_s, contradiction. Let a_j and a_{j-1} be the two elements that are exchanged by τ_s. Define the sequence $\Sigma^<$ with transpositions $\tau_1, \ldots, \tau_{s-1}, \tau_{s+1}, \ldots, \tau_{z-1}, \sigma^j$, the effect of removing τ_s is that now a_{j-1} follows the path of a_j in Σ' and vice versa. Only at the end a_j and a_{j-1} are transposed by σ^j. The allowable sequence $\Sigma^<$ contains the same number of critical transpositions but one confined transposition less. Repeat this step until a sequence Σ without confined transpositions is reached. □

In the *liberation sequence* λ corresponding to Σ we list all a's and c's in the order in which they become free. We assume that Σ contains no confined transpositions (Lemma 4.14) which implies that the a's appear in λ in increasing order, i.e., a_i precedes a_j in λ iff $i < j$, and the same for the c's.

For $i \leq j$, let $[a_j, \sigma_i]$ be the first transposition of type σ_i that moves a_j in Σ. Define $[a_j, \sigma^i]$, $[c_j, \sigma^i]$ and $[c_j, \sigma_i]$ accordingly.

All the transpositions thus assigned to the a's move the defining element to the right, so they are pairwise distinct. The c's are moved left by the transpositions assigned to them. Therefore, only two cases are left for *duplication*:

$[a_j, \sigma_i] = [c_l, \sigma_i]$. This requires $i \leq \min(j, l)$, moreover, the transposition takes place while a_j is confined and c_l is free. Assign this duplication to c_l.

$[a_j, \sigma^i] = [c_l, \sigma^i]$. Again $i \leq \min(j, l)$, such a duplication takes place while c_l is confined and a_j is free. Assign this duplication to a_j.

Claim a. The number of duplications assigned to an a is upper bounded by each of the following quantities:

(i) One plus the number of a's preceding it in the liberation sequence.

(ii) The number of c's behind it in the liberation sequence.

Proof. The a's appear in λ in increasing order, therefore, the value from (i) just gives the index j for the a. The index is a bound since $i \leq j$ in $[a_j, \sigma^i]$. For (ii) recall that when we assign a duplication to a_j, then a_j is free and c_l confined, therefore, c_l is behind a_j in the liberation sequence. △

Claim c. The number of duplications assigned to a c is upper bounded by each of the following quantities:

(i) One plus the number of c's preceding it in the liberation sequence.

(ii) The number of a's behind it in the liberation sequence

The proof is as above. Given a liberation sequence we let $\mu_\lambda(a_j)$ and $\mu_\lambda(c_l)$ be the bounds from the above claims, i.e., if there are s of the c's behind a_j, then $\mu_\lambda(a_j) = \min(j, s)$.

Lemma 4.15 *If λ is a liberation sequence, then*

$$\sum_{j=1}^{k} \big(\mu_\lambda(a_j) + \mu_\lambda(c_j)\big) = \binom{k+1}{2}.$$

Proof. If $\lambda = \langle a_1, a_2, \ldots, a_k, c_1, c_2, \ldots, c_k\rangle$, then $\mu_\lambda(a_j) = j$ and $\mu_\lambda(c_j) = 0$ and the equation holds. Starting from this sequence every other sequence (with the same letters and the a's and c's in order) can be reached with a series of adjacent transpositions. We claim that the sum is invariant under adjacent transpositions. It is, obviously, enough to show this for an exchange of an a and a c which are adjacent. Consider $\lambda = \rho_1 * \langle a_j, c_l\rangle * \rho_2$ and $\lambda' = \rho_1 * \langle c_l, a_j\rangle * \rho_2$. Observe that $\mu_\lambda(x) = \mu_{\lambda'}(x)$ for $x \neq a_j, c_l$. We distinguish two cases:

$j + l \leq k$: From $j \leq k - l$ it is evident that $\mu_\lambda(a_j) = \min(j, k-l+1) = j = \min(j, k-l) = \mu_{\lambda'}(a_j)$. Similarly, $l \leq k-j$ implies $\mu_\lambda(c_l) = \min(l, k-j) = l = \min(l, k-j+1) = \mu_{\lambda'}(c_l)$.

$j + l > k$: From $j \geq k - l + 1$ it is evident that $\mu_\lambda(a_j) = k - l + 1$ and $\mu_{\lambda'}(a_j) = k - l$. Similarly, $l \geq k - j + 1$ implies $\mu_\lambda(c_l) = k - j$ and $\mu_{\lambda'}(c_l) = k - j + 1$.

In both cases $\mu_\lambda(a_j) + \mu_\lambda(c_l) = \mu_{\lambda'}(a_j) + \mu_{\lambda'}(c_l)$ □

From the lemma it follows that there are at most $\binom{k+1}{2}$ many duplicate transpositions in the family $\{[a_j, \sigma_i], [a_j, \sigma^i], [c_j, \sigma^i], [c_j, \sigma_i] : 1 \leq i \leq j \leq k\}$. Therefore, the number of critical transpositions is at least $3\binom{k+1}{2}$. Stating the result more explicitly:

Theorem 4.16 *If Σ is an allowable sequence on n elements and $k < n/2$, then Σ contains at least $3\binom{k+1}{2}$ transpositions from the following set:*
$\{(1,2), (2,3), \ldots, (k, k+1), (n-k, n-k+1), (n-k+1, n-k+2), \ldots, (n-1, n)\}$.

The next figure indicates a construction that can be used to show that the bound given of the theorem is tight for all $k \leq n/3$.

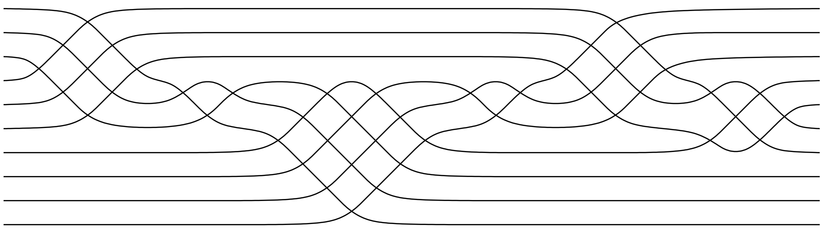


Figure 4.8 An allowable sequence corresponding to this arrangement shows that the theorem is tight for $n = 10$ and $k = 1, 2, 3$.

4.4 Notes and References

According to Lovász [133] it was Simmons who asked for the maximum number of halving lines. Lovász proves the upper bound $2n\sqrt{2n}$ and remarks that Straus constructed a set with $cn\log(n)$ halving lines. A paper by Erdős, Lovász, Simmons and Straus [76] contains upper and lower bounds for k-sets, respectively k-edges, which are of order $O(n\sqrt{k})$ and $\Omega(n\log(k))$. Up to the first proof of Proposition 4.4 this chapter presents the classical approach from [76].

The k-set bounds from [76] were later rediscovered by Edelsbrunner and Welzl [68]. A nice selection of proofs for the $O(n\sqrt{k})$ upper bound was published by Agarwal, Aronov and Sharir [3]. Pach, Steiger and Szemerédi [152] obtained a slightly improved bound of $O(\frac{n\sqrt{k}}{\log^*(k)})$. Significant progress was made by Dey [59] in 1997, he improved the upper bound to $O(nk^{1/3})$. Dey's proof combined the concept of convex chains, introduced by Agarwal et al [3], with an application of the Crossing Lemma. The proof of Theorem 4.5 given here follows ideas of Welzl, see Andrzejak et al. [13].

Tóth [198] gave a construction that improved the lower bound to $\Omega(n \exp(c\sqrt{\log(k)}))$. Tóth's examples can also be found in the book of Matoušek [138].

The dual question is to bound the complexity of the k-th level of an arrangement, i.e., what is the maximum number of vertices v of an arrangement of n lines such that exactly k lines pass below v. In this setting there is an obvious generalization to arrangements of pseudolines (cf. Chapter 6), still the known bounds are the same as for k-sets. The proof of the lower bound for $\leq k$-sets (Theorem 4.16) actually is a proof in the generalized dual setting. Theorem 4.16 was obtained by Lovász, Vesztergombi, Wagner and Welzl [135]. The bound is best possible for the number of $\leq k$-sets in point configurations with $n \geq 3k$. The precise upper bound kn for $\leq k$-sets of n points was obtained by Alon and Györi [11] (cf. Proposition 6.15).

Welzl [211] proves the bound $O(n\sqrt{\sum_{k\in K} k})$ for the number of k-sets with $k \in K$. In [13] it is mentioned that the square-root in this bound can be improved to a third-root. Special classes of sets which allow smaller bounds on k-sets have been studied by Edelsbrunner, Valtr and Welzl [67] and Alt et al. [12]. The expected number of k-sets of a set drawn uniformly from a convex polygon was studied by Bárány and Steiger [20]. Edelman [63] investigates the expected size of the k-level of allowable sequences.

The complexity of the k-th level in arrangements of other objects, like segments or pseudocircles has been studied intensely over the past few years. These issues are covered in a survey of Agarwal and Sharir [4].

Bounds for k-sets have widespread applications. Among the most surprising ones is the connection with parametric matroid optimization first described by Eppstein [73]. As an example consider a graph with edge-weights $w(e) = a_e t + b_e$ which are time dependent and linear and let S_t be the minimum spanning tree at time t. The upper bound for k-sets implies that there are at most $O(mn^{1/3})$ many different trees S_t.

The study of k-sets in higher dimensions was initiated in a paper by Bárány, Füredi and Lovász [19]. The best known bound is of order $O(n^{\lfloor \frac{d}{2} \rfloor} k^{\lceil \frac{d}{2} \rceil - \varepsilon_d})$, where ε_d decreases exponentially with d, see Agarwal, Aronov, Chan and Sharir [1]. Nevertheless, this ε_d is the crucial point, since, for $\leq k$-facets there is an upper bound of order $O(n^{\lfloor \frac{d}{2} \rfloor} k^{\lceil \frac{d}{2} \rceil})$. Andrzejak and Welzl [14] give a beautiful proof for this result, it combines random sampling and the Upper Bound Theorem for polytopes.

The paper of Bárány et al. [19] already contains a higher-dimensional Lovász Lemma. Several improvements were made before Welzl [212] proved the exact version (Theorem 4.7). Welzl describes mappings between pairs $(\mathcal{P}, \ell)$, where $\mathcal{P}$ is a set of n points in $\mathbb{R}^d$ and ℓ a directed line and simplicial polytopes Q with n vertices in $\mathbb{R}^{n-d-1}$, such that $\bar{h}_j(\mathcal{P}, \ell) = h_j(Q)$ where (h_j) is the h-vector of Q (for terminology of polytopes, see [219]). Via this mapping, which is essentially the Gale transform, Theorem 4.7 corresponds to the Upper Bound Theorem for convex polytopes, which is known to be best possible. Using the Generalized Lower Bound Theorem (GLBT) for polytopes in $\mathbb{R}^d$ with at most $d+4$ vertices Welzl can show that in $\mathbb{R}^3$ the number of $\leq k$-facets is minimized by point sets in convex position. More connections in particular with the GLBT are made by Sharir and Welzl in [178].

In $\mathbb{R}^3$ a bound close to the bound of Theorem 4.8, namely $O(n^{8/3} \log^{5/3} n)$ was first obtained by Aronov et al. [15]. Dey and Edelsbrunner [60] gave a more direct argument and got rid of the log-factors. Substantial progress was made by Sharir, Smorodinsky and

Tardos [177], they prove a $O(nk^{3/2})$ bound. This improved bound in based on a more clever definition of graphs G_p. These graphs contain edges and rays to infinity, therefore, the Crossing Lemma is not directly applicable. The ideas are outlined in the book of Matoušek [138].

Crossing numbers of complete graphs. Related to the conjecture of Zarankiewicz for the crossing number of complete bipartite graphs there is a conjecture for the crossing number of complete graphs:

$$\mathrm{cr}(K_n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

The conjecture and a matching upper bound construction was popularized by Guy [110]. Guy also describes a proof of the conjecture for $n \leq 10$. Ringel comments on this: 'the proofs for $7 \leq n \leq 10$ are very uncomfortable'. Moon [143] suggested the following construction for drawings of K_n: Choose a set $\mathcal{P}$ of n points in general position from the unit sphere. For each of the points decide independently with probability 1/2 to keep the point or replace it by its antipodal, let $\hat{\mathcal{P}}$ be this random point set. Connect every pair of points from $\hat{\mathcal{P}}$ by the shorter arc on the great cycle defined by them. Consider a four element subset A of $\mathcal{P}$ and the six great cycles defined by A. These cycles intersect in the points of A, their antipodals and six additional crossings. Each of these additional crossings requires a specific choice of $\hat{A}$ for its appearance in the drawing $D(\hat{\mathcal{P}})$. Hence, the probability that $\hat{A}$ determines a crossing is 6/16. Linearity of expectation implies that the expected number of crossings in the drawing is $\frac{3}{8}\binom{n}{4}$. Accordingly, there must exist a drawing with at most that number of crossings. Note that this construction reaches the order of magnitude of Guy's conjecture.

The rectilinear crossing number $\overline{\mathrm{cr}}(K_n)$ is, in general, larger than $\mathrm{cr}(K_n)$. As shown by Guy [110] this is first true for $n = 8$ with $\mathrm{cr}(K_8) = 18$ and $\overline{\mathrm{cr}}(K_8) = 19$. Exact values for $\overline{\mathrm{cr}}(K_n)$ have been determined for n up to 12. The latest progress, by Aichholzer, Aurenhammer and Krasser [5], is a consequence of the generation of all order types of point sets with $n \leq 10$. The known rectilinear crossing numbers have been used by Aichholzer et al. [5] to give the following asymptotic estimates:

$$0,3115\binom{n}{4} < \overline{\mathrm{cr}}(K_n) < 0,3807\binom{n}{4}$$

In 1865 Sylvester asked for the probability that four points chosen at random in a set $R \subset \mathbb{R}^2$ have a convex hull which is a quadrilateral. Depending on R and the probability distribution used to pick points from R, a number of different solutions are possible, cf. the web-page on Sylvester's Four-Point Problem [210]. Let R be open and of finite area Scheinerman and Wilf [171] consider $q_* = \inf_R \square(R)$, where $\square(R)$ is the probability that four points drawn independently from the uniform distribution on R are in convex position. They show $q_* = \lim_n \overline{\mathrm{cr}}(K_n)/\binom{n}{4}$.

Uli Wagner [209] improved the lower bound for q_* from $0,3115$ to $0,3288$. In this proof Wagner used an object called *staircase of encounter* which encodes information about the number of points on the left side of a rotating line through p. Further improvement from $0,3288$ to $3/8 + \varepsilon = 0.375 + \varepsilon$ was obtained by Lovász, Vesztergombi, Wagner and Welzl [135]. Section 4.3 is based on that paper. Note that we have not included the improvement from $3/8$ to $3/8 + \varepsilon$. This ε, however, is required for the observation that the difference between $\overline{\mathrm{cr}}(K_n)$ and $\mathrm{cr}(K_n)$ is of order $\Omega(n^4)$. The ε improvement is made possible by using a second bound for E_k. The bound given in Theorem 4.16 is only tight for $k \leq n/3$. Lovász et al. give a second bound which is better for $k > 0.495n$.

5 Combinatorial Problems for Sets of Points and Lines

In this chapter we study some fundamental questions of combinatorial geometry. The objects of this study are finite sets of points and finite arrangements, i.e., finite sets of lines or hyperplanes. As an introduction let us look at three classical contributions to this area.

- 1826 Steiner [188] enumerated the regions of Euclidean arrangements of lines and planes.
- 1893 Sylvester [192] asked for a proof of the following: If n points in the plane are not collinear then there is a line containing exactly two of the points.
- 1926 Levi [132] proved: A set of n lines in the projective plane determines at least n triangles.

Steiner's result (Theorem 5.1) is taken here as a warm-up in arrangements. Section 5.1 continues with an elementary discussion of planar geometry: We relate Euclidean and projective planes and explain the important concept of duality between point configurations and arrangements of lines. Sylvester's Problem and the stronger Kelly–Moser Theorem (Theorem 5.9) are the subject of Section 5.2. This is followed by a lower bound for the number of lines spanned by n points. Finally, in Section 5.4 we deal with Levi's Problem for the Euclidean plane. The result is that every simple Euclidean arrangement of n lines contains $n-2$ triangles. This is substantially harder than the question for the projective plane.

5.1 Arrangements, Planes, Duality

An *arrangement of lines* is a collection $\mathcal{A}$ of n lines $\ell_1, \ldots, \ell_n$ in the plane. The arrangement is called *trivial* if there exists a point p incident to all the lines ℓ_i of $\mathcal{A}$. If there are no parallel lines and no point belongs to more than two lines of $\mathcal{A}$ the arrangement is called *simple*.

An arrangement partitions the plane into convex faces of dimensions 0, 1 or 2. The faces of dimension 0 are the intersection points of lines, these are the *vertices* of the arrangement. Maximal vertex free pieces of lines are the *edges* and *cells* are the connected pieces of the plane after removal of the lines. Similar notions are used for arrangements of hyperplanes in d dimensions, in particular, the d-dimensional faces of such arrangements are again called cells. An arrangement of hyperplanes in d dimensions is *simple* if the intersection of any k hyperplanes is of dimension $d-k$, for $k = 1, \ldots, d$ and no point belongs to more than d hyperplanes. The following theorem gives Steiner's counting result.

Theorem 5.1 *The number of cells of a simple arrangement of n hyperplanes in $\mathbb{R}^d$ is*

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{d}.$$

Proof. We prove the formula by induction. The truth for $d = 1$ is easy, there is a line and n points on it. Removing the points leaves $n + 1$ connected pieces of line. Now let $\mathcal{H}$ be a simple arrangement in $\mathbb{R}^d$. Let H be an additional hyperplane, we will think of H as being horizontal and assume the following properties:

(a) H is intersected by every hyperplane in $\mathcal{H}$.

(b) All the vertices of $\mathcal{H}$ are strictly above H.

Place one marble in every cell of $\mathcal{H}$ and activate gravity so that a marble moves to the lowest vertex of its cell or is inhibited from disappearing vertically to $-\infty$ by the plane H (we assume that all marbles originally were placed above H). Property (b) and the assumption about $\mathcal{H}$ imply that the arrangement $\mathcal{H}_H$ induced by $\mathcal{H}$ on $H \equiv \mathbb{R}^{d-1}$ is simple. Hence, by induction, the number of marbles that hit H is $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{d-1}$. It remains to account for the marbles stopped by vertices of $\mathcal{H}$. A vertex of $\mathcal{H}$ is the intersection of d-hyperplanes and every set of d hyperplanes intersects in a vertex, hence, there are $\binom{n}{d}$ vertices in $\mathcal{H}$. Since there are no horizontal hyperplanes in $\mathcal{H}$ every vertex is the lowest point of exactly one cell. The number of marbles and hence the number of cells is $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{d}$. □

Before going ahead it seems appropriate to discuss different geometries hosting configurations of points and arrangements. The conception of the plane we have been working with has two main aspects. We want that a piece of paper represents a generic finite piece of plane, this is the basis for the extensive use of figures to illustrate and support conclusions. The second fundamental aspect is the coordinate system which allows to identify the plane with $\mathbb{R}^2$. Together these two aspects make a good intuitive model of the *Euclidean plane.*

Embed the Euclidean plane, i.e., $\mathbb{R}^2$, as an affine plane A not containing the origin into $\mathbb{R}^3$. This plane is described as $A = \{x \in \mathbb{R}^3 : \langle x - a, a\rangle = 0\}$ for some nonzero vector $a \in \mathbb{R}^3$ and the inner product $\langle a, b\rangle = \sum_i a_i b_i$. Every point in plane A spans a line through the origin of $\mathbb{R}^3$, intersecting this line with the unit sphere S^2 maps a point of A to a pair of antipodal points on S^2. The image of a line in the plane under this mapping is a great circle on S^2. The equator E_A of S^2 relative to A is the great circle of intersection between S^2 and the plane $A_0 = \{x \in \mathbb{R}^3 : \langle x, a\rangle = 0\}$, this is the plane parallel to A which contains the origin. Every pair of antipodal points corresponding to a point of A is separated in S^2 by E_A. Therefore, one of the open hemispheres of $S^2 \setminus E_A$ with half great circles as lines is a finite model of the Euclidean plane. Conceptually, the same thing can be obtained by identifying antipodal points of $S^2 \setminus E_A$.

The *projective plane* is obtained from S^2 by identifying all pairs of antipodal points, that is, by treating all pairs of antipodal points alike. Referring to our original plane A we can think of the projective plane as A enhanced by the pairs of antipodal points on E_A which form the line at infinity.

These considerations enable us to move a configuration of points and lines between different planes. The general ambient space for such configurations is S^2 or equivalently modulo identification of antipodal points the projective plane. By choosing an equator and a plane parallel to this equator the configuration can be mapped into Euclidean space. Figure 5.1 shows a configuration with two Euclidean representations. The important fact about these transformations is that they preserve incidences between points and lines as well as collinearity of points. Concurrency of lines is a bit more delicate as a set of concurrent lines can become parallel. In this case part of the configuration (the intersection point of the parallel lines) has to disappear with the line at infinity.

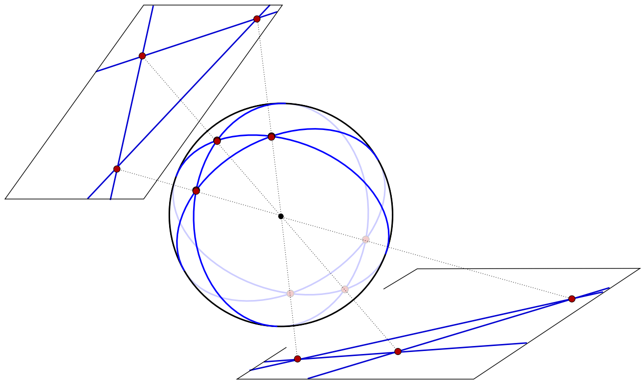


Figure 5.1 A spherical configuration and two Euclidean representations.

A very pleasing aspect of the geometry of antipodal points and great circles on S^2 is the natural notion of *duality*. With an antipodal pair $\{v, -v\}$ on S^2 associate the great circle $C_v = \{x \in S^2 : \langle x, v\rangle = 0\}$ and conversely with a great circle C associate $\{v \in S^2 : \langle x, v\rangle = 0 \text{ for all } x \in C\}$, this set is the pair $\{v, -v\}$ with $C = C_v$. Hence, duality $v \leftrightarrow C_v$ is a bijective mapping between points and great circles. Applying this duality mapping to each element of a configuration of points and lines gives a dual configuration. The importance of duality transformations originates from the fact that it preserves incidences and maps collinear points to concurrent lines and vice versa. By choosing affine planes A and B as primal and dual planes duality on the sphere can be used to map a configuration in plane A to a dual configuration on plane B. With Figure 5.2 we try to illustrate this construction of a dual configuration.

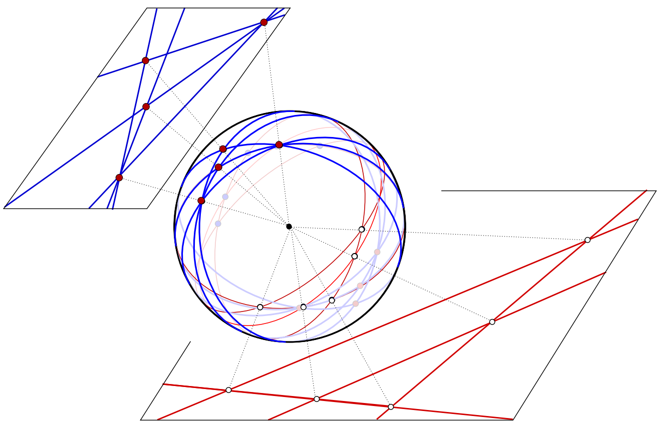


Figure 5.2 A configuration and its dual projected to different Euclidean planes.

We have kept these considerations intuitive and informal. It seems appropriate to conclude with a concrete example. Let A be the plane $z = 1$ in $\mathbb{R}^3$, this choice of A corresponds to the representation of points of $\mathbb{R}^2$ by homogeneous coordinates. Corresponding to the point $p = (p_x, p_y)$ in $\mathbb{R}^2$ we have $(p_x, p_y, 1) \in A$. Dual to this point in $\mathbb{R}^3$ is the plane $\{(x, y, z) : p_x x + p_y y + z = 0\}$. Intersecting with A we obtain the line $p_x x + p_y y = -1$ as the dual to p. Conversely, the dual of a line ℓ is obtained by bringing ℓ into the form $ax + by = -1$ and taking $p_\ell = (a, b, 1)$. Of course, the origin of A and all lines through it are exempt from this game, they map to the line at infinity. Things get even nicer if we take the plane $z = -1$ for B and identify points of A and B if they have the same x and y coordinates. Now the dual of $p = (p_x, p_y)$ is the line $\ell_p = \{(x, y) : p_x x + p_y y = 1\}$. With this duality $p \in \ell_p$ iff p is on the unit circle. This kind of dual mapping is called *polarity* with respect to the unit circle. In Figure 5.3 we indicate a plane geometric construction for this polarity. Similar constructions can be made with respect to other conics, e.g. the polarity at the parabola $y = x^2$ is frequently used.

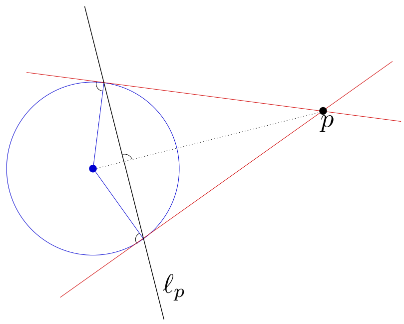


Figure 5.3 A plane construction for the polarity $p \leftrightarrow \ell_p$.

5.2 Sylvester's Problem

For a configuration $\mathcal{P}$ of points an *ordinary line* is a line containing exactly two of the points. Sylvester asked for a proof that every configuration of n points, not all on a line, admits an ordinary line. In the 1930's the problem was revived by Erdős and others. The first solution by Gallai was followed by several other proofs based on different ideas. We include the amazingly short proof due to Kelly. A remarkable feature of this proof is that it yields the conclusion for point sets in arbitrary dimensions.

Proposition 5.2 (Sylvester–Gallai)
Every configuration $\mathcal{P}$ of n points, not all on a line, admits an ordinary line.

Proof. Consider the set $\mathcal{L}$ of all lines containing at least two points of $\mathcal{P}$. Among all point-line pairs $(p, L) \in \mathcal{P} \times \mathcal{L}$ with p not on L let (p^*, L^*) be one which minimizes the distance between the point and the line. The claim is that L^* is ordinary. Let q be the closest point to p^* on L^* and suppose L^* contains at least three points, then there are two points p_1 and p_2 on the same side of q on L^*, Figure 5.4 shows the situation.

Let L be the line spanned by p^* and p_2. Clearly the distance of p_1 and L is smaller than the distance p^* and L^*, this contradicts the choice of the pair (p^*, L^*). $\square$

By duality the role of points and lines in incidence statements may be interchanged. The dual version of Proposition 5.2 is: *Every arrangement $\mathcal{A}$ of n lines, not intersecting in a single point, and not all parallel, admits an ordinary point, i.e., a point contained in exactly two of the lines.*

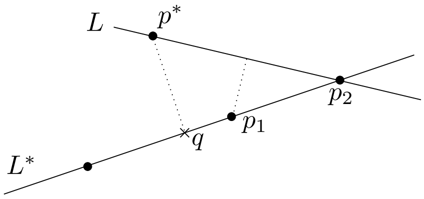


Figure 5.4 Illustration for Kelly's distance argument: $\mathsf{dist}(p^*, L^*) \geq \mathsf{dist}(p_1, L)$.

Below we give a nice proof for this dual version of Sylvester's problem whose main ingredient is Euler's formula for arrangements of lines living in the projective plane.

Proposition 5.3 (Euler)
If $\mathcal{A}$ is a projective arrangement with f_0 vertices, f_1 edges and f_2 faces, then

$$f_0 - f_1 + f_2 = 1.$$

Proof. The proof is by induction, if there are two lines, we have $f_0 = 1$, $f_1 = 2$ and $f_2 = 2$. Upon addition of a new line L to an arrangement of at least two lines some vertices and edges are created on L. Every edge on L splits an old face into two new faces and every vertex on L splits an old edge into two new edges. Therefore, the value of the alternating sum is not effected by the insertion of L. □

With a projective arrangement $\mathcal{A}$ and $j \geq 2$, $k \geq 3$ associate the following statistics:

$$\begin{aligned} t_j &= \# \text{ vertices where } j \text{ lines cross,} \\ p_k &= \# \text{ faces surrounded by } k \text{ edges } (k\text{-faces}), \end{aligned}$$

Following Melchior we cleverly define

$$Y = \sum_{j \geq 2} (3 - j) t_j + \sum_{k \geq 3} (3 - k) p_k.$$

Clearly $\sum_{j \geq 2} t_j = f_0$ and $\sum_{k \geq 3} p_k = f_2$ (this last equation only holds in the absence of 2-faces, i.e., if the arrangement has at least two vertices). Every edge is incident with two faces, therefore $\sum_{k \geq 3} k \cdot p_k = 2f_1$. A vertex where j lines cross is incident to $2j$ edges and every edge is incident to two vertices, therefore $\sum_{j \geq 2} j \cdot t_j = f_1$. Substituting the sums in the definition of Y by these formulas and applying Proposition 5.3 yields:

$$Y = (3f_0 - f_1) + (3f_2 - 2f_1) = 3(f_0 - f_1 + f_2) = 3.$$

Observe that in the definition of Y only the coefficient of t_2 is positive, it is 1. This gives a strengthening of Proposition 5.2 in its dual form. We follow the tradition and state Proposition 5.4 in the Euclidean version, the assumption that there are no parallel lines ensures that we loose no ordinary point on the line at infinity.

Proposition 5.4 *If $\mathcal{A}$ is an Euclidean arrangement of lines with at least two vertices and without parallel lines, then*

$$t_2(\mathcal{A}) \geq 3.$$

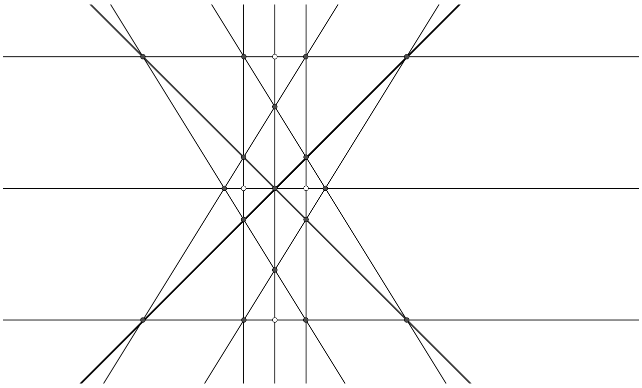


Figure 5.5 A projective arrangement of 13 lines (the 13th line is the line at infinity). The arrangement has 6 ordinary points: The 4 ordinary points shown in white and 2 more ordinary points where the bold lines intersect the line at infinity.

It is natural to ask for the minimum number of ordinary points in arrangements of n lines. A family of examples with few ordinary points is obtained from regular n-gons with n odd. Take the n supporting lines of the n-gon and add the n lines of symmetry. This arrangement of $2n$ lines only has n ordinary points, the midpoints of the edges of the n-gon. Two exceptional arrangements are known with an even smaller number of ordinary points, one with $n = 7$ and $t_2 = 3$ and a second one with $n = 13$ and $t_2 = 6$, see Figure 5.5.

Conjecture 5.5 *For $n \geq 14$ every non-trivial arrangement of n lines has at least $n/2$ ordinary points.*

The strongest result known today is $t_2 \geq 6n/13$. We proceed to show the weaker bound $t_2 \geq 3n/7$ which was first obtained by Kelly and Moser.

Let $\mathcal{A}$ be an arrangement of $n > 3$ lines in the projective plane. Further assume that every line of $\mathcal{A}$ contains at least three vertices. This assumption is legitimized by the observation that otherwise, either all lines intersect in a single point, or there are at least $n - 2$ ordinary points in $\mathcal{A}$. The general plan is to associate ordinary points to the lines and finally use a double counting argument. An ordinary point p is *attached* to a line L not containing p, if L together with the two lines crossing in p form a triangular face T of the arrangement. Sometimes it is useful to be more precise and say p is attached to L through T.

Lemma 5.6 *Let T be a triangle formed by three lines of $\mathcal{A}$. Let L be one of the defining lines of T and $[p, q]$ be the interval of intersection of L and T. If T is not a cell of $\mathcal{A}$ and every line intersecting T also intersects $[p, q]$ and there are no ordinary points on the interval $[p, q]$, then there exists an ordinary point x attached to L through some triangle T_x contained in T.*

Proof. We use a distance argument á la Kelly (cf. Proposition 5.2). Let x be the vertex of $\mathcal{A}$ in T but not on L, which has the smallest distance to L. Suppose that there are three lines L_1, L_2, L_3 intersecting in x, let v_1, v_2, v_3 be their intersection points with L. By the assumptions all the v_i are in the interval $[p, q]$ we assume that v_2 is between v_1 and v_3.

Since v_2 is not ordinary there is a line $M \neq L_2$ entering T at v_2. Clearly, the intersection of M and L_1 or of M and L_2 is of smaller distance to L than x, a contradiction. □

Figure 5.6 shows a situation conforming with the assumptions of the lemma and two ordinary points attached to line L .

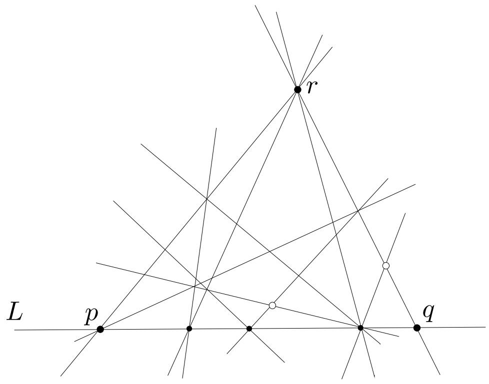


Figure 5.6 Two ordinary points attached to L in T.

Lemma 5.7 *If a line L of $\mathcal{A}$ contains no ordinary point, then there are at least three ordinary points attached to L.*

Proof. Since L contains no ordinary points a distance argument reveals that the vertex x of $\mathcal{A}$ with the smallest positive distance to L is an ordinary point attached to L. Let M and N be the lines crossing in x and let p and q be their intersection points with L. The three lines L, M and N partition the plane into four triangular cells. Let T_1 and T_2

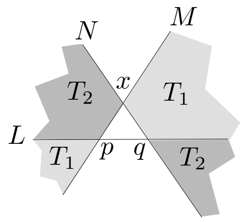


Figure 5.7 Illustration for Lemma 5.7.

be the two cells indicated in Figure 5.7. Both triangles have a side which is an edge of $\mathcal{A}$, px for T_2 and qx for T_1. Let v be a third vertex on L. All the lines crossing L at v intersect the interior of T_1 and T_2. Therefore, Lemma 5.6 implies that we find ordinary points attached to L in T_1 and T_2. Together with x this makes for at least three ordinary points attached to L. □

Lemma 5.8 *If a line L of $\mathcal{A}$ contains exactly one ordinary point, then there are at least two ordinary points attached to L.*

Proof. Let p be the ordinary point on L and let r be a neighbor of p on the other line M through p. Let N be a second line through r and q be the point of intersection of N and L. Line L is partitioned by p and q into the intervals $[p, q]$ and $[q, p]$. If both intervals contain further vertices of $\mathcal{A}$, then we consider the two triangular regions T_1 and T_2 as indicated in Figure 5.8. Both triangles have a side pr which is an edge of $\mathcal{A}$. With Lemma 5.6 we find ordinary points attached to L in T_1 and T_2. As second case we

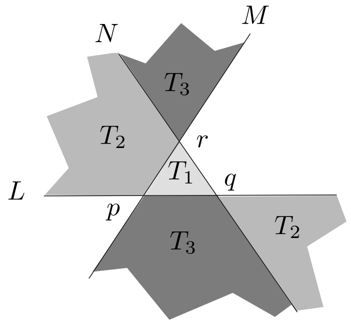


Figure 5.8 Illustration for Lemma 5.8.

have to consider the situation where all the vertices of L fall into one interval $[p, q]$. This time the triangular regions T_1 and T_3 as indicated in Figure 5.8 are of use. To verify the assumptions of Lemma 5.6 for T_3 note that a line intersecting T_3 but not the interval $[p, q]$ would have to intersect L in $[q, p]$, contradicting the assumptions for this case. Hence, again with Lemma 5.6 we find two ordinary points attached to L in T_1 and T_3. □

Theorem 5.9 (Kelly–Moser)
If $\mathcal{A}$ is a projective arrangement of $n \geq 4$ lines with at least two vertices, then

$$t_2(\mathcal{A}) \geq 3n/7.$$

Proof. Consider a matrix A with n rows corresponding to the lines of $\mathcal{A}$ and $t = t_2(\mathcal{A})$ columns corresponding to the ordinary points of $\mathcal{A}$. The entry $A(L, p)$ is $3/2$ if $p \in L$, it is 1 if p is attached to L and 0 otherwise. With Lemmas 5.7 and 5.8 it follows that $\sum_p A(L, p) \geq 3$ for every line L.

An ordinary point p is contained in two lines. Because p is incident to only four faces it can be attached to at most four lines. Therefore $\sum_L A(L, p) \leq 7$ for every ordinary point.

The combination of the two bounds on the entries of A completes the proof:

$$3n \leq \sum_L \sum_p A(L, p) = \sum_p \sum_L A(L, p) \leq 7t.$$

□

5.3 How many Lines are Spanned by n Points?

In this section we treat another question raised by Erdős in the early 1930's. Given a configuration $\mathcal{P}$ of n points, not all on a line, how many lines are determined by the points? The *near-pencil* is the configuration with $n-1$ points on one line and one additional point. The near-pencil shows that we cannot hope for more than n lines. An easy inductive proof for the fact that there are at least n lines can be based on the existence of ordinary lines.

Theorem 5.10 *Every configuration $\mathcal{P}$ of $n \geq 3$ points in the plane, not all on a line, determines at least n lines each passing through at least 2 of the points.*

Proof. For $n = 3$ the result is obvious, assume that it is true for $n-1$ points. Let $\mathcal{P}$ consist of n points and let ℓ be an ordinary line spanned by two points of $\mathcal{P}$. Let p be one of the points on ℓ. Removing p from $\mathcal{P}$ there are two possible cases. Either all points of $\mathcal{P} \setminus p$ are collinear, in this case $\mathcal{P}$ is a near-pencil with n lines. Otherwise, the set $\mathcal{P} \setminus p$ determines at least $n-1$ lines, by induction, together with line ℓ which is not spanned by $\mathcal{P} \setminus p$ this makes for a total of at least n lines. $\square$

The conclusion of Theorem 5.10 is valid in many more general situations where it is known under different names, e.g. de Bruijn-Erdős theorem or Fisher's inequality. Consequently there are proofs using only elementary facts, incidence and counting. Here we concentrate on a sharpening of the result for point configurations. Again it was Erdős who conjectured that, if we exclude the near-pencil and some small exceptional configurations, a set of n points will span $2n-4$ lines.

Theorem 5.11 *If $\mathcal{P}$ is a set of $n \geq 27$ points not all on a line and not forming a near-pencil, then $\mathcal{P}$ determines at least $2n-4$ lines.*

Lemma 5.12 *If $\mathcal{P}$ is a configuration of points such that there is a line ℓ containing exactly $n-s$ points of $\mathcal{P}$, then the number of lines determined by $\mathcal{P}$ is at least*

$$s\,(n-s) - \binom{s}{2} + 1.$$

Proof. Let A be the set of points on ℓ and B be the complementary set. For every pair (a, b) with $a \in A$ and $b \in B$ there is a connecting line. Not all of these $s\,(n-s)$ lines have to be different. However, for every pair (b, b') there is at most one a such that the three points a, b, b' are collinear. Therefore, there are at least $s\,(n-s) - \binom{s}{2}$ lines containing points from A and B. Adding one, to account for line ℓ, completes the proof. $\square$

Fixing n the expression from the lemma is a polynomial $f(s)$ in s of degree two. The equation $f(s) = 2n-4$ has the solutions 2 and $(2n-5)/3$, for all values of s between these solutions $f(s) > 2n-4$. Since we have excluded the near-pencil, i.e., the configuration with $s = 1$, we obtain: If $\mathcal{P}$ is a configuration of n points conforming with the theorem and there is a line with at least $n - (2n-5)/3 = (n+5)/3$ points, then $\mathcal{P}$ determines at least $2n-4$ lines.

Now assume that $\mathcal{P}$ contains two points p and q each of degree at most 5. Let ℓ be the line connecting p and q and consider the set B of points not on ℓ. Every point $b \in B$ is the intersection point of a line through p and a line through q. The degree conditions for p and q imply $|B| \leq 16$. Therefore line ℓ contains at least $n-16$ points. But $n-16 < (n+5)/3$ only if $n \leq 26$, hence, non-trivial configurations of $n \geq 27$ points with two points of degree at most 5 determine at least $2n-4$ lines.

Now let $\mathcal{P}$ contain at most one point incident to at most 5 lines. As a side product of the proof of Proposition 5.4 we had the following equation for arrangements

$$3 = \sum_{j\geq 2}(3-j)t_j + \sum_{k\geq 3}(3-k)p_k.$$

Disregarding the non-positive contribution of the second sum we obtain Melchior's inequality

$$3 \leq \sum_{j\geq 2}(3-j)t_j.$$

In the context of arrangements t_j was counting the number of vertices where j lines cross. Dually the formula also holds for a point configuration with

$$t_j \quad = \quad \#\text{ lines containing } j \text{ points of } \mathcal{P},\ j \geq 2.$$

Note that $t = \sum_j t_j$ is the number of lines determined by $\mathcal{P}$. For the points we have the counting coefficients

$$r_j \quad = \quad \#\text{ points of } \mathcal{P} \text{ of degree } j,\ j \geq 2.$$

Double counting yields the equation $\sum_j jt_j = \sum_j jr_j$. Rewriting Melchior's inequality and using the assumption that there is at most one point of degree ≤ 5 we obtain:

$$3t - 3 \geq \sum_{j\geq 2} j\cdot t_j = \sum_{j\geq 2} j\cdot r_j \geq \sum_{j\geq 6} j\cdot r_j \geq 6(n-1).$$

Hence, the number t of lines spanned is at least $2n-1$ in this case. This completes the proof of the theorem. □

The following infinite family of examples shows that Theorem 5.11 is best possible: Consider two lines, place one point at their intersection two additional points on the first line and $n-3$ points on the second line. This configuration of n points spans $2n-4$ lines.

5.4 Triangles in Arrangements

Ordinary points are vertices of an arrangement with minimal degree, i.e., where a minimal number of lines cross. In non-trivial arrangements the faces of minimal degree, i.e., faces with a minimal number of surrounding lines, are the triangles. In this section we discuss bounds on the number p_3 of triangles. An easy inductive argument shows that in the projective plane every arrangement which is not a star contains a triangle: Add a line L to an arrangement $\mathcal{A}$. If $\mathcal{A}$ is a star then it either remains a star or some triangles are created. Now suppose that $\mathcal{A}$ contains a triangle T. If T is not crossed by L it remains a triangle. If L is cutting through T, then T is either subdivided into a triangle and a quadrangle or into two triangles.

Using Euler's formula for projective arrangements a small improvement is possible. As for the proof of Proposition 5.4 we begin with a clever definition.

$$Z = \sum_{j\geq 2}(4-2j)t_j + \sum_{k\geq 3}(4-k)p_k.$$

Recall the four elementary equations $\sum_{j\geq 2} t_j = f_0$ and $\sum_{k\geq 3} p_k = f_2$ and $\sum_{j\geq 2} j\cdot t_j = f_1$ and $\sum_{k\geq 3} k\cdot p_k = 2f_1$. These formulas together with Euler's formula yield:

$$Z = (4f_0 - 2f_1) + (4f_2 - 2f_1) = 4(f_0 - f_1 + f_2) = 4.$$

In the definition of Z only the coefficient of p_3 is positive, it is 1. Hence, $p_3 \geq 4$ or more accurately:

$$p_3 \geq 4 + \sum_{k \geq 5} (k-4) p_k.$$

Levi [132] has obtained a sharp lower bound on p_3 in the projective case.

Proposition 5.13 *For every non-trivial arrangement $\mathcal{A}$ of n lines in the projective plane we have $p_3(\mathcal{A}) \geq n$.*

Proof. Let L be a line of $\mathcal{A}$ and consider L as the line at infinity. Denote by V_L the set of vertices of $\mathcal{A}$ which are not on L. If the convex hull of V_L is degenerate, i.e., a single point or all vertices of V_L are collinear, then there are $2n-2$ triangles in the arrangement. If the convex hull is not degenerate it consists of at least three vertices. Together with line L a vertex v of this convex hull with $t(v) = j$ determines $j-1$ triangular faces of $\mathcal{A}$. Therefore, there are at least three triangles incident to L.

Since the line L was chosen arbitrarily, every line is incident with at least three triangles. Every triangle is incident with exactly three lines. Therefore, $p_3 \geq n$. □

The result is best possible. To see this take the n supporting lines of the edges of a regular n-gon for $n \geq 4$. This arrangement of lines has $p_3 = n$.

Levi noted that his theorem does not make use of the straightness of the lines. He coined the term *pseudoline* to denote a curve whose intersection behavior with respect to some other pseudolines is as one would expect it from lines and proved his theorem in this more general context. We will frequently work with arrangements of pseudolines, here is a precise definition for these objects.

An *arrangement $\mathcal{B}$ of pseudolines* in the projective plane $\mathcal{P}$ is a family of simple closed curves, called *pseudolines*, in $\mathcal{P}$. Each pair of the pseudolines has exactly one point in common and they cross at this common point. With this notation we can state Levi's Theorem:

Theorem 5.14 *For every non-trivial arrangement $\mathcal{A}$ of n pseudolines in the projective plane we have $p_3(\mathcal{A}) \geq n$.*

Let $\mathcal{B} = \{P_0, P_1, \ldots, P_n\}$ be an arrangement of pseudolines in $\mathcal{P}$. Specifying a pseudoline P_0 in $\mathcal{B}$ as the line at infinity induces the arrangement $\mathcal{B}_{P_0}$ of pseudolines $\{P_1, \ldots, P_n\}$ in $\mathcal{P} \setminus P_0$. Since $\mathcal{P} \setminus P_0$ is homeomorphic to the Euclidean plane we may regard $\mathcal{B}_{P_0}$ as an arrangement in this plane.

It has already been noted by Levi that arrangements of pseudolines are a proper generalization of arrangements of lines. This is due to the existence of incidence laws in plane geometry. E.g., the reason for the non-stretchability of the arrangement shown in Figure 5.9 is the Theorem of Pappus. Arrangements of pseudolines have received attention since they provide a generic model for oriented matroids of rank 3. In this context questions of stretchability are of considerable interest. For more about these connections the reader may consult the 'bible of oriented matroids' [28].

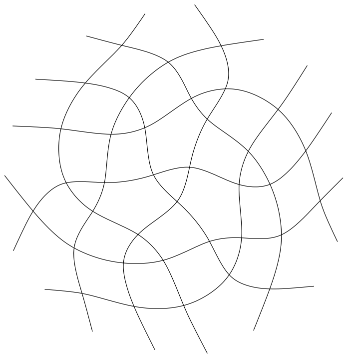


Figure 5.9 A simple arrangement of 9 pseudolines. This is the smallest non-stretchable arrangement.

Closely related is the problem of counting triangles in Euclidean arrangements. Again, the problem is old, in 1889 Roberts asserted that an arrangement of n lines in the plane contains at least $n-2$ triangles if it is simple, i.e., if there is no point of intersection of three or more lines. However, the proof was flawed, the problem remained unsolved for a long time. Ninety years later Shannon [176] proved Roberts' Theorem using dual configurations. Actually, he proved the analog of Roberts' Theorem for arbitrary dimensions: Every non-trivial arrangement of n hyperplanes in $\mathbb{R}^d$ contains at least $n-d$ simplicial d-cells, i.e., cells with the structure of a d-simplex.

Here we give a proof for the pseudoline case. As shown by Figure 5.10 in order to obtain Roberts' Bound it is actually necessary to require that the arrangement is simple in this case. The example belongs to an infinite family of non-simple arrangements with $3n$ lines and $2n$ triangles. In fact, it can be shown that every arrangement of pseudolines has at least $\lceil 2n/3 \rceil$ triangles. The bound is best possible as shown by a family of examples whose member with $n = 12$ is shown in Figure 5.10.

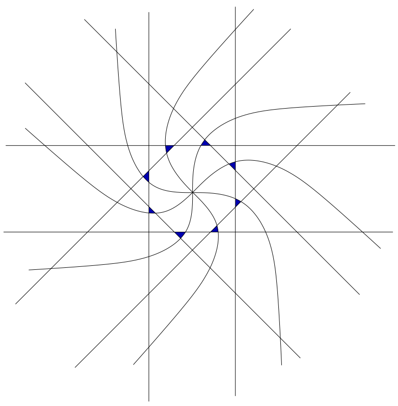


Figure 5.10 A non-simple arrangement of 12 pseudolines with only 8 triangles.

Theorem 5.15 *If $\mathcal{A}$ is a simple arrangement of n lines or pseudolines in the Euclidean plane then $p_3(\mathcal{A}) \geq n-2$.*

Proof. We consider the finite part of $\mathcal{A}$ as a planar graph. Let f_0 be the number of vertices, f_1^b be the number of bounded edges and f_2^b be the number of bounded faces. Since we only consider simple arrangements these statistics can be expressed as functions of the number n of pseudolines.

$$f_0 = \binom{n}{2}, \qquad f_1^b = n\,(n-2), \qquad f_2^b = \binom{n-1}{2}.$$

In this setting Euler's formula is $f_0 - f_1^b + f_2^b = 1$.

We assign labels $\oplus$ or $\ominus$ to the two sides of every edge. Let F be one of the two (possibly unbounded) faces bounded by e and let e' and e'' be the edge-neighbors of e along F. Let l, l' and l'' be the supporting pseudolines of e, e' and e'' respectively. The label of e on the side of F is $\oplus$ if F is contained in the finite triangle T of the arrangement $\{l, l', l''\}$ otherwise the label is $\ominus$. See Figure 5.11 for an illustration of the definition.

Lemma 5.16 *Every edge e of a simple arrangement has a $\oplus$ and a $\ominus$ label.*

Proof. Let F_1 and F_2 be the two faces bounded by e and let e_1', e_1'' and e_2', e_2'' be the edge-neighbors of e in these two faces. Since the arrangement is simple the supporting lines $\{l_1', l_1''\}$ of both pairs of edges are the same. The finite triangular region T of the arrangement $\{l, l', l''\}$ has edge e on its boundary. Therefore, exactly one of the two faces F_1 and F_2 is contained in T. $\triangle$

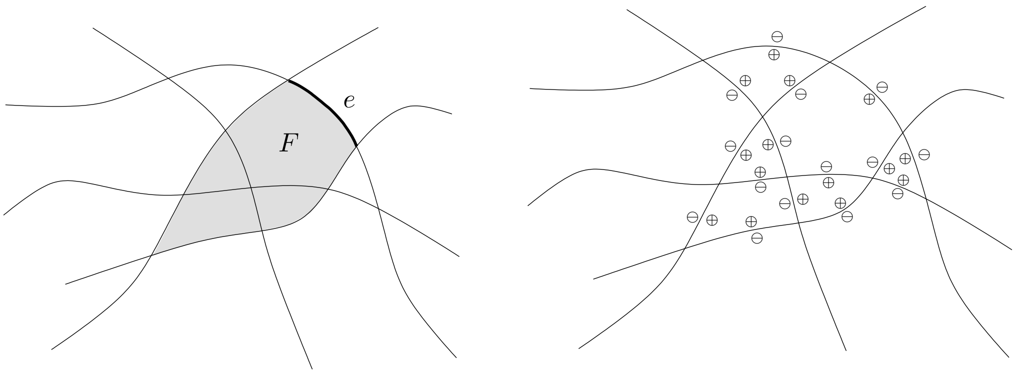


Figure 5.11 The label of e in F is $\oplus$ since F is contained in the shaded triangle. This rule leads to the completed labeling of edges shown on the right.

As seen in the proof of the lemma, the triangular region T used to define the edge label of e on the side of F is independent of F. This allows to adopt the notation $T(e)$ for this region.

Lemma 5.17 *All three edge labels in a triangle are $\oplus$. A quadrangle contains two $\oplus$ and two $\ominus$ labels. For $k \geq 5$ a k-sided face contains at most two $\oplus$ labels.*

Proof. If F is a triangle, then for each of its edges e the triangular region $T(e)$ is F itself.

Let F be a quadrangle and e, $\overline{e}$ be a pair of opposite edges of F. Both edges have the same neighboring edges, hence, two of the lines bounding the triangles $T(e)$ and $T(\overline{e})$ are equal. Either $T(e) = F \cup T(\overline{e})$ or $T(\overline{e}) = F \cup T(e)$. In the first case e has label $\oplus$ and $\overline{e}$ has label $\ominus$ in F, in the second case the labels are exchanged. The second pair of opposite edges also has one label $\oplus$ and the other $\ominus$.

It remains to consider the case where F is a face with $k \geq 5$ sides.

Claim. *Any two edges with label $\oplus$ in F are neighbors, i.e., share a common vertex.*

Let $e_1, e_2, \ldots, e_k$ be the edges of F numbered in counterclockwise direction along F and let l_i be the supporting line of e_i. Let e_1 have label $\oplus$ and consider an edge e_i with $4 \leq i \leq k-2$. We show that the label of e_i is $\ominus$:

Face F is contained in $T(e_1)$ and line l_i has to leave $T(e_1) \setminus F$ through l_k and l_2. Figure 5.12 is a generic sketch of the situation.

Consider line l_{i-1}. This line enters the region R_1 bounded by l_2, l_i and the chain of edges $e_3, e_4, \ldots, e_{i-1}$ at the vertex $e_{i-1} \cap e_i$. To leave region R_1 line l_{i-1} has to cross l_2. Therefore, l_{i-1} has to leave the region R_2 bounded by l_i, l_2 and l_k through l_k. Symmetrically, l_{i+1} has a crossing with l_k to leave the region bounded by l_k, l_i and the chain of edges $e_{i+1}, e_{i+2}, \ldots, e_k$. Therefore, to leave region R_2 line l_{i+1} has to cross l_2. This shows that l_{i-1} and l_{i+1} cross inside region R_2. Hence, $T(e_i)$ is contained in R_2 and e_i has label $\ominus$ in F.

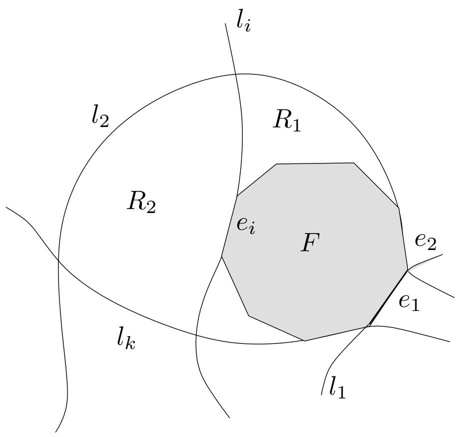


Figure 5.12 Edge e_1 has label $\oplus$ in F so e_i must have $\ominus$.

It remains to show that if e_1 is labeled $\oplus$ then neither e_3 nor e_{k-1} are. Considering the crossing of lines l_4 and l_2 observe that $T(e_3)$ is contained in $T(e_1) \setminus F$. Hence, the label of e_3 in F is $\ominus$. A symmetric argument applies to e_{k-1}. This completes the proof of the claim. $\triangle$

We use the two lemmas to count the number of $\oplus$ labels in different ways:

$$f_1^b = \sum_F \#\{\oplus \text{ labels in } F\} \leq 2f_2^b + p_3.$$

With $f_1^b = n(n-2)$ and $2f_2^b = (n-1)(n-2)$ this implies

$$p_3 \geq n-2.$$

$\square$

Back to straight lines we now give a proof for the 2-dimensional case of Shannon's Theorem. For simple arrangements the result is contained in the more general Theorem 5.15. However, the proof extending to non-simple straight line arrangements is really lovely and it contains all the ingredients required for Shannon's Theorem in all dimensions.

Theorem 5.18 *Every non-trivial Euclidean arrangement $\mathcal{A}$ of n lines contains at least $n-2$ triangles.*

Proof. Let line L_i of the arrangement be given by the equation $a_i x + b_i y = c_i$, for $i = 1, \ldots, n$. The intersection of lines L_i and L_j is at the point $e_{ij} = \left(\frac{c_i b_j - b_i c_j}{a_i b_j - b_i a_j}, \frac{a_i c_j - c_i a_j}{a_i b_j - b_i a_j}\right)$. The area of a triangle T formed by lines L_i, L_j, L_k can be expressed by a determinant:

$$\operatorname{Area}(T) = \frac{g(a,b)}{3} \begin{vmatrix} c_i b_j - b_i c_j & c_j b_k - b_j c_k & c_i b_k - b_i c_k \\ a_i c_j - c_i a_j & a_j c_k - c_j a_k & a_i c_k - c_i a_k \\ a_i b_j - b_i a_j & a_j b_k - b_j a_k & a_i b_k - b_i a_k \end{vmatrix}.$$

Here $g(a,b)$ is the product of the expressions in the last row of the matrix. Via some calculations this can be transformed to the much nicer formula

$$\operatorname{Area}(T) = \frac{g(a,b)}{3} \begin{vmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{vmatrix}^2.$$

Assign some real s_i to every line L_i and imagine that starting from its initial position in $\mathcal{A}$ line L_i is shifting parallel with speed s_i. For every real t this gives an arrangement $\mathcal{A}(t)$ whose lines are given by the equations $a_i x + b_i y = c_i + s_i t$. By linearity of the determinant the area of the triangle $T(t)$ formed in $\mathcal{A}(t)$ by lines $L_i(t), L_j(t), L_k(t)$ is given by

$$\operatorname{Area}(T(t)) = \frac{g(a,b)}{3} \left(\begin{vmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{vmatrix} + t \begin{vmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ s_i & s_j & s_k \end{vmatrix} \right)^2.$$

Thus $\operatorname{Area}(T(t))$ is a quadratic function of t and it will be constant if and only if the coefficient of t in the above expression vanishes. The condition

$$\begin{vmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ s_i & s_j & s_k \end{vmatrix} = 0$$

is a homogeneous linear equation in the variables s_i, s_j, s_k with coefficients determined by the equations of L_i, L_j, L_k.

Now assume that $\mathcal{A} = \mathcal{A}(0)$ has fewer than $n-2$ triangles. Requiring that the area of these triangular faces remains constant in all $\mathcal{A}(t)$ gives at most $n-3$ equations in the n variables $s_1, \ldots, s_n$. Adding the two equations $s_1 = 0$ and $s_2 = 0$ the homogeneous system still has a non-trivial solution $s_1, \ldots, s_n$. In the following we will lead this to a contradiction.

Consider the parameterized family $\mathcal{A}(t)$ of arrangements. By construction, the area of every triangle T of $\mathcal{A} = \mathcal{A}(0)$ remains constant, i.e., $\operatorname{Area}(T(t)) = \operatorname{Area}(T(0))$ for all t.

The lines L_1 and L_2 remain fixed ($s_1 = 0 = s_2$) but since we have chosen a non-trivial solution some lines are moving. In particular for every moving line L_i there is a t such that in $\mathcal{A}(t)$ the crossing of L_1 and L_2 is on $L_i(t)$. Let $t^* > 0$ be the least t with the property that some set of at least three lines which are not concurrent in $\mathcal{A}(0)$ intersect at a vertex of $\mathcal{A}(t)$. By the choice of t^* the combinatorial type of all arrangements $\mathcal{A}(t)$ with t in the open interval $(0, t^*)$ is the same. Most importantly, the set of triangles of these arrangements is the same.

It is tempting to claim that the area of all triangles in $\mathcal{A}(t)$, $t \in (0, t^*)$ is constant. However, there may be sets of $m \geq 3$ concurrent lines in $\mathcal{A}(0)$ which move apart. For a triangle formed by lines which were concurrent in $\mathcal{A}(0)$ the area must be increasing.

Now consider the set $\mathcal{L}$ of newly concurrent lines in $\mathcal{A}(t^*)$ and let $\mathcal{A}_{\mathcal{L}}(t)$ be the sub-arrangement formed by these lines at time t. By induction the sub-arrangement $\mathcal{A}_{\mathcal{L}}(t)$, $t \in (0, t^*)$, contains some triangles. Since the combinatorial type of all arrangements $\mathcal{A}(t)$ with t in the open interval $(0, t^*)$ is the same, these triangles are triangular faces of $\mathcal{A}(t)$. For t close to t^* the area of these triangles has to decrease.

The contradiction is that there is at least one triangle whose area is simultaneously decreasing and non-decreasing when t is approaching t^*. The contradiction was reached by assuming that $\mathcal{A}$ has fewer than $n-2$ triangles. $\square$

5.5 Notes and References

Detailed summaries of the state of knowledge about arrangements in the early 1970s have been given by Grünbaum. In [107] he surveys arrangements in arbitrary dimensions and in the valuable monograph [108] he presents a vast number of results and problems for the two-dimensional case.

Recently, survey articles focusing on different aspects of point configurations and arrangements have appeared in several handbooks. Erdős and Purdy [77] have a rich collection of problems and more than 250 references. A more compact collection is due to Pach [147]. Goodman [101] gives a good overview of the state of the art in arrangements of pseudolines. Agarwal and Sharir [4] emphasize on the complexity of substructures of arrangements, the survey is enriched by a huge bibliography.

Steiner's result on the number of cells in simple arrangements of hyperplanes has found generalizations in many directions. In the past 20 years the interest in the complexity of arrangements and substructures of arrangements was stimulated by the observation that these quantities have immediate consequences in the analysis of algorithms working with geometric structures. These connections between computational and combinatorial geometry are emphasized in the book of Edelsbrunner [65] and in [4]. To hint on the importance of results like Theorem 5.1 we point to another connection. For general set systems there is the notion of *VC-dimension*. In many cases bounds for approximation results or complexity of algorithms can be given in terms of the VC-dimension. Let $\Phi_d(n)$ denote the maximum cardinality of a set system on n elements with VC-dimension at most d, then $\Phi_d(n)$ is exactly the quantity of Theorem 5.1. Gärtner and Welzl [97] show that set systems, realizing the $\Phi_d(n)$ bound, are geometric. More about VC-dimension can be found in the books of Matoušek [137] or Agarwal and Pach [148].

Chazelle, Guibas and Lee [48] exemplify the power of geometric duality by some nice applications in computational geometry. For further reading concerning transformations of planar configurations and duality transforms we refer to Coxeter [52] and Preparata and Shamos [159].

Sylvester was probably motivated to pose his problem by the observation that the statement is false in the complex projective plane. In this plane there are configurations of 9 points arising in connection with cubic curves which have the property that every line containing two of them contains a third point. In finite geometry this structure is now known as the affine plane of order 3. The affirmative answer to Sylvester's problem shows that it is impossible to draw this finite geometry in the plane using only straight lines. Grünbaum [108] has many references for contributions related to Sylvester's problem. A beautiful theorem about colored configurations is due to Chakerian [46]: A non-trivial two-colored configuration always has a monochromatic line. Pach and Pinchasi [149] deal with the colored analog of Sylvester's problem. They show that in a non-trivial two-colored configuration there need not be a monochromatic ordinary line.

Motzkin [144] gave the first non-constant bound for the number of ordinary lines, he proves a lower bound of order $\sqrt{n}$. Kelly and Moser [120] proved the $3n/7$ lower bound. The example shown in Figure 5.5 found by McKee was published in 1968. In 1981 S. Hansen claimed to have a proof for the conjectured $n/2$ result, except for the known special configurations. The work of Hansen, however, was in general incomprehensible and contained errors. Csima and Sawyer [53], see also [54], found a proof for the weaker result: $t_2 \geq 6n/13$.

As already mentioned the statement that n points define at least n lines is valid in many more general situations: Let $\mathcal{L}$ be a set of subsets of a set P. We call the elements of $\mathcal{L}$ lines and the elements of P points. If any two lines from $\mathcal{L}$ have at most one point in common and every pair of points is covered by exactly one line of $\mathcal{L}$, then either $\mathcal{L}$ consists of a single line covering all points or $|\mathcal{L}| \geq n$. A sharpening of this result together with many references is given in [94]. Two classical proofs of the result can be found in The Book [8].

The proof presented for Theorem 5.11 is essentially due to Kelly and Moser. Elliott [70] has an improvement, he showed that the conclusion holds for $n \geq 10$. An example of Kelly and Moser, see Figure 5.13, shows that Elliott's result is best possible.

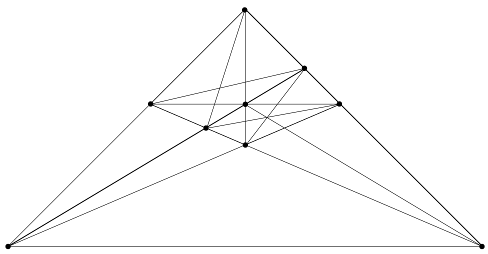


Figure 5.13 A configuration of 9 points spanning only 13 lines.

Along the lines of the given proof it is possible to show that if at most $n-k$ points of $\mathcal{P}$ are collinear then the number of lines is at least $kn-p(k)$, whenever $n \geq q(k)$, for suitable quadratic functions $p(k)$ and $q(k)$. Comprehensive surveys on Sylvester's problem and its relatives are Borwein and Moser [33] and a chapter by Brass and Pach in their announced monograph [35].

To summarize today's knowledge about the number p_3 of triangles in arrangements we first consider the projective case. If $\mathcal{A}$ is a projective arrangement, then:

(1) Every pseudoline is incident with at least three triangles. Therefore, $p_3(\mathcal{A}) \geq n$. Equality is possible for all $n \geq 4$.

(2) $p_3(\mathcal{A}) \leq \frac{1}{3}n(n-1)$ for $n \geq 9$. Equality holds for some arrangements of n lines for infinitely many values of n.

The lower bound is due to Levi [132], see Theorem 5.14. Grünbaum [108] has the following argument for $p_3 \leq \frac{1}{3}n(n-1)$ in simple arrangements: Since $\mathcal{A}$ is simple only one of the cells bounded by an edge can be a triangle. There are $n(n-1)$ edges and every triangle uses three of them. Grünbaum conjectured that the bound remains true for non-simple arrangements of lines with sufficiently large n. A series of papers proving weaker bounds or special cases was published before Roudneff [167] proved Grünbaum's conjecture for $n \geq 9$. Arrangements of pseudolines achieving equality were constructed by Harborth [112]. Recently, Forge and Ramírez-Alfonsín [91] managed to construct families of straight line examples achieving the bound.

If $\mathcal{A}$ is an arrangement of n pseudolines in the Euclidean plane, then:

(1) If $\mathcal{A}$ is simple or stretchable then $p_3(\mathcal{A}) \geq n-2$. Equality is possible for all $n \geq 3$.

(2) If $n \geq 6$ then $p_3(\mathcal{A}) \geq \frac{2}{3}n$. Equality is possible for all $n \equiv 0 \pmod 3$.

(3) $p_3(\mathcal{A}) \leq \frac{1}{3}n(n-2)$. Equality is possible for infinitely many values of n.

The stretchable case of (1) is the 2-dimensional case of Shannon's theorem [176]. With Theorem 5.18 we gave a proof of this fact. The idea for the proof is due to A.Ya Belov, we have learned it through the writeup of Grünbaum [109]. Felsner and Kriegel [85] proved the pseudoline case of (1), Theorem 5.15, as well as (2). The upper bound (3) can be obtained along the lines of Roudneff's proof for the projective case. The examples achieving the bound can also be borrowed from the projective case.

Among the many problems in the area one of the most challenging goes back to Dirac [62]. Let $r^*(n)$ be the smallest integer such that in every nontrivial arrangement of n lines there is a line with at least $r^*(n)$ vertices. Dirac observed that $r^*(n) > \sqrt{n}$ and conjectured $r^*(n) \geq \lceil n/2 \rceil$. Grünbaum [108] (page 25) gives counterexamples for values of n up to 37. There is a family of arrangements showing that $r^*(n) \leq \lceil n/2 \rceil - 2$ for all n of the form $n = 12k+7$, the member of this family with $k = 2$ is shown in Figure 5.14. A reasonable adaption of Dirac's conjecture could be as follows:

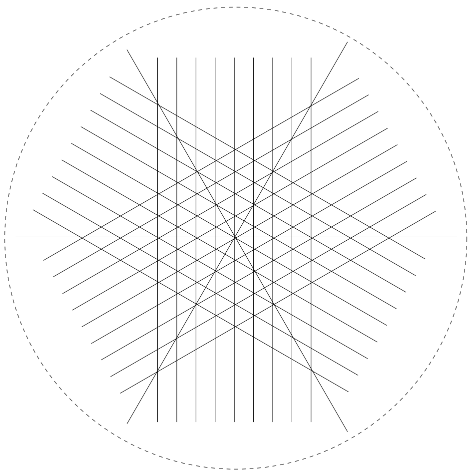


Figure 5.14 An arrangement of 31 lines (including the line at infinity) and at most 14 vertices on each line.

$$\liminf_{n \to \infty} \frac{r^*(n)}{n} = \frac{1}{2}.$$

Erdős had a much weaker question, he asked for the existence of $c > 0$ such that $r^*(n) > c\,n$. This was answered affirmatively by Szemerédi and Trotter [195] as an application of their incidence theorem (Theorem 3.6), see also Beck [22].

6 Combinatorial Representations of Arrangements of Pseudolines

It can be very useful to have combinatorial representations of geometric objects. The combinatorial structure of such an encoding may be easier to analyze and manipulate than the original object.

In this chapter we focus on combinatorial representations for arrangements of pseudolines. Corresponding results for arrangements of lines and configurations of points follow by specialization and dualization. Working with arrangements of pseudolines instead of arrangements of lines is advisable because it is possible to decide efficiently whether a given combinatorial structure represents an arrangement of pseudolines, but the corresponding question for arrangements of lines is an NP-complete problem.

The workhorse in our approach to combinatorial encodings of arrangements is the Sweeping Lemma (Lemma 6.1). Section 6.2 introduces allowable sequences and wiring diagrams. The Slope Theorem (Theorem 6.4) is a surprising application of this encoding. In Section 6.3 we discuss variants of local sequences and give an application to the enumeration of isomorphism classes of marked arrangements. The study of zonotopal tilings in Section 6.4 yields a standardized way of drawing arrangements of pseudolines. Section 6.5 deals with a representation for arrangements by triangle sign functions. These sign functions induce an order relation on the set of all Euclidean arrangements of n pseudolines. The notion of a signotope is introduced as a generalization of triangle sign functions. In Section 6.6 we study an order $S_r(n)$ on all r-signotopes on n elements.

6.1 Marked Arrangements and Sweeps

An Euclidean arrangement of pseudolines can be defined using the projective definition. To do so take a projective arrangement of $n+1$ pseudolines and declare one of the pseudolines as the line at infinity. Assume that the chosen line intersects the other lines in n different points. Removing this 'line at infinity' from the projective plane leaves an object homeomorphic to the Euclidean plane carrying n pseudolines, an Euclidean arrangement of pseudolines.

Probably, more convenient, is a direct definition: A *pseudoline* in the Euclidean plane is a simple curve which approaches a point at infinity in either direction. An *arrangement of pseudolines* is a family of pseudolines with the property that each pair of pseudolines has a unique point of intersection, where the two pseudolines cross. Throughout this chapter arrangements of pseudolines lacking further specification will be assumed to be Euclidean.

An arrangement $\mathcal{A}$ is *simple* if no three pseudolines of $\mathcal{A}$ have a common point of intersection. An arrangement partitions the plane into cells of dimensions 0, 1 or 2, the *vertices*, *edges* and *faces* of the arrangement. Two arrangements are *isomorphic* (*combinatorially equivalent*) if there is an isomorphism (incidence and dimension preserving bijection) between the induced cell decompositions. Edges and faces of the arrangement may either be bounded or unbounded. Let F be an unbounded face of an arrangement

$\mathcal{A}$ and let $\overline{F}$ be the *complementary face of* F, i.e., the face separated from F by all pseudolines. We may orient all pseudolines such that F is in the left half-space and $\overline{F}$ in the right half-space of every line. This orientation of pseudolines induces an orientation of the edges of the arrangement. The pair $(\mathcal{A}, F)$ is a *marked arrangement* or an arrangement with *north-face* F and *south-face* $\overline{F}$. If there is no explicit reference to the north-face of a marked arrangement $\mathcal{A}$ embedded in a coordinatized plane we assume that there is a unique unbounded cell containing a ray to $(0, \infty)$ and this cell is the north-face. Two marked arrangements are *isomorphic* if there is an isomorphism of the induced cell decompositions which maps north-face to north-face and respects the induced orientation of edges. See Figure 6.1 for an illustration.

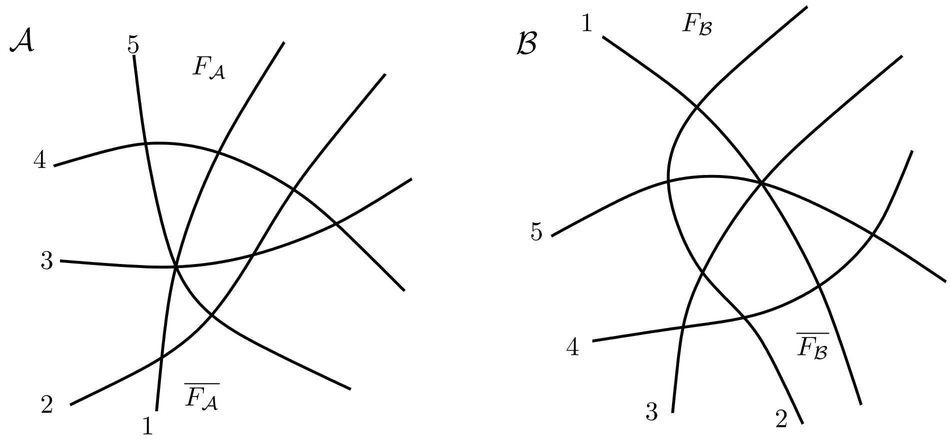


Figure 6.1 Arrangements $\mathcal{A}$ and $\mathcal{B}$ are isomorphic as arrangements but non-isomorphic as marked arrangements.

Sweeping is an important paradigm in algorithmic geometry. The vision is that a line is swept over the plane. In the course of the sweep movement the line will visit all the items of interest, e.g. points, line segments, crossings. In our context the sweeping line is replaced by a pseudoline. This is a variant known as topological sweep. With this idea in mind we begin with a formalization.

Let $(\mathcal{A}, F)$ be a marked arrangement. A *sweep* of $\mathcal{A}$ with north-pole in F is a sequence $c_0, c_1, \ldots c_r$ of curves such that each curve c_i has the same endpoints $\overline{x} \in \overline{F}$ and $x \in F$. Further requirements are:

(1) None of the curves c_i contains a vertex of thearrangement $\mathcal{A}$.

(2) Each curve c_i has exactly one point of intersection with each line L_j of $\mathcal{A}$.

(3) Any two curves c_i and c_j are interiorly disjoint.

(4) For any two consecutive curves c_i, c_{i+1} of the sequence there is exactly one vertex of arrangement $\mathcal{A}$ between them, i.e., in the interior of the closed curve $c_i \cup c_{i+1}$.

(5) Every vertex of the arrangement is between a unique pair of consecutive curves, hence, the interior of the closed curve $c_0 \cup c_r$ contains all vertices of $\mathcal{A}$.

See Figure 6.2 for an example of a sweep for the arrangement $\mathcal{A}$ of Figure 6.1.

Note that if $c_0, \ldots, c_r$ is a sweep for $\mathcal{A}$ then the reversed sequence is also a sweep for $\mathcal{A}$. One of these sweeps is from left to right and the other from right to left. As usual we

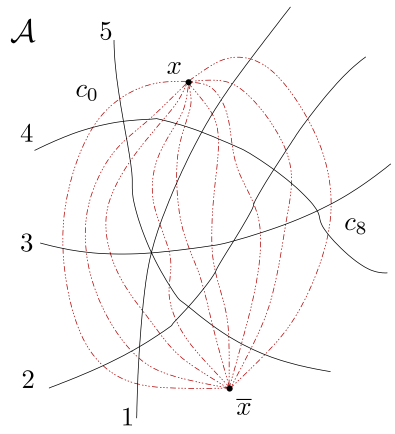


Figure 6.2 A sweep for arrangement $\mathcal{A}$.

will always think of a sweep as a left to right sweep. A discrete sweep as defined here can be transformed into a continuous sweep by appropriate interpolation between each pair c_i, c_{i+1} of curves.

Lemma 6.1 (Sweeping Lemma)
There is a sweep sequence of curves for every marked Euclidean arrangement $(\mathcal{A}, F)$ *of pseudolines, i.e.,* $\mathcal{A}$ *can be swept.*

Proof. Let $G = (V, E)$ be the graph such that the vertices V of G are the vertices of $\mathcal{A}$ and the edges of G are the bounded edges of the arrangement $\mathcal{A}$. Let $\overrightarrow{E}$ be the orientation of the edges of G induced by the orientation of pseudolines (the north-face is in the left halfplane of each pseudoline).

Claim A. The orientation $\overrightarrow{E}$ is an acyclic orientation of G.

Walking 'at infinity' and clockwise from $\overline{F}$ to F all pseudolines of $\mathcal{A}$ are met. Let π be the list of lines in the order they are met.

The claim is proved by contradiction: Assume that $\overrightarrow{E}$ is not acyclic and choose a cycle C such that the area enclosed by the corresponding curve is minimal. By this choice C corresponds to the boundary of a face of $\mathcal{A}$. With respect to this face the cycle C may be oriented clockwise or counterclockwise. We consider the first case (clockwise), the other is symmetric.

Let $e_1, e_2, \ldots, e_k$ be edges of C in clockwise order and let L_j be the supporting pseudoline of e_j. Since e_j and e_{j+1} are consecutive on C lines L_j and L_{j+1} cross at a vertex of C. From the definition of π and the clockwise orientation of C it follows that $L_j <_\pi L_{j+1}$ (see Figure 6.3). Hence $L_1 <_\pi L_2 <_\pi \ldots <_\pi L_k <_\pi L_1$, a contradiction. $\triangle$

Since $\overrightarrow{G} = (V, \overrightarrow{E})$ is acyclic there exists a *topological sorting* of $\overrightarrow{G}$, i.e., an ordering $v_1, v_2, \ldots, v_r$ of the vertices of the graph such that all edges are directed from left to right. Formally, $i < j$ for every directed edge $v_i \to v_j$ of $\overrightarrow{G}$.

Claim B. There are curves $c_0, c_1, \ldots, c_r$ such that vertices $v_1, \ldots, v_i$ are to the left of c_i and vertices $v_{i+1}, \ldots, v_r$ are to the right of c_i for all $i = 1, \ldots, r$.

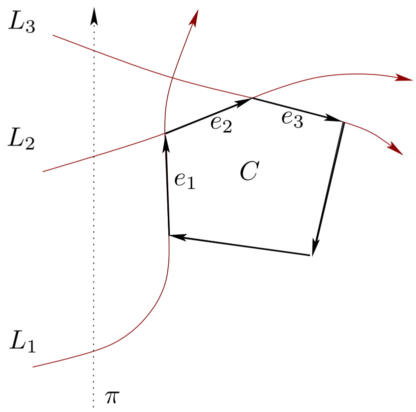


Figure 6.3 Assuming an oriented cycle.

Select arbitrary points $x \in F$ and $\overline{x} \in \overline{F}$. Let R be the union of the closed bounded cells of $\mathcal{A}$. Disregarding some trivial cases the region R is homeomorphic to a disk. Define c_0 as the union of three curves. The first and the second connect x to R within F and $\overline{x}$ to R within $\overline{F}$, the third is the left boundary of an ϵ-tube of the left boundary of R and connected to the two other curves. For an appropriate ϵ this gives a curve as required.

Now suppose that for some $i \leq r$ curve c_{i-1} is already defined. Consider vertex v_i (the numbering of vertices is a topological sort) and let $L_1^i, \ldots, L_t^i$ be the lines of $\mathcal{A}$ containing v_i such that $L_1^i <_\pi \ldots <_\pi L_t^i$. Let T be the triangle defined by curve c_{i-1} and the two lines L_1^i and L_t^i. Since v_i is a source (minimal) in the restriction of $\overrightarrow{G}$ to $v_i, \ldots, v_r$ and $v_1, \ldots, v_{i-1}$ are left of c_{i-1} vertex v_i is the unique vertex of $\mathcal{A}$ in the triangular region T. Define c_i as the right boundary of an ϵ-tube around c_{i-1} and T. For an appropriate ϵ this gives a curve as required, see Figure 6.4. $\triangle$

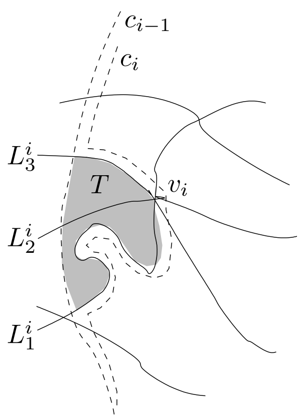


Figure 6.4 Defining c_i based on c_{i-1} and the shaded triangular region T.

The curves $c_0, c_1, \ldots, c_r$ constructed according to Claim B have the five properties of a sweep of $\mathcal{A}$ with north-pole F. This concludes the proof of the lemma. $\square$

Levi's Extension Lemma

As an application of the Sweeping Lemma we derive Levi's extension lemma. This is a fundamental lemma and the proof is a nice game swinging between the projective and the Euclidean plane.

Lemma 6.2 (Levi's extension lemma)
Let $\mathcal{A}$ be an arrangement of n pseudolines and let p, q be two points in the plane which are not both contained in a single line of $\mathcal{A}$. Then there is a pseudoline c containing p and q such that $\mathcal{A} \cup c$ is an arrangement of $n + 1$ pseudolines.

Proof. We detail the proof for the case where p and q are not contained in lines of $\mathcal{A}$.

Let p be contained in the face F_p of $\mathcal{A}$. Let $L_1, \ldots, L_n$ be the pseudolines of $\mathcal{A}$ and assume that L_1 contains an edge e of the boundary of F_p. Add the line at infinity L_∞ to the arrangement and map it back to Euclidean space so that L_1 becomes the line at infinity thus obtaining an arrangement $\mathcal{A}'$ with lines $L_\infty, L_2, \ldots, L_n$. Mark $\mathcal{A}'$ such that $p \in F_p$ is the north-pole. Apply the Sweeping Lemma to find a curve c crossing the face F_q containing q. Line c can be bent in F_q to make q a point on c. Extending c from the poles to infinity we obtain an arrangement $\mathcal{A}' \cup c$ of $n + 1$ lines. Together with L_1 we have a projective arrangement of $n+2$ lines which can be mapped back to the Euclidean plane using L_∞ as line at infinity. This results in an Euclidean arrangement with lines $L_1, \ldots, L_n, c$ such that the points p and q are both on pseudoline c. □

6.2 Allowable Sequences and Wiring Diagrams

Let $c_0, c_1, \ldots, c_r$ be a sweep for the marked arrangement $(\mathcal{A}, F)$ of n pseudolines. We assume that the lines are labeled $1, \ldots, n$ and identify the lines with their labels. Traversing curve c_i from $\overline{x}$ to x we meet the lines of $\mathcal{A}$ in some order. Since each line is met by c_i exactly once the order of the crossings corresponds to a permutation π_i of $[n]$. Relabeling the lines of $\mathcal{A}$ appropriately we may assume that π_0 is the identity permutation. This labeling of the lines of $\mathcal{A}$ is the *standard labeling*.

Consider the labels of lines crossing at vertex v_i. Since the region T defined in the proof of Claim B is empty of vertices of $\mathcal{A}$ and by property 2 of the sweep curve c_i the lines $L_1^i, \ldots, L_t^i$ containing vertex v_i are a consecutive substring of π_{i-1}. Moreover, in permutation π_{i-1} these lines are in the reversed order and this is the only difference between π_{i-1} and π_i.

Example A. The sequence of permutations obtained from the sweep of Figure 6.2 is
$(1,2,3,4,5) \xrightarrow{4,5} (1,2,3,5,4) \xrightarrow{1,2} (2,1,3,5,4) \xrightarrow{1,3,5} (2,5,3,1,4) \xrightarrow{2,5} (5,2,3,1,4) \xrightarrow{1,4}$
$(5,2,3,4,1) \xrightarrow{2,3} (5,3,2,4,1) \xrightarrow{2,4} (5,3,4,2,1) \xrightarrow{3,4} (5,4,3,2,1)$.

Assuming the standard labeling for $\mathcal{A}$ the sequence $\pi_0, \ldots, \pi_r$ has the following properties:

(1) π_0 is the identity permutation and π_r is the reverse permutation on $[n]$.

(2) Each permutation π_i, $1 \leq i \leq r$ is obtained by the reversal of a consecutive substring M_i from the preceding permutation π_{i-1}.

(3) Any two elements $x, y \in [n]$ are joint members of exactly one *move* M_i, i.e., reverse their order exactly once.

A sequence $\Sigma = \pi_0, \ldots, \pi_r$ of permutations with properties (1), (2) and (3) is called an *allowable sequence of permutations*. If each move from π_{i-1} to π_i consists in the reversal of just one pair of elements, i.e., an adjacent transposition, we have $r = \binom{n}{2}$ and the sequence Σ is a *simple allowable sequence*. The existence of sweeps implies that there is an allowable sequence of permutations for every marked arrangement $(\mathcal{A}, F)$. However, more can be said:

Every topological sorting of the graph $\overrightarrow{G}$ of $(\mathcal{A}, F)$ induces an allowable sequence. Consider the allowable sequences Σ and Σ' corresponding to topological sortings σ and σ' of $\overrightarrow{G}$ with the property that $\sigma = v_1, \ldots, v_i, v_{i+1}, \ldots, v_r$ and $\sigma' = v_1, \ldots, v_{i+1}, v_i, \ldots, v_r$, i.e., σ and σ' differ in an adjacent transposition. It follows that v_i and v_{i+1} are both minimal elements in the restriction of $\overrightarrow{G}$ to $\{v_i, v_{i+1}, v_{i+2}, \ldots, v_r\}$. Hence, there is no line in $\mathcal{A}$ that contains vertices v_i and v_{i+1} and the labels of lines involved in the moves $M_i : \pi_{i-1} \to \pi_i$ and $M_{i+1} : \pi_i \to \pi_{i+1}$ in Σ are disjoint. In fact for $j \neq i, i+1$ the permutations π_j and π'_j in Σ and Σ' coincide and $M'_i = M_{i+1}$ and $M'_{i+1} = M_i$. Call two allowable sequences Σ and Σ' *elementary equivalent* if Σ can be transformed into Σ' by interchanging two disjoint adjacent moves. Two allowable sequences Σ and Σ' are called *equivalent* if there exists a sequence $\Sigma = \Sigma_1, \Sigma_2, \ldots, \Sigma_m = \Sigma'$ such that Σ_i and Σ_{i+1} are elementary equivalent for $1 \leq i < m$. It is well known that it is possible to transform any topological sorting of a directed acyclic graph $\overrightarrow{G}$ into any other by a sequence of adjacent transpositions, i.e., reversals of adjacent pairs of unrelated vertices. Corresponding to the topological sortings of $\overrightarrow{G}$ there is an equivalence class of allowable sequences of the arrangement $(\mathcal{A}, F)$. Actually, every allowable sequence obtained from a sweep of $(\mathcal{A}, F)$ belongs to this class.

Theorem 6.3 *There is a bijection between equivalence classes of allowable sequences and marked arrangements of pseudolines. Moreover, this bijection maps simple allowable sequences to simple arrangements.*

Proof. We have already seen how to use $\overrightarrow{G}$ to define an equivalence class of allowable sequences corresponding to a marked arrangement $(\mathcal{A}, F)$.

Let Σ be an allowable sequence. The following construction yields an arrangement $\mathcal{A}$ such that Σ corresponds to a sweep of $\mathcal{A}$: Start drawing n horizontal lines called *wires* and let p_j be the vertical line at $x = j$. On the ith wire from below label the crossing with p_j with $\pi_j(i)$, for $j = 0, \ldots, r$. Draw pseudoline i such that it interpolates the crossings labeled i. For an example see Figure 6.5.

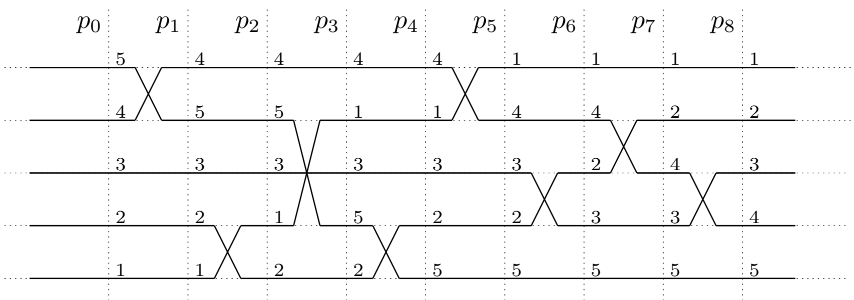


Figure 6.5 A wiring diagram for the arrangement of Figure 6.2

The arrangement thus obtained is the *wiring diagram* for Σ. Since the vertical lines $p_0, \ldots, p_r$ essentially are a sweep sequence of curves for the wiring diagram we see that the mapping from arrangements to allowable sequences is surjective. It remains to show injectivity. Let $(\mathcal{A}, F)$ be any marked arrangement such that Σ corresponds to a sweep of $c_0, \ldots, c_r$ of $\mathcal{A}$. It is obvious that the part of $\mathcal{A}$ between c_{i-1} and c_i is isomorphic to the part of the wiring diagram between p_{i-1} and p_i. These isomorphisms for $i = 1, \ldots, r$ can be glued together to an isomorphism of the arrangements. This proves the first part of the theorem.

For the second part recall that a move $M : \pi_i \to \pi_{i+1}$ reverts as many elements as there are lines crossing at the corresponding vertex in $\mathcal{A}$. $\square$

Application of Allowable Sequences: Ungar's Slope Theorem

From Section 5.3 we know that n points, not all on a line, determine at least n lines. In the Euclidean plane we can ask for the number of parallel classes (slopes) of the connecting lines. The problem was raised by Scott. He conjectured that $2k$ points, not all on a line, determine at least $2k$ slopes. If true, this trivially implies that $2k+1$ points not all on a line also determine at least $2k$ slopes. Configurations of n points (n odd) which determine only $n-1$ slopes exist, they are called *slope critical*. The simplest example, the *bipencil*, consists of $k-1$ points on the positive x-axis, their reflections to the negative x-axis and the 3 additional points $(0,-1), (0,0), (0,1)$. Two other examples taken from infinite families of slope critical configurations are shown in Figure 6.6. The example on the left consists of the vertices of a regular $2k$-gon together with its center. The example on the right is from the family of exponential crosses.

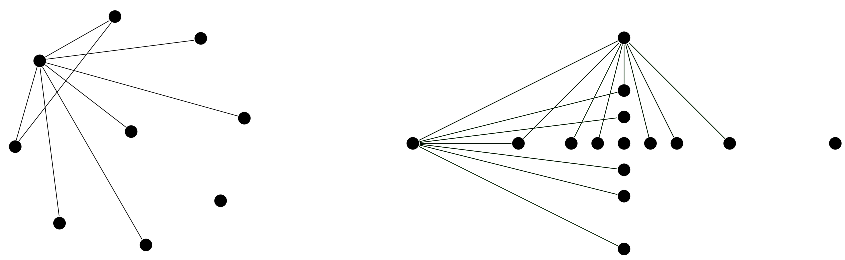


Figure 6.6 Two slope critical configurations.

Theorem 6.4 (Ungar)
A configuration of $2n$ points, not all on a line, determines at least $2n$ slopes.

Proof. A configuration X of points may be dualized yielding an arrangement $\mathcal{A}_X$ of lines. Connecting lines of the points in X dualize to vertices of $\mathcal{A}_X$. Now assume that the duality transform is the polarity at the parabola $y = x^2$. This transformation maps a point (a, b) to the line $y = 2ax - b$ and vice versa. In particular lines with the same slope are mapped to vertices with the same x-coordinate. Using a vertical line to sweep the arrangement $\mathcal{A}_X$ we obtain a sequence of permutations $\Sigma_X = \pi_0 \ldots \pi_r$. This sequence is not quite an allowable sequences, it obeys properties (1) and (3) of allowable sequences, but property (2) is not exactly true. The appropriate statement in the given setting is the following:

(2′) Each permutation π_i, $1 \leq i \leq r$ is obtained by the reversal of one or more disjoint consecutive substrings from the preceding permutation π_{i-1}.

We call a sequence Σ with (1), (2′) and (3) a *generalized allowable sequence*. Actually, this is the original definition of an allowable sequence. The reason is that when coming from point configurations, as Goodman and Pollack did, property (2′) is the natural choice. Property (2) is convenient for pseudoline arrangements and has the advantage that the notion of equivalence of sequences becomes simpler.

Back to the theorem. With point set X we have associated a generalized allowable sequence $\Sigma_X = \pi_0 \ldots \pi_r$ such that the number of slopes determined by X is r, i.e., the number of moves that transform π_0 (the identity) into π_r (the reverse). The following proposition is enough to complete the proof of the theorem.

Proposition 6.5 *Let $\Sigma = \pi_0, \pi_1, \ldots, \pi_r$ be a generalized allowable sequence of permutations of $[2n]$, if $r > 1$ then $r \geq 2n$.*

Proof. Let the *middle-barrier* separate each permutation π_i of $[2n]$ between the first n elements and the last n elements. A move $\pi_i \longrightarrow \pi_{i+1}$ is *crossing* if one of the substrings reversed in the move contains the middle-barrier, i.e., if the move brings some elements from one side of the middle-barrier to the other. A crossing move has *order d* if it brings $2d$ elements across the middle-barrier, see Figure 6.7.

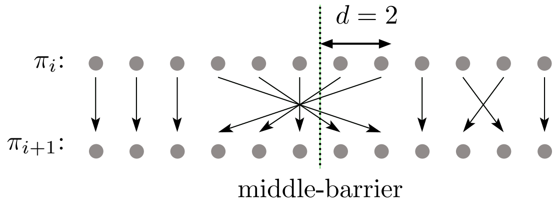


Figure 6.7 A crossing move of order 2.

An allowable sequence with more than one move contains at least two crossing moves. Suppose that there are $t \geq 2$ crossing moves $m_1, \ldots, m_t$ and let d_i be the order of m_i. Each element has to cross the middle-barrier in some crossing move, therefore

$$\sum_{i=1}^{t} 2d_i \geq 2n.$$

Claim. Between consecutive crossing moves m_i and m_{i+1} there are at least $d_i + d_{i+1} - 1$ non-crossing moves.

The basis for the proof is the following simple observation:

($\star$) A move always transforms increasing substrings into decreasing ones.

After move m_i there is a decreasing block of length d_i on either side of the middle-barrier. Alike, before move m_{i+1} there is an increasing block of length d_{i+1} on either side of the middle-barrier. With $\star$ the following statements follow:

- Between m_i and m_{i+1} there is a (unique) move m^* touching the middle barrier, such that the middle pair (i.e., the pair of elements left and right of the middle barrier) is changed from decreasing to increasing by m^*.

- Every move between m_i and m^* can strip off at most one element from each of the decreasing blocks on the two sides of the middle-barrier. After m_i the length of the decreasing block is d_i. Move m^* requires that the length of a decreasing block touching the barrier is one.
- Every move between m^* and m_{i+1} can stick on at most one element to each of the increasing blocks on the two sides of the middle-barrier. After m^* the length of such a block is one. Move m_{i+1} requires that the length of this block has grown to d_{i+1}.

Together this shows that there are at least $1 + (d_i - 1) + (d_{i+1} - 1) = d_i + d_{i+1} - 1$ non-crossing moves between m_i and m_{i+1}. △

Claim. Before m_1 and after m_t there are additional $d_1 + d_t - 1$ non-crossing moves.

We reduce this to the previous claim: Let m_1 be the move $\pi_i \longrightarrow \pi_{i+1}$ in the generalized allowable sequence $\Sigma = \pi_0, \pi_1, \ldots, \pi_r$. Consider the mapping taking π to $\pi' = \pi_{i+1}^{-1} \circ \pi$, this is a relabeling such that π'_{i+1} is the identity. Recall that $\overline{\pi}$ is the reverse of π. and consider the sequence $\Sigma' = \pi'_{i+1}, \pi'_{i+2}, \ldots, \pi'_r, \overline{\pi'_1}, \ldots, \overline{\pi'_i}, \overline{\pi'_{i+1}}$. This is a generalized allowable sequence and there is an obvious correspondence between the moves of this sequence and the moves of Σ. In particular, the two last crossing moves m'_{t-1} and m'_t and the moves between them in Σ' are in bijection with m_t and m_1 and the moves before m_1 and after m_t in Σ. The claim now follows from the previous claim applied to Σ'. △

For the total number r of moves we have the estimate:

$$r \geq t + (d_1 + d_t - 1) + \sum_{i=1}^{t-1} (d_i + d_{i+1} - 1) = \sum_{i=1}^{t} 2d_i \geq 2n.$$

□

6.3 Local Sequences

Representing an arrangement by an allowable sequence can be seen as an encoding by an ordered sequence of vertical cuts through the arrangement. A representation of simple arrangements by a sequence of horizontal cuts can be obtained by associating with line i the permutation α_i of $\{1, .., n\} \setminus i$ reporting the order from left to right in which the other pseudolines cross line i. The family $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is called the family of *local sequences* of the arrangement. In case of non-simple arrangements local sequences are slightly more general structures than permutations since several lines can cross line i in the same point. For the arrangement of Figure 6.2 the local sequences can be coded as $\alpha_1 = [2, \{3, 5\}, 4]$, $\alpha_2 = [1, 5, 3, 4]$, $\alpha_3 = [\{1, 5\}, 2, 4]$, $\alpha_4 = [5, 1, 2, 3]$ and $\alpha_5 = [4, \{1, 3\}, 2]$.

Later, in Theorem 6.17, we characterize those $(\alpha_i)_{i=1..n}$ corresponding to simple marked arrangements. The next goal (Theorem 6.6), however, is to show that in order to obtain a representation of simple marked arrangements with the standard labeling it is sufficient to know for each i, which entries of α_i are larger than i and which are smaller.

With the local sequence $\alpha_i = (a_1^i, a_2^i, \ldots, a_{n-1}^i)$ corresponding to line i associate a binary vector $\beta_i = (b_1^i, b_2^i, \ldots, b_{n-1}^i)$ such that $b_j^i = 1$ if $a_j^i > i$ and $b_j^i = 0$ if $a_j^i < i$. Since α_i is a permutation of $\{1, .., n\} \setminus i$ exactly $n - i$ entries of β_i are 1, i.e., $\sum_{j=1}^{n-1} b_j^i = n - i$ for all i.

Assume that $\mathcal{A}$ is given with the standard labeling of lines, i.e., $i < j$ implies that far enough to the left the line with label i is below the line with label j. We can use this to reinterpret the meaning of b^i_j: If the j-th crossing on line i in the wiring diagram is a move of line i up into the next wire, then $b^i_j = 1$ and if line i is moving down at its j-th crossing, then $b^i_j = 0$. Hence we can get the value of b^i_j directly from the arrangement, $b^i_j = 1$ iff at its j-th crossing line i is crossing another line from below, i.e., line i is moving up.

Let $\mathcal{B}(n)$ be the set of n-tuples $(\beta_1, \beta_2, \ldots, \beta_n)$ with $\beta_i = (b^i_1, b^i_2, \ldots, b^i_{n-1})$ a binary vector and $\sum_{j=1}^{n-1} b^i_j = n - i$ for all i. Above we have described a mapping Φ from arrangements of n pseudolines to $\mathcal{B}(n)$. Figure 6.8 shows an example.

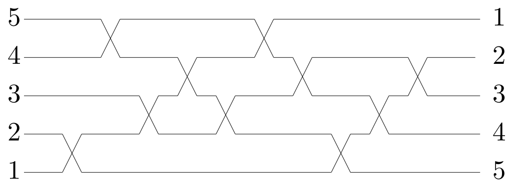


Figure 6.8 A wiring diagram $\mathcal{A}$ of five lines. The corresponding sequence of five binary vectors is $\Phi(\mathcal{A}) = \big((1,1,1,1),(0,1,1,1),(0,1,1,0),(1,0,0,0),(0,0,0,0)\big)$.

Not all elements of $\mathcal{B}(n)$ correspond to arrangements. For $n = 4$ there are 9 elements in $\mathcal{B}(4)$ but only 8 arrangements. The element of $\mathcal{B}(4)$ which is not in the image of Φ is $\big((1,1,1),(1,0,1),(0,1,0),(0,0,0)\big)$.

Theorem 6.6 *The mapping Φ is an injective map from simple arrangements of n pseudolines to $\mathcal{B}(n)$.*

Proof. Below we describe an algorithm *bit-sweep* which takes an element of $\mathcal{B}(n)$ as input and constructs a sequence of permutations of $[n]$. If the element of $\mathcal{B}(n)$ is the $\Phi(\mathcal{A})$ for some simple arrangement $\mathcal{A}$, this sequence of permutations is a simple allowable sequence Σ for $\mathcal{A}$. Since Σ uniquely determines the marked arrangement $\mathcal{A}$, this is also true for $\Phi(\mathcal{A})$. In particular Φ is injective.

Let $(\beta_1, \beta_2, \ldots, \beta_n)$ be the input for the bit-sweep algorithm. The first permutation in the output is the identity, $\pi_0 = (1, 2, \ldots, n)$. For each i there is a *position counter* $p(i)$ which is initialized as $p(i) = 1$. The *state* of the algorithm is a vector $S = (s_1, s_2, \ldots, s_n)$. The state vector is initialized by the first bit of each of the β_i, i.e., $(s_1, s_2, \ldots, s_n) = (b^1_1, b^2_1, \ldots, b^n_1)$. After k steps, i.e., after the output of π_k, the ith entry of the state is

$$s_i = b^{\pi_k(i)}_{p(\pi_k(i))}$$

this is the bit at the actual position $p(\pi_k(i))$ on the line which is the ith element of π_k. In step $k+1$ the algorithm takes the least index i with $s_i = 1$ and $s_{i+1} = 0$. If there is no such index the algorithm stops. If i exists, then π_{k+1} is obtained from the current permutation π_k by an adjacent transposition at positions i and $i+1$. The new permutation is $\pi_{k+1} = (\pi_k(1), \ldots, \pi_k(i-1), \pi_k(i+1), \pi_k(i), \pi_k(i+2), \ldots, \pi_k(n))$. To complete the step the position counters $p(\pi_k(i))$ and $p(\pi_k(i+1))$ are incremented by one and the state vector is updated accordingly, i.e., with new entries s_i and s_{i+1} taken from the appropriate positions of $\beta_{\pi_{k+1}(i)}$ and $\beta_{\pi_{k+1}(i+1)}$. If a position counter reaches the value n the β-vector has no corresponding entry, in that case write a symbol ∞ into the state vector. Figure 6.9 indicates a run of the algorithm.

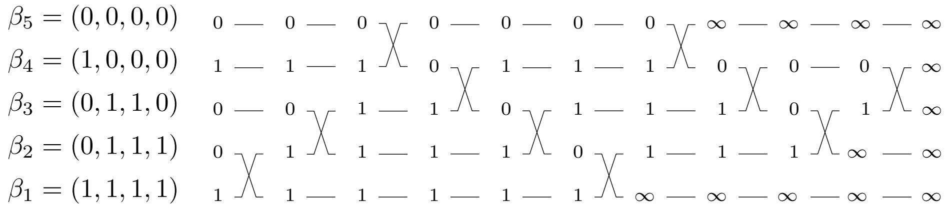


Figure 6.9 The columns show the state vectors in a run of the bit-sweep algorithm.

The *canonical allowable sequence* of an arrangement $\mathcal{A}$ is the allowable sequence which results from a sweep which always picks the lowest admissible vertex for the advance. The claim is that with input $\Phi(\mathcal{A})$ the bit-sweep algorithm produces the canonical allowable sequence of $\mathcal{A}$.

The proof is by induction. We observe both algorithms, the canonical sweep and the bit-sweep algorithm with input $\Phi(\mathcal{A})$ step by step. Each step for both algorithms consists in the move from one output permutation to the next. The invariant for the proof is that after k steps:

(1) The current permutation in both algorithms is the same π_k.

(2) The sweep has passed $p(i) - 1$ vertices on line i for each $i \in [n]$.

(3) $s_i = 1$ iff at the next crossing the line $\pi_k(i)$ is moving up.

This is obvious for $k = 0$, both algorithms start with the identity permutation. Assume the invariant after k steps. To show that it holds after $k + 1$ steps we need that

($\star$) lines $\pi_k(i)$ and $\pi_k(i+1)$ cross at a vertex which is admissible for the sweep if and only if $s_i = 1$ and $s_{i+1} = 0$.

If $s_i = 1$ and $s_{i+1} = 0$ then (by definition and invariant) line $\pi_k(i)$ is moving up at its next crossing while line $\pi_k(i + 1)$ is moving down at its next crossing. Since line $\pi_k(i)$ is below line $\pi_k(i + 1)$ and they border a common face in $\mathcal{A}$, the vertex where they cross is admissible for the sweep. Conversely, if lines $\pi_k(i)$ and $\pi_k(i + 1)$ cross at an admissible vertex, then line $\pi_k(i)$ is moving up at its next crossing and line $\pi_k(i + 1)$ is moving down at its crossing. Therefore, $s_i = 1$ and $s_{i+1} = 0$. □

Application of Local Sequences: Counting Arrangements

Let B_n be the number of simple marked arrangements of n pseudolines. Combinatorial encodings are a tool to get hand on this number. With today's methods exact enumeration is by far out of reach, but there are some asymptotic bounds.

Let A_n be the number of simple allowable sequences. Stanley found a remarkable exact formula for A_n. From that formula the asymptotic growth of A_n is known to be $2^{\Theta(n^2 \log n)}$. Clearly $B_n < A_n$, we will show that B_n is substantially smaller than A_n.

Theorem 6.7 *The number B_n of non-isomorphic simple marked arrangements of n pseudolines is at most $2^{0.72n^2}$.*

Proof. By Theorem 6.6 we have an injective mapping from arrangements of n lines to $\mathcal{B}(n)$, i.e., $B_n \leq |\mathcal{B}(n)|$. Counting elements of $\mathcal{B}(n)$ is a simple task:

$$|\mathcal{B}(n)| = \binom{n-1}{0}\binom{n-1}{1}\binom{n-1}{2}\cdots\binom{n-1}{n-1}.$$

Let $f(n) = |\mathcal{B}(n)|$, from $\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k}$ we obtain the recursion $f(n+1) = \frac{n^n}{n!}f(n)$. Applying Stirling's formula and taking logarithms gives

$$\log_2 f(n+1) = n\log_2 e + \log_2 f(n) + O(\log n) = \sum_{k=1}^{n} k\log_2 e + O(n\log n).$$

With a table lookup we find that $\binom{n}{2}\log_2 e \approx 0.7213(n^2 - n)$. □

To improve the bound on the number of arrangements it would be interesting to have some tools to discriminate between those members from $\mathcal{B}(n)$ which are in the image of Φ and those which are not. At this time we have little more than the bit-sweep algorithm from the proof of Theorem 6.6. We can take an arbitrary element $B \in \mathcal{B}(n)$ as input to this algorithm. The two possible outcomes are:

(1) The algorithm gets stuck before $\binom{n}{2}$ moves have been made, i.e., in the current state $(s_1, \ldots, s_n)$ there is no index i with $s_i = 1$ and $s_{i+1} = 0$.

(2) B indeed corresponds to an arrangement.

Recall the element $B = \big((1,1,1),(1,0,1),(0,1,0),(0,0,0)\big)$ of $\mathcal{B}(4)$ not in the image of Φ. Trying to sweep B we get stuck after three moves. At the second move we may already note that something goes wrong, the lines involved in the crossing of the first move cross back. An *immediate back-cross* is a situation where two lines cross twice in a row. When sweeping $B \in \mathcal{B}(n)$, as in the proof of Theorem 6.6, we recognize an immediate back-cross when the pair $(s_i, s_{i+1}) = (1,0)$ of the move is replaced by $(s'_i, s'_{i+1}) = (1,0)$, i.e., the state vectors S and S' before and after the move are identical.

The sweep-like algorithm used for the proof of Theorem 6.6 took an element $B \in \mathcal{B}(n)$ as input and in case $B = \Phi(\mathcal{A})$ for some arrangement $\mathcal{A}$ it produced the canonical allowable sequence for $\mathcal{A}$ as output. Internally, the algorithm also produced a sequence of state vectors. It is quite obvious that the algorithm can be modified such that it takes the sequence $S_0, S_1, \ldots, S_{\binom{n}{2}}$ of state vectors as input and the output produced is the same.

Two successive state vectors S_{i-1} and S_i only differ in two positions, namely, in the least two positions on which the entries of S_{i-1} are $(1,0)$. Therefore, S_i is completely determined by two new bits, call the pair $w_i = (w_i^1, w_i^2)$ of these bits the *replace pair* for the transition from S_{i-1} to S_i.

Hence, the sweep corresponding to $B \in \mathcal{B}(n)$ is completely determined by the initial state $S = S_0$ and a sequence of replace pairs $w_1, w_2, \ldots, w_{(n^2-n)/2}$. A sequence of replace pairs leads to an immediate back-cross exactly iff one of the pairs w_j is $(1,0)$. The number of back-cross free elements of $\mathcal{B}(n)$ and, hence, the number of arrangements can be estimated from above by the number of initial states S and the number of $(1,0)$-free sequences of replace pairs. For S there are $\leq 2^n$ choices and for each pair w_j there remain 3 choices*, therefore:

$$B_n \leq 2^n 3^{\binom{n}{2}} = 2^{\frac{\log_2(3)}{2}\,n^2 + O(n)} = 2^{0.7924\,n^2 + O(n)}.$$

* Here we disregard the special entries ∞ in the state vectors. Actually, these special entries are not necessary, see the proof of Lemma 6.8.

This new bound is weaker than the bound from Theorem 6.7. However, the new bound was only using the forbidden immediate back-crossings. The result of Theorem 6.7 only made use of the number of zeros and ones in each β_j. With the replace matrix we next define a representation that takes care of both aspects.

A *replace matrix* is a binary $n \times n$ matrix M with two properties:

(1) $\sum_{j=1}^{n} m_{ij} = n - i$ for $i = 1, .., n$,

(2) $m_{ij} \geq m_{ji}$ for all $i < j$.

Lemma 6.8 *There is an injective mapping Ψ from simple arrangements of n pseudolines to $n \times n$ replace matrices.*

Proof. Consider the local bit encoding $\Phi(\mathcal{A})$ (see Theorem 6.6) and let $m_{ii} = b_1^i$, that is, we record the initial state vector S of the sweep along the diagonal of M. If in the kth step of the sweep of $\Phi(\mathcal{A})$ lines i and j cross we define $m_{ij} = 1$ if the next crossing (after the crossing with line j) of line i goes up and $m_{ij} = 0$ if the next crossing of line i goes down. If $i < j$ then at their crossing line i is going up and line j is going down. Since there is no immediate back-cross of lines i and j we conclude $(m_{ij}, m_{ji}) \neq (0, 1)$ or equivalently $m_{ij} \geq m_{ji}$. Having completed the sweep of $\Phi(\mathcal{A})$ we remain with a single undefined entry in each row of M. The undefined entry is m_{ij} when line j was the last line crossing line i, define $m_{ij} = 0$ in this case.

Property (1) of replace matrices is easily seen to hold for M as defined above. The entries in row i of M are a permutation of the entries of β_i and an additional 0. Property (2) is only in question at a pair i, j with $i < j$ and $m_{ij} = 0$ because it was the last undefined entry of its row. In this case, after crossing line j from below line i was not involved in further crossings. Suppose that line j has a crossing after the crossing with line i. At the first of these later crossings line j has to move down. This is because the position above j is occupied by i, hence, $m_{ji} = 0$. The other possibility is that line j has no further crossings and again $m_{ji} = 0$. In both cases $m_{ij} \geq m_{ji}$.

Hence, $M = \Psi(\mathcal{A})$ is a well defined replace matrix. The mapping Ψ is injective because $M = \Psi(\mathcal{A})$ can guide a sweep to reconstruct (the canonical allowable sequence of) $\mathcal{A}$. □

The replace matrix corresponding to the arrangement of Figure 6.8 illustrates this encoding.

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

To obtain an estimate for the number of replace matrices, a probabilistic argument can be used. Consider the probability space Ω of all binary $n \times n$ matrices with $\sum_{j=1}^{n} m_{ij} = n - i$ for $i = 1, \ldots, n$ and let M be a uniformly distributed random matrix from Ω. Let p_i be the probability that a fixed entry in row i of M is 0, i.e., $p_i = \frac{i}{n}$, and $q_i = 1 - p_i$ be the probability that this entry is 1, i.e., $q_i = \frac{n-i}{n}$.

For $i < j$ let E_{ij} be the event $m_{ij} \geq m_{ji}$. Since $m_{ij} \not\geq m_{ji}$ iff $(m_{ij}, m_{ji}) = (0, 1)$ the probability of event E_{ij} is $\text{Prob}[E_{ij}] = (1 - p_i q_j)$. The number R_n of replace matrices can be written as $R_n = |\Omega| \text{Prob}[\bigwedge_{i<j} E_{ij}]$.

For independent events E_{ij} we would have $\mathrm{Prob}[\bigwedge_{i<j} E_{ij}] = \prod_{i<j} \mathrm{Prob}[E_{ij}]$. This would allow to estimate R_n as $\prod_{k=0}^{n-1} \binom{n}{k} \prod_{i<j}(1 - \frac{i(n-j)}{n^2})$. The base 2 logarithm of this function behaves like $0.66n^2$.

Unfortunately, due to the fixed row sums of replace matrices the E_{ij} are not independent. There are positively and negatively correlated pairs. Therefore, it is not obvious in which direction the error made by ignoring dependencies goes. Nevertheless, this probabilistic approach can be exploited to prove:

Theorem 6.9 *The number R_n of $n \times n$ replace matrices and, hence, the number B_n of non-isomorphic simple marked arrangements of n pseudolines is at most $2^{0.69n^2}$.*

6.4 Zonotopal Tilings

A particularly nice representation of arrangements of pseudolines is the representation by 'zonotopal tilings'. This can be considered as a standardized drawing of the 'dual graph' of the arrangement. Figure 6.10 shows an example.

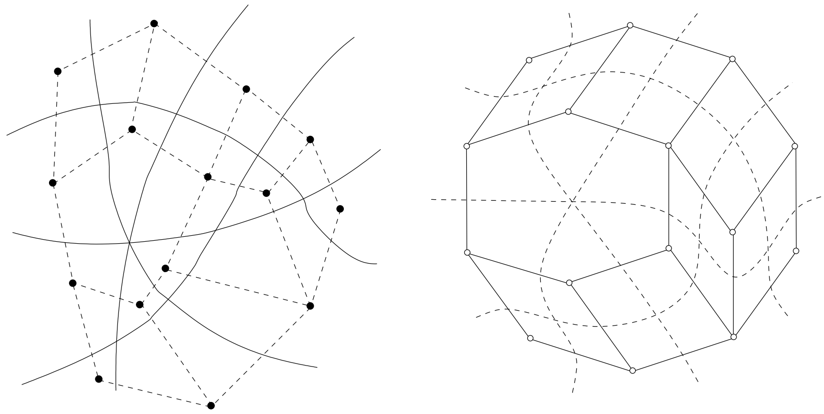


Figure 6.10 An arrangement with its dual graph and the dual graph as zonotopal tiling.

A 2-dimensional *zonotope* is the Minkowski sum of a set of n line segments in $\mathbb{R}^2$, this is a centrally symmetric $2n$-gon. With a vector v_i associate the line segment $[-v_i, +v_i]$, the Minkowski sum of the line segments corresponding to $V = \{v_1, \ldots, v_n\}$ is the set

$$Z(V) = \left\{ \sum_{i=1}^{n} c_i\, v_i : -1 \leq c_i \leq 1 \text{ for all } 1 \leq i \leq n \right\}.$$

A *zonotopal tiling* $\mathcal{T}$ is a tiling of $Z(V)$ by translates of zonotopes $Z(V_i)$ with $V_i \subset V$. A zonotopal tiling is a *simple zonotopal tiling* if all tiles are rhombi, i.e., $|V_i| = 2$ for all i. A zonotopal tiling together with a distinguished vertex x of the boundary of $Z(V)$ is a *marked zonotopal tiling.*

The next theorem is a precise statement for the correspondence suggested by Figure 6.10. The proof of the theorem is based on a Sweeping Lemma for zonotopal tilings.

Theorem 6.10 *Let V be a set of n pairwise non-collinear vectors in $\mathbb{R}^2$.*

(1) There is a bijection between marked zonotopal tilings of $Z(V)$ and marked arrangements of n pseudolines.

(2) Via this bijection simple tilings correspond to simple arrangements.

We first give the mapping from zonotopal tilings to equivalence classes of allowable sequences. Let $Z(V)$ be a marked zonotope generated by a set V of n pairwise non-collinear vectors. The zonotope $Z = Z(V)$ is a centrally symmetric $2n$-gon. Rotate Z such that the distinguished vertex x is the unique highest vertex of Z, in particular the boundary of Z has no horizontal edge. Assume a labeling of the vectors in V, such that along the left boundary of Z, i.e., on the left path from the lowest vertex $\overline{x}$ to x, the segments correspond to $v_1, v_2, \ldots, v_n$ in this order. This labeling is the *standard labeling* of V.

Given a zonotopal tiling $\mathcal{T}$ of Z, consider the set of y-monotone paths along segments of $\mathcal{T}$ from $\overline{x}$ to x. We define a *sweep of $\mathcal{T}$ with north-pole x* as a sequence $p_0, p_1, \ldots, p_r$ of y-monotone paths from $\overline{x}$ to x in $\mathcal{T}$ with the following properties.

(1) Any two consecutive paths p_i, p_{i+1} of the sequence have exactly one tile T_i of the tiling $\mathcal{T}$ between them, i.e., in the interior of the closed curve $p_i \cup p_{i+1}$.

(2) Every tile is between a unique pair of consecutive paths, therefore, $p_0 \cup p_r$ is the boundary of $Z(V)$.

As we did for sweeps of arrangements, we further assume that the sweep of $\mathcal{T}$ is from left to right, i.e., p_0 is the left boundary of $Z(V)$.

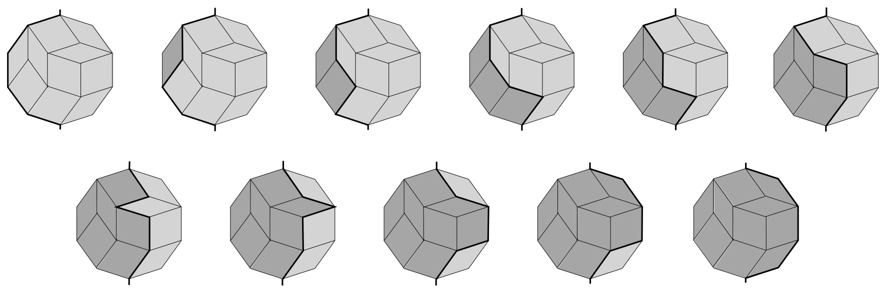


Figure 6.11 A sweep of a zonotopal tiling.

Remark. There is some interest in the maximum number $m(n)$ of y-monotone $\overline{x}$ to x paths a marked zonotopal tiling can have. Knuth ([123] page 39) conjectures that $m(n) \leq n2^{n-2}$. By an inductive argument this would imply that the number of marked arrangements of n pseudolines is bounded by $\Pi_{k=1}^{n} m(k)$. Therefore, the conjectured bound would show that this number is at most $2^{n^2/2+o(n^2)}$, an improvement over the bound of Theorem 6.9.

A sweep of tiling $\mathcal{T}$ induces a total order $T_1, T_2, \ldots, T_r$ on the tiles of $\mathcal{T}$ with the property that any initial segment $T_1, \ldots, T_i$ can be separated from the remaining tiles $T_{i+1}, \ldots, T_r$ by a horizontal translation. Conversely, an order $T_1, T_2, \ldots, T_r$ of the tiles with this separation property corresponds to a sweep: Define path p_i as the right boundary of the union of $T_1, \dots, T_i$. The following lemma implies that a zonotopal tiling $\mathcal{T}$ can be swept.

Lemma 6.11 *Let $\mathcal{C}$ be a family of disjoint convex sets in the plane. There is a $C \in \mathcal{C}$ that can be translated to the left without colliding with another object from $\mathcal{C}$.*

Proof. Define a directed graph $G_\mathcal{C}$ whose vertices are the sets in $\mathcal{C}$. Arcs of $G_\mathcal{C}$ are the pairs (C, C') such that there is a horizontal line segment in $\mathbb{R}^2 \setminus \mathcal{C}$ with left endpoint at C and right endpoint at C', in other words, C' is visible from C horizontally to the right. By definition, elements that can be translated collision free to the left are exactly the sources of $G_\mathcal{C}$.

Imagine the plane being slanted such that a marble placed at a point (x, y) would roll in direction of $(x, -\infty)$ and think of the elements of $\mathcal{C}$ as elevated obstacles. Choose $C \in \mathcal{C}$ arbitrarily, place a marble at the lowest point of C and let it roll, this procedure yields a y-monotone path $p^\downarrow$ connecting C to vertical $-\infty$, (note that here we use the convexity of the obstacles). A symmetrical procedure yields a path $p^\uparrow$ from the highest point of C to vertical $+\infty$. Let p be the concatenation of $p^\uparrow$, a line segment through C and $p^\downarrow$, this is a y-monotone path. Let $\mathcal{C}^l$ be the set of elements of $\mathcal{C}$ which are left of p and $\mathcal{C}^r$ the subset right of p. All arcs of $G_\mathcal{C}$ go left to right in the ordered partition $[\mathcal{C}^l, C, \mathcal{C}^r]$ of $\mathcal{C}$. Therefore either $\mathcal{C}^l = \emptyset$ and C is a source of $G_\mathcal{C}$ or induction yields a source C' of $G_\mathcal{C}[\mathcal{C}^l]$. This element C' is also a source of $G_\mathcal{C}$. △

A total ordering $C_1, C_2, \dots, C_r$ of the elements of $\mathcal{C}$ has the property that $C_1 \dots, C_i$ can be separated from $C_{i+1}, \dots, C_r$ by a horizontal translation exactly if $C_1, C_2, \dots, C_r$ is a topological sorting of $G_\mathcal{C}$.

The tiles of a tiling $\mathcal{T}$ are a family of disjoint convex sets. The graph $G_\mathcal{T}$ has the tiles of $\mathcal{T}$ as vertices. The arcs of $G_\mathcal{T}$ are pairs (T, T') of tiles sharing a common segment and oriented from the tile T on the left side of the segment $T \cap T'$ to the tile on the right side. The previous considerations are summarized with a proposition:

Proposition 6.12 *Let $\mathcal{T}$ be a zonotopal tiling of a marked zonotope Z. Sweeps of $\mathcal{T}$ bijectively correspond to topological sortings of $G_\mathcal{T}$. In particular, every marked zonotopal tiling $\mathcal{T}$ can be swept.*

The next lemma is the 'zonotopal equivalent' of Theorem 6.3. Actually, together with the bijection from Theorem 6.3 this lemma implies Theorem 6.10.

Lemma 6.13 *There is a bijection between marked zonotopal tilings and equivalence classes of allowable sequences. Moreover, this bijection maps simple zonotopal tilings to classes of simple allowable sequences.*

Proof. First we show how to associate an allowable sequence to every sweep of a zonotopal tiling. Given a sweep sequence $p_0, \dots, p_r$ of paths we associate to each path p_i a sequence π_i recording the labels of the vectors which define the segments along the path in the order of the path from $\overline{x}$ to x. The sequence π_0 is a permutation, the identity. Any two consecutive sequences π_i and π_{i+1} only differ in a substring where path p_i takes the left boundary and path p_{i+1} takes the right boundary of tile T_i. Since T_i is a zonotope, the same labels appear on both boundaries but in reversed order. Hence, all π_i are permutations, moreover, $\pi_i \to \pi_{i+1}$ is a move as in part (2) of the definition of allowable sequences. We also note that π_r is the reverse permutation.

It remains to prove property (3) of allowable sequences: Any two elements $i, j \in [n]$ are reversed in exactly one move or equivalently, that there is a unique tile T in the tiling $\mathcal{T}$ which has boundary segments corresponding to v_i and v_j. The labeling of the vectors is such that on the left border of $Z = Z(v_1, \ldots, v_n)$ we have v_i below v_j iff $i < j$. On the right border of Z the vectors appear in the reverse order, therefore, every pair i, j has to be exchanged by some move. To prove that every pair is exchanged exactly once we use the observation that a sub-zonotope $Z(W)$ with $W \subset V$ also has the vectors in increasing order on its left border and in decreasing order on its right border. The tiles $T \in \mathcal{T}$ are sub-zonotopes $Z(W)$ of Z for suitable $W \subset V$. Hence, the move corresponding to a tile $T \in \mathcal{T}$ is the replacement of an increasing substring by its reverse decreasing substring. If $i < j$ and the order of i and j has been changed by a move, then these two elements are a decreasing pair. Therefore, they cannot participate in an increasing substring and, hence, cannot be exchanged by later moves induced by tiles $T \in \mathcal{T}$.

We know (Proposition 6.12) that the sweeps of $\mathcal{T}$ are in one-to-one correspondence with topological sortings of $G_{\mathcal{T}}$. On the other hand, an equivalence class of allowable sequences corresponds to the topological sortings of the graph $\overrightarrow{G}$ of a marked arrangement (page 92). It is not hard to verify that $G_{\mathcal{T}}$ and $\overrightarrow{G}$ are isomorphic.

For the inverse mapping we have to associate a marked zonotopal tiling to an equivalence class of allowable sequences. Let $\Sigma = \pi_0, \ldots, \pi_r$ be any member of the equivalence class and build the tiling from left to right starting with the left boundary of $Z(V)$. After placing i tiles, three properties remain invariant:

(1) The union of the already placed tiles together with the left boundary of Z is a simply connected region.

(2) The right boundary of this region is a y-monotone path p_i.

(3) The segments along path p_i are in the order given by π_i.

The crucial observation is that when it comes to place tile T_{i+1} path p_i contains a part s with the shape of the left border of T_{i+1}. Only this part of the path is affected by the placement of T_{i+1}. This effects in the replacement of the points on s by points which are further right. From the invariant it follows that the tiles are placed without overlap. Since the last permutation π_r is the reverse of the identity, path p_r is the right boundary of $Z(V)$ and zonotope $Z(V)$ is completely covered by tiles. Therefore, the placement of tiles $T_1, \ldots, T_r$ is a tiling $\mathcal{T}$ of $Z(V)$. □

Application of Zonotopal Tilings: k-Sets and $<k$-Sets

Let $\mathcal{A}$ be a marked Euclidean arrangement of pseudolines. Define the kth level of an arrangement $\mathcal{A}$ of n lines as the set of points p in the plane such that exactly k lines of $\mathcal{A}$ pass below p, i.e., p is separated from the south-face $\overline{F}$ by k lines. We are mainly interested in the vertices of $\mathcal{A}$ which belong to the kth level. In the wiring diagram of a simple arrangement these are the vertices of crossings between wires $k+1$ and $k+2$. Recall from Chapter 4, Lemma 4.13, that the k-edges of a set of n points in general position correspond dually to the vertices of the kth level and the $(n-k-2)$th level of the dual arrangement.

Let V_k be the set of vertices of the kth level, the number $|V_k|$ is the complexity of the kth level. We always assume $k \leq n/2$, by symmetry every bound on $|V_k|$ is a bound on $|V_{n-k-2}|$ and vice versa.

A lower bound for the complexity of the kth level is easily obtained. Lines $1, \ldots, k$ start below the kth level and end above it. Assuming that $\mathcal{A}$ is simple every vertex can only bring one of the lines across, therefore $|V_k| \geq k$.

For an upper bound we consider the kth level of a zonotopal tiling of a zonotope $Z = Z(v_1, \ldots, v_n)$. As generating vectors we choose $v_i = (i, 1)$, for $i = 1, \ldots, n$. All tiles of a simple zonotopal tiling of Z have height two. Let the belt of a tile be the horizontal segment at height one, i.e., the widest horizontal segment that fits into the tile. Let $\mathcal{T}$ be the zonotopal tiling of Z corresponding to a simple arrangement $\mathcal{A}$ of n pseudolines.

($\star$) Vertices of the kth level of $\mathcal{A}$ correspond bijectively to tiles of $\mathcal{T}$ whose belts lie on the horizontal line $y = k + 1$, see Figure 6.12.

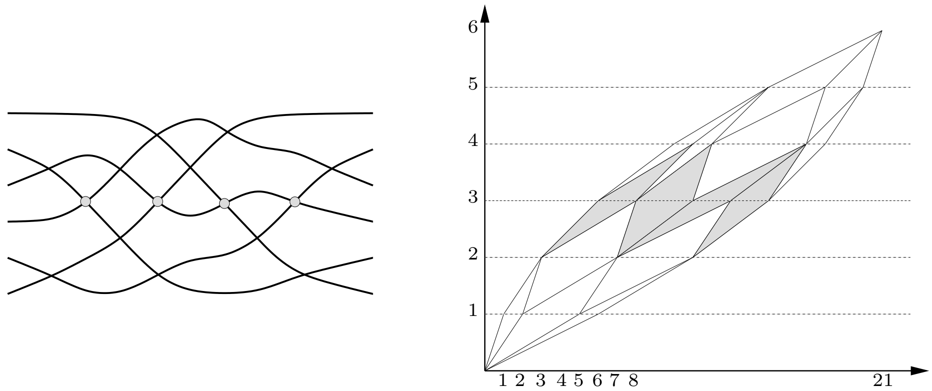


Figure 6.12 An arrangement with a corresponding zonotopal tiling, the 2nd level emphasized.

This observation allows to derive an upper bound for the complexity of the kth level from an one-dimensional packing problem: How many belts of different tiles can be packed disjointly in the interval of intersection of Z with a given horizontal line?

In the following we investigate this later quantity. We obtain a bound for the complexity of the $(k-1)$st level by considering the width of tiles and comparing this to the width of Z at height k.

- The intersection of Z with the line $y = k$ is the interval reaching from $1+2+\ldots+k$ to $n + (n-1) + \ldots + (n-k+1)$. We call this interval the k-interval of Z, it has length $kn - k^2$.
- The width $w(T)$ of a tile T is the length of its belt. A simple zonotopal tiling of Z contains exactly $n - i$ tiles of width i, for $i = 1, \ldots, n-1$, these are the tiles $Z(v_{j_1}, v_{j_2})$ with $|j_1 - j_2| = i$.

Let $T_1, T_2, \ldots T_{\binom{n}{2}}$ be a list of all tiles sorted by increasing width. Take the belts of the tiles in this order and place them disjointly in an interval of length $nk - k^2$. Let T_{t_k} be the last tile whose belt can be used in this process. This index t_k is the bound for $|V_{k-1}|$ we strive for. Formally, t_k is the largest value such that

$$\sum_{i=1}^{t_k} w(T_i) \leq kn - k^2.$$

For $i = 1, \dots, n-1$ let B_i be the set of tiles of width i. The width of tiles in B_i together is $ni - i^2$. The width of tiles in $\bigcup_{i \le j} B_i$ is $\sum_{i \le j}(ni - i^2) > n\frac{j^2}{2} - \frac{j^3}{3}$. Since $n\frac{j^2}{2} - \frac{j^3}{3} \ge kn - k^2$ for $j = \sqrt{2k}$, there will be no tiles of width $\sqrt{2k}$ or more in the packing. Therefore $t_k \le \sum_{i<\sqrt{2k}} |B_i| = \sum_{i<\sqrt{2k}}(n-i) \le \sqrt{2k}\, n - k$. This proves the theorem:

Theorem 6.14 *The complexity of the kth level of an arrangement of n pseudolines is at most $\sqrt{2(k+1)}\, n - (k+1)$.*

The argument above can be used to bound the complexity of a union of levels. Let $K \subset \{0, 1, \dots, \lfloor \frac{n}{2} \rfloor\}$. The tiles whose belts are packed into the k-intervals, $k \in K$, are all distinct. Let t_K be maximal such that $\sum_{i=1}^{t_K} w(T_i) \le \sum_{k \in K}(k+1)n - (k+1)^2$. A simple computation yields a bound of $O(n\sqrt{\sum_{k \in K} k})$ for the total complexity $|\bigcup_{k \in K} V_k|$ of the levels.

A particularly nice case is $K = \{0, 1, \dots, k-1\}$ this means that we are interested in the complexity of the $<k$-levels. Recall that the width of all tiles in B_i is $ni - i^2$ and this equals the length of the i-interval. Therefore, the number of tiles that can be placed on the $<k$-levels is at most $|\bigcup_{i \le k} B_i| = \sum_{i \le k} n - i = kn - \binom{k+1}{2}$. This bound is tight as can be seen from the arrangement generated by the lines $y = -ax + a^2$ for $a = 1, \dots, n$. This is an example of a *cyclic arrangement*.

Proposition 6.15 *The complexity of the $<k$-levels of an arrangement of n pseudolines is at most $kn - \binom{k+1}{2}$ and this bound is tight.*

6.5 Triangle Signs

So far we have studied arrangements of pseudolines as individual objects. Now we change the focus and consider the set of all arrangements. We consider a graph $\mathcal{G}_n$ whose vertices are all combinatorially different simple marked arrangements of n pseudolines in the Euclidean plane. The edges of $\mathcal{G}_n$ correspond to triangular flips: Let T be a triangular face of an arrangement $\mathcal{A}$. Cutting closely around T the triangle can be replaced by a triangle with the opposite orientation T (see Figure 6.13), this replacement is the *triangular flip* at triangle T. Figure 6.14 shows the graph $\mathcal{G}_n$ for $n = 5$, in the figure the arrangements are represented by their corresponding zonotopal tilings.

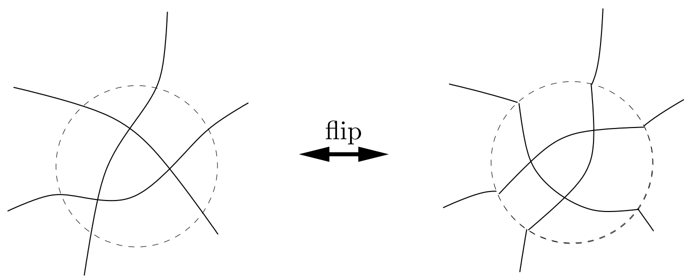


Figure 6.13 A triangular flip.

Below we study an encoding of arrangements by triangle orientations. This encoding imposes a natural orientation on $\mathcal{G}_n$. In Section 6.6 we generalize the patterns and define an order $S_r(n)$, for all $1 \leq r \leq n$, such that $S_1(n)$ is the Boolean lattice, $S_2(n)$ is the weak Bruhat order of the symmetric group, i.e., the elements of $S_2(n)$ are the permutations of $[n]$, and $S_3(n)$ is the abovementioned orientation of $\mathcal{G}_n$. The representation theorem (Theorem 6.16) for arrangements of pseudolines is obtained as a byproduct of a more general correspondence between maximum chains in $S_{r-1}(n)$ and elements of $S_r(n)$ (Theorem 6.21).

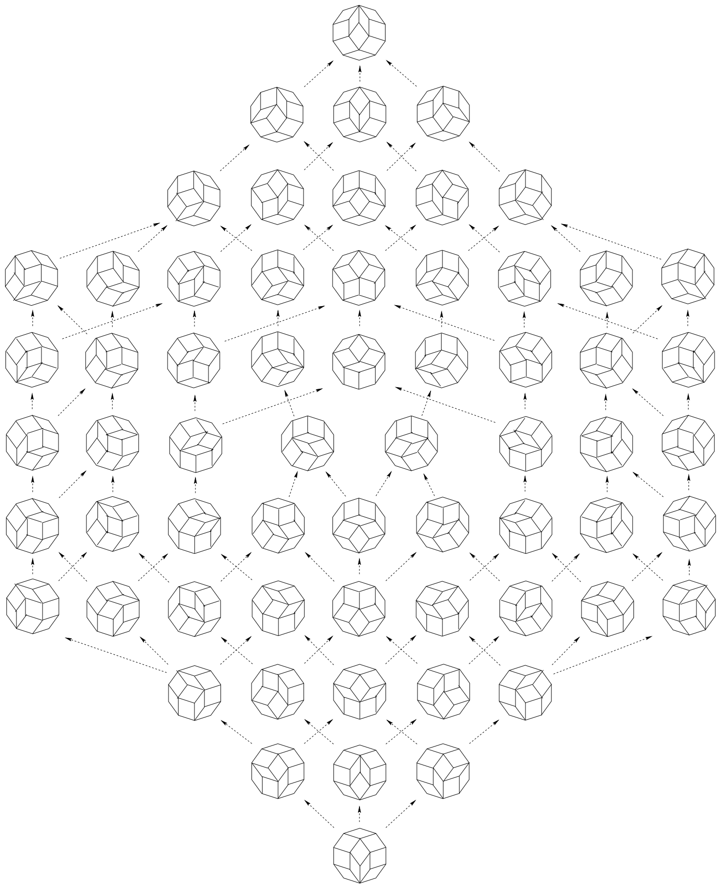


Figure 6.14 The graph $\mathcal{G}_5$ oriented as diagram of the signotope order $S_3(n)$.

Encoding Arrangements by Triangle Orientations

Flips are nicely described in the different encodings of arrangements. In the encoding by zonotopal tilings the projection of a cube is replaced by the view of the cube from the other side. In the encoding by local sequences an adjacent transposition of elements i and j is applied to the local sequence α_k of line k and similarly to local sequences α_i and α_j when the flip-triangle is bounded by lines i, j and k.

Let $(\mathcal{A}, F)$ be a simple marked arrangement of n pseudolines. Consider the arrangement induced by a triple $\{i, j, k\}$ of lines of $\mathcal{A}$, where we assume $i < j < k$. These three lines can induce two combinatorial different arrangements. Either the crossing of lines i and k is above line j, denote this by the symbol $-$, or the crossing is below line j, denoted by $+$. With this convention a function $f : \binom{[n]}{3} \to \{-, +\}$ mapping triples of lines to signs is associated with every marked simple arrangement. This function is called the *triangle-sign function* of the arrangement.

For $i < j$ and all $k \neq i, j$ we have $f(\{i, j, k\}) = +$ iff on line k, the crossing with line j precedes the crossing with line i, i.e., on the local sequence α_k the pair (i, j) is an inversion. Permutations are uniquely determined by their sets of inversions (see e.g., [184]). Therefore, the local sequences of an arrangement are uniquely determined by the triangle sign function. From Theorem 6.6 we know that local sequences encode marked simple arrangements, i.e., arrangements with the same local sequences are isomorphic. It follows that triangle-sign functions $f : \binom{[n]}{3} \to \{-, +\}$ also encode marked simple arrangements of pseudolines.

Not every possible sign pattern $f : \binom{[n]}{3} \to \{-, +\}$ can correspond to an arrangement, there are simply too many such functions. Below we derive an obvious necessary condition on the sign patterns of arrangements. Later it will be shown that this condition is also sufficient.

Consider a quadruple of pseudolines h, i, j, k of $\mathcal{A}$. These lines induce a marked arrangement of four pseudolines. There is only one (unmarked) simple arrangement of four lines. This arrangement has eight unbounded faces, each of these can be chosen for the marking. The eight resulting marked simple arrangements of four lines and their triangle-sign functions are shown in Figure 6.15. The sign functions are given as a vector showing the signs of 3-sets in lexicographic order, i.e, as $\big(\text{sign}(1,2,3), \text{sign}(1,2,4), \text{sign}(1,3,4), \text{sign}(2,3,4)\big)$.

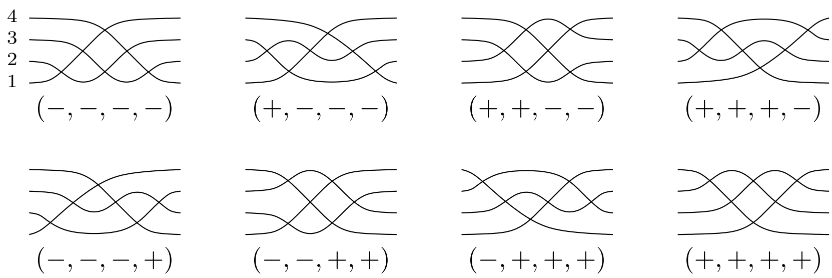


Figure 6.15 The marked simple arrangements of four lines and their triangle-sign functions.

For $A \in \binom{[n]}{4}$ and $1 \leq i \leq 4$ let $A^{\lfloor i \rfloor}$ denote the set A minus the ith element of A in sorted order, e.g., $\{2, 4, 5, 9\}^{\lfloor 3 \rfloor} = \{2, 4, 9\}$. Restricting an arrangement $\mathcal{A}$ to a subset A of four lines we obtain a restricted sign-pattern $\big(\operatorname{sign} A^{\lfloor 4 \rfloor}, \operatorname{sign} A^{\lfloor 3 \rfloor}, \operatorname{sign} A^{\lfloor 2 \rfloor}, \operatorname{sign} A^{\lfloor 1 \rfloor}\big)$. This pattern has to be one of the eight triangle-sign functions from Figure 6.15. Order the set $\{-, +\}$ of signs by $- \prec +$. Inspecting the above list we see that the legal sign patterns are characterized by the following property: For every 4 element subset A of $[n]$ either $f(A^{\lfloor 1 \rfloor}) \preceq f(A^{\lfloor 2 \rfloor}) \preceq f(A^{\lfloor 3 \rfloor}) \preceq f(A^{\lfloor 4 \rfloor})$ or $f(A^{\lfloor 1 \rfloor}) \succeq f(A^{\lfloor 2 \rfloor}) \succeq f(A^{\lfloor 3 \rfloor}) \succeq f(A^{\lfloor 4 \rfloor})$. This property is called *monotonicity*.

Theorem 6.16 *A function $f : \binom{[n]}{3} \to \{-, +\}$ is the triangle-sign function of a marked simple arrangement $\mathcal{A}$ of n pseudolines if and only if f is monotone.*

This theorem is a special instance of the more general Theorem 6.21 about signotopes. Translated back, the proof of Theorem 6.21 shows how to construct a simple allowable sequence from a monotone function $f : \binom{[n]}{3} \to \{-,+\}$. Since allowable sequences encode arrangements this gives a mapping $f \to \mathcal{A}$ from monotone functions $f : \binom{[n]}{3} \to \{-,+\}$ to arrangements. This mapping is the inverse of the mapping from $\mathcal{A}$ to the triangle-sign function of $\mathcal{A}$.

Theorem 6.17 *A set $(\alpha_i)_{i=1..n}$ with α_i a permutation of $[n] \setminus \{i\}$ is the set of local sequences of a simple marked arrangement of order n if and only if for all $i < j < k$ the pairs (i,j), (i,k), (j,k) are inversions in α_k, α_j, α_i or they are all three non-inversions.*

Proof. The necessity of the condition on local sequences can be checked by considering sub-arrangements of three lines.

For the sufficiency we proceed in two steps. First, the property on triples is exactly the property required to associate a triangle-sign function f with $(\alpha_i)_{i=1..n}$ such that $f(i,j,k) = +$ iff (i,j) is an inversion of α_k. In the second step it has to be verified that the function f is monotone on 4-element subsets of $[n]$. If not, then there is an h and $i < j < k$ such that either $f(h,i,j) = +$, $f(h,i,k) = -$, $f(h,j,k) = +$ or $f(h,i,j) = -$, $f(h,i,k) = +$, $f(h,j,k) = -$. Hence, either α_h has inversions i,j and j,k but i,k is a non-inversion or α_h has inversion i,k but non-inversions i,j and j,k. In both cases a contradiction is obtained, since there is no permutation α_h with appropriate inversions and non-inversions. $\square$

6.6 Signotopes and their Orders

In this section we generalize the concept of triangle-sign functions. Let $[n] = \{1,\ldots,n\}$ be equipped with the natural linear order. The set of all r element subsets of $[n]$ is $\binom{[n]}{r}$. For $A \in \binom{[n]}{r}$ with $r \geq i$ we let $A^{\lfloor i \rfloor}$ denote the set A minus the ith element of A in sorted order. The set $\{-,+\}$ of signs is ordered by $- \prec +$.

For integers $1 \leq r \leq n$ an *r–signotope* on $[n]$ is a function σ from the r element subsets of $[n]$ to $\{-,+\}$ such that for every $r+1$ element subset P of $[n]$ and all $1 \leq i < j < k \leq r+1$ either $\sigma(P^{\lfloor i \rfloor}) \preceq \sigma(P^{\lfloor j \rfloor}) \preceq \sigma(P^{\lfloor k \rfloor})$ or $\sigma(P^{\lfloor i \rfloor}) \succeq \sigma(P^{\lfloor j \rfloor}) \succeq \sigma(P^{\lfloor k \rfloor})$. We refer to this property as *monotonicity*.

Let $S_r(n)$ denote the set of all r-signotopes on $[n]$ equipped with the order relation $\sigma \leq \tau$ if $\sigma(A) \preceq \tau(A)$ for all $A \in \binom{[n]}{r}$. Call $S_r(n)$ the *r–signotope order*.

Note that for $r = 3$ the definitions reflect our observations for the encodings of marked simple arrangements of pseudolines made in the previous section. In particular Theorem 6.16 implies that $S_3(n)$ is a partial order on the set of marked arrangements of n pseudolines. Indeed $S_3(n)$ is an orientation of the graph $\mathcal{G}_n$, see Figure 6.14.

The list below collects some other special cases and easy observations.

(1) For $r = 1$ monotonicity is vacuous and $S_1(n)$ is just the lattice of subsets of $[n]$, i.e., the Boolean lattice.

(2) For all $n \geq r \geq 1$ there is a unique minimal and a unique maximal element in $S_r(n)$, namely the constant $-$ and the constant $+$ function.

(3) The diagram of $S_r(r+1)$ is a $(2r+2)$-gon for all $r \geq 1$.

(4) There is a natural correspondence between 2-signotopes on $[n]$ and permutations of n. Permutation π and 2-signotope σ correspond to each other if a pair (i,j) is an inversion of π iff $\sigma(i,j) = +$. For the proof that this is a bijection, note that monotonicity of σ corresponds to transitivity of the inversion relation and transitivity of the non-inversion relation for π (see e.g., [95] or [78]). In the *weak Bruhat order* of the symmetric group, the permutations are ordered by inclusion of their inversion sets. With the indicated correspondence between 2-signotopes and permutations, $S_2(n)$ equals the weak Bruhat order of S_n.

The following two constructions of a new signotope from a given one are useful.

Let σ be an r-signotope on a set $X \subset \mathbb{N}$ with $|X| \geq r+1$. For $x \in X$ the *deletion* $\sigma{\uparrow_x}$ is the induced function on $\binom{X\setminus x}{r}$, i.e., $\sigma{\uparrow_x}(A) = \sigma(A)$. This is an r-signotope on $X \setminus x$. The *contraction* of $x \in X$ in σ is the function $\sigma{\downarrow_x}$ on $\binom{X\setminus x}{r-1}$ with $\sigma{\downarrow_x}(A) = \sigma(A \cup x)$. This is an $(r-1)$-signotope on $X \setminus x$.

Maximum Chains of Signotopes

With an r-signotope σ on $[n]$ associate a directed graph. The vertices are the $r-1$ element subsets of $[n]$. Given two $(r-1)$-subsets A and A' of $[n]$ let $P = A \cup A'$. If $|P| > r$ then there is no edge between A and A'. Otherwise $|A \cap A'| = r-2$ and there are i, j such that $A = P^{\lfloor i \rfloor}$ and $A' = P^{\lfloor j \rfloor}$, we assume that $i < j$. If $\sigma(P) = +$, orient the edge as $P^{\lfloor i \rfloor} \xrightarrow{\sigma} P^{\lfloor j \rfloor}$ and otherwise, if $\sigma(P) = -$, orient it as $P^{\lfloor j \rfloor} \xrightarrow{\sigma} P^{\lfloor i \rfloor}$.

Lemma 6.18 *Let $r \geq 2$ and σ be an r-signotope on $[n]$. The graph with vertices $\binom{[n]}{r-1}$ and edges $\xrightarrow{\sigma}$ is acyclic.*

Proof. For $r = 2$ and arbitrary n, relation $\xrightarrow{\sigma}$ is the transitive tournament corresponding to the permutation whose inversion set is the set of pairs (i,j) with $\sigma(i,j) = +$.

For $n = r$: If $\sigma([r]) = -$, then relation $\xrightarrow{\sigma}$ is the lexicographic order on the $(r-1)$-subsets of $[r]$. Otherwise, $\sigma([r]) = +$ and $\xrightarrow{\sigma}$ it is the reverse-lexicographic order.

Let $n > r > 2$ and let $\tau = \sigma{\uparrow_n}$ be the signotope obtained from σ by deletion of n. By induction $\xrightarrow{\tau}$ is acyclic on $\binom{[n-1]}{r-1}$. Let $\gamma = \sigma{\downarrow_n}$ be the signotope obtained from σ by contraction of n. The vertices of the graph $\xrightarrow{\gamma}$ are the elements of $\binom{[n-1]}{r-2}$. By induction $\xrightarrow{\gamma}$ is acyclic. In the following we use a copy of $\xrightarrow{\gamma}$ on the vertex set $Y = \{A \in \binom{[n]}{r-1} : n \in A\}$. For emphasis we repeat: n is an element of every $A \in Y$.

Let $X^- = \{A \in \binom{[n-1]}{r-1} : \sigma(A \cup \{n\}) = -\}$ and $X^+ = \{A \in \binom{[n-1]}{r-1} : \sigma(A \cup \{n\}) = +\}$. The three sets X^-, X^+, Y partition the $r-1$ element subsets of $[n]$. The subgraph of $\xrightarrow{\sigma}$ induced by each of the three blocks of the partition is acyclic: It agrees with the subgraph induced by $\xrightarrow{\tau}$ in case of X^- and X^+ and with the subgraph induced by $\xrightarrow{\gamma}$ in the case of Y. Now consider the edges of $\xrightarrow{\sigma}$ between the blocks. By definition of X^- all edges with one end in X^- and the other end in Y are oriented from X^- to Y. Also all edges with one end in X^+ and the other end in Y are oriented from Y to X^+. The following claim implies that there are no edges from X^+ to X^- in $\xrightarrow{\tau}$. Since on $\binom{[n-1]}{r-1} = X^+ \cup X^-$ the graphs $\xrightarrow{\tau}$ and $\xrightarrow{\sigma}$ agree the proof of the claim completes the proof of the lemma.

Claim. $A \in X^-$ and $B \xrightarrow{\tau} A$ implies $B \in X^-$, i.e., X^- is an ideal in the partial order defined by the transitive closure of $\xrightarrow{\tau}$.

From $B \xrightarrow{\tau} A$ it follows that $P = A \cup B$ is an r-subset of $[n]$. Let i, j be such that $B = P^{\lfloor i \rfloor}$ and $A = P^{\lfloor j \rfloor}$. For $Q = P \cup \{n\}$ we then obtain $Q^{\lfloor i \rfloor} = B \cup \{n\}$, $Q^{\lfloor j \rfloor} = A \cup \{n\}$ and $Q^{\lfloor r+1 \rfloor} = A \cup B = P$. We use the monotonicity of σ on Q and distinguish two cases:
(1) If $i < j$, then $B \xrightarrow{\tau} A$ implies $\tau(P) = \sigma(Q^{\lfloor r+1 \rfloor}) = +$. From $A \in X^-$ it follows that $\sigma(Q^{\lfloor j \rfloor}) = \sigma(A \cup \{n\}) = -$. Monotonicity forces $\sigma(Q^{\lfloor i \rfloor}) = \sigma(B \cup \{n\}) = -$, i.e., $B \in X^-$.
(2) If $j < i$, then $B \xrightarrow{\tau} A$ implies $\tau(P) = \sigma(Q^{\lfloor r+1 \rfloor}) = -$. From $A \in X^-$ it follows that $\sigma(Q^{\lfloor j \rfloor}) = \sigma(A \cup \{n\}) = -$. Monotonicity forces $\sigma(Q^{\lfloor i \rfloor}) = \sigma(B \cup \{n\}) = -$, i.e., $B \in X^-$. □

Let σ be an r-signotope on $[n]$ and $A_1, A_2, \ldots, A_{\binom{n}{r-1}}$ be a topological sorting of $\xrightarrow{\sigma}$. For $0 \leq t \leq \binom{n}{r-1}$ define $\tau_t : \binom{[n]}{r-1} \to \{+, -\}$ by $\tau_t(A) = -$ if $A = A_i$ for some $i > t$ and $\tau_t(A) = +$ if $A = A_i$ for some $i \leq t$.

Proposition 6.19 *For an r-signotope σ on $[n]$ the above construction yields a chain $\tau_0 < \tau_1 < \ldots < \tau_{\binom{n}{r-1}}$ of $(r-1)$-signotopes in $S_{r-1}(n)$ such that for $t = 1, \ldots, \binom{n}{r-1}$ the signs of τ_{t-1} and τ_t differ at a single $(r-1)$-set.*

Proof. By definition the signs of τ_{t-1} and τ_t only differ at the $(r-1)$-set A_t where the sign of τ_{t-1} is $-$ and the sign of τ_t is $+$. Given that the τ_t are signotopes they form a chain as claimed.

It remains to show that each τ_t is an $(r-1)$-signotope. For every r element set P and all i, j, k with $1 \leq i < j < k \leq r$ we either have $P^{\lfloor i \rfloor} \xrightarrow{\sigma} P^{\lfloor j \rfloor} \xrightarrow{\sigma} P^{\lfloor k \rfloor}$ or $P^{\lfloor k \rfloor} \xrightarrow{\sigma} P^{\lfloor j \rfloor} \xrightarrow{\sigma} P^{\lfloor i \rfloor}$. In the first case we have $\tau_t(P^{\lfloor i \rfloor}) \succeq \tau_t(P^{\lfloor j \rfloor}) \succeq \tau_t(P^{\lfloor k \rfloor})$ for all t and in the second case $\tau_t(P^{\lfloor i \rfloor}) \preceq \tau_t(P^{\lfloor j \rfloor}) \preceq \tau_t(P^{\lfloor k \rfloor})$ for all t. This proves monotonicity for τ_t. □

With this preparation we are ready for the proof of Theorem 6.16.

Proof. [Theorem 6.16] Let σ be a 3-signotope, i.e., a function $\sigma : \binom{[n]}{3} \to \{-, +\}$ obeying monotonicity on 4-subsets of $[n]$. By Proposition 6.19 there is a chain $\tau_0, \ldots, \tau_{\binom{n}{2}}$ in $S_2(n)$ corresponding to σ. Each τ_t, $t = 0, \ldots, \binom{n}{2}$, encodes a permutation of $[n]$. τ_0 is the identity permutation and $\tau_{\binom{n}{2}}$ is the reverse. Moreover, two permutations τ_t and τ_{t+1} differ in a single sign where τ_t is $-$ and τ_{t+1} is $+$. This means that there is a single pair (i, j) which is a non-inversion of τ_t but an inversion of τ_{t+1}. This pair has to be an adjacent pair of both permutations. We conclude that $\tau_0, \ldots, \tau_{\binom{n}{2}}$ is a simple allowable sequence. By Theorem 6.3 this allowable sequence encodes an arrangement $\mathcal{A}$. From the construction it is easily verified that the triangle induced by lines i, j, k in $\mathcal{A}$ is a $+$ triangle iff $\sigma(ijk) = +$. This shows that the $\mathcal{A}$ only depends on σ and not on the choice of the sequence $\tau_0, \ldots, \tau_{\binom{n}{2}}$ for σ. This proves a bijection between 3-signotopes and arrangements. □

Simple allowable sequences encode simple marked arrangements (Theorem 6.3). This is generalized with the next proposition, it shows that saturated chains of $(r-1)$-signotopes encode r-signotopes.

Proposition 6.20 *Let $1 < r \leq n$ and $\tau_0 < \tau_1 < \ldots < \tau_{\binom{n}{r-1}}$ be a maximum chain in $S_{r-1}(n)$. For $t = 1, \ldots, \binom{n}{r-1}$ let A_t be the unique $(r-1)$-set with $\tau_{t-1}(A_t) = -$ and $\tau_t(A_t) = +$. There exists an r-signotope σ on $[n]$ so that $A_1, \ldots, A_{\binom{n}{r-1}}$ is a topological sorting of $\xrightarrow{\sigma}$.*

Proof. For a set $A \in \binom{[n]}{r-1}$ let $\rho(A)$ be the index of A in the list $A_1, \ldots, A_{\binom{n}{r-1}}$. Note that the monotonicity of all τ_t's implies that for all $D \in \binom{[n]}{r}$ either $\rho(D^{\lfloor 1 \rfloor}) < \rho(D^{\lfloor 2 \rfloor}) < \ldots < \rho(D^{\lfloor r \rfloor})$ or $\rho(D^{\lfloor 1 \rfloor}) > \rho(D^{\lfloor 2 \rfloor}) > \ldots > \rho(D^{\lfloor r \rfloor})$. In the first case let $\sigma(D) = +$ in the second case $\sigma(D) = -$. We have to show that σ is a r-signotope, i.e., that σ is monotone at $r+1$ sets. Let $Q \in \binom{[n]}{r+1}$ and consider indices $1 \leq i < j < k \leq r+1$. Suppose $\sigma(Q^{\lfloor i \rfloor}) = \sigma(Q^{\lfloor k \rfloor}) = +$. Let $Q^{\lfloor i,j \rfloor}$ denote the set Q minus the ith largest and the jth largest element of Q, e.g., $\{1,2,5,7,8\}^{\lfloor 2,3 \rfloor} = \{1,7,8\}$. From $\sigma(Q^{\lfloor i \rfloor}) = +$ we obtain $\rho(Q^{\lfloor i,j \rfloor}) < \rho(Q^{\lfloor i,k \rfloor})$. From $\sigma(Q^{\lfloor k \rfloor}) = +$ we obtain that $\rho(Q^{\lfloor i,k \rfloor}) < \rho(Q^{\lfloor j,k \rfloor})$. Hence $\rho(Q^{\lfloor i,j \rfloor}) < \rho(Q^{\lfloor j,k \rfloor})$ which implies $\sigma(Q^{\lfloor j \rfloor}) = +$ as required. The argument for $\sigma(Q^{\lfloor i \rfloor}) = \sigma(Q^{\lfloor k \rfloor}) = -$ is symmetric. Therefore, σ is an r-signotope. From the definition of σ based on $A_1, \ldots, A_{\binom{[n]}{r-1}}$ it follows that this sequence is a topological sorting of the relation $\xrightarrow{\sigma}$. □

Propositions 6.19 and 6.20 together imply the structure theorem for signotopes:

Theorem 6.21
There is a surjective mapping from maximum chains in $S_{r-1}(n)$ to $S_r(n)$.

It seems appropriate to point to two special cases of this theorem:

(1) The case $r = 2$: There is a surjective mapping from maximum chains in the Boolean lattice of subsets of $[n]$ to permutations of $[n]$. Actually, this mapping is a bijection.

(2) Maximum chains in $S_2(n)$ are the same as simple allowable sequences. The case $r = 3$ of the theorem is a reformulation of the statement: Every simple allowable sequence of permutations of $[n]$ encodes a simple marked arrangement of n pseudolines.

6.7 Notes and References

With this chapter I have tried to illustrate the beauty and usefulness of combinatorial encodings of geometric data. For the sake of a compactness and to avoid confusion the presentation was essentially restricted to arrangements of pseudolines in the Euclidean plane. Encodings of projective arrangements and configurations of points are closely related. The following is a sample of more or less general sources on the topic: The book of Björner et al. [28] covers oriented matroids in depth. A more concise source on oriented matroids is the handbook article of Richter-Gebert and Ziegler [163]. In his monograph [123] Knuth takes an axiomatic and self contained approach to combinatorial point configurations. Grünbaum's booklet [108], though not emphasizing encodings, has to be mentioned. Closest to the content of this chapter is Goodman's handbook article [101] about arrangements of pseudolines. Large parts of this chapter have been adapted from the author's articles [80, 88].

For the sweeping lemma (Lemma 6.1) there are at least two sources. Snoeyink and Hershberger [180] have a theorem implying the Sweeping Lemma for simple arrangements. In the book on oriented matroids [28] a result equivalent to the Sweeping Lemma is derived as a consequence of Levi's extension lemma. Here we have reverted the direction. Consequently Levi's extension lemma and the Sweeping Lemma are essentially equivalent. Levi [132] originally stated his lemma for projective arrangements, an English transcription can be found in Grünbaum [108]. In higher-dimensional arrangements of pseudoplanes sweeps and extensions are much more involved: It is known that sweeps can get stuck (Richter-Gebert [161]) and extensions need not exist (Goodman and Pollack [102]).

Sequences of permutations as encodings for point configurations were first used by Perrin in 1882. The modern definition of allowable sequences goes back to Goodman and Pollack [103], this paper also contains the connection with local sequences. The wiring diagram is defined in Goodman [100]. An overview on applications of allowable sequences is given by Goodman and Pollack [104] and more recently in Goodman's handbook article [101].

The slope theorem (Theorem 6.4) is due to Ungar [205] 1982. With this application of allowable sequences he confirmed a conjecture of Scott 1970. Scott had shown a lower bound of $\sqrt{2n}$ for the number of slopes determined by n points. A significant improvement was made by Burton and Purdy [42] in 1979, they investigated properties of the zonotopal tiling associated with the arrangement generated by n points and obtain a bound of $\frac{n-1}{2}$ slopes. A survey of the slope problem with emphasis to slope critical configurations was given by Jamison [117].

Simple allowable sequences also turn up in the theory of Coxeter groups. The symmetric group S_n of permutations of $[n]$ is generated by the adjacent transpositions $\sigma_i = (i, i+1)$, $i = 1, \dots, n-1$. These generators satisfy the Coxeter relations:

$$\begin{array}{rlr}
\sigma_i^2 = id & i = 1, \dots, n-1 & \text{(COX 0)} \\
\sigma_i\sigma_j = \sigma_j\sigma_i & |i-j| \geq 2 & \text{(COX 1)} \\
\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & i = 1, \dots, n-2 & \text{(COX 2)}
\end{array}$$

For $\pi \in S_n$ the least k such that π can be written as $\pi = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$ is called the length of π and the word $\sigma_{i_1}, \sigma_{i_2}, \dots, \sigma_{i_k}$ is called a reduced decomposition for π. Two reduced words (decompositions) represent the same permutation iff they are related by a sequence of moves of type COX 1 and COX 2. Note that reduced words of the reverse of the identity and simple allowable sequences are in one-to-one correspondence. Alternatively, these objects can also be seen as maximal chains in the weak Bruhat order of the symmetric group. In this last context their number A_n has been determined by Stanley [183]. His remarkable formula is

$$A_n = \frac{\binom{n}{2}!}{(2n-3) \cdot (2n-5)^2 \cdot (2n-7)^3 \cdot \ldots \cdot 5^{n-3} \cdot 3^{n-2}}.$$

This number A_n is the number of standard Young tableaux of staircase shape. Edelman and Greene [64] prove a combinatorial bijection between reduced decompositions and such tableaux. An alternative proof of that bijection was presented in [82], the argument there involves operations on wiring diagrams. This bijection was used by Edelman [63] to compute the average complexity of the kth level of an allowable sequence.

Exact numbers for different kinds of arrangements and a small number of lines are given by Knuth [123], Björner et al. [28] and Ziegler [218]. The most recent result is the enumeration of order types for $n = 11$ by Aichholzer et al.

Knuth [123] proves lower and upper bounds for the number of arrangements:

$$2^{\frac{n^2}{6} - \frac{5n}{2}} \leq B_n \leq 3^{\binom{n+1}{2}}.$$

This implies $\log_2(B_n) \leq 0.7924\,(n^2+n)$. Knuth also reports on some computations supporting a conjecture of $\log_2(B_n) \leq \binom{n}{2}$ and explains how to derive the slightly weaker bound $\log_2(B_n) \leq 0.7194\,n^2$ from the zone theorem. The bound in Theorem 6.9 is from Felsner [80]. The number of realizable arrangements, i.e., arrangements of lines, is much smaller, Goodman and Pollack [104] show an upper bound in $O(2^{n \log n})$.

Theorem 6.10 is equivalent to the rank 3 version of the Bohne-Dress Theorem which gives a bijection between zonotopal tilings of d-dimensional zonotopes and oriented matroids of rank $d+1$ with a realizable one-element contraction. The correspondence between oriented matroids and arrangements is given by the representation theorem for oriented matroids. This theorem states that oriented matroids of rank $d+1$ are in bijection with arrangements of pseudohyperplanes in d-dimensiononal projective space. An accessible treatment of these connections is given by Ziegler [219]. A more geometric proof of the Bohne-Dress Theorem was given by Richter-Gebert and Ziegler [164]. An elementary proof of the rank 3 version of the Bohne-Dress Theorem is contained in Felsner and Weil [88]. Another proof of that theorem is given by Elnitsky [71] in the context of reduced decompositions.

The bound for the complexity of the k-level of an arrangement given in Theorem 6.14 reproves a bound we have already seen in Chapter 4, e.g. Proposition 4.4. Somehow I feel like defending the decision of including this weak result. Its just, I like the proof. The bound for the complexity of a set K of levels is due to Welzl [211]. In Chapter 4 we have already remarked that this bound also has been improved. An exact bound for $<k$-sets was first obtained by Alon and Györi [11].

Triangle sign encodings of arrangements are related to chirotopes of uniform rank 3 oriented matroids (see [28]). A result similar to Theorem 6.17 was obtained by Streinu [189] in the context of generalized configurations of points.

Manin and Schechtman [136] introduced the *higher Bruhat order* $B(n, r-1)$ which is an order relation on the set of r-signotopes on $[n]$ (the name *signotope*, however, was introduced much later by Felsner and Weil [88]). The higher Bruhat order relation $\leq_{HB}$ is defined as follows: Let σ and τ be two r-signotopes with $\sigma(A) = \tau(A)$ for all r-subsets A with one exception A^* where $\sigma(A^*) = -$ and $\tau(A^*) = +$ in this case we call the pair (σ, τ) a *single-step*. The order relation $\leq_{HB}$ is the transitive closure of the single-step relation, i.e, $\sigma \leq_{HB} \tau$ iff there is a sequence $\sigma = \sigma_0, \sigma_1, \ldots \sigma_t = \tau$ such that for $i = 1, \ldots, t$ the pair (σ_{i-1}, σ_i) is a single-step. Higher Bruhat orders were further studied by Ziegler [218]. In particular, Ziegler showed that the higher Bruhat order $B(n, r-1)$ and the signotope order $S_r(n)$ are not equal in general. His example is $B(8,3) \neq S_4(8)$. For $r \leq 2$, obviously, $B(n, r-1) = S_r(n)$. Ziegler also shows that $B(n, n-k-1) = S_{n-k}(n)$ for $k \leq 3$. Felsner and Weil [87] proved $B(n,2) = S_3(n)$. Theorem 6.21 is the main result on higher Bruhat orders, it can be found in [136, 218, 88].

The combinatorial structure of signotopes and signotope orders has been studied by Ziegler [218] and Felsner and Weil [88]. In these papers it is also shown that r-signotopes encode geometric structures also for r larger than 3. Ziegler [218] gives a geometric interpretation of signotopes as single element extensions of cyclic hyperplane arrangements. In terms of the theory of oriented matroids the pseudohyperplane arrangements associated with signotopes are the adjoints of the duals of Ziegler's single element extensions, see [89]. In the interpretation of Felsner and Weil [88] signotopes correspond to arrangements of pseudohyperplanes in $\mathbb{R}^{r-1}$. However, for $r > 3$ the situation is not as nice as for $r = 3$. In higher dimensions only a restricted class of arrangements of pseudohyperplanes is actually encoded by signotopes. Theorem 6.21 implies these arrangements have the nice property of being sweepable.

7 Triangulations and Flips

Let $\mathcal{P}$ be a set of points in the plane and assume that $\mathcal{P}$ is in general position. A *triangulation* of $\mathcal{P}$ is a maximal non-crossing geometric graph with vertex set $\mathcal{P}$. All bounded faces of a triangulation are triangles (that's why we call it a triangulation), the unbounded face is the outside of the convex hull of $\mathcal{P}$. Triangulations play a prominent role in many applicable and applied disciplines like computational geometry, computer graphics and numerical modeling. In this chapter we discuss combinatorial and geometrical properties of triangulations and of the flip-graph on the set of all triangulations of a point set $\mathcal{P}$.

Let pq be an edge of a triangulation T of $\mathcal{P}$ and assume that pq belongs to two triangles pqr and pqs whose union is a convex quadrangle. The *diagonal flip* of edge pq consists in removing edge pq and replacing it by the other diagonal rs of the quadrangle.

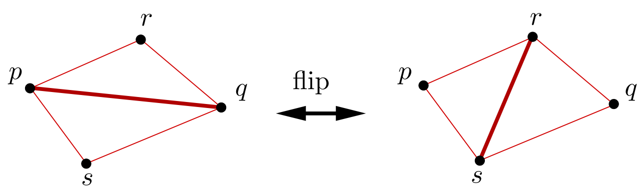


Figure 7.1 A diagonal flip replacing edge pq with rs.

The *flip-graph* $\mathcal{G}(\mathcal{P})$ is the graph whose vertices are the triangulations of $\mathcal{P}$ and two triangulations are adjacent if there is a diagonal flip transforming one into the other. In Section 7.1 we ask about minimal and maximal degree in the flip graph. In Section 7.2 the Delaunay triangulation is introduced. Since every triangulation of $\mathcal{P}$ can be transformed into the Delaunay triangulation of $\mathcal{P}$ by a sequence of Lawson flips (Proposition 7.2) the flip-graph is connected. The proof of Proposition 7.2 makes use of a lifting of $\mathcal{P}$ to a paraboloid in 3-space. In Section 7.3 we study regular triangulations of $\mathcal{P}$, i.e., triangulations that arise as projections of the convex hull of a lifting of $\mathcal{P}$. The secondary polytope of $\mathcal{P}$ is a high-dimensional polytope whose vertices are in bijection with regular triangulations of $\mathcal{P}$. In the special case of a point set $\mathcal{P}$ in convex position the number of triangulations is given by a Catalan number and the secondary polytope is the associahedron, this case is subject of Section 7.4. In the last section we deal with the diameter of the flip-graph of a point set in convex position. The lower bound construction of Sleator, Tarjan and Thurston is obtained by an argument involving volumes of ideal polytopes in hyperbolic space.

7.1 Degrees in the Flip-Graph

Among the simplest questions that can be asked about a graph are the questions about maximal and minimal degree. The degree of a triangulation T in $\mathcal{G}(\mathcal{P})$ strongly depends on the set $\mathcal{P}$ of underlying points. Let us ask for extremal degrees just in terms of the number n of points.

Let T be the triangulation of Figure 7.2. Each line of a pentagram consists of a slightly bent chain of three edges. Only the five hull edges of this triangulation are non-flippable. It can be shown that triangulations of $n \geq 5$ points with a convex hull of only 3 or 4 edges have at least 6 non-flippable edges. Hence, examples of nested pentagons as the one in Figure 7.2 are extremal.

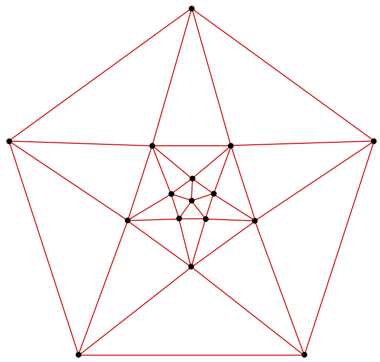


Figure 7.2 A triangulation with only 5 non-flippable edges.

What concerns the minimum degree in flip-graphs the precise answer is given with the following proposition.

Proposition 7.1 *Any triangulation T of a set $\mathcal{P}$ of n points in the plane contains at least $\frac{n}{2} - 2$ flippable edges. The bound is sharp.*

Proof. Let T be a triangulation of a set $\mathcal{P}$ of n points and assume that the convex hull of $\mathcal{P}$ contains γ points. From the counts of edges and faces of maximal planar graphs we deduce that T has $3n-3-\gamma$ edges and $2n-2-\gamma$ triangles, i.e., bounded triangular faces. Clearly, the γ edges of the convex hull are non-flippable. Every non-flippable interior edge e has one endpoint, say u, such that the sum of the two angles adjacent to e at u exceeds π. Define an orientation on the non-flippable interior edges such that e is oriented away from u, see Figure 7.3. We emphasize an important property of this orientation:

- Any two outward oriented edges at a common point u share an angle at u.

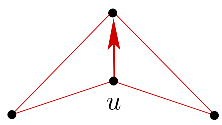


Figure 7.3 The orientation of a non-flippable interior edge.

It follows that a point u can have at most three outward oriented edges and this maximum is only attained by points of degree three in T. Classify the interior points of T by the number of outward oriented edges: Let η_i be the number of interior points with i outward oriented edges, $i = 3, 2, 1, 0$. Clearly

$$\eta_3 + \eta_2 + \eta_1 \leq n - \gamma.$$

A corner u of a triangle Δ in T is the *root* of Δ if the two edges of Δ incident to u are both outward oriented at u. The definition implies that every triangle of T has at most one root.

If u has out-degree two then there is exactly one triangle rooted at u. If u has out-degree three then there are exactly three triangles rooted at u. In all other cases u is not a root. Counting roots of triangles on one side and all triangles of T on the other we obtain

$$3\eta_3 + \eta_2 \leq 2n - 2 - \gamma.$$

Adding the two inequalities with weights, 3/2 for the first and 1/2 for the second, yields:

$$3\eta_3 + 2\eta_2 + \frac{3}{2}\eta_1 \leq \frac{5}{2}n - 2\gamma - 1.$$

Every interior non-flippable edge is oriented. By counting oriented edges at base points their number is seen to be $3\eta_3 + 2\eta_2 + \eta_1$. The above inequality implies that there are at most $\frac{5}{2}n - 2\gamma - 1$ interior non-flippable edges. In total, together with the hull edges, there are at most $\frac{5}{2}n - \gamma - 1$ non-flippable edges. Therefore, the number of flippable edges of T is at least

$$\left(3n - 3 - \gamma\right) - \left(\frac{5}{2}n - \gamma - 1\right) = \frac{1}{2}n - 2.$$

It remains to show that the bound can be attained. Let T be a triangulation of a set of m points in convex position, e.g., of the corners of a regular m-gon. T has $m - 3$ interior edges and $m - 2$ triangles. Subdivide each triangle with a new point connected to the three corners, this gives a triangulation T^* of a set of $n = 2m - 2$ points. The flippable edges are the $m - 3 = \frac{n}{2} - 2$ interior edges of T. Figure 7.4 shows a member from a different family of extremal examples. □

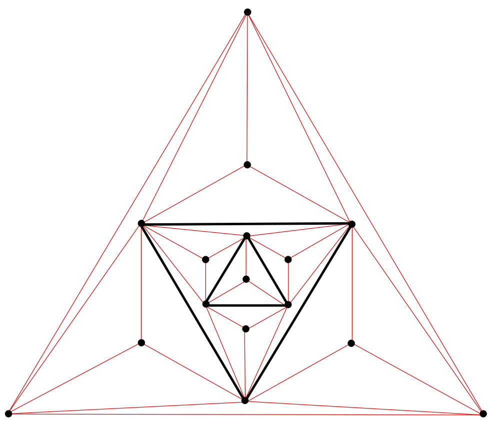


Figure 7.4 A triangulation with $\frac{n}{2} - 2$ flippable (thick) edges.

7.2 Delaunay Triangulations

Let $\mathcal{P}$ be a set of n points in $\mathbb{R}^2$. The *Voronoi region* $V(p)$ of a point $p \in \mathcal{P}$ is the set of all points x that are at least as close to p as to any other point in $\mathcal{P}$; formally

$$V(p) = \{x \in \mathbb{R}^2 : ||x - p|| \leq ||x - q|| \text{ for all } q \in \mathcal{P}\}.$$

Let $V_q(p)$ be the set of points that are at least as close to p as to q. This set $V_q(p)$ is a halfplane defined by the bisecting line of p and q. The Voronoi region of p with respect to $\mathcal{P}$ is the intersection of the halfplanes $V_q(p)$ for $q \in \mathcal{P} \setminus \{p\}$. Therefore, $V(p)$ is a convex polygonal region, possibly unbounded. Every point $x \in \mathbb{R}^2$ has a closest point in $\mathcal{P}$, so it belongs to a Voronoi region. The Voronoi regions of two points lie in opposite halfplanes defined by the bisecting line of the points. It follows that Voronoi regions are internally disjoint. The Voronoi regions together with edges and vertices of their boundaries form the *Voronoi diagram* of $\mathcal{P}$. Figure 7.5 shows an example.

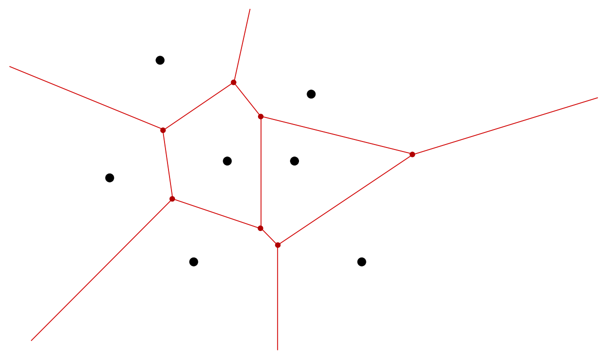


Figure 7.5 The Voronoi diagram of a set of seven points.

If the Voronoi regions of p and q share an edge we call the points $p, q \in \mathcal{P}$ *Delaunay neighbors*. Connecting all pairs of Delaunay neighbors by straight edges we obtain a graph G embedded with vertex set $\mathcal{P}$. This graph is the dual of the planar Voronoi diagram and hence also planar. Duality alone does not yet imply that the embedding of G is plane. However in the embedding of G the edges leaving a vertex p are in the same cyclic order as the dual edges are around $V(p)$. Since the Voronoi edges are non-crossing this indeed implies that G is plane. The faces of G correspond to the vertices of the Voronoi diagram which are of degree 3, usually. The graph G is the *Delaunay triangulation* of $\mathcal{P}$. Figure 7.6 shows an example.

A Voronoi vertex of degree more than 3 corresponds to a point $x \in \mathbb{R}^2$ with more than three nearest neighbors in $\mathcal{P}$. That is there is a circle C with center x such that C contains four or more points from $\mathcal{P}$. This kind of situation can be considered 'unlikely' and we assume that our point sets are free of such degeneracies. A point set is in *general position* if there is no degeneracy, including the degeneracy of 3 or more points on a line.

A triple $p, q, r \in \mathcal{P}$ is a Delaunay triangle if p, q, r are the vertices of a triangular face in the Delaunay triangulation. A circle C is an *empty circle* for $\mathcal{P}$ if there is no point of $\mathcal{P}$ in the interior of C. Dual to a Delaunay triangle p, q, r there is a Voronoi vertex $v = V(p) \cap V(q) \cap V(r)$. Consider the largest empty circle with center v, this circle has p, q and r on the boundary, hence, it is the circumcircle for the triangle p, q, r. Conversely,

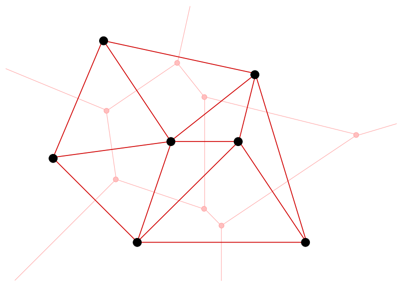


Figure 7.6 The Delaunay triangulation of the point set of Figure 7.5.

let p, q, r be three points of $\mathcal{P}$ with an empty circumcircle. The center v of this circle is in $V(p) \cap V(q) \cap V(r)$, moreover, as a member of three different Voronoi regions v is a Voronoi vertex.

Characterization of Delaunay triangles. Let $\mathcal{P} \subset \mathrm{IR}^2$ be a set of n points in general position. A triple $p, q, r \in \mathcal{P}$ is a Delaunay triangle iff the circumcircle of p, q, r is empty.

A Delaunay edge p, q is dual to a Voronoi edge $V(p) \cap V(q)$. For any point x on that Voronoi edge there is an empty circle with center x and p, q on the boundary. Conversely, let p, q be points of $\mathcal{P}$ such that there is an empty circle C touching both. The center of C is on the bisecting line of p and q. Consider the set X of all points x on this bisector such that there is an empty circle with center x and p, q on the boundary. Not all points of $\mathcal{P}$ are on a line, therefore, X is a proper subset of the bisector and there is an extreme point $x_0 \in X$. The empty circle with center x_0 has a third point $r \in \mathcal{P}$ on the boundary. The circumcircle property implies that p, q, r is a Delaunay triangle, hence, p, q is a Delaunay edge.

Characterization of Delaunay edges. Let $\mathcal{P} \subset \mathrm{IR}^2$ be a set of n points in general position. A pair $p, q \in \mathcal{P}$ is a Delaunay edge iff there is an empty circle with p and q on the boundary.

Let T be a triangulation of $\mathcal{P}$ containing triangles p, q, r and p, q, s such that s is in the interior of the circumcircle of p, q, r. In that case p, q, r, s form a convex quadrangle and the edge p, q is flippable. Note that p, q is not a Delaunay edge. We call p, q a *weak edge* of the triangulation T. The flip of a weak edge in a triangulation is a *Lawson flip*. Figure 7.7 shows an example and illustrates the fact that the edge replacing the weak edge is not weak.

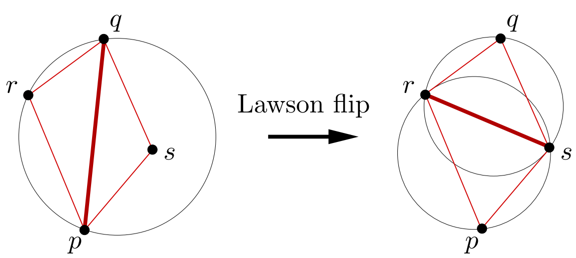


Figure 7.7 A weak edge (p, q) and the Lawson flip replacing (p, q) by (r, s) which is not weak.

Proposition 7.2 *Let T be an arbitrary triangulation of a set $\mathcal{P}$ of n points in general position. Every algorithm that starts with T and repeatedly performs Lawson flips will reach the Delaunay triangulation of $\mathcal{P}$ with at most $\binom{n}{2}$ flips.*

Before proving the proposition we emphasize two implications.

- The Delaunay triangulation is the unique triangulation of $\mathcal{P}$ which has no weak edge.
- The flip-graph $\mathcal{G}(\mathcal{P})$ is connected with diameter at most $n^2 - n$, i.e., given two triangulations T_1, T_2 of $\mathcal{P}$ it takes at most $n^2 - n$ flips to transform T_1 into T_2.

Proof of Proposition 7.2. This works by lifting triangulations into 3-space. A point $p \in \mathcal{P}$ is lifted to $\hat{p} = (p_1, p_2, p_1^2 + p_2^2)$, that is p is lifted to the point $\hat{p}$ on the paraboloid $z = x^2 + y^2$ vertically above p. A triangulation T of $\mathcal{P}$ is lifted to $\hat{T}$ by lifting each triangle p, q, r of T to the spatial triangle with corners $\hat{p}, \hat{q}, \hat{r}$. The crucial property of the lifting is stated in the following lemma whose proof will be given later.

Lemma 7.3 *A point s is in the interior of the circumcircle of p, q, r if and only if $\hat{s}$ is below the plane spanned by $\hat{p}, \hat{q}, \hat{r}$.*

Consider a Lawson flip $T \to T_f$ which replaces p, q by r, s. The lifted triangulations $\hat{T}$ and $\hat{T_f}$ enclose the tetrahedron $\hat{p}, \hat{q}, \hat{r}, \hat{s}$. It follows from the lemma that the lifted triangulation $\hat{T}$ contains the two upper triangles $\hat{p}, \hat{q}, \hat{r}$ and $\hat{p}, \hat{q}, \hat{s}$ of the tetrahedron and $\hat{T_f}$ contains the two lower triangles $\hat{p}, \hat{r}, \hat{s}$ and $\hat{q}, \hat{r}, \hat{s}$. In other words, surface $\hat{T_f}$ is below surface $\hat{T}$ and the edge $\hat{p}, \hat{q}$ of $\hat{T}$ is strictly above $\hat{T_f}$. A sequence of Lawson flips produces a lowering sequence of surfaces and an edge that has once been flipped away can never be reinserted. This implies that there are at most as many Lawson flips as there are edges on a set of n points, namely $\binom{n}{2}$.

It remains to prove that the Delaunay triangulation is the unique Lawson flip free triangulation. We use the characterization of Delaunay triangles. If T is not Delaunay, then T contains a triangle p, q, r which has a circumcircle C with a point $s \in \mathcal{P}$ interior to the cycle. We assume that p, q and r, s are the diagonals of the quadrangle. By Lemma 7.3 the lifted segment $\hat{r}, \hat{s}$ is below $\hat{p}, \hat{q}$. This reveals that $\hat{T}$ is not convex and, hence, contains some non-convex edge $\hat{a}, \hat{b}$. Attached to a, b there are triangles a, b, c and a, b, d. Since $\hat{a}, \hat{b}$ is non-convex the point $\hat{d}$ is below the plane spanned by $\hat{a}, \hat{b}, \hat{c}$. The lemma implies that a, b is a weak edge and allows a Lawson flip. $\square$

Proof of Lemma 7.3. For $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ let $a^+ = (a_1, a_2, a_3, 1)$ be the corresponding homogeneous point in $\mathbb{R}^4$. Four points a, b, c, d in $\mathbb{R}^3$ are coplanar iff the determinant $|a^+, b^+, c^+, d^+|$ vanishes. If $|a^+, b^+, c^+, d^+| < 0$ then looking from d the triangle (a, b, c) appears as a counterclockwise triangle. Let (p, q, r) be a counterclockwise triangle in $\mathbb{R}^2$, with this triangle associate the following mapping $\phi : \mathbb{R}^2 \to \mathbb{R}$

$$\phi(s) = \begin{vmatrix} p_1 & p_2 & p_1^2 + p_2^2 & 1 \\ q_1 & q_2 & q_1^2 + q_2^2 & 1 \\ r_1 & r_2 & r_1^2 + r_2^2 & 1 \\ s_1 & s_2 & s_1^2 + s_2^2 & 1 \end{vmatrix}.$$

This determinant vanishes if the lifted point $\hat{s}$ is in the plane determined by $\hat{p}, \hat{q}, \hat{r}$, otherwise, the sign tells whether $\hat{s}$ is above or below the plane. The lemma is equivalent to the statement $\phi(s) = 0$ iff s is on the circumcircle C of p, q, r and $\phi(s) > 0$ iff s is in the interior of C. Let $m = (m_1, m_2)$ be the center of C and let ϕ_m be the mapping corresponding to the triangle $(p - m, q - m, r - m)$. It follows from basic properties of determinants (linearity and the fact that a determinant with two columns which are multiples of each other vanishes) that $\phi(s) = \phi_m(s - m)$. However, if ρ is the radius of C then

$$\phi_m(s-m) = \begin{vmatrix} p_1 - m_1 & p_2 - m_2 & \rho^2 & 1 \\ q_1 - m_1 & q_2 - m_2 & \rho^2 & 1 \\ r_1 - m_1 & r_2 - m_2 & \rho^2 & 1 \\ s_1 - m_1 & s_2 - m_2 & (s_1 - m_1)^2 + (s_2 - m_2)^2 & 1 \end{vmatrix}.$$

If $s \in C$ then $(s_1 - m_1)^2 + (s_2 - m_2)^2 = \rho^2$ and the last two columns are linearly dependent, i.e., $\phi_m(s - m) = 0$. Since ϕ is a quadric, C is the complete set of zeros of ϕ and the sign of ϕ is the same for all interior points. The value of $\phi(m) = \phi_m(0)$ is ρ^2 times the determinant of the homogenized points p^+, q^+, r^+. Since p, q, r is a counterclockwise triangle, this determinant is positive, hence $\phi(m) = \phi_m(0) > 0$. $\triangle$

The proof of Proposition 7.2 yields a beautiful characterization of the Delaunay triangulation:

Corollary 7.4 *The Delaunay triangulation of a set $\mathcal{P}$ of n points in general position is the vertical projection of the lower convex hull of the point set lifted to the paraboloid, i.e., of $\hat{\mathcal{P}} = \{\, \hat{p} = (p_1, p_2, p_1^2 + p_2^2) : p = (p_1, p_2) \in \mathcal{P}\}$.*

7.3 Regular Triangulations and Secondary Polytopes

In the previous section we have investigated the lifting of point sets to the paraboloid $z = x^2 + y^2$. We now consider more general liftings and use them for the construction of an interesting polytope associated to a set of points.

For a set $\mathcal{P} = \{p_1, \ldots, p_n\}$ of n points in the plane and any numbers $w_i \in \mathbb{R}$, $i = 1, \ldots, n$ consider the lifting that takes p_i to the point $\hat{p_i}$ vertically above p_i at height w_i. Let $\hat{\mathcal{P}_w}$ be the set of lifted points. Suppose that $\hat{\mathcal{P}_w}$ is in general position meaning that no four points are coplanar. The convex hull of $\hat{\mathcal{P}_w}$ is a simplicial polytope, i.e., all facets of this polytope are triangles. This polytope is invariant under the addition of a constant to all the weights w_i, therefore, we can assume that all the w_i are positive. The vertical projection of the lower convex hull, i.e., of the faces that are visible from the plane, is a triangulation with vertices in $\mathcal{P}$. The triangulations that can be obtained with this construction are the *regular triangulations* of $\mathcal{P}$.

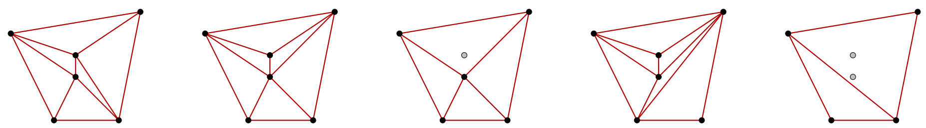


Figure 7.8 Some regular triangulations of a set $\mathcal{P}$ of six points.

- If $p \in \mathcal{P}$ is not a vertex of the convex hull of $\mathcal{P}$, then $\hat{p}$ may be beyond some triangle spanned by points of $\hat{\mathcal{P}_w}$. Consequently, the vertex set $\mathcal{V}$ of a regular triangulation of $\mathcal{P}$ is a subset of $\mathcal{P}$. In fact, every subset $\mathcal{V} \subseteq \mathcal{P}$ that includes all vertices of the convex hull of $\mathcal{P}$ is the vertex set of some regular triangulation.

- The triangulation shown in Figure 7.9 is not regular. Suppose there is a lifting whose lower projection yields the figure. By subtracting a linear function from the weights we can achieve that the inner triangle is in the plane, i.e., $w_4 = w_5 = w_6 = 0$. Clearly $w_1, w_2, w_3 > 0$, the edge needed in the quadrangle p_1, p_2, p_4, p_5 requires $w_2 > w_1$. The two other quadrangles force $w_3 > w_2$ and $w_1 > w_3$. Together these requirements are contradictory.

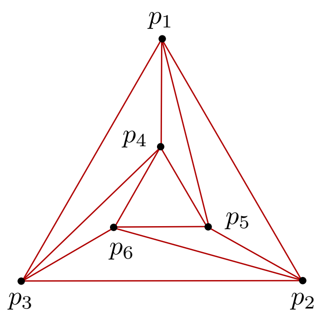


Figure 7.9 A non-regular triangulation.

Let T be a triangulation of $\mathcal{P}$, for a triangle $\Delta \in T$ denote the area with $\text{vol}(\Delta)$. For a point $p \in \mathcal{P}$ let

$$\varphi(p) = \sum_{p \in \Delta \in T} \text{vol}(\Delta).$$

be the sum of the areas of triangles having p as a vertex. The *volume vector* of T is the vector

$$\varphi(T) = \big(\varphi(p_1), \varphi(p_2), \ldots, \varphi(p_n)\big) \in \mathbb{R}^n.$$

The *secondary polytope* $\Sigma(\mathcal{P})$ of $\mathcal{P} = \{p_1, \ldots, p_n\}$ is the convex span of the volume vectors of all triangulations with a vertex set $\mathcal{V} \subseteq \mathcal{P}$ such that all vertices of the convex hull are in $\mathcal{V}$.

Theorem 7.5 *Let $\mathcal{P} = \{p_1, \ldots, p_n\}$ be a set of points in general position and $\Sigma(\mathcal{P})$ be the secondary polytope.*

(1) The dimension of $\Sigma(\mathcal{P})$ is $n - 3$.

(2) The vertices of $\Sigma(\mathcal{P})$ are the volume vectors of regular triangulations of $\mathcal{P}$.

(3) Faces of $\Sigma(\mathcal{P})$ correspond to regular subdivisions of $\mathcal{P}$, in particular the edges of $\Sigma(\mathcal{P})$ correspond to flips of the two types shown in Figure 7.10.

Proof. (1) A lower bound on the dimension of $\Sigma(\mathcal{P})$ can be obtained with induction. If $|\mathcal{P}| = 3$ there is a unique triangulation and $\Sigma(\mathcal{P})$ is a point. If $|\mathcal{P}| = 4$ we may have three or four points on the convex hull. In either case there are two triangulations, Figure 7.10 shows the possible configurations. Hence, if $|\mathcal{P}| = 4$, then the polytope $\Sigma(\mathcal{P})$ is 1-dimensional. Consider $\mathcal{P} = \{p_1, \ldots, p_n\}$ with $n \geq 4$. Suppose p_n is not a vertex of the convex hull $\text{CH}(\mathcal{P})$ of $\mathcal{P}$. Every triangulation of $\mathcal{P} \setminus p_n$ is a triangulation of $\mathcal{P}$ and every triangulation with $\varphi(p_n) = 0$ is a triangulation of $\mathcal{P} \setminus p_n$. Therefore

$\Sigma(\mathcal{P} \setminus p_n) = \Sigma(P) \cap H(x_n = 0)$ where $H(x_n = 0)$ is the hyperplane with last coordinate zero. Since $\Sigma(\mathcal{P})$ is not contained in $H(x_n = 0)$ the increase of dimension from $\Sigma(\mathcal{P} \setminus p_n)$ to $\Sigma(\mathcal{P})$ is at least one. A similar argument applies when p_n is a vertex of the convex hull $\mathrm{CH}(\mathcal{P})$. In this case $\Sigma(\mathcal{P} \setminus p_n) = \Sigma(P) \cap H(x_n = \delta)$ where $\delta = \mathrm{vol}(\mathrm{CH}(\mathcal{P})) - \mathrm{vol}(\mathrm{CH}\,\mathcal{P} \setminus p_n)$.

To show that the dimension of $\Sigma(\mathcal{P})$ is at most $n-3$ we exhibit three independent linear identities satisfied by all volume vectors of triangulations of $\mathcal{P}$. A triangulation T contributing to $\Sigma(\mathcal{P})$ is a triangulation of the interior of the convex hull $\mathrm{CH}(\mathcal{P})$. Therefore, $\mathrm{vol}(\mathrm{CH}(\mathcal{P})) = \sum_{\Delta \in T} \mathrm{vol}(\Delta) = \frac{1}{3} \sum_p \sum_{p \in \Delta \in T} \mathrm{vol}(\Delta)$. Written in terms of the volume vector:

$$3\,\mathrm{vol}(\mathrm{CH}(\mathcal{P})) = \sum_{i=1}^{n} \varphi(p_i).$$

Let b be the barycenter of $\mathrm{CH}(\mathcal{P})$. Think of b as the center of gravity of $\mathrm{CH}(\mathcal{P})$ with the uniform mass distribution. A triangulation T can be used to compute b by concentrating the mass of every triangle Δ of T in the barycenter b_Δ of the triangle. This yields $b = \mathrm{vol}(\mathrm{CH}(\mathcal{P}))^{-1} \sum_{\Delta \in T} b_\Delta \mathrm{vol}(\Delta)$. The barycenter of a triangle $\Delta = \Delta(p_i, p_j, p_k)$ is the point $b_\Delta = \frac{1}{3}(p_i + p_j + p_k)$. Therefore,

$$3\,b\,\mathrm{vol}(\mathrm{CH}(\mathcal{P})) = \sum_{\Delta \in T} (p_i + p_j + p_k)\mathrm{vol}(\Delta) = \sum_{i=1}^{n} p_i \sum_{p_i \in \Delta \in T} \mathrm{vol}(\Delta) = \sum_{i=1}^{n} p_i\, \varphi(p_i).$$

Each of the two coordinates of this vector equation gives an affine subspace of $\mathbb{R}^n$ that contains $\Sigma(\mathcal{P})$. Together we have found three linear identities satisfied by volume vectors of triangulations. If $\mathcal{P}$ is non-degenerate, these identities are independent. Together with the lower bound this shows that the dimension of $\Sigma(\mathcal{P})$ is exactly $n-3$.

(2) Consider a regular triangulation T, we want to prove that $\varphi(T)$ is a vertex of $\Sigma(\mathcal{P})$. This can be done by showing that there is a linear function which attains its unique minimum value over $\Sigma(\mathcal{P})$ at $\varphi(T)$. Since T is regular, there is a vector $w \in \mathbb{R}^n$ such that T is the projection of the lower hull of the lifted point set $\hat{\mathcal{P}_w}$ in $\mathbb{R}^3$. The claim is that this lifting vector w defines the objective function we look for:

$$\langle w, \varphi(T) \rangle < \langle w, \varphi(T') \rangle \text{ for all triangulations } T' \neq T. \tag{7.1}$$

For the proof of the claim we consider 3-dimensional volumes. Enclosed by a triangle $\Delta = \Delta(p_i, p_j, p_k)$ and the lifted triangle $\Delta(\hat{p_i}, \hat{p_j}, \hat{p_k})$ there is a triangular prism with edge length w_i, w_j and w_k. The volume of this prism can be written as $\frac{w_i + w_j + w_k}{3} \mathrm{vol}(\Delta)$. Let T' be a triangulation of $\mathcal{P}$ and $\hat{T'}$ be the lifted triangulated surface. The volume between this surface $\hat{T'}$ and the plane $z = 0$ is the sum of the volumes of triangular prisms and can be written as:

$$\sum_{\substack{i,j,k \\ \Delta(p_i,p_j,p_k) \in T'}} \frac{w_i + w_j + w_k}{3} \mathrm{vol}(\Delta) = \sum_{i=1}^{n} \frac{w_i}{3} \sum_{p_i \in \Delta \in T'} \mathrm{vol}(\Delta) = \sum_{i=1}^{n} \frac{w_i}{3}\, \varphi(p_i) = \frac{1}{3} \langle w, \varphi(T') \rangle.$$

Fix a lifting vector w. The volume below a surface based on the lifted point set $\hat{\mathcal{P}_w}$ is always at least as large as the volume below the lower convex hull of $\hat{\mathcal{P}_w}$. As shown, the volume below a lifted triangulation $\hat{T'}$ is $\frac{1}{3}\langle w, \varphi(T') \rangle$. This implies equation 7.1, hence, all regular triangulations of $\mathcal{P}$ are vertices of $\Sigma(\mathcal{P})$.

Given a vertex v of $\Sigma(\mathcal{P})$ there is some $w \in \mathbb{R}^n$ such that v is the unique minimum of $x \to \langle w, x\rangle$ over $\Sigma(\mathcal{P})$. Lift $\mathcal{P}$ with this w to $\hat{\mathcal{P}_w}$ and let T be the projection of the lower hull of $\hat{\mathcal{P}_w}$. By the above we know that $\langle w, \varphi(T)\rangle \leq \langle w, x\rangle$ for all $x \in \Sigma(\mathcal{P})$. Hence $v = \varphi(T)$. This proves that the vertices of $\Sigma(\mathcal{P})$ are in bijection with the regular triangulations of $\mathcal{P}$.

Figure 7.10 The two types of 'tetrahedral' flips corresponding to the edges of the secondary polytope $\Sigma(\mathcal{P})$.

(3) Let F be a face of $\Sigma(\mathcal{P})$ and $T_1, \dots, T_k$ be the triangulations corresponding to the vertices $\varphi(T_i)$ of F. Let w be the normal of a supporting hyperplane of F. All the lifted triangulations $\hat{T_1}, \dots, \hat{T_k}$ minimize the volume below the surface. Therefore they all coincide with the lower convex hull of $\hat{\mathcal{P}_w}$. Let S be the vertical projection of this hull, S is a subdivision of $\mathcal{P}$ containing all edges of $\textsc{ch}(\mathcal{P})$. The vertices of S are in $\mathcal{P}$ and the faces of S are convex. Every edge of S is an edge of each of the T_i and each (completed) triangulation of S is one of the T_i. This mapping from faces of $\Sigma(\mathcal{P})$ to regular subdivisions of $\mathcal{P}$ is bijective. $\square$

7.4 The Associahedron and Catalan families

A particularly nice family of secondary polytopes are the associahedra. The *associahedron* $\mathcal{A}_n$ is the secondary polytope of a set $\mathcal{P}$ of n points in convex position.

The coordinates of the secondary polytope depend on the coordinates of points of $\mathcal{P}$, however, the combinatorial structure of the polytope remains unaffected by the choice of the set $\mathcal{P}$ of n points in convex position. So the associahedron $\mathcal{A}_n$ is actually an equivalence class of polytopes. The classical realization of $\mathcal{A}_n$ is the realization as secondary polytope of the vertices of a regular n-gon $\mathcal{C}_n$.

A triangulation T of a convex n-gon has $2n-3$ edges. The convex cycle has n edges and $n-3$ edges are interior, we call them *diagonals*. Let $\mathcal{G}_n$ be the flip-graph of $\mathcal{C}_n$, Figure 7.11 displays the graph $\mathcal{G}_6$. All triangulations of $\mathcal{C}_n$ are regular, therefore, $\mathcal{G}_n$ is the graph of the associahedron. The number of triangulations of $\mathcal{C}_n$ is the *Catalan number* C_{n-2}:

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Catalan numbers form a fascinating sequence arising in many counting problems. Stanley [186] (exercise 6.19), gives a list of 66 Catalan families, i.e., combinatorial interpretations of Catalan numbers. Among the most prominent Catalan families we find:

[CF_1] Triangulations of a labeled $(n+2)$-gon.

[CF_2] Binary trees with $n+1$ leaf vertices.

[CF_3] Rooted plane trees with $n+1$ vertices.

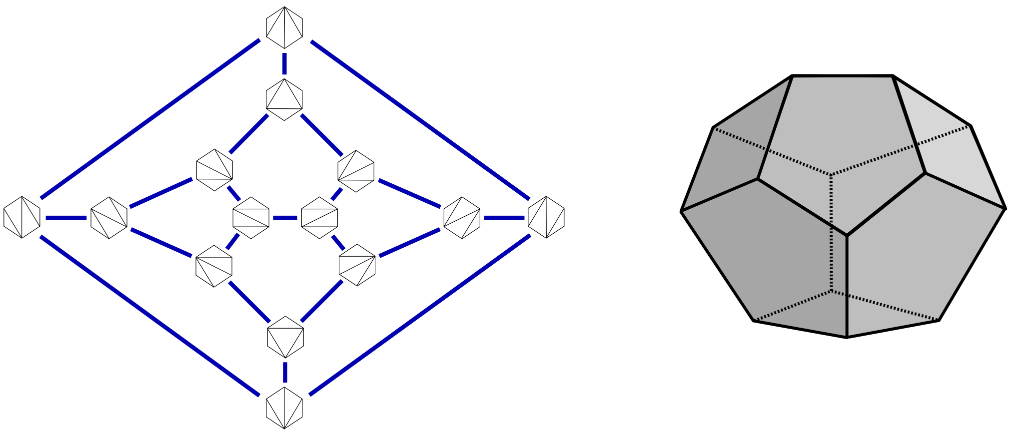


Figure 7.11 The flip-graph $\mathcal{G}_6$ of a hexagon and the associahedron $\mathcal{A}_6$.

[CF_4] Paths in the plane from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$ that never go below the x-axis (*Dyck path*).

[CF_5] Ways to parenthesize a non-associative product $x_0 \cdot x_1 \cdot \ldots \cdot x_n$ with n pairs of parentheses, e.g., $(((x_0 \cdot x_1) \cdot x_2) \cdot ((x_3 \cdot (x_4 \cdot x_5)) \cdot x_6))$.

Figure 7.12 indicates bijections between the first four of these Catalan families. The plane binary tree of the figure is the evaluation tree for the product illustrating family CF_5, this hints a bijection between CF_2 and CF_5.

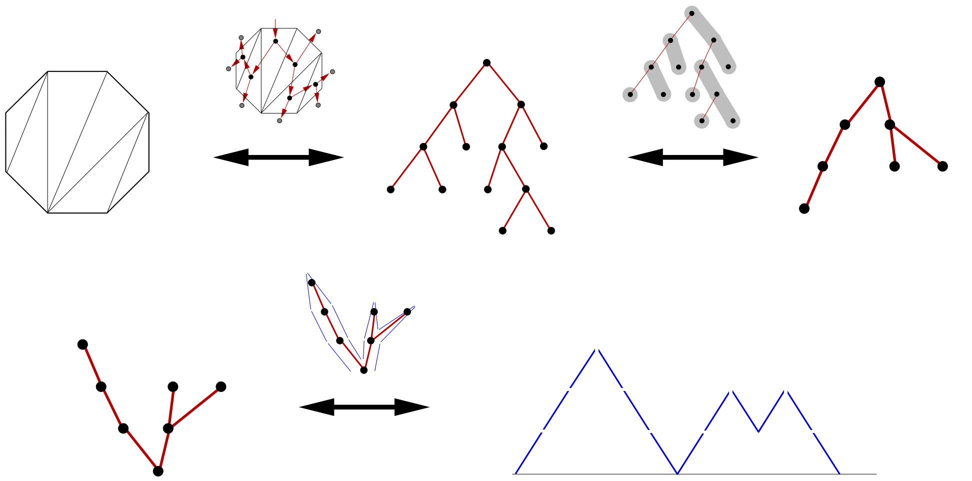


Figure 7.12 Bijections between four Catalan families.

The faces of dimension k of the associahedron $\mathcal{A}_n$ correspond to sets of $n-3-k$ non-crossing diagonals in $\mathcal{C}_n$, this is a consequence of part 3 of Theorem 7.5. The number C_n^k of ways to draw k non-crossing diagonals in a convex n-gon is known to be:

$$C_n^k = \frac{1}{n+k}\binom{n+k}{k+1}\binom{n-3}{k}.$$

7.5 The Diameter of $\mathcal{G}_n$ and Hyperbolic Geometry

We have used Lawson flips and the Delaunay triangulation to show that for every set $\mathcal{P}$ of n points the diameter of $\mathcal{G}(\mathcal{P})$ is at most $n^2 - n$. In the particular case of points in convex position there is a substantially smaller bound:

Proposition 7.6 *The diameter of the flip-graph $\mathcal{G}_n$ of n points in convex position is at most $2n - 10 + \lfloor \frac{12}{n} \rfloor$, hence, at most $2n - 10$ for $n \geq 13$.*

Proof. Let T_1, T_2 be triangulations of $\mathcal{C}_n$. The degree $d_i(x)$ of a point x in T_i is the number of diagonals incident to x. If $d_i(x) < n - 3$ then the degree of x can be increased by an appropriate flip $T_i \to T_i'$. Therefore, T_i can be transformed into the star triangulation S_x which has all its $n - 3$ diagonals incident to x. The number of flips required to get from T_i to S_x is $n - 3 - d_i(x)$. The number of flips required to get from T_1 to T_2 via S_x is at most $2n - 6 - d_1(x) - d_2(x)$. Consequently, T_1 can be transformed into T_2 with no more than $\min_x(2n - 6 - d_1(x) - d_2(x)) = 2n - 6 - \max_x(d_1(x) + d_2(x))$ flips.

A bound on $\max_x(d_1(x)+d_2(x))$ is obtained from the average of $d_1(x)+d_2(x)$ which is $\frac{1}{n}\sum_x(d_1(x) + d_2(x)) = \frac{1}{n}(4n - 12)$. Together this gives the upper bound $2n - 10 + \lfloor \frac{12}{n} \rfloor$, as claimed. □

The bound on $\mathrm{diam}(\mathcal{G}_n)$ given in the proposition is tight for small $n \leq 18$. For $n \leq 8$ this is doable by hand, for the larger values a computer search is reported. Surprisingly, the bound $2n-10$ is also known to be tight for large n. The lower bound was obtained by Sleator, Tarjan and Thurston [179] in 1988. Their exciting analysis is based on volume estimates for hyperbolic polytopes. The full argument is too complex for our context. We constrain the exposition to an outline of the beautiful proof.

(1) If T_1, T_2 is a pair of triangulations which maximizes the flip-distance $\mathrm{dist}(T_1, T_2)$, then the triangulations have no diagonal in common. Henceforth, we assume that T_1, T_2 is such a pair.

(2) The union of T_1 and T_2 is a maximal planar graph $G = G(T_1, T_2)$ and, hence, 3-connected. The Theorem of Steinitz implies that G is the skeleton graph of a convex polytope in $\mathbb{R}^3$. Let P_G be such a polytope with skeleton G.

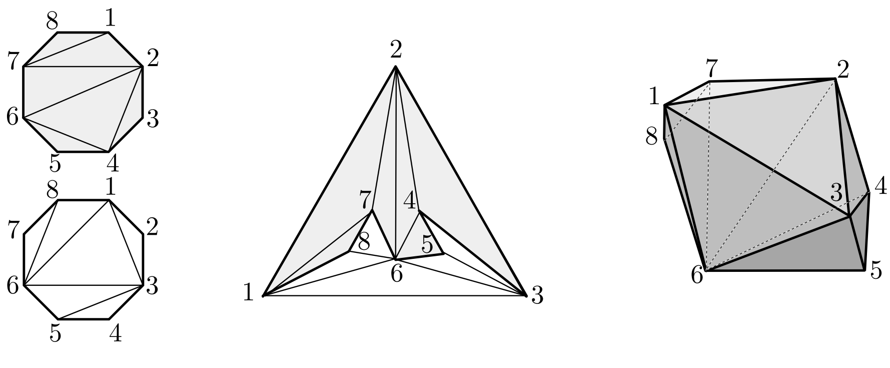


Figure 7.13 From a pair of triangulations to a polytope.

(3) Let $T_1 \to T_1'$ be a flip replacing edge (p, q) by edge (r, s). Let $\hat{p}, \hat{q}, \hat{r}, \hat{s}$ be the corresponding vertices of the polytope P_G. Cutting off the tetrahedron τ spanned by $\hat{p}, \hat{q}, \hat{r}, \hat{s}$

from the polytope P_G leaves a polytope P' such that the skeleton graph of P' is the union of T_1' and T_2.

The vision is to iterate this process: Use each flip of a flip-sequence from T_1 to T_2 to cut off a tetrahedron from the polytope. This should associate a tetrahedral decomposition of P_G with every flip sequence transforming T_1 to T_2.

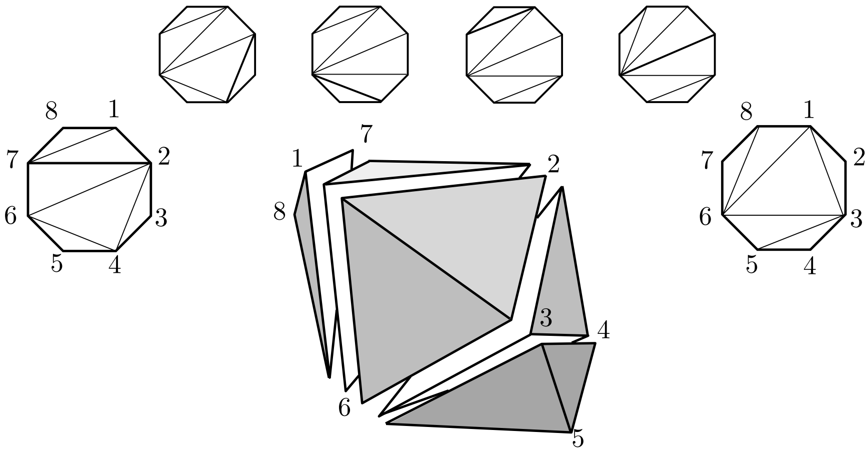


Figure 7.14 A tetrahedral decomposition induced by a flip sequence.

The truth is more delicate: Cutting off a tetrahedron can make the polytope non-convex and later flips may flip away non-convex edges of the polytope. In that case the tetrahedron corresponding to the flip can be glued onto the polytope so that the polytope retains a skeleton graph as required. Actually, this is again a simplification. Gluing a tetrahedron on a non-convex polytope may cause self-intersections. These problems can be bypassed with the use of an appropriate notion of pseudo-polytope.

(4) A flip sequence from T_1 to T_2 corresponds to a sequence of tetrahedral operations (cut off or glue on) transforming the initial polytope P_G with $G = G(T_1, T_2)$ into the polytope with skeleton $G(T_2, T_2)$. The later is a degenerate polytope, actually, it is only a union of triangles, it has no interior points and hence no volume. Let the flip sequence consist of t flips and let $\tau_1 \dots \tau_t$ be the tetrahedra corresponding to the flips, then $P_G = \bigcup_i \tau_i$. Let $\mathrm{vol}(P)$ denote the volume of a 3-dimensional polytope P. Our considerations imply the inequality

$$\mathrm{vol}(P_G) \leq \sum_{i=1}^{t} \mathrm{vol}(\tau_i).$$

If V_Δ is the maximum volume of a tetrahedron that can be inscribed in the polytope P_G, then every covering of P_G with tetrahedra will require at least $\mathrm{vol}(P_G)/V_\Delta$ many tetrahedra. Since a flip sequence from T_1 to T_2 induces a tetrahedral cover we conclude

$$\mathrm{dist}(T_1, T_2) \geq \mathrm{vol}(P_G)/V_\Delta.$$

(5) At first, this bound for the flip distance of two triangulations seems to be extremely poor. Every polytope P in $\mathbb{R}^3$ has an inscribed tetrahedron of volume $c \cdot \mathrm{vol}(P)$ for a small constant c.

Volumes of polytopes in hyperbolic space behave completely different. In hyperbolic space there is a constant V_0 which is an upper bound for the volume of all tetrahedra. We digress for the introduction of some elements of hyperbolic geometry.

Hyperbolic geometry. In hyperbolic geometry there are many parallels to a line through every given point not on the line. There are several models of hyperbolic space within Euclidean space. There are three important models for the hyperbolic plane.

($\mathbb{K}$) The *Klein model*. The points are the points of an open unit disk D. The lines are chords, i.e., straight in the Euclidean sense. This model has the advantage that distances are easy to compute. Given points a and b consider the line spanned by a and b and let α and β be the limit points on the boundary circle ∂D, see Figure 7.15. If $\overline{pq}$ is the Euclidean distance of p and q, then the hyperbolic distance of a and b can be expressed as:

$$\mathrm{dist}_{\mathbb{K}}(a,b) = \frac{1}{2}\left|\log\Big(\frac{\overline{a\alpha}\cdot\overline{b\beta}}{\overline{b\alpha}\cdot\overline{a\beta}}\Big)\right|.$$

($\mathbb{P}$) The *Poincaré model*. The points are the points of a disk D. The lines are the diameter of the disc and open arcs of circles orthogonal to ∂D. This model is conformal, i.e., the angle of two intersecting lines is the Euclidean angle of the tangents at the point of intersection.

($\mathbb{H}$) The *half-space model*. The points are all points above the x-axis in the Euclidean plane. Lines are vertical rays emanating upward from a point on the x-axis and halfcircles with center on the x-axis. This model is again conformal.

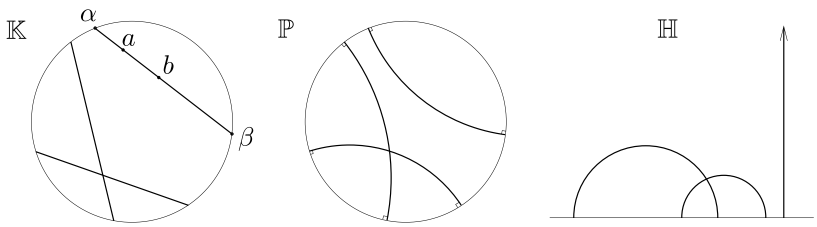


Figure 7.15 Models for the hyperbolic plane.

In Section 5.1 we have used the sphere S^2 as a hub to connect between different geometries. This can be done again. Figure 7.16 should transport the idea of how to construct points $p_{\mathbb{K}}$, $p_{\mathbb{P}}$, $p_{\mathbb{H}}$ in the three models if any of them is given.

Area and volume. Triangles in hyperbolic space have angle sum less than π. In fact, the Gauß-Bonnet theorem states that the area of a triangle with angles α, β and γ is exactly $\pi - \alpha - \beta - \gamma$.

Think of hyperbolic space as enhanced by the sphere at infinity. In the half-space model of the plane this adds the x-axis and a point v_∞ which is an endpoint of all vertical lines. In the 3-dimensional half-space model the sphere at infinity consists of the plane $z = 0$ and a point v_∞.

An *ideal triangle* is one with three corners in the sphere at infinity. Ideal triangles have area π, they are triangles of maximal area. Another amazing fact about ideal triangles is that they are all congruent, i.e., they can be transformed into each other by an isometry.

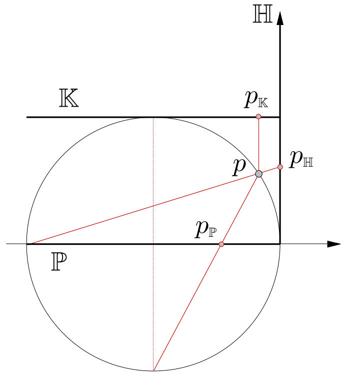


Figure 7.16 A point p on the half-sphere and the corresponding points in $\mathbb{K}$, $\mathbb{P}$ and $\mathbb{H}$.

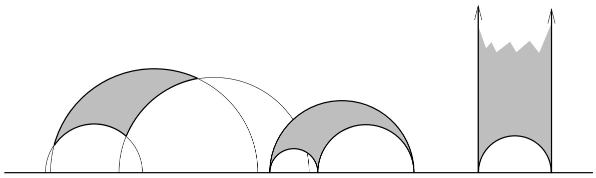


Figure 7.17 A triangle of area 0.78 and two ideal triangles in $\mathbb{H}$.

An *ideal tetrahedron* has all its four vertices on the sphere at infinity. In the half-space model it can be assumed that one vertex of an ideal tetrahedron T is at v_∞ and the other three vertices are in the plane $z = 0$. Let α, β and γ be the angles of the base triangle spanned by the three points in the plane. Let $\mathrm{Lob}(t) = -\int_0^t \log(|2\sin(u)|)du$ Milnor proved that $\mathrm{vol}(T) = \mathrm{Lob}(\alpha) + \mathrm{Lob}(\beta) + \mathrm{Lob}(\gamma)$. This function attains its maximum when the base triangle is equilateral. The maximum is $V_0 = 3 \cdot \mathrm{Lob}(\frac{\pi}{3}) = 1.0149416$.

An *ideal polytope* is a polytope which has all its vertices on the sphere at infinity. Again it can be assumed that v_∞ is one of the vertices of an ideal polytope P in the half-space model. The skeleton of P is just the Delaunay triangulation of the vertices of P in the plane, together with all the edges connecting convex hull vertices with v_∞. The volume of the ideal polytope P is obtained as the sum of the volumina of the ideal tetrahedra spanned by v_∞ and a triangle of the Delaunay triangulation.

Proposition 7.7 *For each k and $n = k^2 + 1$ there exist triangulations T_1, T_2 of $\mathcal{C}_n$ with* $\mathrm{dist}(T_1, T_2) \geq 2n - 4\sqrt{n} + O(1)$.

Proof. Consider the section of the triangular grid shown in Figure 7.18. This is the Delaunay triangulation of a set S of k^2 points. All the $2(k-1)^2$ triangles in this triangulation are equilateral. The set S together with v_∞ is the vertex set of an ideal polytope P in the half-space model. This polytope P has $n = k^2 + 1$ vertices and volume $\mathrm{vol}(P) = 2(k-1)^2 V_0$ where V_0 is the volume of an ideal tetrahedron over an equilateral base. Since V_0 is the maximal volume of a hyperbolic tetrahedron every covering of P by tetrahedra will require at least $2(k-1)^2 = 2n - 4\sqrt{n} + O(1)$ tetrahedra.

Given the polytope P from the preceding paragraph, it remains to find triangulations T_1, T_2 of $\mathcal{C}_n$ such that $P = P_G$ with $G = G(T_1, T_2)$. The existence of such triangulations is

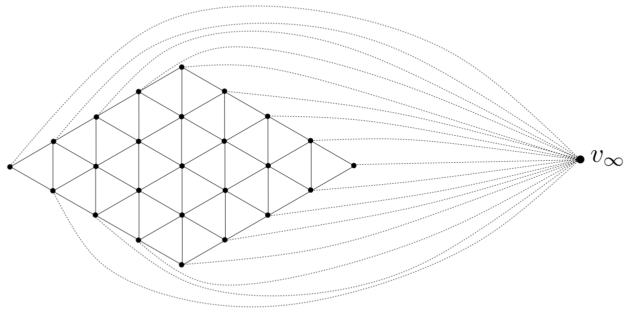


Figure 7.18 Sketch of an ideal polytope composed by $2(k-1)^2$ tetrahedra of volume V_0.

implied by a theorem of Whitney: Every 4-connected planar triangulation has a Hamilton cycle. A Hamilton cycle of the skeleton of P can be identified with the hull of $\mathcal{C}_n$. The cycle induces a partition of the remaining edges: In a planar drawing of G this partition corresponds to the partition into edges in the interior of the cycle and edges in the exterior. Together with the cycle each of these two sets of edges is (equivalent to) a triangulation of $\mathcal{C}_n$. In the case of the graph of Figure 7.18 it is easy to describe a Hamilton cycle explicitly. □

Let P be a simplicial polytope which has a vertex of degree Δ. From the proof of Proposition 7.6 it follows that two triangulations whose union is isomorphic to the skeleton of P have flip-distance at most $2n - 4 - \Delta$. In the example used for the proof of Proposition 7.7 the degree of v_∞ is $\Delta_\infty \approx 4\sqrt{n}$.

To improve the lower bound on the flip-distance, polytopes with smaller maximum degree are required. The construction of [179] is based on the icosahedron. Each face of this polytope is subdivided into k^2 equilateral triangles. The resulting polytope has $20k^2$ triangles and $n = 10k^2 + 2$ vertices. The ideal polyhedron $P(k)$ is obtained by projecting the edges of the icosahedron out to the sphere at infinity and using a conformal mapping to send the faces of the icosahedron to the corresponding spherical triangles.

By a technically demanding analysis Sleator, Tarjan and Thurston show that for large n the tetrahedralization of $P(k)$ requiring the least number of tetrahedra is of cone type. That is there is a vertex belonging to all the tetrahedra, consequently there are $2n - 10$ tetrahedra in a minimal tetrahedralization. Via the Theorem of Whitney this implies that the diameter of $\mathcal{G}_n$ is $2n - 10$ for n sufficiently large and of the form $n = 10k^2 + 2$.

7.6 Notes and References

Triangulations span the full range of geometry from pure to applied and from high- to low-dimensional. This span may be illustrated by the following recent contributions: Santos [169] investigates concepts of triangulations for oriented matroids. Edelsbrunner [66] presents algorithmic and structural aspects of triangulations relevant for mesh generation. Remarkably, flips play a prominent role in both. The bound on the number of flippable edges, Proposition 7.1, is from Hurtado, Noy and Urrutia [116]. This paper contains examples of triangulations of a point set which are at flip distance $\Theta(n^2)$. Similar examples are discussed by Santos and Seidel [170] in the context of counting triangulations.

Voronoi diagrams and Delaunay triangulations are named after the Russian mathematicians Georgii Feodosevich Voronoi (1868-1908) and Boris Nikolaevich Delone (1890-1980). The concepts themselves have been studied earlier among others by Dirichlet, Gauß and Descartes. The quadrangular flip (*Lawson flip*) was studied in a paper by Lawson [129]. He proved that the flip graph $\mathcal{G}(\mathcal{P})$ is connected for all point sets $\mathcal{P}$ in the plane. More comprehensive accounts on Delaunay triangulations including the analysis of various algorithms for their construction can be found in the books of Edelsbrunner [65] and [66] and in the handbook article of Aurenhammer and Klein [18]. These sources also contain generalizations, e.g., higher order Voronoi diagrams and power diagrams.

Secondary polytopes where introduced by Gel'fand et al. [98] in the context of generalized hypergeometric functions. A self-contained study of these polytopes is Billera et al. [27]. In Ziegler's book on polytopes [219] secondary polytopes are investigated as a subclass of fiber polytopes. The name *associahedron* was coined by Lee [130]. Some authors remark that associahedra already appear in work of Stasheff around 1960 and call them *Stasheff polytopes*. Lee gave explicit coordinates for the associahedron $\mathcal{A}_n$, he also investigates the number $f_j = C_n^{n-3-j}$ of j-dimensional faces of $\mathcal{A}_n$. A nice derivation of the formula for C_n^k is given by Stanley [185]. Actually, these formulas have been known for more then hundred years, Stanley [185] contains references to work of Kirkman, Prouhet and Cayley.

We make no attempt to extract references from the vast literature on Catalan numbers. The extensive collection for pointers to this topic is the second volume of Richard Stanley's Enumerative Combinatorics [186].

The bounds for the diameter of the flip-graph $\mathcal{G}_n$ of a convex polygon were obtained by Sleator, Tarjan and Thurston [179]. Most of our exposition is close to the lines of their brilliant paper. For additional aspects of hyperbolic geometry we recommend Milnor [141] and Cannon et al. [44].

8 Rigidity and Pseudotriangulations

A *framework* $G[\mathbf{p}]$ is a graph $G = (V, E)$ and an embedding $\mathbf{p} : V \to \mathbb{R}^d$. The straight edges of the framework are thought of as rigid bars connecting vertices (joints) where incident bars are connected flexibly. An important problem for civil engineers is the question: "Is a given framework rigid?"

The first part of this chapter is about rigidity of plane frameworks. The concept of infinitesimal motions allows to deal with the problem in terms of linear algebra. Stress is introduced as the dual notion of motion. Theorem 8.9 collects three characterizations for minimal generically rigid graphs, mgr-graphs. These characterizations nicely generalize well-known characterizations of trees which happen to be the mgr-graphs in one dimension.

In Section 8.2 we define pseudotriangulations and show that minimal pseudotriangulations constitute a particular class of planar mgr-graphs. A special class of motions, expansive motions, are used in Section 8.3 to define a polyhedron whose vertices are in bijection to minimal pseudotriangulations. The edges of the polyhedron correspond to edge flips between two minimal pseudotriangulations. As an application of this structure we indicate a solution to the Carpenter's Rule Problem: "Can a linkage in the plane be moved continuously to a position where all its vertices are on a line, so that during the motion the linkage remains non-crossing and edge lengths are preserved?"

8.1 Rigidity, Motion and Stress

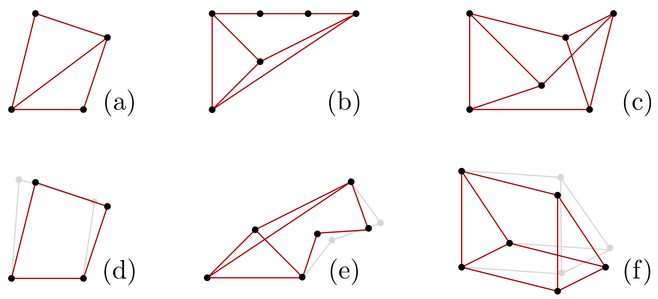


Figure 8.1 Frameworks (a,b,c) are rigid, (d,e,f) are flexible.

Figure 8.1 shows that there are some subtleties to questions of rigidity: At first note that the notion of rigidity depends on the ambient space. All six frameworks are flexible as frameworks in space but as plane frameworks those in the upper row are rigid. The underlying graph for frameworks (b) and (e) is the same, even the length of the edges are equal, still (b) is rigid and (e) flexible, i.e., non-rigid. The graphs for (c) and (f) are isomorphic and again one of the frameworks is rigid the other flexible. The difference between the two pairs is that almost all embeddings of (e) are flexible while almost all embeddings of (c) are rigid.

Let a framework $G[\mathbf{p}]$ be given. If $G[\mathbf{p}]$ is flexible, then there is a motion which moves vertices $\mathbf{p}_i = (x_i, y_i)$ of the framework along paths $\mathbf{p}_i(t) = (x_i(t), y_i(t))$. Throughout the motion the length $||\mathbf{p}_i - \mathbf{p}_j||$ of every edge $\{i, j\}$ has to remain constant and the same holds for the square of the length, $||\mathbf{p}_i - \mathbf{p}_j||^2 = \langle\, \mathbf{p}_i(t) - \mathbf{p}_j(t)\,,\, \mathbf{p}_i(t) - \mathbf{p}_j(t)\,\rangle$. By differentiation, this condition becomes

$$\frac{d}{dt}\langle\, \mathbf{p}_i(t) - \mathbf{p}_j(t)\,,\, \mathbf{p}_i(t) - \mathbf{p}_j(t)\,\rangle = 2\,\langle\, \mathbf{p}_i(t) - \mathbf{p}_j(t)\,,\, \mathbf{p}'_i(t) - \mathbf{p}'_j(t)\,\rangle = 0.$$

In particular the initial velocities $\mathbf{v}_i := \mathbf{p}'_i(0)$ of the motion satisfy

$$\langle\, \mathbf{p}_i - \mathbf{p}_j\,,\, \mathbf{v}_i - \mathbf{v}_j\,\rangle = 0 \quad \text{for all} \quad \{i, j\} \in E.$$

We turn the observation into a definition. An *infinitesimal motion* of a plane framework is an assignment of a velocity $\mathbf{v}_i \in \mathbb{R}^2$ to each vertex i such that for each edge $\{i, j\} \in E$ we have $\langle\, \mathbf{p}_i - \mathbf{p}_j\,,\, \mathbf{v}_i - \mathbf{v}_j\,\rangle = 0$. Rewriting the equation as $\langle\, \mathbf{p}_i - \mathbf{p}_j\,,\, \mathbf{v}_i\,\rangle = \langle\, \mathbf{p}_i - \mathbf{p}_j\,,\, \mathbf{v}_j\,\rangle$ we note that the condition means that $\mathbf{v}_i$ and $\mathbf{v}_j$ have the same projection on $\mathbf{p}_i - \mathbf{p}_j$.

A *trivial motion* is a motion which comes from a rigid transformation of the whole plane, i.e., translations and rotations. A framework $G[\mathbf{p}] = (V, E, \mathbf{p})$ is *infinitesimally rigid* if every infinitesimal motion of $G[\mathbf{p}]$ is trivial. Figure 8.2 illustrates non-trivial infinitesimal motions. Note that the frameworks of examples (b) and (c) are rigid, still they admit infinitesimal motions. Our aim is to understand infinitesimal rigidity of frameworks.

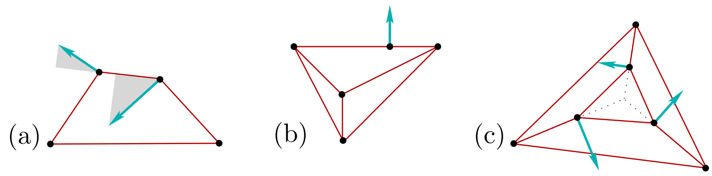


Figure 8.2 Arrows indicate the velocity vectors of non-trivial infinitesimal motions.

The *rigidity matrix* of a plane framework $G[\mathbf{p}] = (V, E, \mathbf{p})$ with $n = |V|$ and $m = |E|$ is an $m \times 2n$ matrix $\mathbf{R}_{G[\mathbf{p}]}$. The rows of $\mathbf{R}_{G[\mathbf{p}]}$ are indexed by the edges of G. Each vertex of G has two columns in $\mathbf{R}_{G[\mathbf{p}]}$ representing the two coordinates. Let $\mathbf{p}_i = (x_i, y_i)$ and $\mathbf{p}_j = (x_j, y_j)$ and suppose $\{i, j\}$ is an edge. The row of $\mathbf{R}_{G[\mathbf{p}]}$ corresponding to edge $\{i, j\}$ is

$$[\, 0\,0 \ldots 0\,0\ (x_i - x_j)\ (y_i - y_j)\ 0\,0 \ldots 0\,0\ (x_j - x_i)\ (y_j - y_i)\ 0\,0 \ldots 0\,0\,]$$

The rigidity matrix enables us to write the conditions for an infinitesimal motion $\mathbf{v} : V \to \mathbb{R}^2$ in a compact form: $\mathbf{R}_{G[\mathbf{p}]} \cdot \mathbf{v} = \mathbf{0}$. Hence, infinitesimal motions are exactly the elements of the kernel of $\mathbf{R}_{G[\mathbf{p}]}$. The following velocity vectors correspond to trivial motions of $G[\mathbf{p}]$:

$$\mathbf{t}_x = [\,1\,0\,1\,0 \ldots 1\,0\,]\,, \quad \mathbf{t}_y = [\,0\,1\,0\,1 \ldots 0\,1\,]\,, \quad \mathbf{t}_r = [\,-y_1\ x_1\ -y_2\ x_2 \ldots\ -y_n\ x_n\,].$$

The vectors $\mathbf{t}_x$ and $\mathbf{t}_y$ are translations in x- and y-direction, $\mathbf{t}_r$ is a counterclockwise rotation around $\mathbf{0}$. Check that these vectors satisfy $\langle\, \mathbf{r}\,,\, \mathbf{t}\,\rangle = 0$ for every row $\mathbf{r}$ of $\mathbf{R}_{G[\mathbf{p}]}$. It follows that the rank of $\mathbf{R}_{G[\mathbf{p}]}$ is at most $2n - 3$.

The rigid transformations of $\mathbb{R}^2$ form a 3-dimensional vector space. Hence, the vectors $\mathbf{t}_x, \mathbf{t}_y, \mathbf{t}_r$ constitute a basis of the space of trivial motions for the framework $G[\mathbf{p}]$. This implies that $G[\mathbf{p}]$ is infinitesimally rigid iff $\dim(\mathsf{ker}(\mathbf{R}_{G[\mathbf{p}]})) = 3$. Recall the dimension-formula from linear algebra:

$$\dim(\mathsf{ker}(\mathbf{R}_{G[\mathbf{p}]})) + \dim(\mathsf{im}(\mathbf{R}_{G[\mathbf{p}]})) = \dim(\mathsf{domain}(\mathbf{R}_{G[\mathbf{p}]})) = 2n.$$

If $G[\mathbf{p}]$ is infinitesimally rigid the formula implies $\mathsf{rank}(\mathbf{R}_{G[\mathbf{p}]})) = \dim(\mathsf{im}(\mathbf{R}_{G[\mathbf{p}]})) = 2n-3$. If $|E| > 2n-3$ then it is possible to delete a row from $\mathbf{R}_{G[\mathbf{p}]}$ without affecting the rank. Therefore, a minimal infinitesimally rigid graph G has exactly $2n-3$ edges. We summarize the findings:

Proposition 8.1 *The rank of the rigidity matrix $\mathbf{R}_{G[\mathbf{p}]}$ of a framework $G[\mathbf{p}] = (V, E, \mathbf{p})$ is at most $2|V|-3$. Moreover, $G[\mathbf{p}]$ is minimal infinitesimally rigid iff $|E| = 2|V|-3$ and $\mathsf{rank}(\mathbf{R}_{G[\mathbf{p}]}) = 2|V|-3$.*

Row- and column-rank of a matrix are the same. In the context of the row-rank of the rigidity matrix some additional terminology is in use. A *self-stress* on a framework $G[\mathbf{p}] = (V, E, \mathbf{p})$ is an assignment of forces $\omega : E \to \mathrm{I\!R}$ to the edges, subject to the condition that all vertices remain in equilibrium:

$$\sum_{\substack{j \\ \{i,j\}\in E}} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = 0 \quad \text{for all } i \in V.$$

Self-stresses of $G[\mathbf{p}]$ are the solutions of $\omega \cdot \mathbf{R}_{G[\mathbf{p}]} = \mathbf{R}^T_{G[\mathbf{p}]} \cdot \omega = \mathbf{0}$.

A subset E' of edges of $G[\mathbf{p}] = (V, E, \mathbf{p})$ is *dependent* if there is a non-trivial self-stress ω with support in E', i.e., $\omega(e) = 0$ for all $e \in E \setminus E'$. If E' is not dependent it is *independent.* If $E' = E$, we also speak of a dependent or independent graph or framework.

The following operation $\mathbf{H}_2$ allows to build larger independent graphs from smaller ones.

($\mathbf{H}_2$) The graph $G^+ = (V^+, E^+)$ is produced from $G = (V, E)$ by an $\mathbf{H}_2$-addition if $V^+ = V \cup \{v_0\}$ with $v_0 \notin V$ and there are two vertices v_i, v_j in V such that $E^+ = E \cup \{(v_0, v_i), (v_0, v_j)\}$. If $G[\mathbf{p}]$ is a framework with graph G, we require that the point $\mathbf{p}_0$ of $G^+[\mathbf{p}^+]$ is chosen such that $\mathbf{p}_0, \mathbf{p}_i, \mathbf{p}_j$ are not collinear.

Lemma 8.2 *Let $G^+[\mathbf{p}^+]$ be produced by a sequence of $\mathbf{H}_2$-additions from $G[\mathbf{p}]$. The framework $G^+[\mathbf{p}^+]$ is independent iff $G[\mathbf{p}]$ is independent.*

Proof. Let ω be a self-stress for $G^+[\mathbf{p}^+]$. The equilibrium condition for vertex v_0 is the equation $\omega_{0i}(\mathbf{p}_0 - \mathbf{p}_i) + \omega_{0j}(\mathbf{p}_0 - \mathbf{p}_j) = \mathbf{0}$. Since $\mathbf{p}_0, \mathbf{p}_i, \mathbf{p}_j$ are not collinear, $(\mathbf{p}_0 - \mathbf{p}_i)$ and $(\mathbf{p}_0 - \mathbf{p}_j)$ are linearly independent. This enforces $\omega_{0i} = \omega_{0j} = 0$. Therefore, ω is a non-trivial self-stress on $G^+[\mathbf{p}^+]$ only if its restriction to $G[\mathbf{p}]$ is a non-trivial self-stress.

Conversely, augmenting a self-stress ω of $G[\mathbf{p}]$ with $\omega_{0i} = 0$ and $\omega_{0j} = 0$ gives a self-stress of $G^+[\mathbf{p}^+]$. □

Proposition 8.3 *Let $G[\mathbf{p}] = (V, E, \mathbf{p})$ be generated from a single edge by a sequence of $\mathbf{H}_2$-additions. The framework $G[\mathbf{p}]$ is independent and $\mathsf{rank}(\mathbf{R}_{G[\mathbf{p}]}) = |E| = 2|V|-3$.*

Proof. The rigidity matrix for a single edge is a 1×4 matrix. The rank is 1 and there is no non-trivial self-stress.

Let $G[\mathbf{p}]$ be generated from a single edge by a sequence of $\mathbf{H}_2$-additions. Each $\mathbf{H}_2$-step adds two edges, thus $|E| = 2|V|-3$. Inductive application of Lemma 8.2 shows that the rows of $\mathbf{R}_{G[\mathbf{p}]}$ are linearly independent. Differently stated: The kernel of $\mathbf{R}^T_{G[\mathbf{p}]} : \mathrm{I\!R}^{|E|} \to \mathrm{I\!R}^{2|V|}$ is trivial, hence, $\mathsf{rank}(\mathbf{R}_{G[\mathbf{p}]}) = \mathsf{rank}(\mathbf{R}^T_{G[\mathbf{p}]}) = |E| = 2|V|-3$. □

By definition, a framework $G[\mathbf{p}]$ is independent iff $\dim(\mathsf{ker}(\mathbf{R}^T_{G[\mathbf{p}]})) = 0$. From

$$\dim(\mathsf{ker}(\mathbf{R}^T_{G[\mathbf{p}]})) + \dim(\mathsf{im}(\mathbf{R}^T_{G[\mathbf{p}]})) = \dim(\mathsf{domain}(\mathbf{R}^T_{G[\mathbf{p}]})) = |E|$$

it follows that $\dim(\mathsf{ker}(\mathbf{R}^T_{G[\mathbf{p}]})) = |E| - \mathsf{rank}(\mathbf{R}^T_{G[\mathbf{p}]})$. Since $\mathsf{rank}(\mathbf{R}^T_{G[\mathbf{p}]}) \leq 2|V| - 3$ (Proposition 8.1) every graph with $|E| > 2|V| - 3$ is dependent. Assume that E is independent and $|E| < 2|V| - 3$. Let $\mathbf{p}$ be in general position* so that the rank of the rigidity matrix of the complete graph on $\mathbf{p}$ is $2|V| - 3$. In that case it is possible to add edges to E so that with each added edge the rank of $\mathbf{R}^T_{G[\mathbf{p}]}$ increases by one, until $|E| = 2|V| - 3$. This proves a dual to Proposition 8.1:

Proposition 8.4 *Let $\mathbf{p}$ be in general position. A framework $G[\mathbf{p}]$ is maximal independent iff $\mathsf{rank}(\mathbf{R}_{G[\mathbf{p}]}) = 2|V| - 3$ and $|E| = 2|V| - 3$.*

Let G be a graph with n vertices. With a given embedding $\mathbf{p}$ we have $\mathsf{rank}(\mathbf{R}_{G[\mathbf{p}]}) = 2n - 3$ iff the determinant of a $(2n-3) \times (2n-3)$ submatrix is non-zero. Each of these determinants is a polynomial in the coordinates of the embedding. Consequently, each of the determinants is either the zero polynomial or its set of non-zeros is open and dense in $\mathbb{R}^{2n}$. Therefore, the graph G either has $\mathsf{rank}(\mathbf{R}_{G[\mathbf{p}]}) < 2n - 3$ for all embeddings $\mathbf{p}$ or for almost all embeddings $\mathbf{p}$ of G the rigidity matrix has rank $2n - 3$. In the later case we call the graph *generically rigid*. The nice thing about generic rigidity is that it is a property of the graph G alone. If there is need for an embedding $\mathbf{p}$ of G, this can be chosen *generically*, i.e., such that the ranks of all submatrices of $\mathbf{R}_{G[\mathbf{p}]}$ are constant in some neighborhood of $\mathbf{p}$. Recall the examples from Figure 8.1. From the graphs of (a) and (d) the first is rigid, the second non-rigid, the graph of (c) is generically rigid but the graph of (b) is not generically rigid.

Minimal generically rigid graphs in the plane, mgr-graphs for short, are generically rigid, but they lose this property upon removal of any edge. We summarize our knowledge about mgr-graphs:

(I) G is an mgr-graph and $\mathbf{p}$ a generic embedding $\Longrightarrow$ $G[\mathbf{p}]$ is minimal infinitesimally rigid (Proposition 8.1).

(II) G is an mgr-graph and $\mathbf{p}$ a generic embedding $\Longrightarrow$ $G[\mathbf{p}]$ is maximal independent, i.e., self-stress free (Proposition 8.4).

A necessary condition for mgr-graphs is given with the next proposition.

Proposition 8.5 (Edge count)
If $G = (V, E)$ is an mgr-graph, then $|E| = 2|V| - 3$ and $|E'| \leq 2|V[E']| - 3$ for all $\emptyset \neq E' \subset E$ and the set $V[E']$ of all vertices incident to edges in E'.

Proof. By definition $G = (V, E)$ is a mgr-graph iff R_G has generic rank $2|V| - 3$ and E is minimal, i.e., $\mathsf{rank}(R_{G'}) < 2|V| - 3$ for all $G' = (V, E')$ with $E' \subset E$. This implies $|E| = 2|V| - 3$.

If there exists an $E' \subset E$ with $|E'| > 2|V[E']| - 3$, then there is a non-trivial self-stress ω' on $(V[E'], E')$. With $\omega_e = 0$ for all $e \in E \setminus E'$ this extends to a non-trivial self-stress ω on G. However, in (II) we have noted that the mgr-graph G is independent, i.e., has no non-trivial self-stress. □

* It suffices that no line contains all points of $\mathbf{p}$.

Figure 8.3 shows mgr-graphs which cannot be constructed from a single edge with $\mathbf{H}_2$-additions. There is a second operation $\mathbf{H}_3$, such that the two operations together suffice to generate every mgr-graph from a single edge.

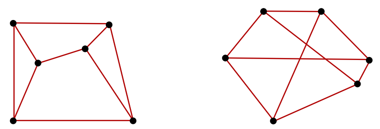


Figure 8.3 Minimal rigid graphs which cannot be constructed from $\mathbf{H}_2$-additions.

($\mathbf{H}_3$) Given a graph $G = (V, E)$ with an edge $e = \{v_i, v_j\}$ and a vertex $v_k \neq v_i, v_j$. A new graph $G^+ = (V^+, E^+)$ is produced from G by a $\mathbf{H}_3$-addition if $V^+ = V \cup \{v_0\}$ with $v_0 \notin V$ and $E^+ = \big(E \setminus e\big) \cup \{(v_0, v_i), (v_0, v_j), (v_0, v_k)\}$.

Let $G^+ = (V, E)$ be a graph and suppose that G^+ has a vertex v_0 of degree three adjacent to v_i, v_j, v_k. Let G_{ij} be the (multi)-graph obtained by deleting v_0 and its edges and adding the edge $\{v_i, v_j\}$, graphs G_{ik} and G_{jk} are defined alike.

Lemma 8.6 *The graph G^+ is (generically) independent iff at least one of G_{ij}, G_{ik} and G_{jk} is (generically) independent.*

Proof. Suppose G^+ is independent and has m edges. A generic embedding $\mathbf{p}$ of G^+ has $\mathsf{rank}(\mathbf{R}_{G^+[\mathbf{p}]}) = m$. If the rank of the rigidity matrix of one of G_{ij}, G_{ik}, G_{jk} is $m-2$, then this graph is independent.

Assume that all these ranks are smaller, then the frameworks have non-trivial self-stresses α, β and γ, such that:

$$\alpha_{ij} R_{ij} = \sum_{e \in E^*} \alpha_e R_e\ , \qquad \beta_{ik} R_{ik} = \sum_{e \in E^*} \beta_e R_e\ , \qquad \gamma_{jk} R_{jk} = \sum_{e \in E^*} \gamma_e R_e\ .$$

Here $E^* = E \setminus \{(v_0, v_i), (v_0, v_j)(v_0, v_k)\}$ and $\alpha_{ij} \neq 0$, $\beta_{ik} \neq 0$ and $\gamma_{jk} \neq 0$ because G^+ is independent.

Consider the complete graph on $\{v_0, v_i, v_j, v_k\}$ embedded at $\mathbf{p}_0, \mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k$. This graph has 4 vertices and 6 edges, since $6 > 2 \cdot 4 - 3$ there is a non-trivial self-stress

$$\omega_{0i} R_{0i} + \omega_{0j} R_{0j} + \omega_{0k} R_{0k} + \omega_{ij} R_{ij} + \omega_{ik} R_{ik} + \omega_{jk} R_{jk} = \mathbf{0}.$$

Substituting from above, yields:

$$\omega_{0i} R_{0i} + \omega_{0j} R_{0j} + \omega_{0k} R_{0k} + \sum_{e \in E^*} \big(\frac{\omega_{ij}}{\alpha_{ij}} \alpha_e + \frac{\omega_{ik}}{\beta_{ik}} \beta_e + \frac{\omega_{jk}}{\gamma_{jk}} \gamma_e\big) R_e = \mathbf{0}.$$

This is a non-trivial self-stress on $G^+[\mathbf{p}]$, contradicting our assumption. Therefore, at least one of G_{ij}, G_{ik} and G_{jk} has a rigidity matrix of rank $m-2$ and is independent.

For the converse suppose that G_{ij} is independent and let $\mathbf{p}$ be a generic embedding of G_{ij} so that $\mathsf{rank}(\mathbf{R}_{G_{ij}[\mathbf{p}]}) = m-2$. Extend $\mathbf{p}$ by specifying $\mathbf{p}_0 = \frac{\mathbf{p}_i + \mathbf{p}_j}{2}$, i.e., place vertex v_0 on the midpoint of $\{\mathbf{p}_i, \mathbf{p}_j\}$.

Assume that G_{ij} is independent and G^+ dependent, then $\mathsf{rank}(\mathbf{R}_{G^+[\mathbf{p}]}) < m$ and there is a non-trivial self-stress ω. The equation for vertex v_0 reads:

$$\omega_{0i}(\mathbf{p}_0 - \mathbf{p}_i) + \omega_{0j}(\mathbf{p}_0 - \mathbf{p}_j) + \omega_{0k}(\mathbf{p}_0 - \mathbf{p}_k) = 0.$$

Since $(\mathbf{p}_0 - \mathbf{p}_i)$ and $(\mathbf{p}_0 - \mathbf{p}_j)$ are parallel and $(\mathbf{p}_0 - \mathbf{p}_k)$ is a different direction we conclude that $\omega_{0k} = 0$ and $-\omega_{0i} = \omega_{0j}$. Define ω' for G_{ij} by $\omega'_{ij} = \omega_{0j}/2$ and $\omega'_e = \omega_e$ for all other edges e of G_{ij}. The definition is such that $\omega'_{ij}(\mathbf{p}_i - \mathbf{p}_j) = \omega_{0j}(\mathbf{p}_0 - \mathbf{p}_j)$. This and the assumption that ω is non-trivial implies that ω' is a non-trivial self-stress of G_{ij}, a contradiction. □

A graph G has a *Henneberg construction* iff G can be produced from a single edge by a sequence of $\mathbf{H}_2$ and $\mathbf{H}_3$-additions.

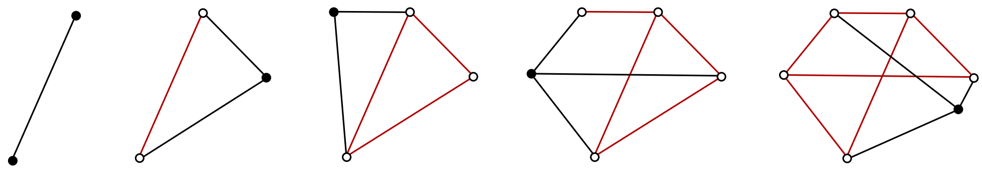


Figure 8.4 A Henneberg construction for $K_{3,3}$.

Theorem 8.7 *A graph $G = (V, E)$ is* mgr *iff G has a Henneberg construction.*

Proof. Suppose there is a Henneberg construction for G. A single edge is independent and $\mathbf{H}_2$ and $\mathbf{H}_3$-additions preserve generic independence (Lemma 8.2 and 8.6). Hence, G is generically independent. Since $|E| = 2|V| - 3$ the graph is mgr.

For the converse, let $G = (V, E)$ be an mgr-graph and $d(v_i)$ be the degree of vertex v_i. From $\sum d(v_i) = 2|E| = 4|V| - 6$ it follows that there is a vertex of degree at most 3 in G. A vertex of degree 0 or 1 is impossible in an rigid graph. Let v_0 be a vertex with $d(v_0) = 2$ or 3.

If $d(v_0) = 2$, then by Lemma 8.2 the graph G' induced by $V' = V \setminus \{v_0\}$ is independent, since it has two edges less than G it is mgr. A Henneberg construction of G' is guaranteed by induction. The $\mathbf{H}_2$-addition of v_0 to G' gives a Henneberg construction of G.

If $d(v_0) = 3$, then by Lemma 8.6 there are neighbors v_i, v_j of v_0 such that the graph G_{ij} is mgr. A Henneberg construction of G_{ij} is extended by the $\mathbf{H}_3$-addition of v_0 to G_{ij} which replaces the edge $\{v_i, v_j\}$. This gives, again with induction, a Henneberg construction of G. □

Actually it is possible to prescribe the two vertices of the initial edge of a Henneberg construction. This additional property will be helpful later.

Lemma 8.8 *Let $G = (V, E)$ be a* mgr*-graph and v_i, v_j be any two vertices of G, then there is a Henneberg construction of G beginning with the edge $\{v_i, v_j\}$.*

Proof. In the proof of Theorem 8.7 the vertex v_0 was selected subject to the condition that $d(v_0) = 2$ or 3. From $\sum d(v_i) = 2|E| = 4|V| - 6$ it follows that there are at least three vertices with this property. Therefore, it is possible to choose a vertex different from v_i and v_j in every step. At the end of the recursion there is a single edge which must be $\{v_i, v_j\}$. □

The following theorem gives two additional characterizations of mgr-graphs.

Theorem 8.9 *Each of the following conditions on a graph $G = (V, E)$ is a characterization of* mgr*-graphs in the plane:*

(1) G can be produced from a single edge by a sequence of $\mathbf{H}_2$- and $\mathbf{H}_3$-additions. (Henneberg construction)

(2) For any two vertices $v \neq w$ the (multi)-graph with edges $E \cup \{(v, w)\}$ is the union of two disjoint spanning trees. (Recski Theorem)

(3) $|E| = 2|V| - 3$ and $|E'| \leq 2|V[E']| - 3$ for all $\emptyset \neq E' \subset E$. (Laman Condition)

Proof. We know that (1) is a characterization of mgr-graphs (Theorem 8.7), it remains to prove that conditions (1), (2) and (3) are equivalent.

(1⇒2) The idea is to build trees T_1 and T_2 along a Henneberg construction. From Lemma 8.8 we take the existence of a Henneberg construction of G starting with edge $\{v, w\}$. Let G_0 be this initial graph, duplicate edge $\{v, w\}$. At this stage $T_1 = \{(v, w)\}$ and $T_2 = \{(v, w)\}$ is a decomposition of $E(G_0) \cup \{(v, w)\}$ into two spanning trees.

Let $G^+ = (V \cup \{v_0\}, E^+)$ with $E^+ = E \cup \{(v_0, v_i), (v_0, v_j)\}$ be a $\mathbf{H}_2$-addition from $G = (V, E)$. Assume that $E \cup \{(v, w)\}$ is the union of two spanning trees T_1 and T_2. Define $T_1^+ = T_1 \cup \{(v_0, v_i)\}$ and $T_2^+ = T_2 \cup \{(v_0, v_j)\}$, this is a partition of $E^+ \cup \{(v, w)\}$ into two spanning trees.

Let $G^+ = (V \cup \{v_0\}, E^+)$ with $E^+ = \big(E \setminus \{(v_i, v_j)\}\big) \cup \{(v_0, v_i), (v_0, v_j), (v_0, v_k)\}$ be a $\mathbf{H}_3$-addition from $G = (V, E)$. Assume that $E \cup \{v, w\}$ is the union of two spanning trees T_1 and T_2 and that the edge $\{v_i, v_j\}$ is in T_1. Let $T_1^+ = \big(T_1 \setminus \{(v_i, v_j)\}\big) \cup \{(v_0, v_i), (v_0, v_j)\}$ and $T_2^+ = T_2 \cup \{(v_0, v_k)\}$, this is a partition of $E^+ \cup \{(v, w)\}$ into two spanning trees.

The inductive definition yields a partition of $E \cup \{(v, w)\}$ into two spanning trees T_1 and T_2.

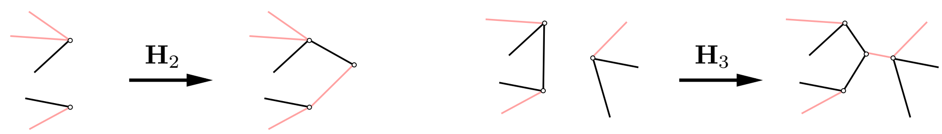


Figure 8.5 $\mathbf{H}_2$- and $\mathbf{H}_3$-addition and the growth of the two trees.

(2⇒3) A spanning tree of G has $|V| - 1$ edges, hence, $|E| = 2|V| - 3$. We verify the inequality for E': Let $\{v, w\} \in E'$ and let T_1, T_2 be a partition of $E \cup \{(v, w)\}$ into two spanning trees. The duplicate edge $\{v, w\}$ is contained in both trees. Delete $\{v, w\}$ from T_2. This splits T_2 into two trees, call them S_2 and S_3 and let $S_1 = T_1$. Since S_2 and S_3 are vertex disjoint we have:

- Each vertex v is incident to exactly two of the three trees S_1, S_2, S_3.

Back to E'. Let $F_i = E' \cap S_i$ and $V_i = V[E'] \cap S_i$. Since $v, w \in V[E']$ our construction guarantees that $|V_i| \geq 1$ for $i = 1, 2, 3$. Each (V_i, F_i) is a forest with non-empty vertex-set, therefore, $|F_i| \leq |V_i| - 1$. Each vertex of $V[E']$ is contained in at most two of the forests, therefore, $\sum |V_i| \leq 2|V[E']|$. This gives the inequality

$$|E'| = \sum |F_i| \leq \sum (|V_i| - 1) \leq 2|V[E']| - 3.$$

(3⇒1) Let G be a graph with the Laman Property. The claim is that there is a graph G' with one vertex less, such that G' has the Laman Property and G can be obtained from G' by a $\mathbf{H}_2$- or a $\mathbf{H}_3$-addition.

The Laman Property implies (see the proof of Theorem 8.7) that G has a vertex v_0 with $d(v_0) = 2$ or 3.

If $d(v_0) = 2$, then removing v_0 and the two incident edges gives a graph G' with the Laman Property. Clearly, G is obtained from G' by $\mathbf{H}_2$-addition of v_0.

If $d(v_0) = 3$, then consider the (multi)-graphs G_{ij}, G_{ik} and G_{jk} as before. If one of them, say G_{ij}, has the Laman Property, we choose $G' = G_{ij}$ and G can be obtained from G' by $\mathbf{H}_3$-addition of v_0.

Suppose none of G_{ij}, G_{ik} and G_{jk} has the property. Choose $E_i \subseteq E(G_{jk})$ as a minimal set violating the Laman Property. That is $|E_i| > 2|V[E_i]| - 3$ and $|E_i'| \leq 2|V[E_i']| - 3$ for all $E_i' \subsetneqq E_i$. It follows that $|E_i| = 2|V[E_i]| - 2$ and $\{v_j v_k\} \in E_i$. Sets $E_j \subseteq E(G_{ik})$ and $E_k \subseteq E(G_{ij})$ are chosen alike.

First assume that $V[E_i]$, $V[E_j]$ and $V[E_k]$ have pairwise exactly one vertex in common. This implies $|V[E_i]| + |V[E_j]| + |V[E_k]| = |V[E_i] \cup V[E_j] \cup V[E_k]| + 3 = |V[E_i \cup E_j \cup E_k]| + 3$. It follows that $|E_i \cup E_j \cup E_k| = |E_i| + |E_j| + |E_k| = 2(|V[E_i]| + |V[E_j]| + |V[E_k]|) - 6 = 2|V[E_i \cup E_j \cup E_k]|$. In G we consider the set $E^* = (E_i \cup E_j \cup E_k) \setminus \{\{v_i, v_j\}, \{v_i, v_k\}, \{v_j, v_k\}\} \cup \{\{v_0, v_i\}, \{v_0, v_j\}, \{v_0, v_k\}\}$. This set E^* contains v_0, hence, $|V[E^*]| = |V[E_i \cup E_j \cup E_k]| + 1$. For $|E^*|$ we compute: $|E^*| = |E_i \cup E_j \cup E_k| = 2|V[E_i \cup E_j \cup E_k]| = 2|V[E^*]| - 2$. Consequently, E^* violates the Laman Property of G, a contradiction.

For the other case we prepare with the following observation:

($\star$) Suppose A and B are sets of edges with $|V[A] \cap V[B]| \geq 2$. Suppose further that $|A| = 2|V[A]| - a$ and $|B| = 2|V[B]| - b$ and $|A \cap B| \leq 2|V[A \cap B]| - c$. Then $|A \cup B| \geq 2|V[A \cup B]| - a - b + c$.

For the proof of ($\star$) note that $|V[A \cap B]| \leq |V[A] \cap V[B]|$ and compute: $|A \cup B| = |A| + |B| - |A \cap B| \geq (2|V[A]| - a) + (2|V[B]| - b) - (2|V[A \cap B]| - c) \geq 2(|V[A]| + |V[B]| - |V[A] \cap V[B]|) - a - b + c = 2|V[A] \cup V[B]| - a - b + c = 2|V[A \cup B]| - a - b + c$.

Assume that the two sets $V[E_i]$ and $V[E_j]$ have at least two vertices in common. Use ($\star$), with $A = E_i$, $a = 2$, $B = E_j$, $b = 2$ and $c = 3$, this choice of c is legitimized by the fact that G is Laman. This yields $|E_i \cup E_j| \geq 2|V[E_i \cup E_j]| - 1$. Since G is Laman and $E_i \cup E_j$ has exactly two edges not from G, the inequality is tight: $|E_i \cup E_j| = 2|V[E_i \cup E_j]| - 1$.

Using ($\star$), with $A = E_i \cup E_j$, $a = 1$, $B = E_k$, $b = 2$ and $c = 3$, gives $|E_i \cup E_j \cup E_k| \geq 2|V[E_i \cup E_j \cup E_k]|$. Since G is Laman and $E_i \cup E_j \cup E_k$ has exactly three edges not from G, the inequality is again tight.

As before $E^* = (E_i \cup E_j \cup E_k) \setminus \{\{v_i, v_j\}, \{v_i, v_k\}, \{v_j, v_k\}\} \cup \{\{v_0, v_i\}, \{v_0, v_j\}, \{v_0, v_k\}\}$, the set E^* contains v_0, hence, $|V[E^*]| = |V[E_i \cup E_j \cup E_k]| + 1$. For $|E^*|$ we compute: $|E^*| = |E_i \cup E_j \cup E_k| = 2|V[E_i \cup E_j \cup E_k]| = 2|V[E^*]| - 2$. This is again contradicting the Laman Property of G. □

The 1-Dimensional Case

It is instructive to compare the result for 2-dimensional rigidity with the much simpler case of 1-dimensional rigidity. In the 1-dimensional setting a framework $G[\mathbf{p}]$ is a graph $G = (V, E)$ with an embedding $\mathbf{p} : V \to \mathbb{R}$.

An assignment of velocities $\mathbf{v} : V \to \mathbb{R}$ is an infinitesimal motion of $G[\mathbf{p}]$ iff $\mathbf{v}_i = \mathbf{v}_j$ for every edge $\{i, j\} \in E$. Trivial motions of the line are translations, therefore, $\mathbf{v}$ is a trivial infinitesimal motion iff all velocities $\mathbf{v}_i$ are equal. With these observations it is easy to deduce:

- The rank of the rigidity matrix of a 1-dimensional framework $G[\mathbf{p}] = (V, E, \mathbf{p})$ is at most $|V| - 1$.
- A 1-dimensional framework $G[\mathbf{p}]$ is (infinitesimally) rigid iff the graph G of the framework is connected.
- A 1-dimensional framework $G[\mathbf{p}] = (V, E, \mathbf{p})$ is minimally rigid iff G is a spanning tree of V.

A self-stress of $G[\mathbf{p}]$ is an assignment of forces $\omega : E \to \mathrm{I\!R}$ to the edges so that all vertices remain in equilibrium: $\sum_{\{i,j\}\in E} \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i) = 0 \quad$ for all $i \in V$. If i is a vertex with $d(i) = 1$ and $\{i, j\}$ is the edge incident to i then $\omega_{ij} = 0$ in every self-stress. If C is a cycle of G, then there is a self-stress with $\omega_e \neq 0$ for all $e \in C$. This gives the result corresponding to Proposition 8.4.

- A 1-dimensional framework $G[\mathbf{p}] = (V, E, \mathbf{p})$ is maximal independent if G is cycle free and $|E| = |V| - 1$.

On the line every embedding that puts all vertices to different points is generic. The characterization of mgr-graphs for the line is very easy, they are trees.

With the following definition we can give a complete analog to Theorem 8.9.

($\mathbf{H}_1$) The graph $G^+ = (V^+, E^+)$ is produced from $G = (V, E)$ by a $\mathbf{H}_1$-addition if $V^+ = V \cup \{v_0\}$ with $v_0 \notin V$ and there is a vertex v_i in V such that $E^+ = E \cup \{(v_0, v_i)\}$.

Theorem 8.10 *Each of the following conditions is a characterization of* mgr*-graphs in one dimension:*

(1) G can be produced from a single vertex by a sequence of $\mathbf{H}_1$-additions.

(2) G is a tree.

(3) $|E| = |V| - 1$ and $|E'| \leq |V[E']| - 1$ for all $\emptyset \neq E' \subset E$.

8.2 Pseudotriangles and Pseudotriangulations

A *pseudotriangle* is a simple polygon with exactly three convex vertices. The convex vertices are called *corners* of the pseudotriangle. Between any two corners there is a polygonal chain with only concave vertices, i.e., a chain bent towards the interior of the pseudotriangle, see Figure 8.6.

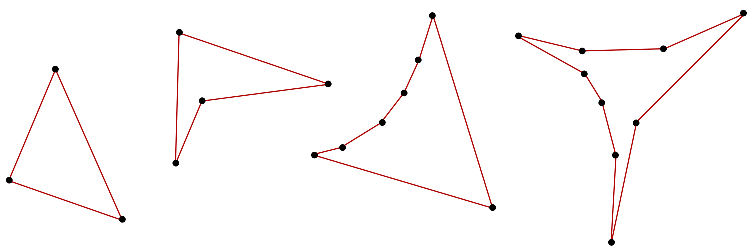


Figure 8.6 A collection of pseudotriangles.

Let $\mathcal{P}$ be a set of points in the plane. A *pseudotriangulation* of $\mathcal{P}$ is a non-crossing geometric graph with vertex set $\mathcal{P}$ which includes all edges of the convex hull and such that all the bounded faces are pseudotriangles (implying that there are no isolated points $p \in \mathcal{P}$ in the pseudotriangles). In this section we are mainly interested in minimum pseudotriangulations, i.e., pseudotriangulations with a minimal number of edges. In Theorem 8.11 it is shown that minimum pseudotriangulations are a particular class of mgr-graphs.

A geometric graph is *pointed at a vertex* v if one of the angles between neighboring edges at v is of size at least π. A geometric graph is *pointed* if it is pointed at each vertex. Figure 8.7 illustrates these definitions.

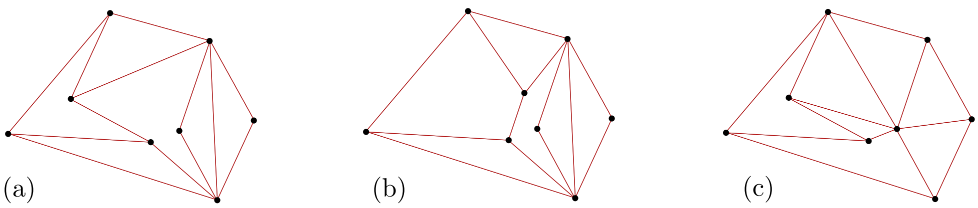


Figure 8.7 A pointed pseudotriangulation (a), the same graph but not a pseudotriangulation (b) and a non-pointed pseudotriangulation (c).

Note that the pseudotriangulation (c) in the figure is minimal (it has no removable edge) but not minimum (with the same number of vertices it has more edges than pseudotriangulation (a)).

Theorem 8.11 *Let G be a geometric graph on a set $\mathcal{P}$ of n points in general position. The following properties are equivalent:*

(1) G is a pointed pseudotriangulation of $\mathcal{P}$.

(2) G is a pseudotriangulation of $\mathcal{P}$ with $2n-3$ edges.

(3) G is a non-crossing and pointed graph on $\mathcal{P}$ with $2n-3$ edges.

(4) G is a minimum pseudotriangulation of $\mathcal{P}$.

(5) G is a non-crossing, pointed mgr*-graph on $\mathcal{P}$.*

(6) G is maximal non-crossing and pointed, i.e., upon addition of an edge the graph will lose one of the properties.

Before going into the proof of the theorem we have a lemma.

Lemma 8.12 *Let G be a non-crossing geometric graph with n vertices and e edges. Any two of the following three properties imply the third,*

(1) G is pointed.

(2) G is a pseudotriangulation.

(3) $e = 2n-3$

Proof. Euler's formula gives: $e = n + f^b - 1$, where f^b is the number of bounded faces. Recall the fact that every simple polygon and hence every bounded face of G has at least three convex corners.

Let $c_{\mathbf{p}}$ be the number of angles of size less than π between neighboring edges at $\mathbf{p}$, i.e., $c_{\mathbf{p}}$ is the number of faces which have $\mathbf{p}$ as a convex corner. Let $d_{\mathbf{p}}$ be the degree of vertex $\mathbf{p}$. Note that $c_{\mathbf{p}} = d_{\mathbf{p}} - 1$ if $\mathbf{p}$ is pointed and otherwise $c_{\mathbf{p}} = d_{\mathbf{p}}$. The total number of pointed vertices is n^*. Then

$$2e = \sum_{\mathbf{p}} d_{\mathbf{p}} = \sum_{\mathbf{p}} c_{\mathbf{p}} + n^* = \sum_{F} (\text{\# convex corners of } F) + n^* \geq 3f^b + n^*.$$

Equality holds iff G is a pseudotriangulation. Subtracting the inequality from Euler's formula multiplied by three gives:

$$e \leq 3n - n^* - 3.$$

Again, equality holds iff G is a pseudotriangulation. Based on this the three implications required for the lemma are immediate. △

Proof of Theorem 8.11. The lemma implies that (1), (2) and (3) are equivalent. Moreover, a pseudotriangulation has $3n - n^* - 3 \geq 2n - 3$ edges. Therefore, a pseudotriangulation with $e = 3n - 3$ is a minimum pseudotriangulation, i.e., (2⇒4). For the converse we need that a pseudotriangulation of $\mathcal{P}$ with $2n-3$ edges exists. This can be shown by induction: Let $\mathbf{p}_0$ be a point from the convex hull of $\mathcal{P}$. Consider a pseudotriangulation of $\mathcal{P} \setminus \{\mathbf{p}_0\}$ with $2(n-1) - 3$ edges and add $\mathbf{p}_0$ back, together with the two tangents to the convex hull of $\mathrm{CH}(\mathcal{P} \setminus \{\mathbf{p}_0\})$.

The characterization of mgr-graphs (Theorem 8.9) implies $e = 2n - 3$, hence, (5⇒3). For the converse consider a set E' of edges of a pointed pseudotriangulation G. The graph $G' = (V[E'], E')$ is planar and pointed. The inequality $e \leq 3n - n^* - 3$ from the proof of the lemma implies $|E'| \leq 2|V[E']| - 3$. This is the Laman condition, hence, G is a mgr-graph.

A pointed planar graph has $e \leq 2n - 3$, therefore, (3⇒6). For the converse assume that G is maximal planar and pointed with $e < 2n - 3$. This implies that there is a bounded face F with at least four convex corners. Consider a geodesic, that is a shortest path, in the interior of F connecting two convex corners which are not adjacent in the cyclic order of convex corners induced by the boundary cycle of F, see Figure 8.8. Such a geodesic contains a straight segment s traversing the interior of F. The two endpoints of s are vertices of G. Moreover, either these endpoints are the endpoints of the path or

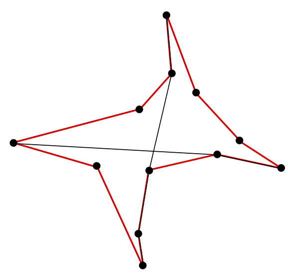


Figure 8.8 A pseudoquadrilateral with two geodesic paths, between pairs of opposite corners.

the path continues with a boundary edge of F such that the angle between this edge and s is more than π, i.e., upon addition of s this vertex of s remains pointed. This shows that s can be added to G as an edge. This addition preserves planarity and pointedness and, therefore, contradicts the maximality of G. □

As in the context of triangulations, flips play a prominent role in studies of pointed pseudotriangulations. Let e be an arbitrary interior edge of a pointed pseudotriangulation P. Removing e from P we have a planar pointed graph with $2n-4$ edges. A combination of Euler's formula with a double counting of corners (as in the proof of Lemma 8.12) implies that the new face created by the removal of e must be a pseudoquadrilateral.

In a pseudoquadrilateral there are the two geodesics each connecting a pair of opposite corners, Figure 8.8. Each of these two geodesics contains a single line segment which can be added such that the pseudoquadrilateral is subdivided into two pseudotriangles. The exchange between these two edges is the *flip*-operation in the context of pseudotriangulations, for examples see Figure 8.9.

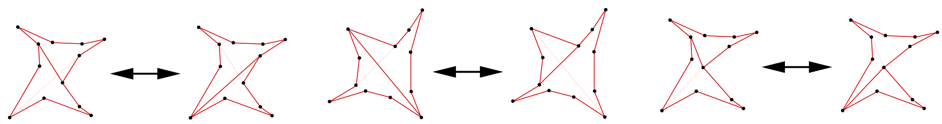


Figure 8.9 Three examples for a flip of edges.

The *flip-graph* $\mathcal{G}^{\curlyvee}(\mathcal{P})$ is the graph whose vertices are the pointed pseudotriangulations of $\mathcal{P}$, the edges of $\mathcal{G}^{\curlyvee}(\mathcal{P})$ are pairs of pseudotriangulations such that there is a flip operation transforming one into the other. An example of a flip-graph is shown in Figure 8.10.

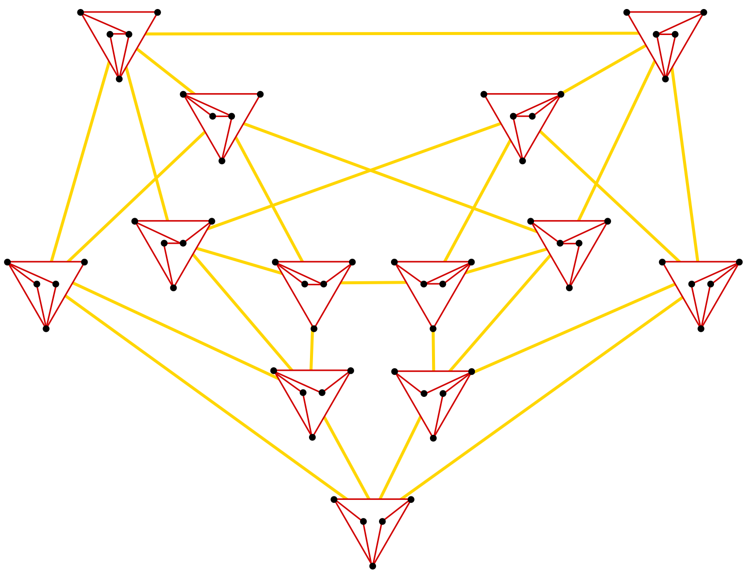


Figure 8.10 The flip-graph $\mathcal{G}^{\curlyvee}(\mathcal{P})$ of pointed pseudotriangulations of a set $\mathcal{P}$ of 5 points.

Proposition 8.13 *Let $\mathcal{G}^{\curlyvee}(\mathcal{P})$ be the flip-graph of pointed pseudotriangulations of a set $\mathcal{P}$ of n points in general position.*

(a) $\mathcal{G}^{\curlyvee}(\mathcal{P})$ is regular of degree $2n-3-k$, where k is the number of points on $\text{CH}(\mathcal{P})$.

(b) $\mathcal{G}^{\curlyvee}(\mathcal{P})$ is connected.

Proof. We have already observed that every interior edge of a pointed pseudotriangulation P of $\mathcal{P}$ can be flipped. By Theorem 8.11 P has exactly $2n-3$ edges and k of these edges are on the convex hull, hence, not interior and non-flippable. This proves part (a).

For (b) we use an inductive argument. Let $\mathbf{p}_0$ be a point from the convex hull of $\mathcal{P}$. Every pointed pseudotriangulation of $\mathcal{P} \setminus \mathbf{p}_0$ can be extended by adding $\mathbf{p}_0$ and the two tangents connecting $\mathbf{p}_0$ to the convex hull $\text{CH}(\mathcal{P} \setminus \mathbf{p}_0)$. We assume, by induction, that $\mathcal{G}^{\curlyvee}(\mathcal{P} \setminus \mathbf{p}_0)$ is connected. To complete the proof we show that starting from an arbitrary pointed pseudotriangulation we can perform flips to reduce the degree of $\mathbf{p}_0$ until this degree is two: Let e be an interior edge incident to $\mathbf{p}_0$. The pseudoquadrilateral created by removing e has $\mathbf{p}_0$ as one of its four corners. The geodesic connecting the two neighboring corners of $\mathbf{p}_0$ avoids $\mathbf{p}_0$, hence, the edge e' replacing e after the flip also avoids $\mathbf{p}_0$. □

From the results of the following section it will follow that there is a $(2n-3)-k$-dimensional polytope whose vertices correspond to pointed pseudotriangulations of $\mathcal{P}$ and whose skeleton-graph is the flip-graph $\mathcal{G}^{\curlyvee}(\mathcal{P})$. Here, of course, $\mathcal{P}$ denotes a set of n points in general position in the plane. In particular the graph of Figure 8.10 is the graph of a 4-dimensional polytope.

8.3 Expansive Motions

An expansive motion on a point set is a motion which is increasing (or at least non-decreasing) on the distance of every pair of points. An *infinitesimal expansive motion* of a set $\mathcal{P}$ of n points in the plane is an assignment $\mathbf{v} : \mathcal{P} \to \mathbb{R}^2$ of a velocity to each point such that:

$$\langle\, \mathbf{p}_i - \mathbf{p}_j\,,\, \mathbf{v}_i - \mathbf{v}_j \,\rangle \geq 0 \quad \text{for all } 1 \leq i < j \leq n.$$

In terms of the rigidity matrix $\mathbf{R}_{\mathcal{P}}$ of the complete graph on $\mathcal{P}$ this reads $\mathbf{R}_{\mathcal{P}} \cdot \mathbf{v} \geq 0$. Taking the velocities as variables this system of inequalities defines a polyhedral cone. This cone contains the trivial motions, i.e., those corresponding to rigid transformations of the whole plane. These trivial motions can be excluded if we fix three velocity variables:

$$v_1^1 = v_1^2 = v_2^1 = 0 \quad \text{in other words:} \quad \mathbf{v}_1 = (0,0)\,,\ \ \mathbf{v}_2 = (0, v_2^2).$$

The normalized cone thus obtained is the *expansion cone* of $\mathcal{P}$, we denote this cone by $\bar{X}_0(\mathcal{P})$.

Lemma 8.14 *Let $\mathcal{P}$ be a set of n points in general position. The expansion cone $\bar{X}_0(\mathcal{P})$ is a pointed $(2n-3)$-dimensional cone.*

Proof. Translate $\mathcal{P}$ such that $\mathbf{p}_1 = (0,0)$ and $\mathbf{p}_2 = (0, y_2)$. The motion $\mathbf{v}$ with $\mathbf{v}_i = \mathbf{p}_i$ for all i is strictly expanding on all i,j with $1 \leq i < j \leq n$. Therefore, $\mathbf{v}$ is an interior point of $\bar{X}_0(\mathcal{P})$ and the cone is full-dimensional, i.e., $(2n-3)$-dimensional.

A cone is pointed if it contains no line. The pointedness of $\bar{X}_0(\mathcal{P})$ follows from the fact that the intersection of its facets is the origin: A motion $\mathbf{v}$ belongs to the facet defined by points $\mathbf{p}_i, \mathbf{p}_j$ iff the distance between these points remains invariant under the motion. Therefore, a motion $\mathbf{v}$ is in the intersection of all facets iff it keeps the length of all edges of the complete graph $K_n(\mathcal{P})$ invariant. Since this graph is rigid only trivial motions keep all edge length invariant but the only trivial motion with $\mathbf{v}_1 = (0,0)$ $\mathbf{v}_2^1 = 0$ corresponds to the origin. □

The expansion cone is a highly degenerate object. It is a polyhedron in $(2n-3)$-dimensional space, yet an extreme ray of the polyhedron is, in general, contained in a quadratic number of facets. Below we will investigate this in detail, an illustration for this fact is given with Figure 8.11. The left part of the figure indicates an expansive motion. The edges shown correspond to rigid pairs, i.e., pairs with $\langle \mathbf{p}_i - \mathbf{p}_j\,, \mathbf{v}_i - \mathbf{v}_j \rangle = 0$. Any additional edge would yield a rigid framework, therefore, this expansive motion corresponds to a ray of the cone. The right part of the figure shows a framework with $2n-4$ edges which determines the same rigid components and hence the same expansive motion.

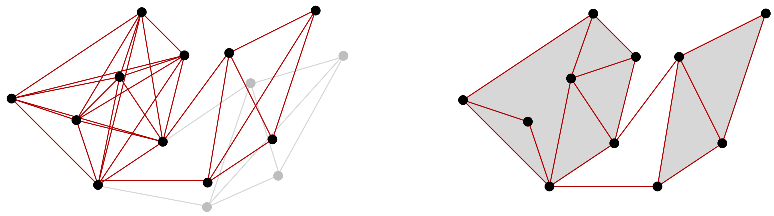


Figure 8.11 An expansive motion corresponding to a ray of the expansion cone and a planar framework with the same flexibility.

8.4 The Polyhedron of Pointed Pseudotriangulations

We study the expansion cone $\bar{X}_0(\mathcal{P})$ and perturbations of this cone. The perturbations will be given by a scalar f_{ij} for every pair i, j and the translated inequalities:

$$\langle \mathbf{p}_i - \mathbf{p}_j\,, \mathbf{v}_i - \mathbf{v}_j \rangle \geq f_{ij} \qquad \text{for all } 1 \leq i < j \leq n.$$

For any choice of scalars f_{ij} the inequalities define a polyhedron $\bar{X}_f(\mathcal{P})$. The expansion cone $\bar{X}_0$ is the special case $f \equiv 0$. The remarkable fact is that there is a choice of f such that the vertices of the perturbed cone $\bar{X}_f$ are in one to one correspondence to pointed pseudotriangulations of $\mathcal{P}$. Given such a choice of f we call the polyhedron $\bar{X}_f(\mathcal{P})$ the *polyhedron of pointed pseudotriangulations* of $\mathcal{P}$.

Lemma 8.15 *Let $\mathcal{P}$ be a set of n points in general position. For every choice of f the perturbed expansion cone $\bar{X}_f(\mathcal{P})$ is $(2n-3)$-dimensional and has at least one vertex.*

Proof. In the proof of Lemma 8.14 we found a strictly interior point of $\bar{X}_0$. Some positive multiple of this point is strictly interior in $\bar{X}_f$.

The recession cone of $\bar{X}_f$ is

$$\operatorname{rec}(\bar{X}_f) = \{\mathbf{y} \in \mathbb{R}^{2n-3} : \mathbf{x} + t\,\mathbf{y} \in \bar{X}_f \text{ for all } \mathbf{x} \in \bar{X}_f, t > 0\}.$$

If $\bar{X}_f$ had no vertex then this would also be true for the recession cone $\operatorname{rec}(\bar{X}_f)^\dagger$. The recession cone $\operatorname{rec}(\bar{X}_f)$ equals $\bar{X}_0$ which has a vertex by Lemma 8.14. □

† Sorry, this is not elementary. Consult [219] for some background.

A point $\mathbf{v} \in \bar{X}_f(\mathcal{P})$ is a solution to the system $\mathbf{R}_{\mathcal{P}} \cdot \mathbf{v} \geq f$ of inequalities. Each inequality corresponds to a pair of points, i.e., an edge. An edge is *rigid* with respect to $\mathbf{v}$ iff $\mathbf{v}$ attains equality in the corresponding inequality. Let $E_{\mathbf{v}}$ denote the set of edges which are rigid with respect to $\mathbf{v}$. A face K of $\bar{X}_f(\mathcal{P})$ determines the set E_K of edges which are rigid for every $\mathbf{v} \in K$.

Lemma 8.16 *Consider the set $E_{\mathbf{v}}$ of rigid edges of a point $\mathbf{v} \in \bar{X}_0(\mathcal{P})$. If $E_{\mathbf{v}}$ contains (a) two crossing edges, (b) a non-pointed vertex (c) a convex subpolygon, then it contains the complete graph on the endpoints of all involved edges. In the case of (c) the complete graph also includes all points of $\mathcal{P}$ which are in the interior of the subpolygon. Figure 8.12 illustrates the three cases.*

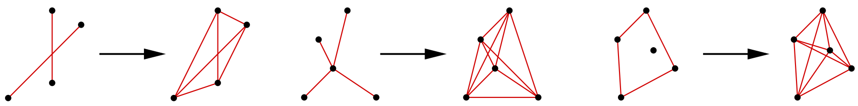


Figure 8.12 The closure for rigid sets of edges $E_{\mathbf{v}}$.

Proof. (a) Let $(\mathbf{p}_i, \mathbf{p}_j)$ and $(\mathbf{p}_k, \mathbf{p}_l)$ be a pair of crossing edges in $E_{\mathbf{v}}$ and let $\mathbf{x}$ be the point of intersection of the two edges. Suppose that the distance between $\mathbf{p}_i$ and $\mathbf{p}_k$ is strictly increasing. Adding a trivial motion to $\mathbf{v}$ we obtain a motion $\mathbf{v}'$ with $\mathbf{v}'_i = \mathbf{v}'_j = (0, 0)$. The motion $\mathbf{v}'$ keeps the edge $(\mathbf{p}_i, \mathbf{p}_j)$ invariant, it is expanding on the pair i, k and non-decreasing on j, k. Since $\mathbf{x}$ is between $\mathbf{p}_i$ and $\mathbf{p}_j$ also $\left\langle\, \mathbf{p}_k - \mathbf{x}\,,\, \mathbf{v}'_k \,\right\rangle > 0$, see Figure 8.13. The vector $\mathbf{p}_k - \mathbf{p}_l$ is a positive scale of $\mathbf{p}_k - \mathbf{x}$, therefore, $\left\langle\, \mathbf{p}_k - \mathbf{p}_l\,,\, \mathbf{v}'_k \,\right\rangle > 0$. Similarly,

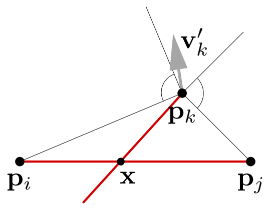


Figure 8.13 Crossing edges in $E_{\mathbf{v}}$.

$\left\langle\, \mathbf{p}_l - \mathbf{p}_k\,,\, \mathbf{v}'_l \,\right\rangle \geq 0$. Together, this implies that the length of $(\mathbf{p}_k, \mathbf{p}_l)$ is not preserved under $\mathbf{v}'$, hence, not under $\mathbf{v}$. This is in contradiction to $(\mathbf{p}_k, \mathbf{p}_l) \in E_{\mathbf{v}}$.

(b) Let $\mathbf{p}_i$ be a non-pointed vertex in $E_{\mathbf{v}}$. Suppose the neighbors of $\mathbf{p}_i$ do not move rigidly with $\mathbf{p}_i$. Since the distances are fixed some angles between neighboring edges at $\mathbf{p}_i$ change. Since the sum of angles around $\mathbf{p}_i$ is constant, a change is only possible if there is a decreasing angle. This, however, implies that the distance between the endpoints decreases, a contradiction.

(c) This time we consider the inner angles at the vertices of a convex polygon in $E_{\mathbf{v}}$. If the polygon does not move rigidly, some of these angles change. Since the sum of the inner angles is constant a change is only possible if one of them decreases. This, however, would imply that the distance between the endpoints decreases. Therefore, the the complete graph on the vertices of the polygon belongs to $E_{\mathbf{v}}$.

Now consider a point interior to the polygon. Adding a trivial motion to $\mathbf{v}$ we obtain a motion $\mathbf{v}'$ which keeps the polygon stationary. If the interior point moves it decreases its distance to one of the polygon edges and, hence, to a polygon vertex. □

Theorem 8.17 *For every set $\mathcal{P}$ of $n \geq 3$ points in general position there exist perturbations f_{ij} such that there is a bijection between the faces of the polyhedron $\bar{X}_f(\mathcal{P})$ and pointed non-crossing graphs on $\mathcal{P}$. The bijection $K \leftrightarrow E_K$ is order reversing, i.e., $K \subset K'$ if and only if $E_K \supset E_{K'}$. In particular:*

(a) Vertices of $\bar{X}_f(\mathcal{P})$ correspond to pointed pseudotriangulations of $\mathcal{P}$.

(b) The graph of bounded edges of $\bar{X}_f(\mathcal{P})$ is the flip-graph $\mathcal{G}^{\curlyvee}$.

(c) Extreme rays correspond to pointed pseudotriangulations with one convex hull edge removed.

A choice of perturbations $f = (f_{ij}) \in \mathbb{R}^{\binom{n}{2}}$ is called *valid* if the statement of Theorem 8.17 becomes true.

Lemma 8.18 *A set of perturbations f_{ij} is valid if and only if the graph with edge set $E_\mathbf{v}$ is non-crossing and pointed for every $\mathbf{v} \in \bar{X}_f(\mathcal{P})$.*

Proof. For every vertex $\mathbf{x}$ of $\bar{X}_f(\mathcal{P})$ the claimed properties of $E_\mathbf{x}$ follow from the definition of valid. If $\mathbf{v} \in \bar{X}_f(\mathcal{P})$ is a convex combination of vertices $\mathbf{x}_i$, $i \in I$, then $E_\mathbf{v} = \bigcap_{i\in I} E_{\mathbf{x}_i}$. This implies that the graph with edge set $E_\mathbf{v}$ is non-crossing and pointed. The harder part is the converse.

The polyhedron $\bar{X}_f$ is $(2n-3)$-dimensional, see Lemma 8.15. Hence, every vertex $\mathbf{v}$ of $\bar{X}_f$ is incident to at least that many facets and these facets correspond to rigid edges, i.e., edges in $E_\mathbf{v}$. The graph with edge set $E_\mathbf{v}$ is pointed and non-crossing by assumption. With Theorem 8.11 it follows that it is a pointed pseudotriangulation, in particular $|E_\mathbf{v}| = 2n-3$. It follows that $\mathbf{v}$ is incident to $2n-3$ facets and, by definition, the polyhedron is a simple polyhedron. Every vertex figure of a simple polyhedron is a simplex. Therefore, the faces incident to a vertex $\mathbf{v}$ of $\bar{X}_f$ are in bijection with subsets of facets incident to $\mathbf{v}$. These subsets of facets in turn correspond to subgraphs of the pointed pseudotriangulation $E_\mathbf{v}$.

Since the polyhedron $\bar{X}_f$ is simple, every vertex $\mathbf{v}$ is incident to $2n-3$ edges of the polyhedron. These edges of $\bar{X}_f$ correspond to the subgraphs of $E_\mathbf{v}$ obtained by deleting a single edge. Let K_{ij} be the polyhedral edge corresponding to the removal of the edge $\{i,j\}$ from $E_\mathbf{v}$. If K_{ij} is bounded, then there is a second pointed pseudotriangulation containing $E_\mathbf{v} \setminus \{i,j\}$, this must be the pointed pseudotriangulation obtained from $E_\mathbf{v}$ by flipping the edge $\{i,j\}$. Hence, the bounded edges of $\bar{X}_f$ at vertex $\mathbf{v}$ correspond to the flips of the pointed pseudotriangulation $E_\mathbf{v}$. Since the flip-graph is connected (Proposition 8.13.b) we can reach every vertex of $\bar{X}_f$ along bounded edges from an arbitrary initial vertex. Since $\bar{X}_f$ has a vertex, this implies that all pointed pseudotriangulations appear as vertices. Moreover, the graph of bounded edges is exactly the flip-graph $\mathcal{G}^{\curlyvee}$. The simplicity of $\bar{X}_f$ implies that all pointed and non-crossing graphs appear as the graphs of rigid edges corresponding to faces of $\bar{X}_f$.

As for the extreme rays, they also correspond to subgraphs of $E_\mathbf{v}$ obtained by deleting a single edge. In this case there is exactly one completion to a pointed pseudotriangulation. Therefore, the removed edge must be an edge from the convex hull. □

The following lemma provides a simple criterion for valid perturbations.

Lemma 8.19 *A set of perturbations $(f_{ij}) \in \mathbb{R}^{\binom{n}{2}}$ is valid if and only if its restriction to every four point subset of $\mathcal{P}$ is valid.*

Proof. The implication "$\Longrightarrow$" is obvious. For the converse we use the previous lemma. There it was shown that $f = (f_{ij})$ is valid iff $E_{\mathbf{v}}$ is non-crossing and pointed for every $\mathbf{v} \in \bar{X}_f$. If this condition is violated then there is a subset $\mathcal{P}'$ of four points in $\mathcal{P}$ such that the subgraph of $E_{\mathbf{v}}$ induced by these four points violates either the non-crossing or the pointedness condition. Let $\mathbf{v}'$ and f' be the restrictions of $\mathbf{v}$ and f to $\mathcal{P}'$. Then $\mathbf{v}' \in \bar{X}_{f'}(\mathcal{P}')$ and the graph with edge set $E_{\mathbf{v}'}$ is crossing or non-pointed, hence f' is not valid for $\mathcal{P}'$. □

The following theorem completes the proof of Theorem 8.17.

Theorem 8.20 *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be any two points in the plane. For every set of n points* $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$ *in general position in the plane the following perturbations $f = (f_{i,j})$ are valid:*

$$f_{ij} = \det \begin{pmatrix} \mathbf{a} & \mathbf{p}_i & \mathbf{p}_j \\ 1 & 1 & 1 \end{pmatrix} \cdot \det \begin{pmatrix} \mathbf{b} & \mathbf{p}_i & \mathbf{p}_j \\ 1 & 1 & 1 \end{pmatrix}.$$

The full proof can be found in the original source, Rote, Streinu and Santos [166]. There it is verified that the perturbations of the theorem are valid on every subset of four points. By Lemma 8.19 this implies that f is valid for all n points. We indicate the main stations in the argument for 4-element point sets. Let $\mathcal{P} = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$ be in general position.

- Let $\gamma_{ij} = \det \begin{pmatrix} \mathbf{p}_i & \mathbf{p}_j & \mathbf{p}_k \\ 1 & 1 & 1 \end{pmatrix} \cdot \det \begin{pmatrix} \mathbf{p}_i & \mathbf{p}_j & \mathbf{p}_l \\ 1 & 1 & 1 \end{pmatrix}$ and $\omega_{ij} = \dfrac{1}{\gamma_{ij}}$, here k and l are the two indices other than i and j. The weights ω_{ij} on the edges of the complete graph K_4 on $\mathcal{P}$ define a self-stress.
- The signs of the determinants in the definition of ω_{ij} correspond to the orientations of the triangles $\{\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k\}$ and $\{\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_l\}$. Therefore, $\omega_{ij} > 0$ if $\{i, j\}$ is a convex hull edge and $\omega_{ij} < 0$ if $\{i, j\}$ is an interior edge. Equivalently: $\omega_{ij} < 0$ iff $K_4 \backslash \{i, j\}$ is a pointed pseudotriangulation.
- Let $R = \sum_{1 \le i < j \le 4} \omega_{ij} f_{ij}$ with ω_{ij} and f_{ij} as above. For every edge $\{k, l\}$ of K_4 the following statements are equivalent:
  (1) The cone $\bar{X}_f(\mathcal{P})$ has a vertex $\mathbf{v}$ such that $E_{\mathbf{v}}$ is $K_4 \setminus \{k, l\}$.
  (2) R and ω_{kl} have opposite signs.
- With the above the claim of the theorem can be reduced to the inequality $R > 0$. For ω_{ij} and f_{ij} as above, the stronger statement $\displaystyle\sum_{1 \le i < j \le 4} \omega_{ij} f_{ij} = 1$ can be verified:

  The idea is to consider $R = \sum_{1 \le i < j \le 4} \omega_{ij} f_{ij}$ as a function $R(\mathbf{a}, \mathbf{b})$ of $\mathbf{a}$ and $\mathbf{b}$. For fixed $\mathbf{a}$ this is an affine function $R(\mathbf{a}, \mathbf{b}) = R_{\mathbf{a}}(\mathbf{b})$ of $\mathbf{b}$. Since $R(\mathbf{p}_i, \mathbf{p}_j) = 1$ for all $i \neq j$, it follows that $R_{\mathbf{p}_i}(\mathbf{b}) \equiv 1$. Exchanging the roles of $\mathbf{a}$ and $\mathbf{b}$ we note that $R(\mathbf{a}, \mathbf{b}) = R_{\mathbf{b}}(\mathbf{a})$ is affine and attains the value 1 whenever $\mathbf{a} = \mathbf{p}_i$, therefore, $R(\mathbf{a}, \mathbf{b}) \equiv 1$.

This completes the sketch for the proof of Theorem 8.20. The conclusion is that the polyhedron of pointed pseudotriangulations $\bar{X}_f$ with properties as listed in Theorem 8.17 exists.

The defining inequalities of $\bar{X}_f$ which correspond to convex hull edges can be set to equality:

$$\left\langle \mathbf{p}_i - \mathbf{p}_j \,,\, \mathbf{v}_i - \mathbf{v}_j \right\rangle = f_{ij} \qquad \text{for all convex hull edges } ij.$$

These equality constraints make the set of solutions $\mathbf{v}$ bounded. But still, with the above choice of perturbations f the vertices of this polytope $X_f(\mathcal{P})$ are in bijection to pointed pseudotriangulations of $\mathcal{P}$. The polytope $X_f(\mathcal{P})$ is the *polytope of pointed pseudotriangulations*. It should be remarked that in the case of a set $\mathcal{P}$ of points in convex position the polytope $X_f(\mathcal{P})$ is an associahedron. More precisely, the combinatorial structure of $X_f(\mathcal{P})$ is the structure of the associahedron, the representation by inequalities is not equivalent to the representation of the associahedron as secondary polytope.

8.5 Expansive Motions and Straightening Linkages

A *linkage* is a planar (non-crossing) framework whose graph is a path. *Straightening a linkage* means to apply a motion to the framework which ends in an embedding of the path on a line. The problem gets interesting by the requirement that throughout the motion the linkage remains planar, that is free of self-intersections.

The Carpenter's Rule Problem: Is it always possible to straighten a linkage?

Based on the theory presented in this chapter the above problem has an elegant affirmative answer. Actually, expansive motions were invented for the proof that every linkage can be straightened. The essential observation in this context is the fact that the application of an expansive motion to a non-crossing framework will preserve planarity.

Let L be a linkage which is not straight and let $\mathcal{P}$ be the points/vertices of L. In the convex hull $\text{CH}(\mathcal{P})$ there is an edge $\{i,j\}$ which does not belong to L. The expansive cone $\bar{X}_0(\mathcal{P})$ contains a ray $\vec{r}$ such that all edges of L belong to $E_{\vec{r}}$, the set of rigid edges of motions in $\vec{r}$, but $\{i,j\} \notin E_{\vec{r}}$. This means that $\{i,j\}$ is infinitesimally expanding.

The crucial result is that there is a real motion corresponding to the infinitesimal motion $\vec{r}$. The real motion is curved, see Figure 8.14. The infinitesimal motion only gives

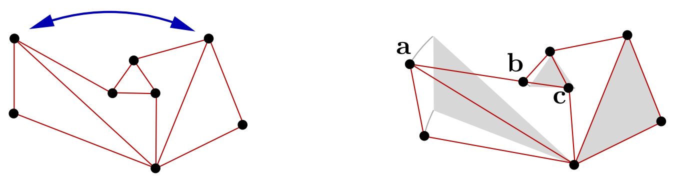


Figure 8.14 An expanding motion, stopped when $(\mathbf{a},\mathbf{b})$ and $(\mathbf{b},\mathbf{c})$ become collinear.

the initial velocities. The motion from the framework in the left part of the figure to the framework at the right can be interpreted as a smooth trajectory belonging to a certain initial infinitesimal motion. In the configuration on the right the edges $\mathbf{a},\mathbf{b}$ and $\mathbf{b},\mathbf{c}$ are collinear and the distance of $\mathbf{a}$ and $\mathbf{c}$ is at its extreme. Still this framework has an expansive motion but with a different set of rigid components. These cursory remarks may serve as an indication of how to combine small expanding motions whose existence comes from infinitesimal expanding motions to a global straightening motion for a linkage.

8.6 Notes and References

The first major result in rigidity theory was a theorem of Cauchy.

> If two combinatorially equivalent 3-polyhedra P and P' are realized in $\mathbb{R}^3$ as convex polytopes such that corresponding faces are congruent, then the two polytopes are congruent.

A nice proof of Cauchy's theorem can be found in The Book [8]. Figure 8.15 illustrates that the conditions in the theorem are necessary. Dropping the convexity condition or the condition on the faces allows non-congruent realizations. In particular it is not enough to require that the edges of P and P' are of the same length to make them congruent.

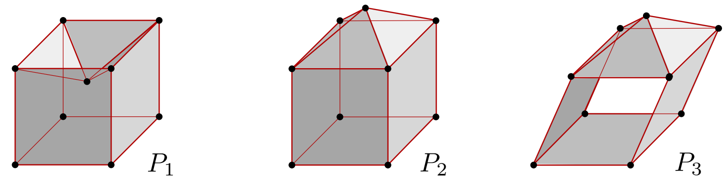


Figure 8.15 Polyhedron P_1 is non-convex, P_2 and P_3 have edges of equal length but non-congruent faces.

Asimow and Roth [17] showed that the 1-skeleton of a strictly convex polytope in 3-space which has at least one non-triangular face is not rigid.

It is not hard to see that the graphs of Asimow and Roth are not generically rigid in $\mathbb{R}^3$. This follows from a generalization of Proposition 8.5: If a graph $G = (V, E)$ is minimally generically rigid in dimension 3, then $|E| = 3|V| - 6$ and $|E'| \leq 3|V_{[E']}| - 6$ for all $E' \subset E$.

Laman [128] proved that in two dimensions the counting condition is also sufficient for generic rigidity. In three dimensions there are graphs with the 'right number of edges' which are non-rigid. It remains a major open problem to find a combinatorial characterization of generic 3-rigidity of graphs.

As shown in Theorem 8.9 both Laman property and Henneberg construction can be interpreted as characterizations of graphs which are the union of two spanning trees after adding any new edge. Frank and Szegő [93] extend this to a characterization of graphs which are the union of k spanning trees after addition of a new edge.

The articles of Whiteley [215] and [216] and the book by Graver, Servatius and Servatius [105] give comprehensive introductions into rigidity theory. These sources also provide lots of pointers into the huge literature on this topic.

A set of edges of a framework $G_{[\mathbf{p}]} = (V, E, \mathbf{p})$ is *independent* iff the corresponding rows of the rigidity matrix are independent. Therefore, the independent subsets of E are a matroid. If the embedding $\mathbf{p}$ is generic, this is the *generic rigidity matroid* of G. The bases of the generic rigidity matroid of a complete graph are exactly the mgr-graphs. In the 1-dimensional case the generic rigidity matroid of a connected graph G coincides with the cycle matroid of G (see page 138).

Pseudotriangulations have been used and studied in the context of ray-shooting by Chazelle et al. [47] and of visibility by Pocchiola and Vegter [158, 157] and others [146]. A paper of Streinu [190] where pseudotriangulations were applied to the problem of straightening linkages made them very popular. Since then new applications in polygon guarding, Speckmann and Tóth [181] and kinetic data structures, Kirkpatrick et al. [122], have been found.

There has also been a lot of research around questions of generating and enumerating pseudotriangulations, see [6, 7, 23, 41, 55]. The most prominent open problem in the area is the question whether every plane set of points has at least as many minimal pseudotriangulations as triangulations.

Kettner et al. [121] have shown that every point set has a pseudotriangulation P such that the maximum degree of a vertex in P is 5, this result is best possible. Bespamyatnikh [24] shows that the diameter of the flip graph is of order $O(n \log n)$.

Haas et al. [111] have the following result: If G is a planar graph with the Laman property, then there is an embedding of G as a pointed pseudotriangulation.

Connelly, Demaine, and Rote [51] proved that linkages in the plane can be straightened and closed polygonal chains can be convexified. Their unlocking motions are expansive. The proof relies on the duality of stresses and motions. The existence of an infinitesimal expansive is shown via the Maxwell-Cremona theorem. The real motion is then obtained as a solution to differential equations.

Streinu [190] proved that a pointed pseudotriangulation is rigid and that removing a convex hull edge yields one degree of freedom (1-DOF) which corresponds to an expansive motion. Adding bars to a given collection of polygonal chains to produce a pointed pseudotriangulation and using flips to get from one pseudotriangulation to another, leads to a piecewise-algebraic straightening motion.

Rote, Streinu and Santos [166] introduce the expansive cone $\bar{X}_0(\mathcal{P})$ and the perturbed cone $\bar{X}_f(\mathcal{P})$ of a point set $\mathcal{P}$. Section 8.3 follows closely along the lines of that paper. The cited paper contains the full proof of Theorem 8.20. Moreover, the authors discuss connections of the polytope of constrained expansions with the associahedron. A surprising appearance of the associahedron is found as follows: They consider a special perturbation of the cone of expansive motions in one dimension. The vertices of this perturbed cone are shown to be in in bijection with non-crossing alternating trees. Non-crossing alternating trees are a Catalan family and indeed by adding a bounding constraint they obtain a polytope isomorphic to the associahedron.

Orden and Santos [145], building on ideas from [166], find a more general polyhedron whose faces correspond to non-crossing marked geometric graphs on a planar point set $\mathcal{P}$. In this context a marking of a graph is a subset of its pointed vertices.

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