Embeddings of Planar Graphs.

1 Drawing (I). (Bipolar Orientations). (22-3-2006)

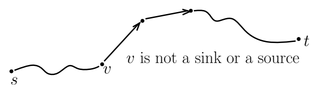
1.1 Definitions and previous lemmas.

Definition 1.1 *Let $G = (V; E)$ be a digraph, a topological numbering of G is $T : V \to \mathbb{N}$ such that $t(u) < t(v)$ for all $(u, v) \in E$. It is a topological sorting if $t : V \to [u]$*

Fact 1.1 *A topological sorting exists iff G is acyclic.*

Definition 1.2 *A digraph G is bipolar if G is acyclic and G has unique source s and sink t.*

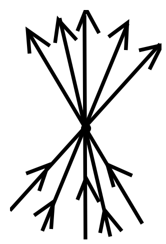
Fact 1.2 *G bipolar $\Rightarrow \forall v \in V$ there is a directed $s - t$ path through v.*

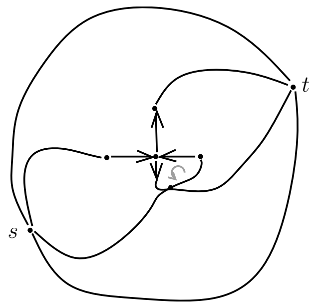


Definition 1.3 *A digraph is planar bipolar if s and t are on the outer face.*

Exercises 1.1 *Bipolar orientation can be found in $O(n + m)$.* ***Q****: When does it exist?*

Lemma 1.4 *G planar bipolar $\Rightarrow$ every vertex shows this pattern:*





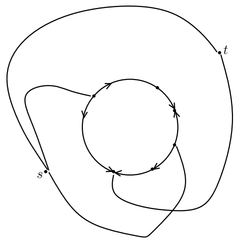
Proof: Suppose this is not the case, so wehave this kind of situation:

There are a direct cyclic (CONTRADICTION!!) (See the last figure). □

Lemma 1.5 *Every face has a unique source and sink.*

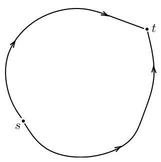
Proof: We know that:

- It has at least one.
- If a face has two sinks then it has two sources and they alternate. So we have a direct cyclic. (CONTRADICTION!!).



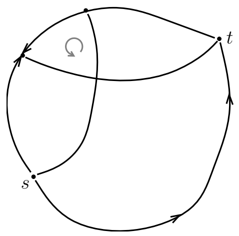
□

Lemma 1.6 *The outer face consists of two $s - t$ path.*

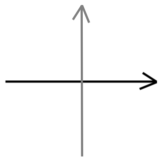


Proof: If not

(CONTRADICTION!!)(See next figure). □



Definition 1.7 *Dual $G^\star$ of a planar bipolar orientation.*

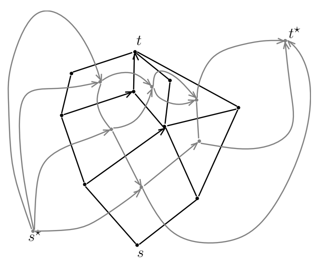


Lemma 1.8 $G^{\star\star} = \overleftarrow{G}$

Proposition 1.1 *$G^\star$ is planar bipolar.*

Proof: It is planar. It has an unique sink and source (lemmas 1.2 and 1.3). It is acyclic (lemma 1.2 unique s and t of G). □

Example 1.1



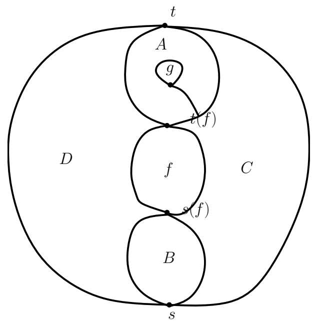
Lemma 1.9 *Let f and g be two faces of a planar bipolar graph G. Exactly one of the following holds:*

(a) G has a directed path from $t(f)$ to $s(g)$.

(b) G has a directed path from $t(g)$ to $s(f)$.

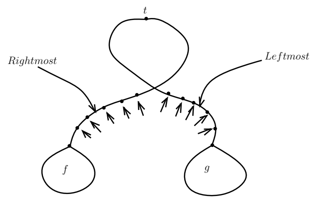
(c) $G^\star$ has a directed path from f to g.

(d) $G^\star$ has a directed path from g to f.



Proof: We consider the $t(f)$ to t path always taking leftmost outgoing edge.

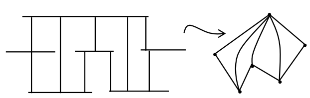
If g is in region A, this is (a). If g is in region B, this is (b). If g is in region C



If g is in region D, similar to region C, but changing f and g. □

1.2 Visibility representation.

- With each vertex associate a horizontal segment (they do not overlap).
- With each edge there is a vertical segment tauching the vertex segment of the edge not intersecting any other vertex segment.



Observation 1.1 *If G has a visibility representation then G is planar.*

Definition 1.10 *G planar bipolar and $G^\star$ its dual:*

- *$rk(v)$=length of a longest s to v path in G.*
- *$rk(f)$=length or a longest $s^\star$ to f path in $G^\star$.*
- $S_v = [(rk(f_v^{left}), rk(v)), (rk(f_v^{right}), rk(v))]$

- $S_f = [(rk(f), rk(s(f))), (rk(f), rk(t(f)))]$

Theorem 1.11 *The segment S_v, S_f are a simultaneous visibility representation of G and $G^\star$.*

Additional. Vertex and face segment are internally disjoint.

Proof:

Claim 1. f segments are disjoint.

Let f, g be faces.

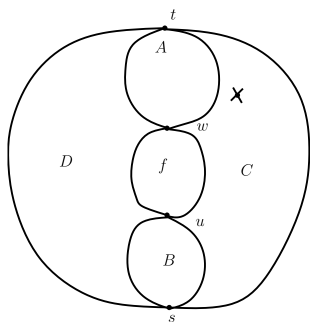
- Cases c and $d \Rightarrow rk(f) \neq rk(g)$.
- Case a $rk(t(f)) \leq rk(s(g))$. (Equal if the vertices are the same).

Claim 2. v segments are disjoint.

Claim 1 + Lemma 4.

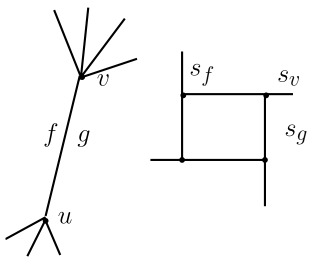
Claim 3. S_f and S_v are internaly disjoint.

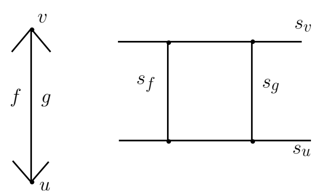
- $v \in A \Rightarrow rk(v) \geq rk(w)$.
- $v \in B \Rightarrow rk(v) \leq rk(u)$.
- $v \in C \Rightarrow rk(f_v^{left}) \geq rk(f)$.



- If $v \in f \Rightarrow$ internally disjoint.

We can represent the edges of G and $G^\star$.

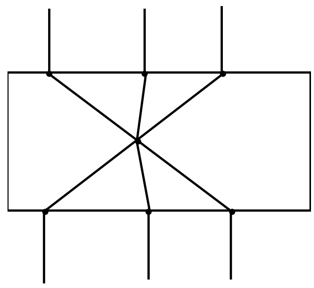




□

Corollary 1.12 *(Visibility Representation.)*
Height $\leq n-1$.
Weidth $\leq 2n-5$.
Computable in $O(n+m)$.

Polyline drawing of this size with at most 2 bends per edge.



1.3 More Compact Visibility Representation.

- Kant Kant He 97

  weidth $\leq \frac{3n-6}{2}$

  weidth $\leq n-1$ if G is 4 connected.

- Lin Lu Sun 03.

  weidth $\leq \frac{22n-42}{15}$ $(1.4\overline{6})$.

- Zang He 03.

  height $\leq \frac{15n}{16}$

  height $\leq \frac{3}{4}n$ if G is 4-connected.