

Embeddings of Planar Graphs. March 20, 2006. 11:00–12:30.

Theorem 1 *If a graph G has a drawing such that any pair of edges has an even number of crossings, then G is planar.*

Proof. We allow the graph to be a multigraph with loops. We prove the theorem by induction on the number of vertices.

The base case is a graph on a single vertex v . The edges then are loops. Two loops a, b cross each other evenly. Therefore there is no “ $abab$ ” in the rotation system of v (=the cyclic order of edges at v) and so we can find a crossing-free drawing of G that has the same rotation as the given drawing.

The induction step: Contract one edge (u, v) such that u remains and the new edges to u run close to the old edge (u, v) . Then any pair of edges in the new graph has an even number of crossings and the induction assumption gives us a crossing-free drawing that keeps the rotation system. This drawing can be easily extended to a crossing-free drawing of G . \square

Theorem 2 (Pach, Toth) *Let Θ be a drawing of a graph G and E^* be the set of even edges (=edges such that their crossing number with every other edge is even). Then there is a drawing $\hat{\Theta}$ of G such that all edges in E^* are free of crossings and there are no new odd intersecting pairs.*

Proof. Again we allow multigraphs with loops and we preserve the rotations. The proof is by induction on the number of edges of E^* that are no loops.

The base case is a graph with no edges in E^* except for loops. Let l be a loop of E^* . Because l is in E^* every other edge starts and ends either inside or outside the loop. So we can classify edges and vertices of G as inner or outer. Now we remove all crossings of l by drawing the inner parts inside the loop and the outer parts outside the loop. (The algorithm is left as an exercise.)

The induction step is the same as in theorem 1. \square

Theorem 3 (Hanani, Tutte) *If a graph G has a drawing such that any pair of non-adjacent edges has an even number of crossings, then G is planar.*

Proof 1. The first proof is by induction on $w(G) := \sum_{v \in V} d(v)^3$. Let $P \subseteq E$ be the set of processed edges. Initially let P be the empty set. Invariant:

$$e \in P \quad \Rightarrow \quad e \text{ is free of crossing and } P - e \text{ is connected} \quad (1)$$

Let e be an edge involved in an odd number of crossings (if no such edge exists, we can apply theorem 1). If e is a bridge, then delete e , find a crossing-free drawing of the components of $G - e$ (the weight of the components is less than $w(G)$) and reinsert e . If e is not a bridge, e is on a cycle C .

case 1 On C there is a vertex u incident to P -edges and to a pair of odd crossing edges e_1, e_2 . It then follows from (1) that u is on a P -cycle and that e_1 and e_2 are both on the same side of this cycle. Now we split u into the two adjacent vertices u_1, u_2 where u_1 is adjacent to all edges inside the cycle and u_2 to all edges outside the cycle. The weight $w(G)$ after splitting is less $((d_1 + 1)^3 + (d_2 + 1)^3 < (d_1 + d_2)^3$), so we can make use of the induction assumption.

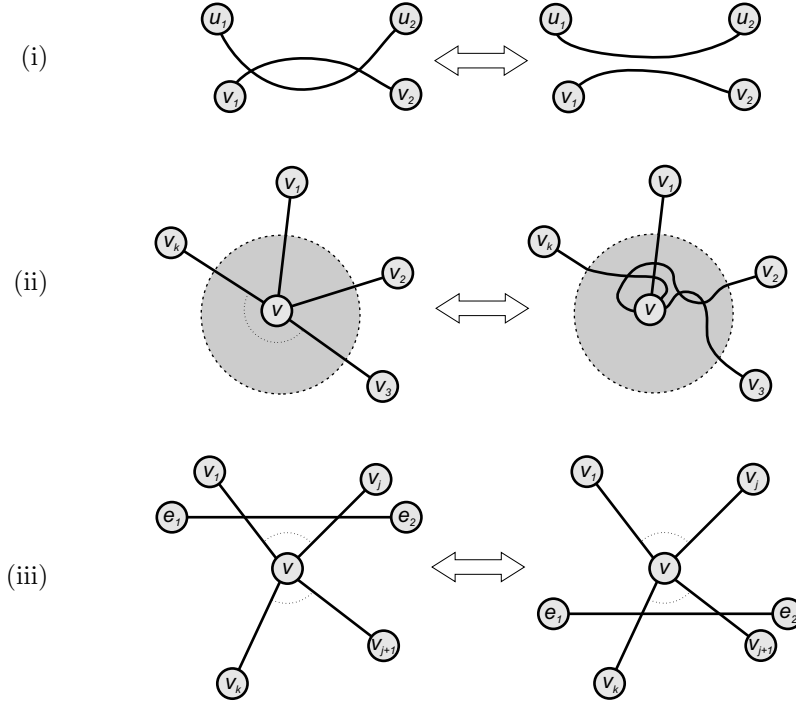
case 2 If case 1 doesn't hold we have for each vertex $u \in C$:

- (i) All edges incident with u are unprocessed (i. e. not in P) or
- (ii) The number of crossings of every two edges incident with u is even.

As long as there are vertices with property (i) but not (ii), we make them (ii) by changing the rotation at u locally. If all vertices on C are of type (ii) it follows from (ii) and the assumption of the theorem that all edges of C are even. Now we can apply theorem 2, put the edges of C in P and continue.

□*Proof 1*

Proof 2. Manipulating drawings: Any two drawings of a graph can be deformed into each other such that all changes in the intersection of edges are of the following three types: (The proof is left as an exercise.)



With a drawing Θ we associate a vector z_Θ with components

$$z_\Theta(e, e') := \begin{cases} 0 & : \text{ number of crossings of } e \text{ and } e' \text{ is even} \\ 1 & : \text{ otherwise} \end{cases}$$

for every pair of non-adjacent edges e, e' . Let $D_G := \{z_\Theta : \Theta \text{ drawing of } G\}$. Then theorem 3 reads as follows: If $0 \in D_G$, then G is planar.

Let us consider what happens with z_Θ if we deform the drawing Θ : (i) doesn't change any parity, (ii) only changes crossings of adjacent edges and (iii) changes the parity $z_\Theta(e, e')$ where e is the edge moving around v and e' is an edge incident with v . It follows that

$$D_G = z_{\Theta_0} + \langle x_{ev} : e \in E, v \in V \rangle$$

where Θ_0 is an arbitrary drawing and $x_{ev}(e', e'') = 1$ if $e' = e$ and e'' incident with v , otherwise $= 0$.

Let G be a non-planar graph. By Kuratowski it has a subdivision of K_5 or $K_{3,3}$. If we can show that $0 \notin D_{K_5}$ and $0 \notin D_{K_{3,3}}$ then it follows that $0 \notin D_G$.

K_5 has a drawing Θ_0 such that z_{Θ_0} has exactly one 1 (e.g. one with only one crossing). Each x_{ev} has exactly two entries 1 (Let e, v be arbitrary. There are exactly two edges at v not adjacent with e). Therefore every vector in D_{K_5} has an odd number of 1's. A similar argument works for $K_{3,3}$.

□*Proof 2*