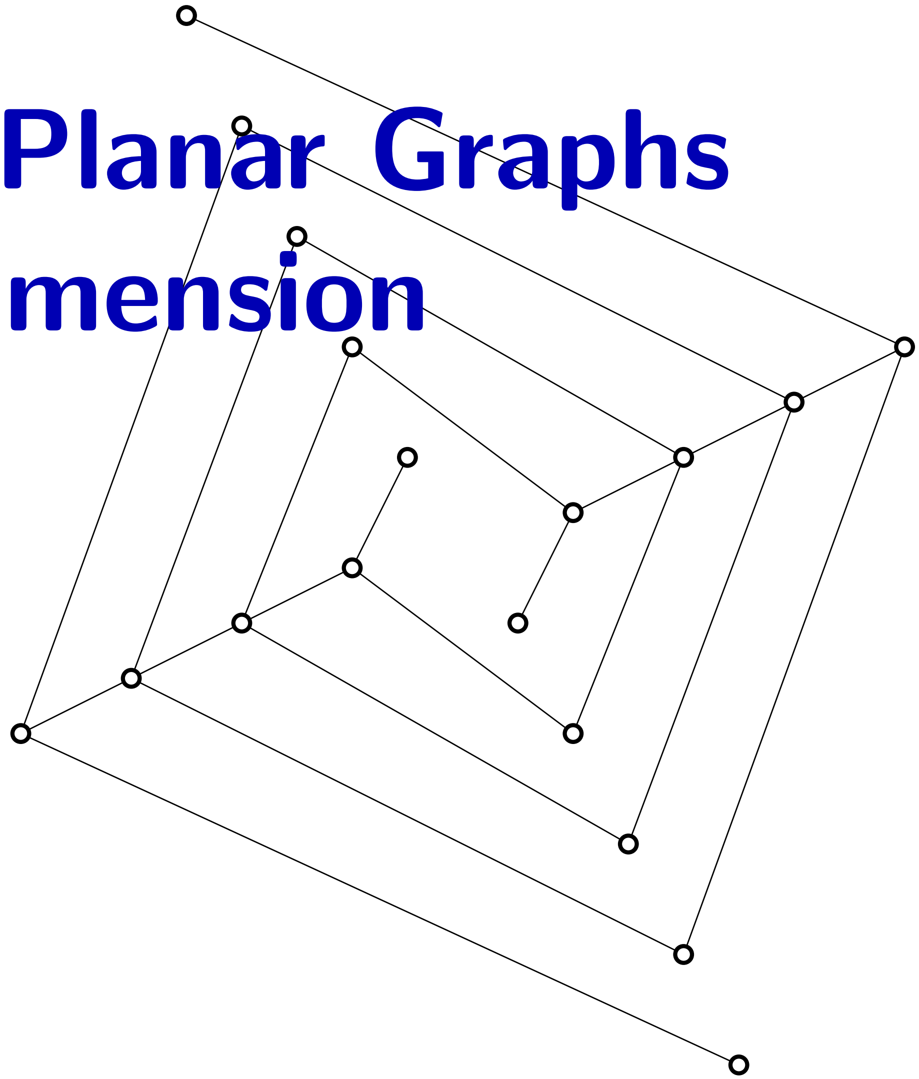
Embeddings of Planar Graphs
Lecture 13 – Dimension



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Modules

Dimension

Orthogonal Surfaces and Dimension

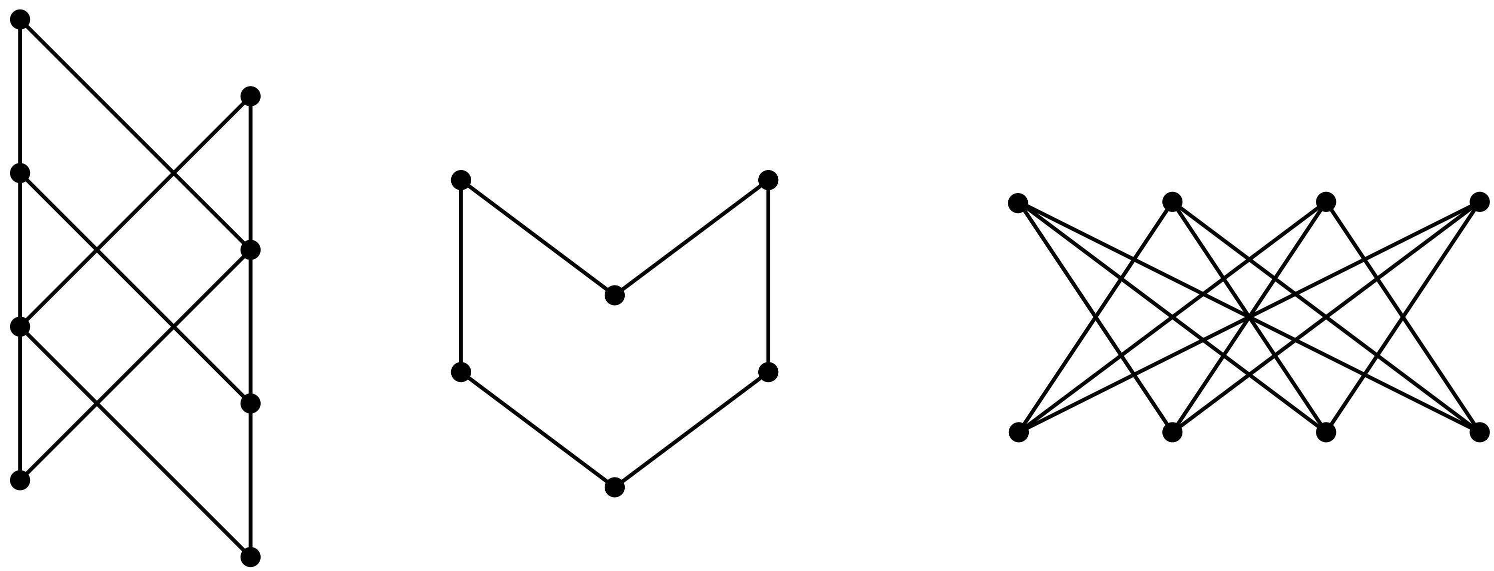
Dimension

Orthogonal Surfaces and Dimension

Finite Orders

$P = (X, <)$ is an order iff

- X finite set
- $<$ transitive and irreflexive relation on X.

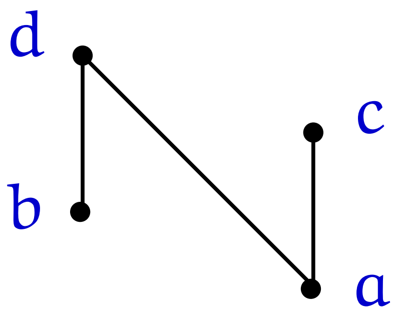


Orders drawn as diagrams.

Linear Extensions

A linear extension of $P = (X, <)$ is a linear order L, such that

- $x <_P y \implies x <_L y$



d	c	d	d	c
c	d	b	c	d
b	b	c	a	a
a	a	a	b	b

Specifying Linear Extensions: Two Recipes

- The generic algorithm
- Acyclic extensions

Dimension of Orders

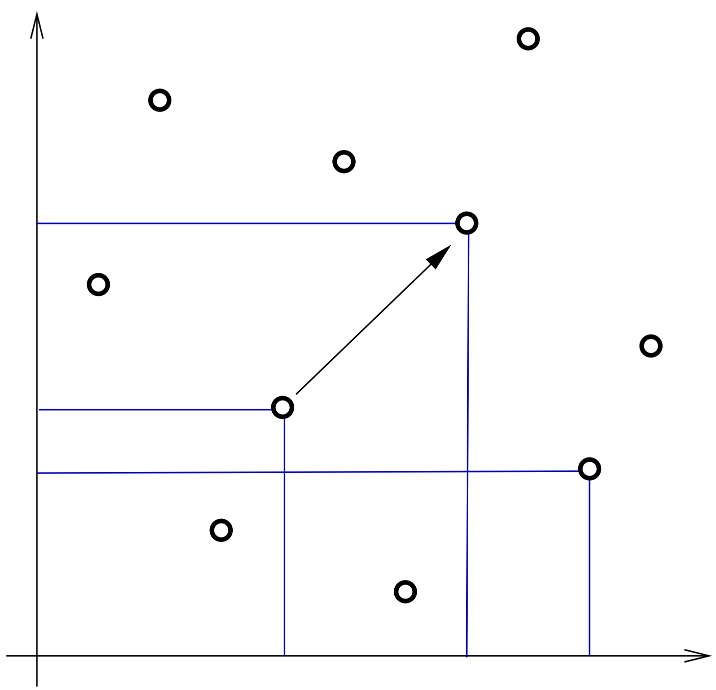
A family $\mathcal{L}$ of linear extensions is a realizer for $P = (X, <)$ provided that

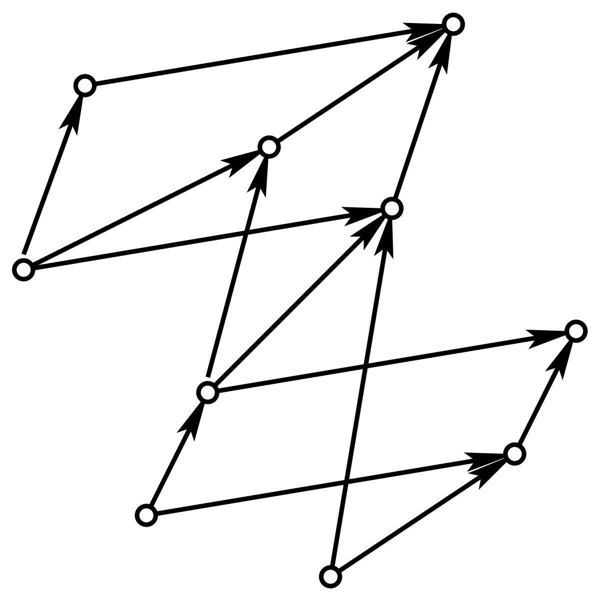
* for every incomparable pair (x, y) there is an $L \in \mathcal{L}$ such that $x < y$ in L.

The dimension, $\dim(P)$, of P is the minimum t, such that there is a realizer $\mathcal{L} = \{L_1, L_2 \ldots, L_t\}$ for P of size t.

Dimension of Orders

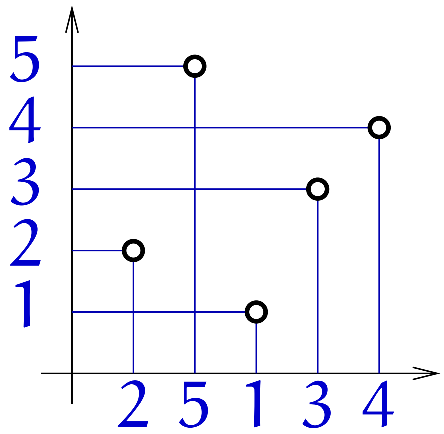
The dimension of an order $P = (X, <)$ is the least t, such that P is isomorphic to a suborder of $\mathbb{R}^t$ with the product ordering.

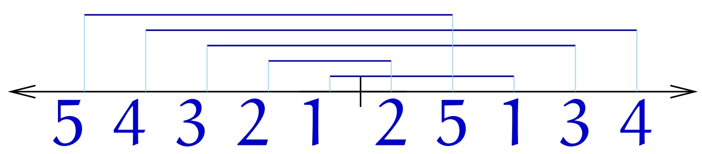




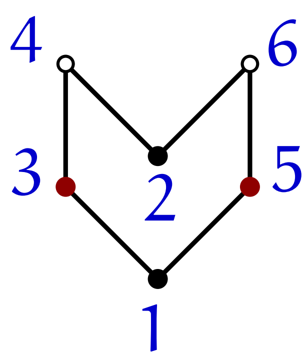
Characterizations

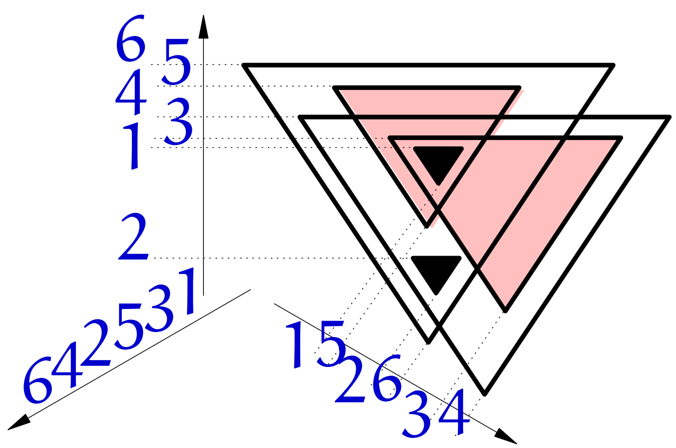
- Dimension 2: Containment orders of intervals.





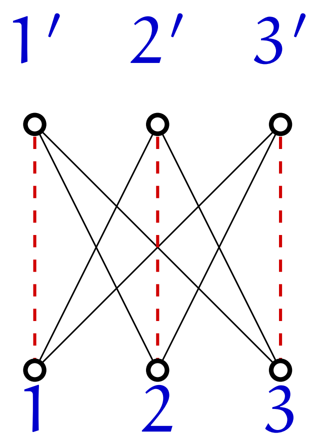
- Dimension 3: Containment orders of triangles.

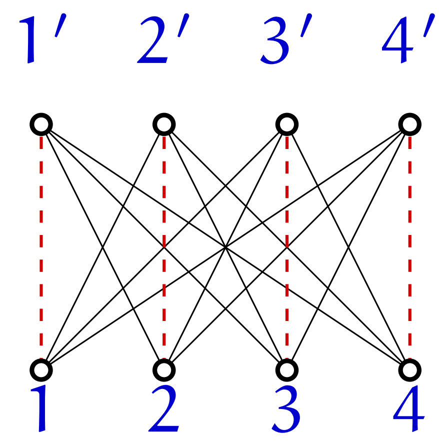




Standard Examples

- Standard example of an n dimensional order:





Orders and Planarity

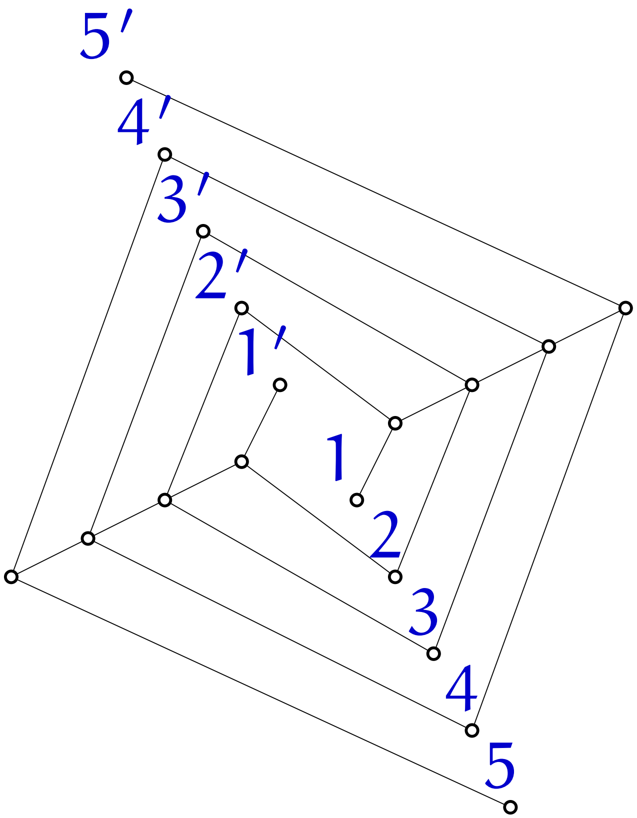
- If an order P has $\mathbf{0}$ and $\mathbf{1}$ and a planar diagram, then $\dim(\mathsf{P}) = 2$.

- If an order P has $\mathbf{0}$ and a planar diagram, then $\dim(\mathsf{P}) \leq 3$.

- The dimension of an order P with a a planar diagram can be unbounded.

Plane Standard Example

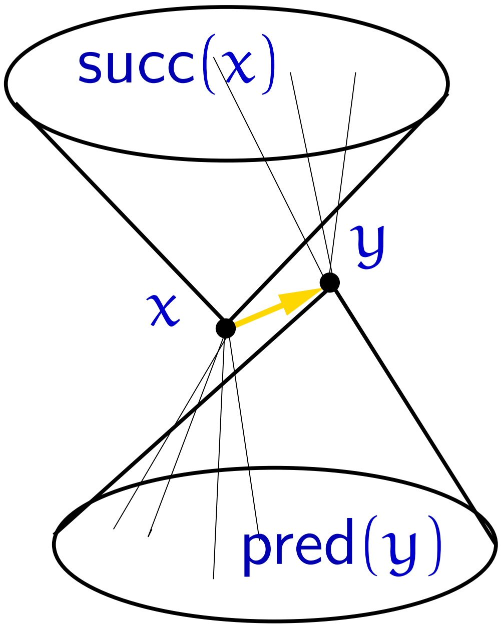
The dimension of an order P with a a planar diagram can be unbounded:



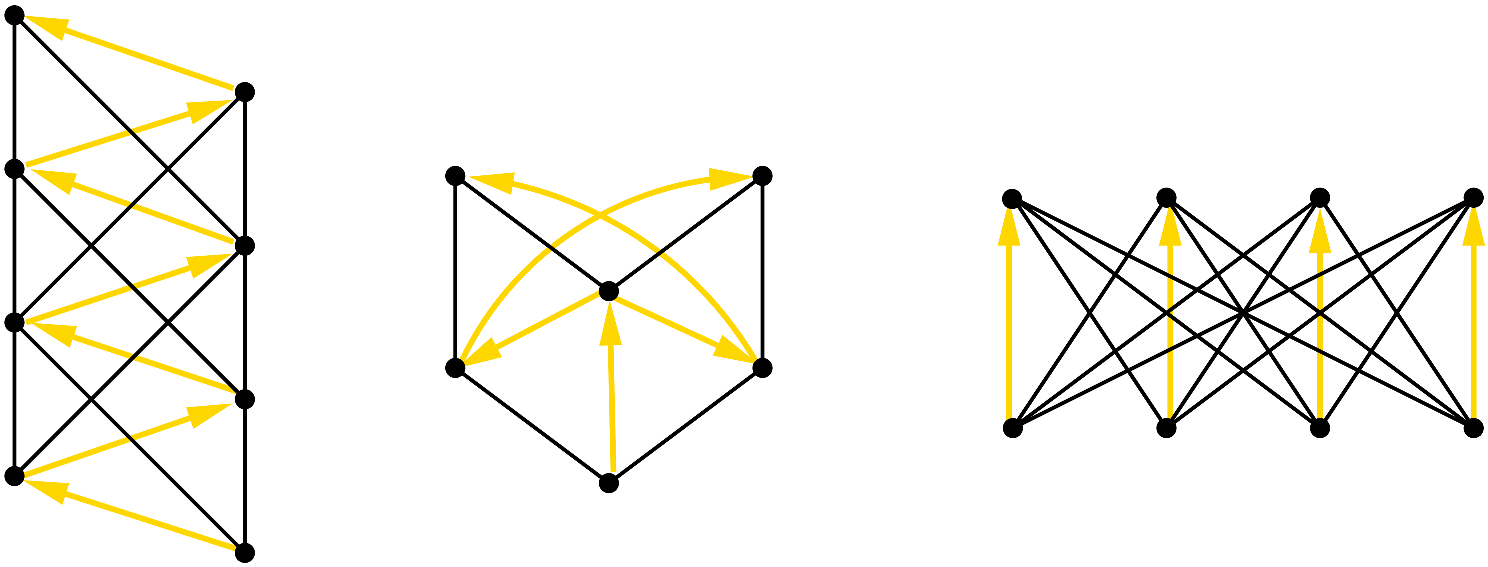
Critical Pairs

Definition. An incomparable pair (x, y) is critical if

- $a < x$ implies $a < y$.
- $y < b$ implies $x < b$.



Critical Pairs



Proposition. A family $\mathcal{R}$ of linear extensions of P is a realizer of P $\iff$ $\mathcal{R}$ reverses all critical pairs.

The Hypergraph of Critical Pairs

Definition. The hypergraph of critical pairs H_P has the set of critical pairs as vertices. A set of critical pairs is an edge if it can't be reversed by a single linear extension and is minimal.

Definition. The graph of critical pairs G_P has the set of critical pairs as vertices. A set of two critical pairs is an edge if it there is no linear extension which reverses both pairs.

Dimension and Chromatic Number

For every order: $\dim(P) = \chi(H_P) \geq \chi(G_P)$.

Theorem [Doignon, Ducamp and Falmagne 1984, F. and Trotter 2000].

$$\dim(P) = 2 \quad \Longleftrightarrow \quad \chi(G_P) = 2.$$

Conjecture. For every $t \geq 3$, there exists an order P with

$$\chi(G_P) = 3 \quad \text{and} \quad \dim(P) = t.$$

Important Examples

- Boolean lattice: $\dim(\mathbf{B}_n) = n$

- Standard example of an n dimensional order: Atoms and coatoms of a Boolean lattice:

$$\dim(\mathbf{B}_n[1, n-1]) = n$$

- First two levels of a Boolean lattice:

$$\dim(\mathbf{B}_n[1, 2]) \sim \log \log n$$

Challenge: Compute $\dim(\mathbf{B}_{100}[1, 2])$.

Important Examples

- Boolean lattice: $\dim(\mathbf{B}_n) = n$
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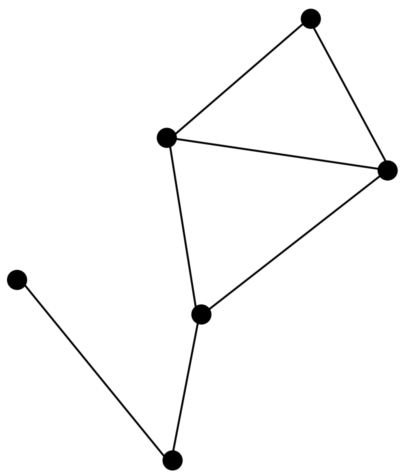
- First two levels of a Boolean lattice:

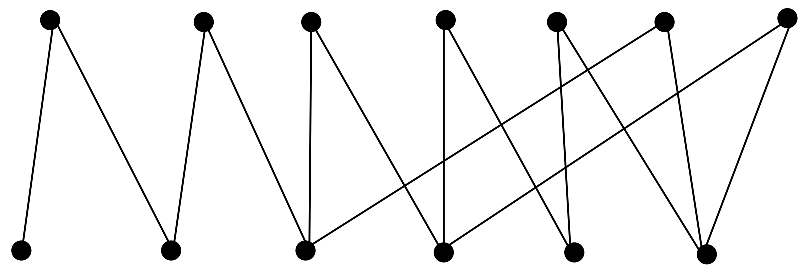
$$\dim(\mathbf{B}_n[1, 2]) \sim \log\log n$$

$\dim(K_n) \leq$	2	3	4	5	6	7	8
$n \leq$	2	4	12	81	2646	1422564	229809982112

Incidence Orders and Dimension

The incidence order P_G of G



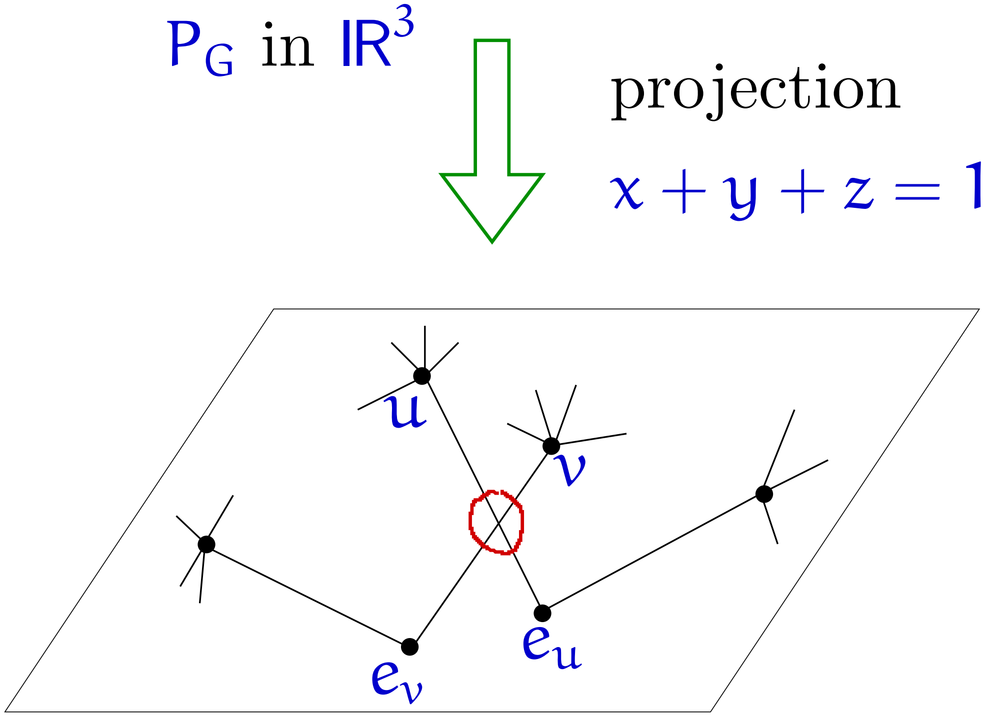


Theorem [Schnyder 1989].
A Graph G is planar $\iff$ $\dim(P_G) \leq 3$.

The Lower Bound

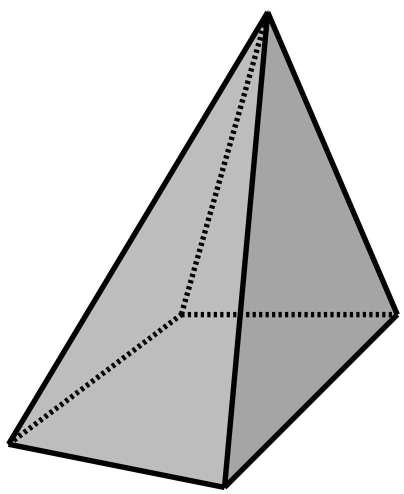
- $\dim(G) \leq 3 \implies G$ planar.

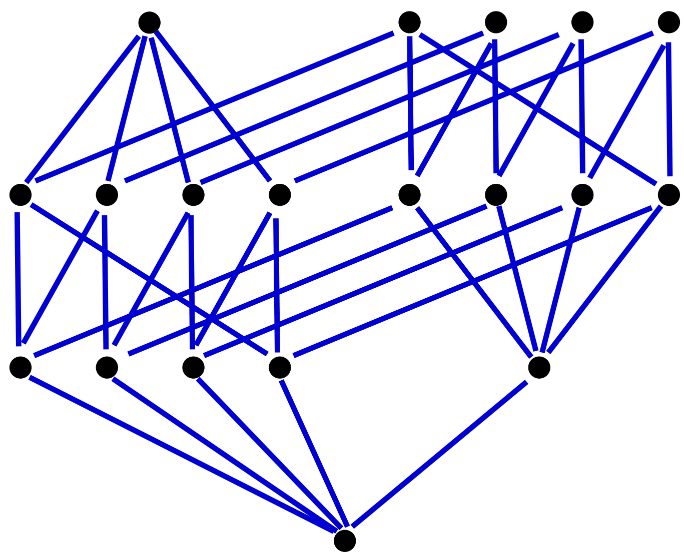
Babai and Duffus 1982



Dimension of Polytopes

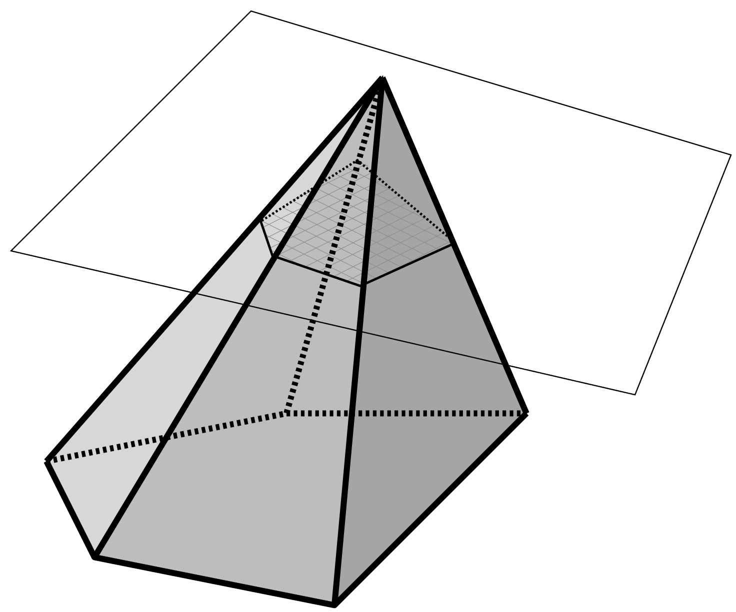
Let $\mathcal{F}_P$ be the face lattice of polytope P.





Dimension of Polytopes: Lower Bound

Theorem [Reuter 1990].
If P is a d-polytope, then $\dim(\mathcal{F}_P) \geq d + 1$.

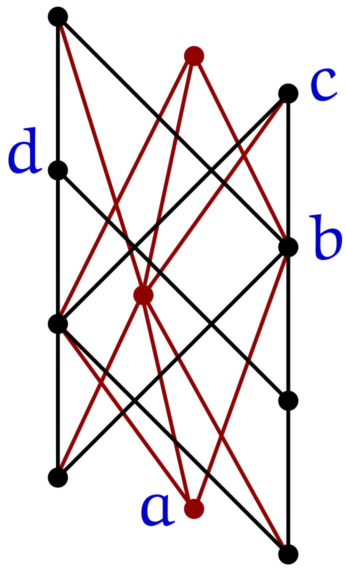


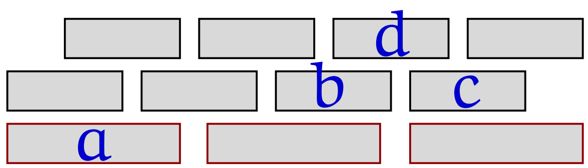
Interval Orders

Definition. An Order $P = (X, <_P)$ is an interval order if there is a family $(I_x)_{x \in X}$ of intervals on $\mathbb{R}$ such that

$$x <_P y \quad \Longleftrightarrow \quad I_x \text{ is left of } I_y.$$

Example.





Interval Dimension

A family $\mathcal{I}$ of interval extensions of P is an interval realizer for $P = (X, <)$ provided that

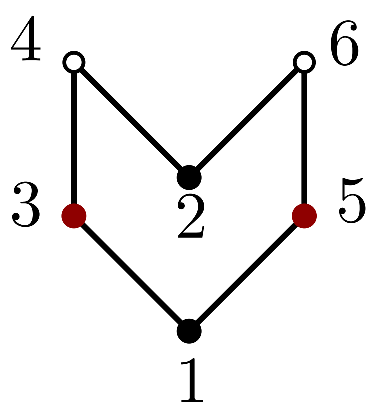
* for every incomparable pair (x, y) there is an $I \in \mathcal{I}$ such that $x \not< y$ in I.

The interval dimension, $\dim_I(P)$, of P is the minimum t, such that there is an interval realizer $\mathcal{I} = \{I_1, I_2 \ldots, I_t\}$ for P of size t.

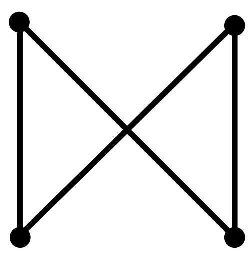
Interval Dimension

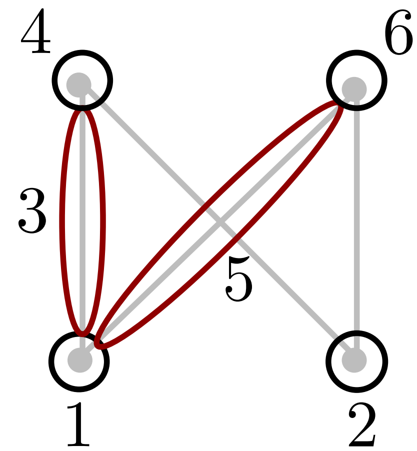
- $\dim_I(P) = \min($ t such that there is an interval realization of P on some Q with $\dim(Q) = t$).

P



Q





Comparing Notions of Dimension

P a bipartite order, then

- $\dim_I(P) \leq \dim(P) \leq \dim_I(P) + 1$.

G a graph, then

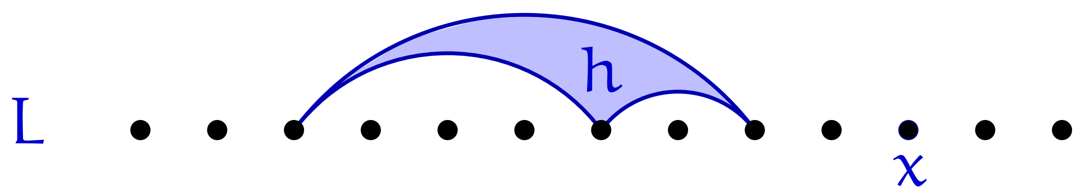
- $\dim(G) = \dim_I(P_G)$.

  If $\delta(G) > 1$, then $\dim(G) = \dim_I(P_G) = \dim(P_G)$.

Dimension of Hypergraphs

A family Γ of permutations of V is a realizer for $H = (V, E)$ provided that

* for every edge h and every $x \in V - h$ there is an $L \in \Gamma$ such that $x > h$ in L.



The dimension, $\dim(H)$, of H is the minimum t, such that there is a realizer $\Gamma = \{L_1, L_2 \ldots, L_t\}$ for H of size t.

Dimension of Polytopes

P a d-polytope.

H_P the hypergraph of faces and $\mathcal{F}_P$ the face lattice.

- $\dim(H_P) = \dim(\mathcal{F}_P) \geq d + 1.$

Theorem [Schnyder 1989].
If P is a simplicial 3-polytope and F is any face of P, then

- $\dim(\mathcal{F}_P \setminus F) = 3$
- $\dim(\mathcal{F}_P) = 4$

∎

Theorem [Brightwell and Trotter 1992].
If P is a 3-polytope and F is any face of P, then

- $\dim(\mathcal{F}_P \setminus F) = 3$
- $\dim(\mathcal{F}_P) = 4$

Dimension and Topological Graphs

- $\dim(G) \leq 2 \iff G$ is a caterpillar.

- $\dim(G) \leq 3 \iff G$ is planar.

- Challenge: Find something meaningful for ≤ 4.

Dimension and Topological Graphs

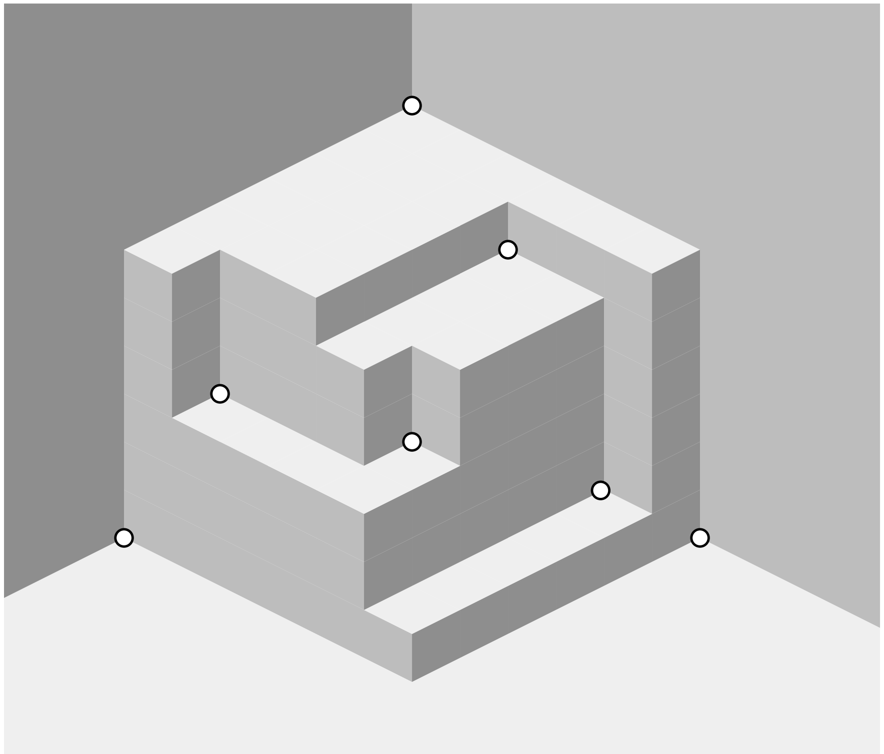
- $\dim(\mathsf{G}) \leq 1\frac{1}{2} \iff \mathsf{G}$ is a path.
- $\dim(\mathsf{G}) \leq 2 \iff \mathsf{G}$ is a caterpillar.
- $\dim(\mathsf{G}) \leq 2\frac{1}{2} \iff \mathsf{G}$ is outerplanar.
- $\dim(\mathsf{G}) \leq 3 \iff \mathsf{G}$ is planar.
- Challenge: Find something meaningful for $\leq 3\frac{1}{2}$.
- Challenge: Find something meaningful for ≤ 4.

Problems

- Graphs of dimension at most 4.
- Dimension of polytopes.
- Given a bipartite order P. Can $\dim(\mathsf{P}) \leq 3$ be decided in polynomial time?

Orthogonal Surfaces and Dimension

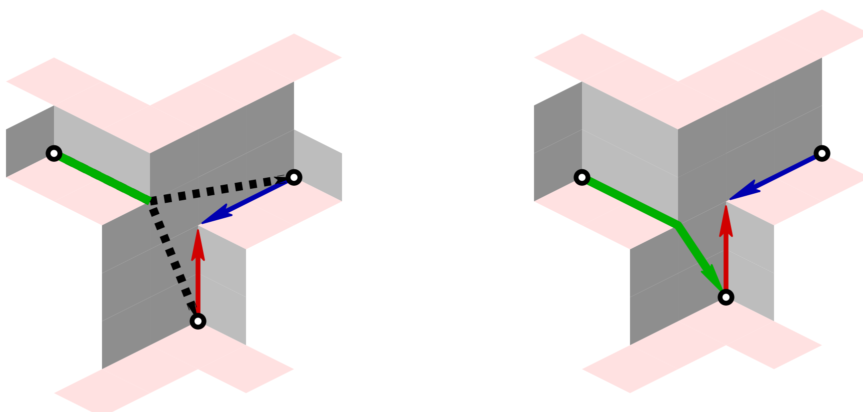
Orthogonal Surfaces



Orthogonal Surfaces and Dimension

A surface $\mathbf{S}_X$ is rigid iff

$$e = (x, y) \in G_S \text{ and } z \neq x, y \text{ implies } z \not\leq e = x \vee y.$$



Remark. A rigid orthogonal surface supports a unique Schnyder wood of a 3-connected planar graph.

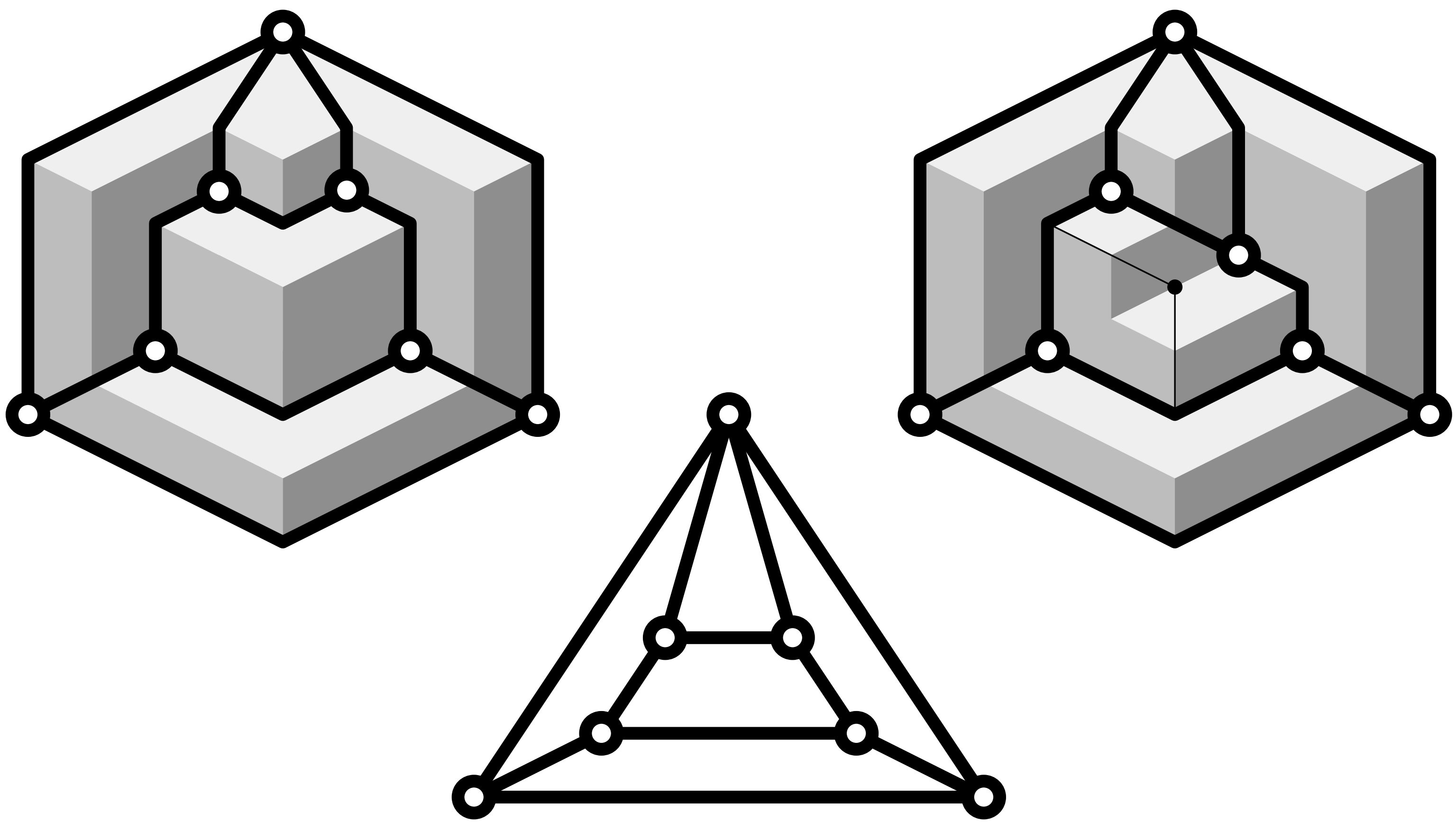
Orthogonal Surfaces and Dimension

Theorem [Miller '02]**.**
If G is the graph of a rigid orthogonal surface and P a 3-polytope with graph G, then $\dim(\mathcal{F}_{\mathsf{P}} \setminus \mathsf{F}_{\infty}) = 3$.

Theorem [Brightwell+Trotter '92]**.**
If P is a 3-polytope and F is any face of P, then

- $\dim(\mathcal{F}_{\mathsf{P}} \setminus \mathsf{F}) = 3$
- $\dim(\mathcal{F}_{\mathsf{P}}) = 4$

Rigidity and the BT Theorem



A New Approach *

Two types of non-rigid edges

resolved by shifting red flats.

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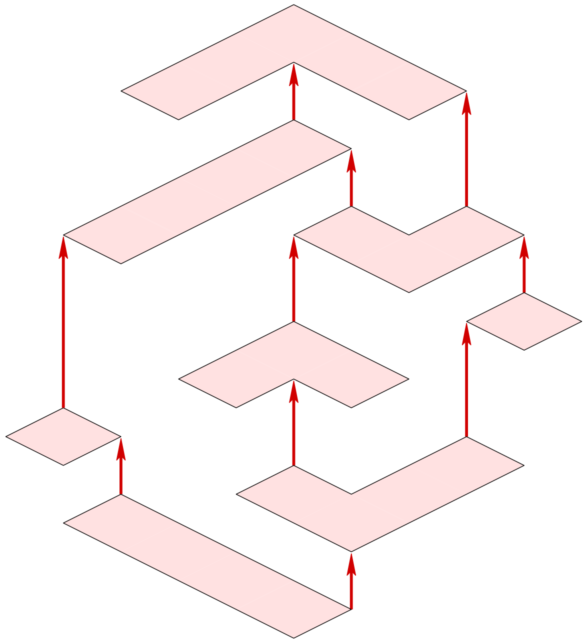
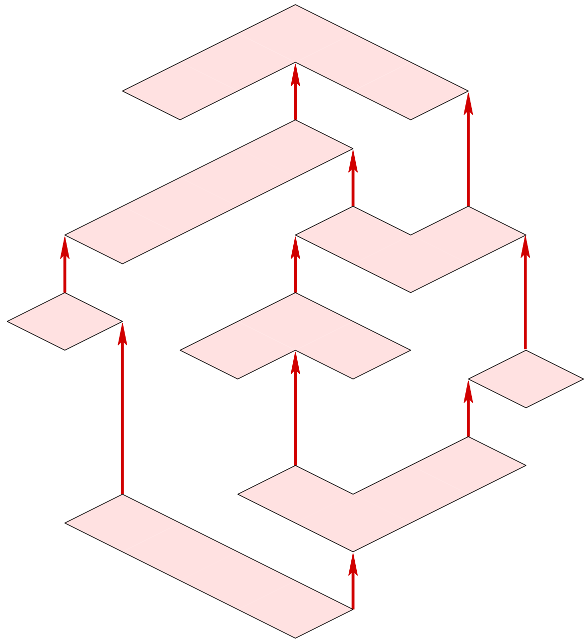
*Joint work with Florian Zickfeld.

Shifting Flats

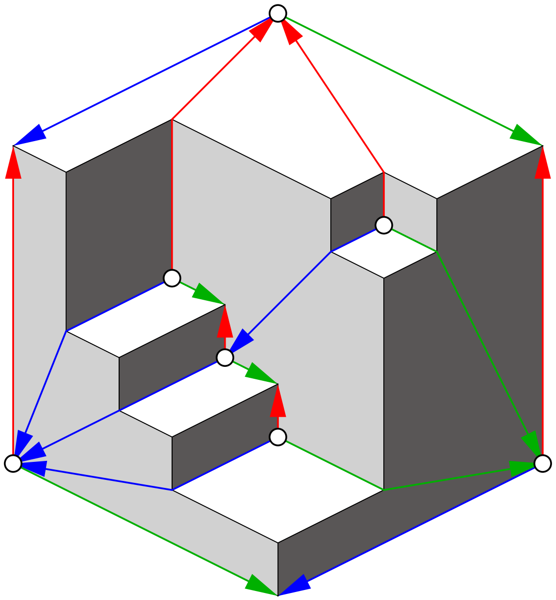
Red flats form a 2-dimensional order $\mathsf{P_r}$. Compatible height functions of red flats correspond to order preserving maps $\phi : \mathsf{P_r} \longrightarrow \mathbb{R}$.

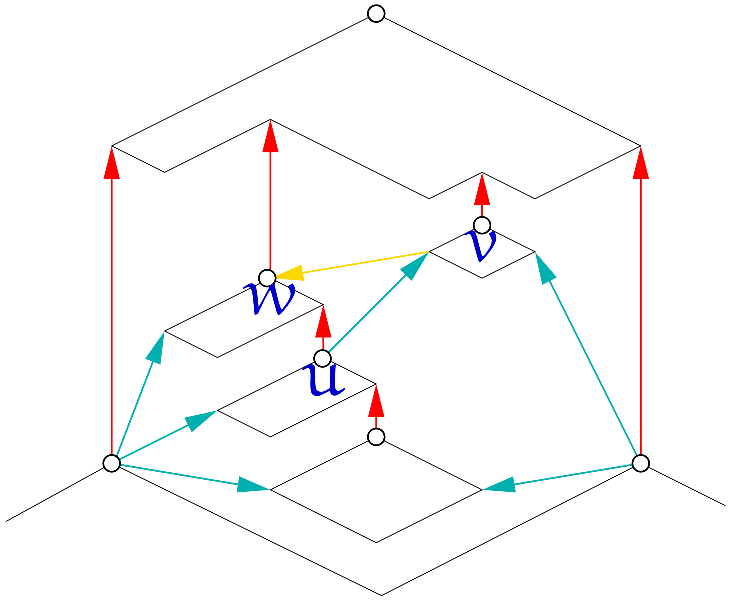




Shifting Flats

Split bidirected edges to prevent non-rigid. This doesn't spoil acyclicity.

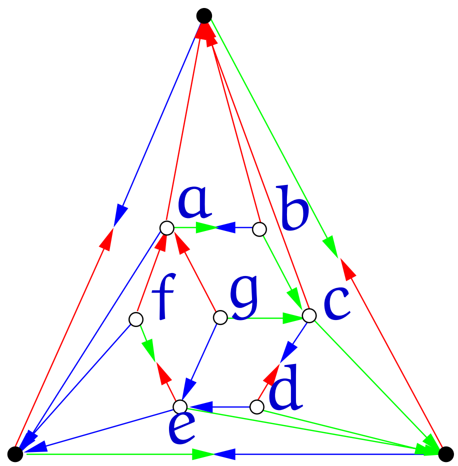


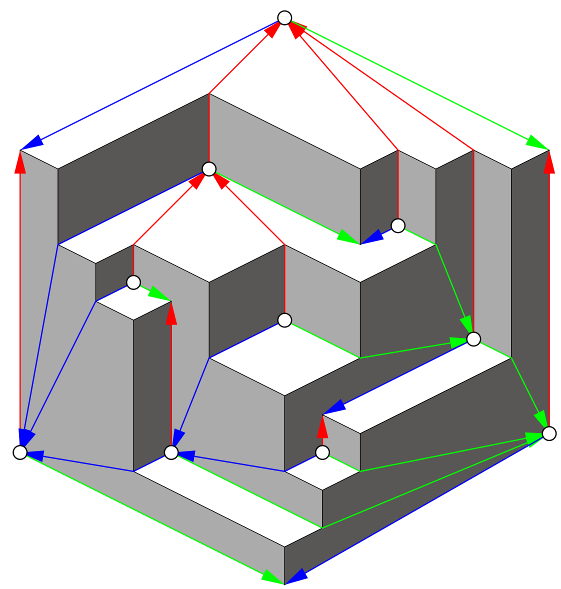


Theorem. There are compatible assignments for the heights of flats of all three colors which together yield a rigid orthogonal surface supporting the original Schnyder wood.

Rigidity and Coplanarity

A Schnyder Wood on a rigid, but not coplanar surface





Coplanarity means that $v_1 + v_2 + v_3 = c = w_1 + w_2 + w_3$ for all $v, w \in V$, hence, $v_i = w_i$ implies $v_{i-1} - w_{i-1} = w_{i+1} - v_{i+1}$. In the given instance rigidity requires $f_1 > g_1$, $b_2 > g_2$ and $d_3 > g_3$.

The End

Thank you.