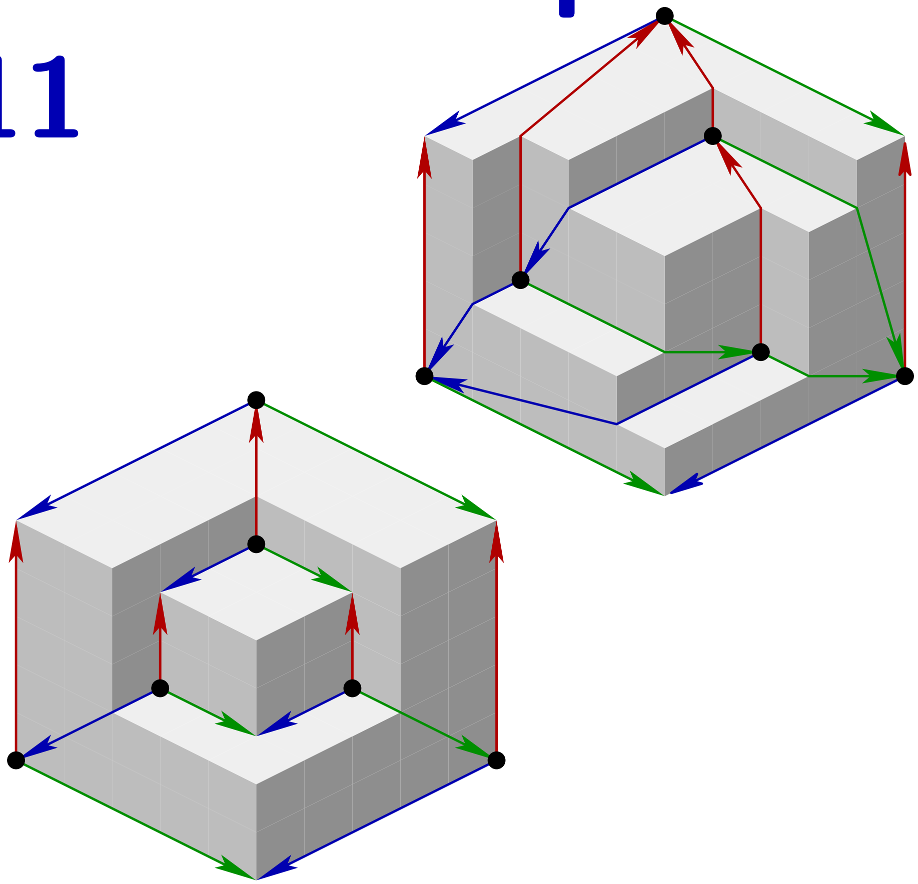
Embeddings of Planar Graphs
Lectures 10 & 11

March 2006



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Modules

Duality and Orthogonal Surfaces

Flips and Lattices

Applications of F & L

More Compact Convex Drawings

Modules

Duality and Orthogonal Surfaces

Flips and Lattices

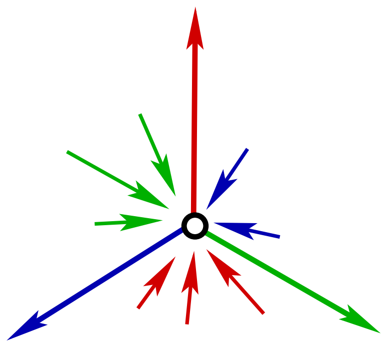
More Compact Convex Drawings

Schnyder Woods

$G = (V, E)$ a plane triangulation,
$F = \{a_1, a_2, a_3\}$ the outer triangle.

A coloring and orientation of the interior edges of G with colors 1,2,3 is a Schnyder wood of G iff

- Inner vertex condition:



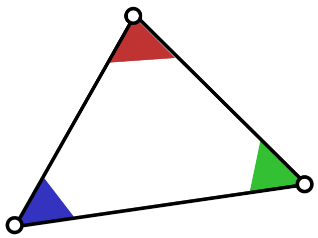
- Edges $\{v, a_i\}$ are oriented $v \to a_i$ in color i.

Schnyder Angle Labelings

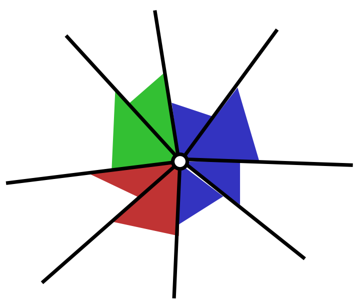
$G = (V, E)$ a plane triangulation,
$F = \{a_1, a_2, a_3\}$ the outer triangle (clock-wise).

A coloring of the angles of G with colors 1,2,3 is a Schnyder angle labeling of G iff

- Inner face condition:

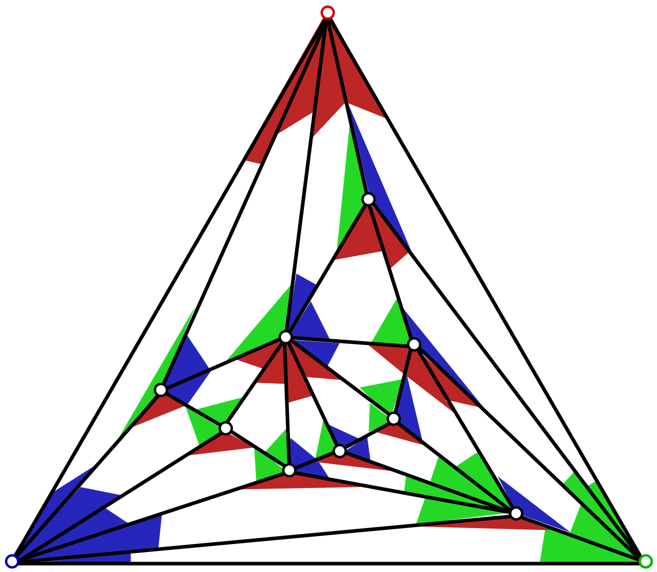


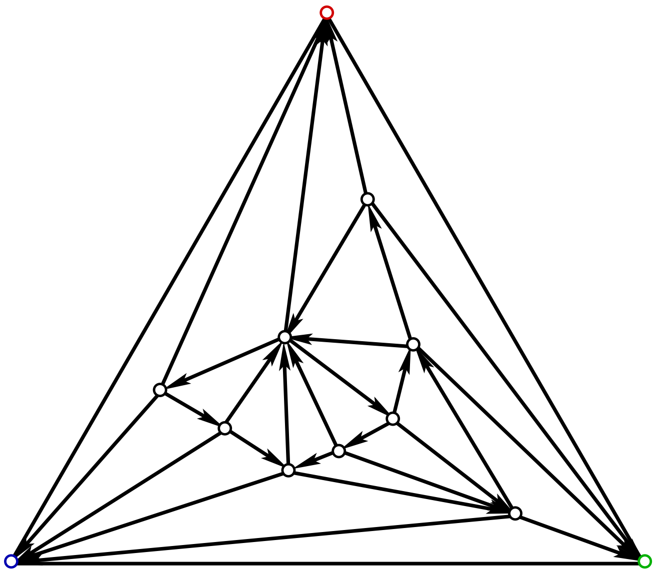
- Inner vertex condition:

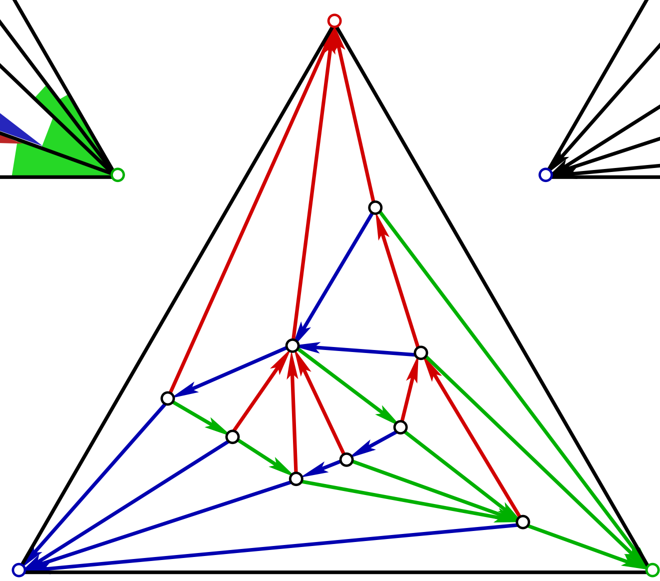


- All angles at a_1 are colored 1,
  angles at a_2 are colored 2
  and angles at a_3 are colored 3.

Three Equivalent Structures

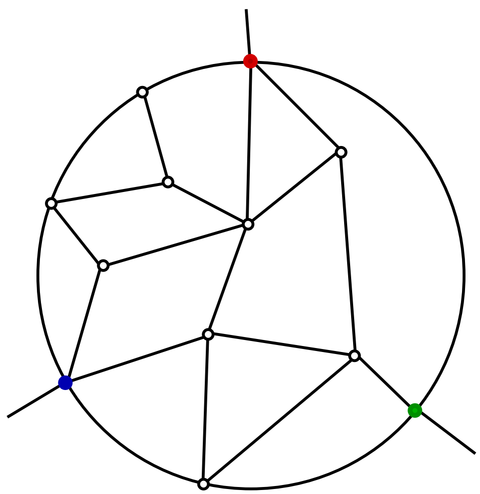






Schnyder Structures for 3-Connected Plane Graphs

We adapt Schnyder's structures for
suspended 3-connected planar graphs.

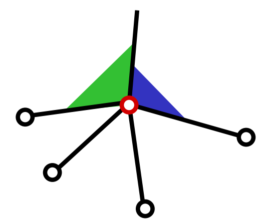


- Schnyder angle labelings,
- Schnyder woods,
- 3-orientations.

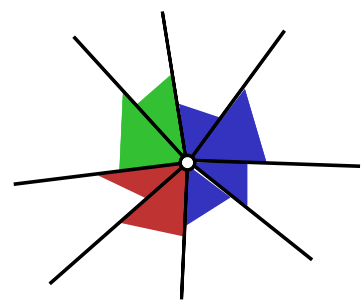
Schnyder Angle Labelings

Axioms for the 3-coloring of angles:

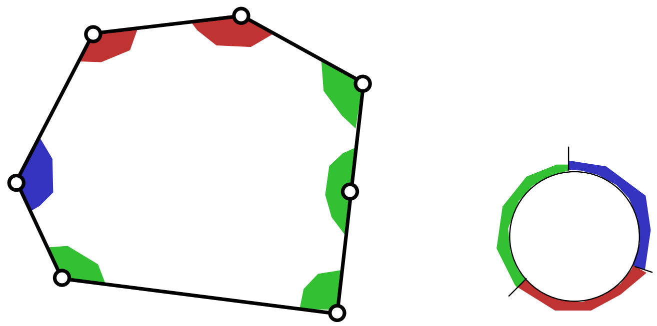
(A1) Angles at the half-edges:



(A2) Rule of vertices:



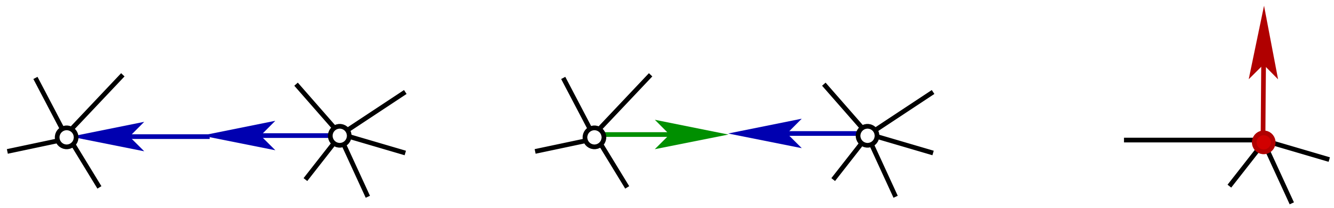
(A3) Rule of faces:



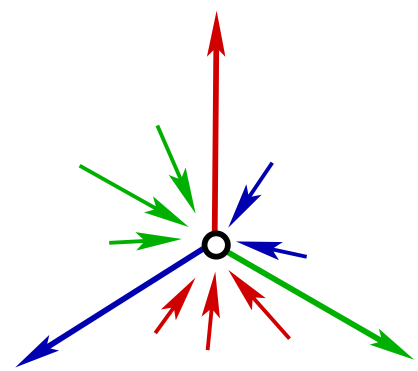
Schnyder Woods

Axioms for 3-coloring and orientation of bi-edges:

(W1 - W2) Rule of edges and half-edges:

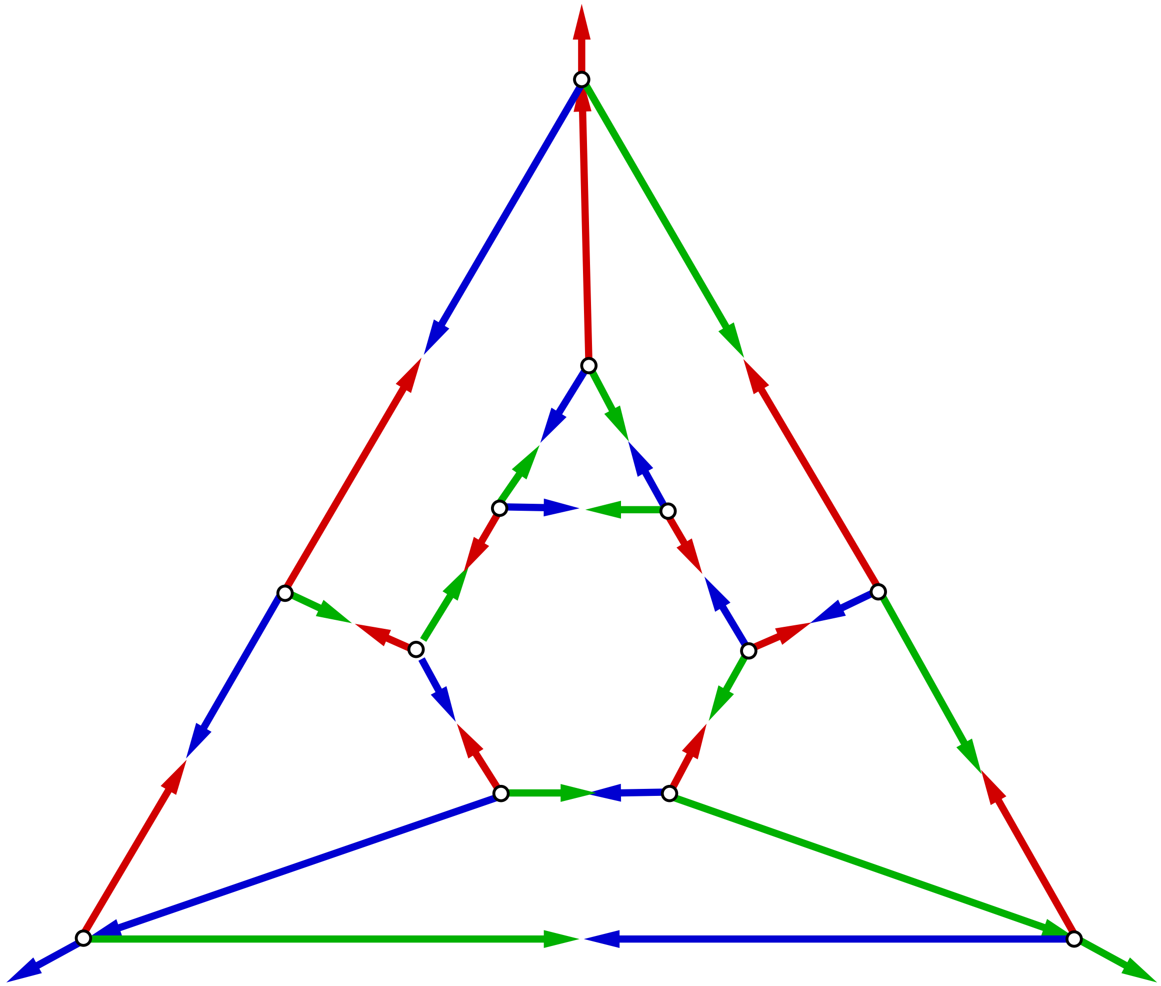


(W3) Rule of vertices:

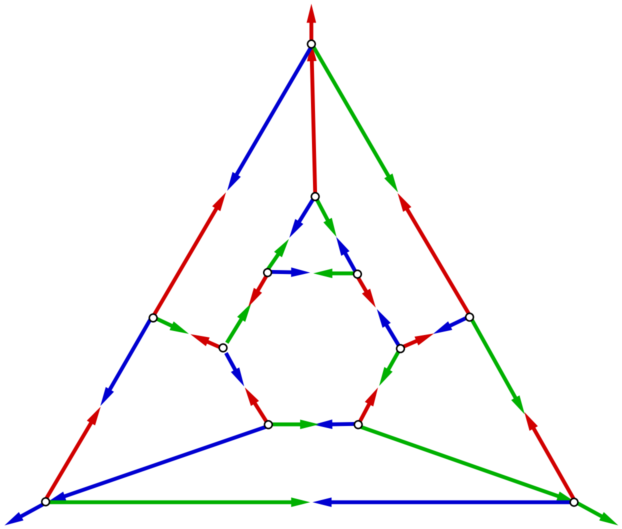


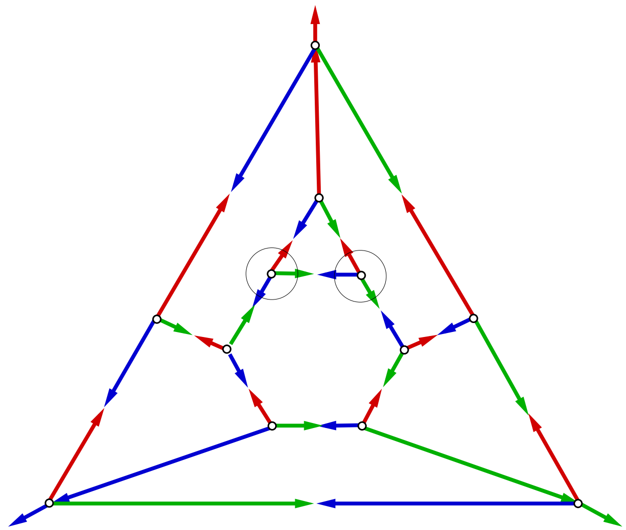
(W4) There is no interior face whose boundary is a directed cycle in one color.

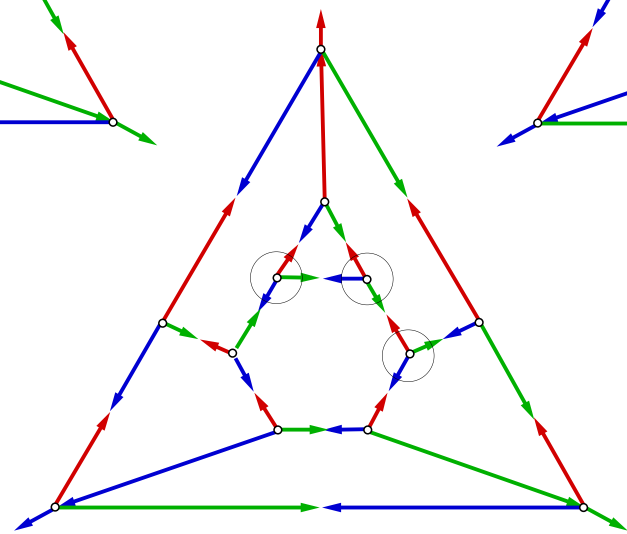
We need W4!



3-Orientations?

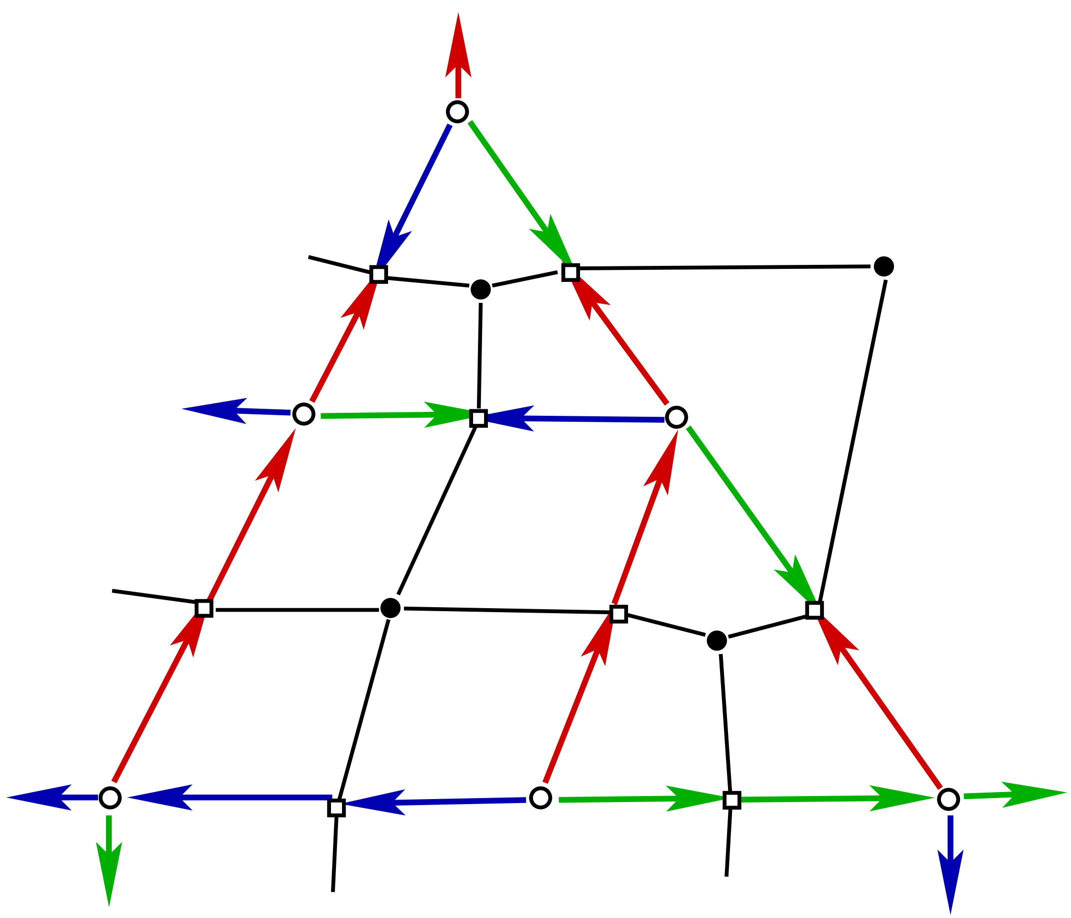






Primal and Dual

A Schnyder coloring of G induces a Schnyder coloring of the dual of G.



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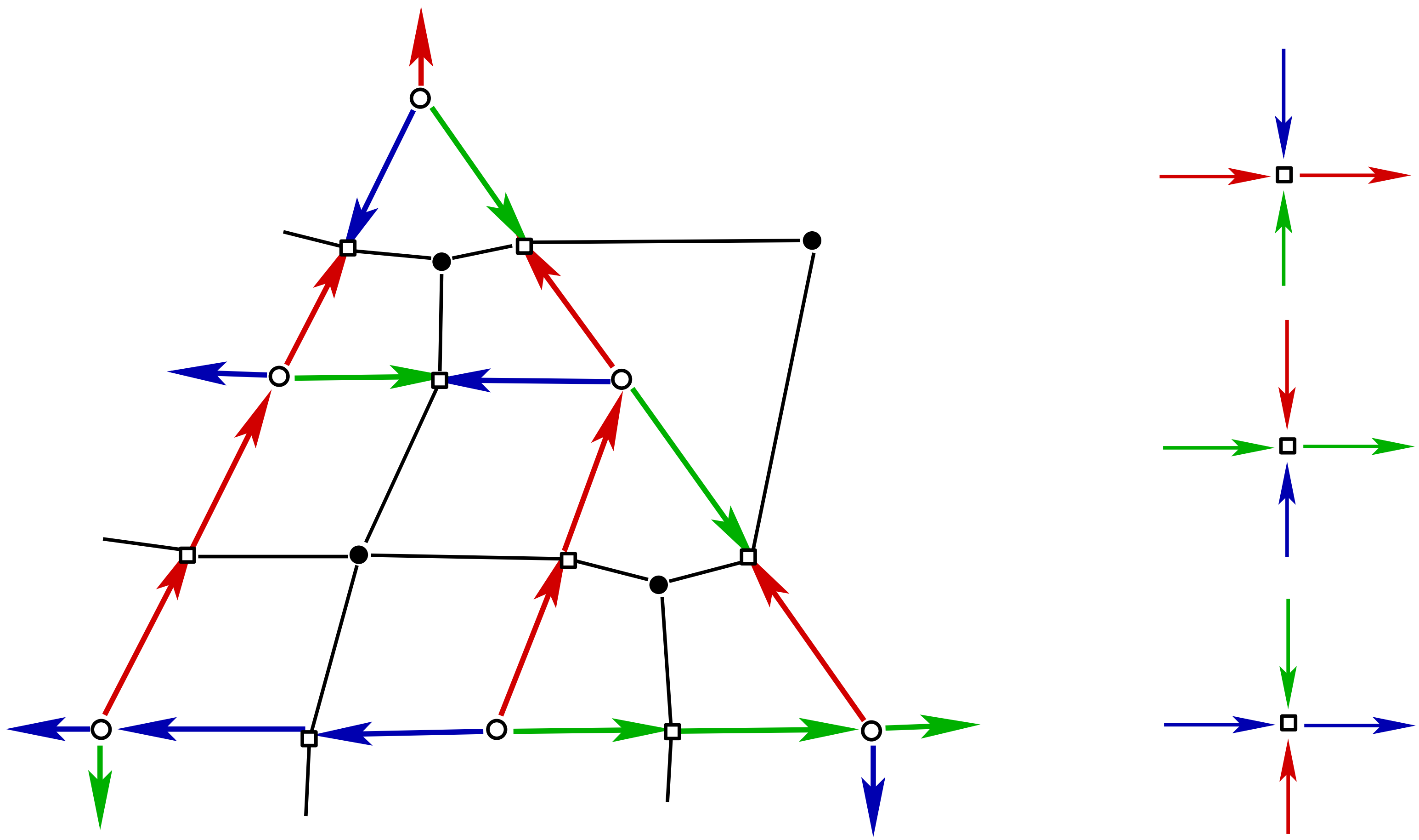

Primal and Dual

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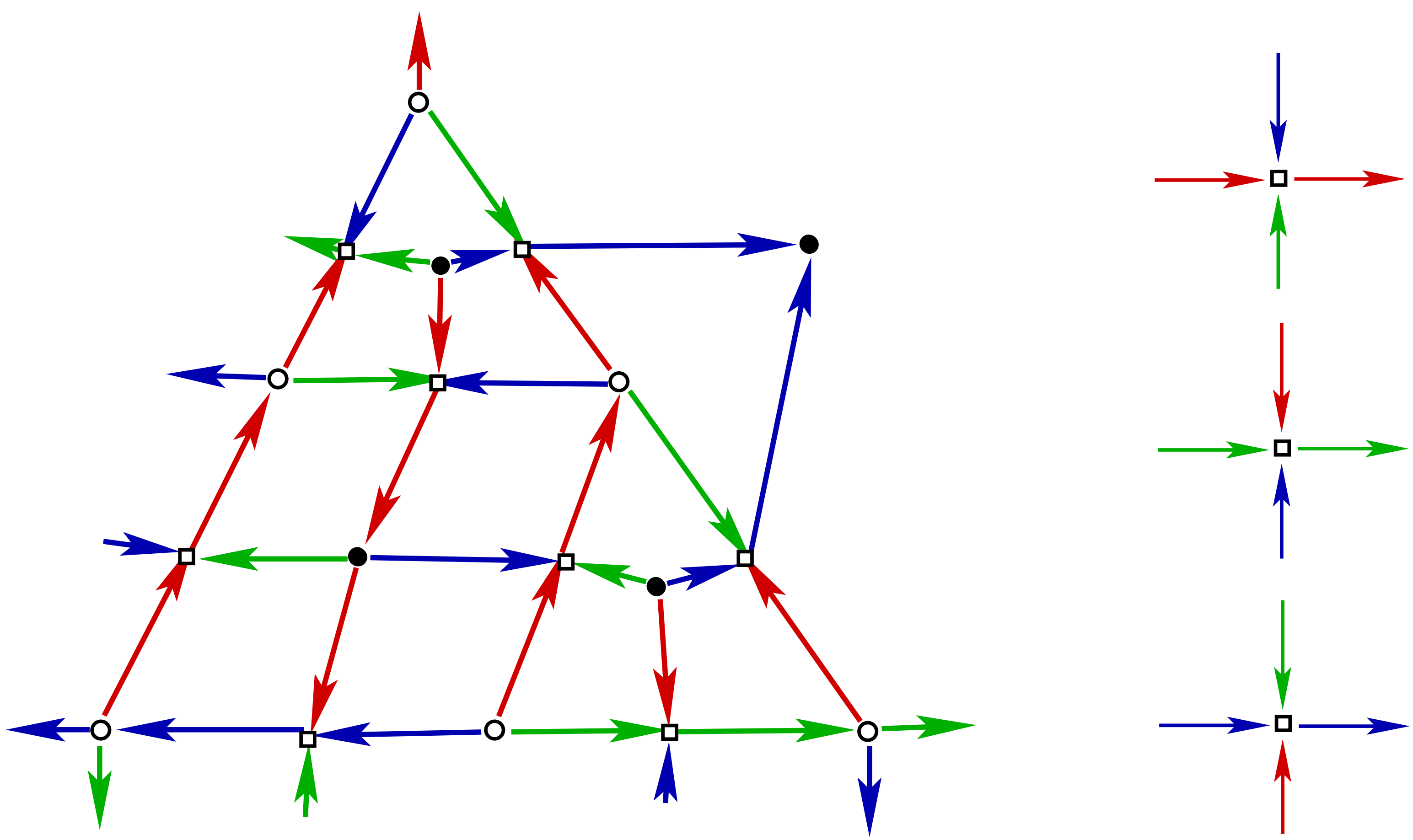
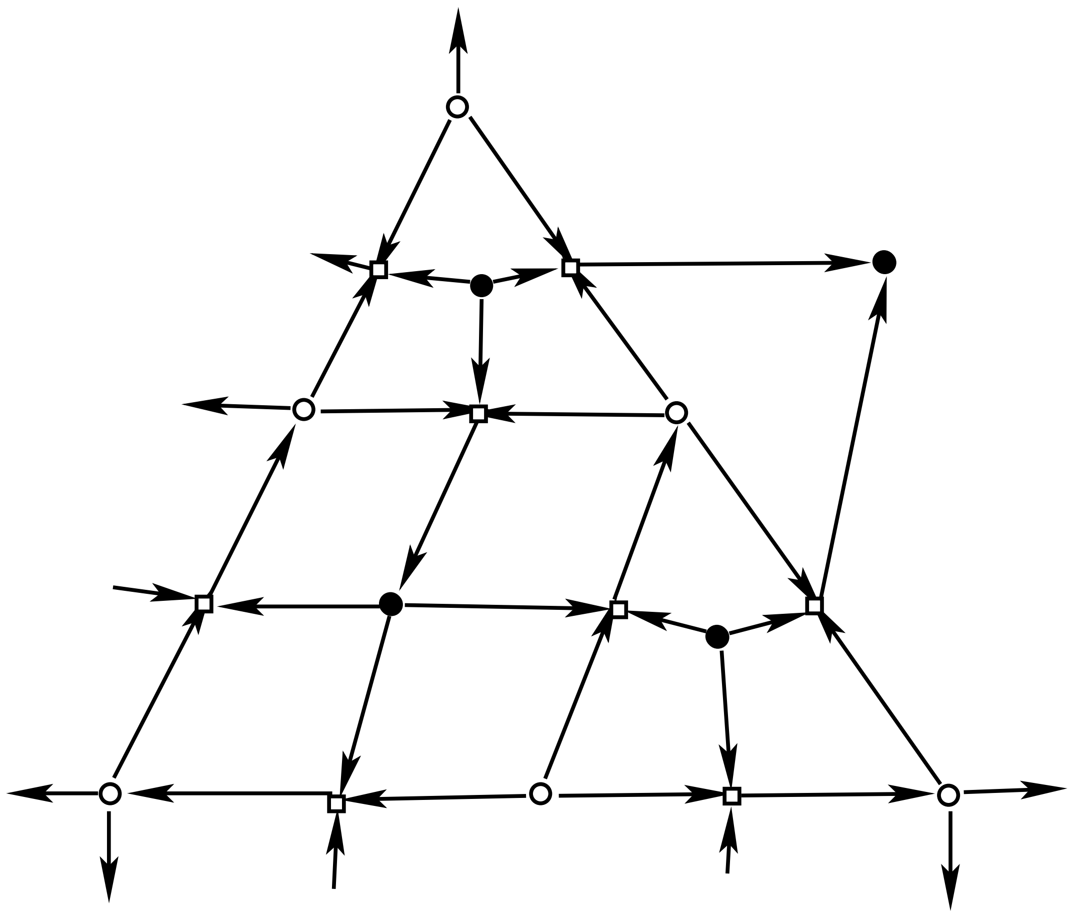


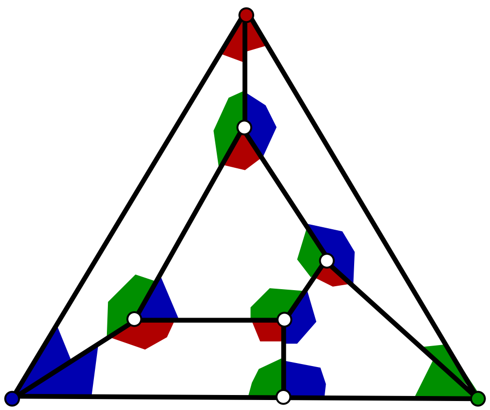
3-Orientations – Rescued

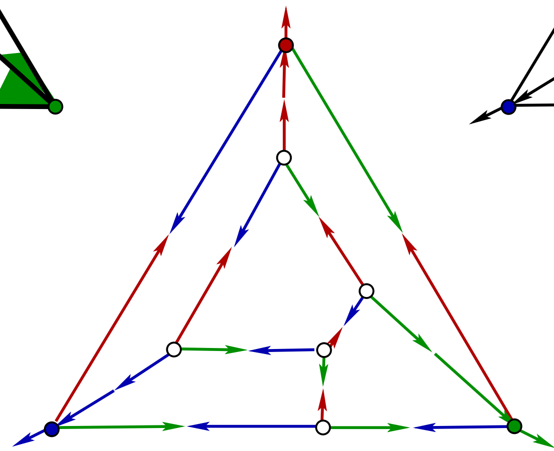
Theorem. Primal dual Schnyder woods are in bijection with primal dual 3-orientations.

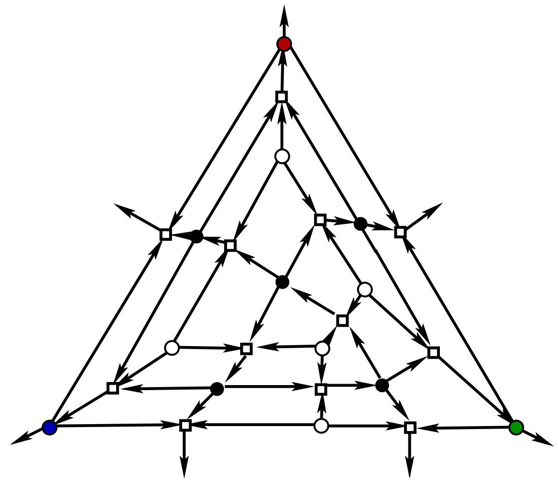


Three Equivalent Structures

- Angle labelings.
- Woods.
- 3-Orientations.

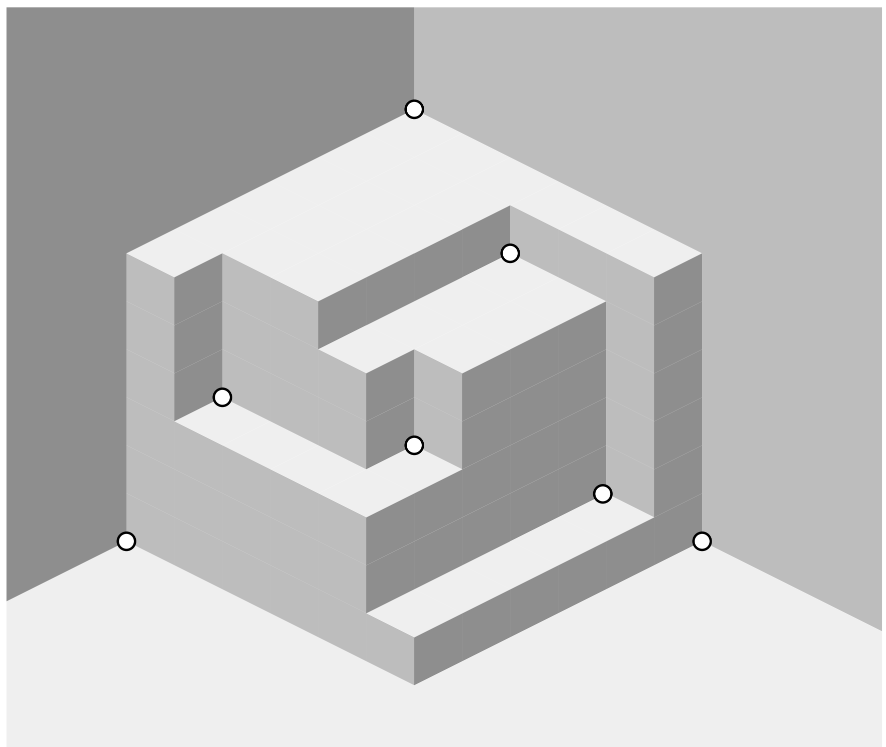






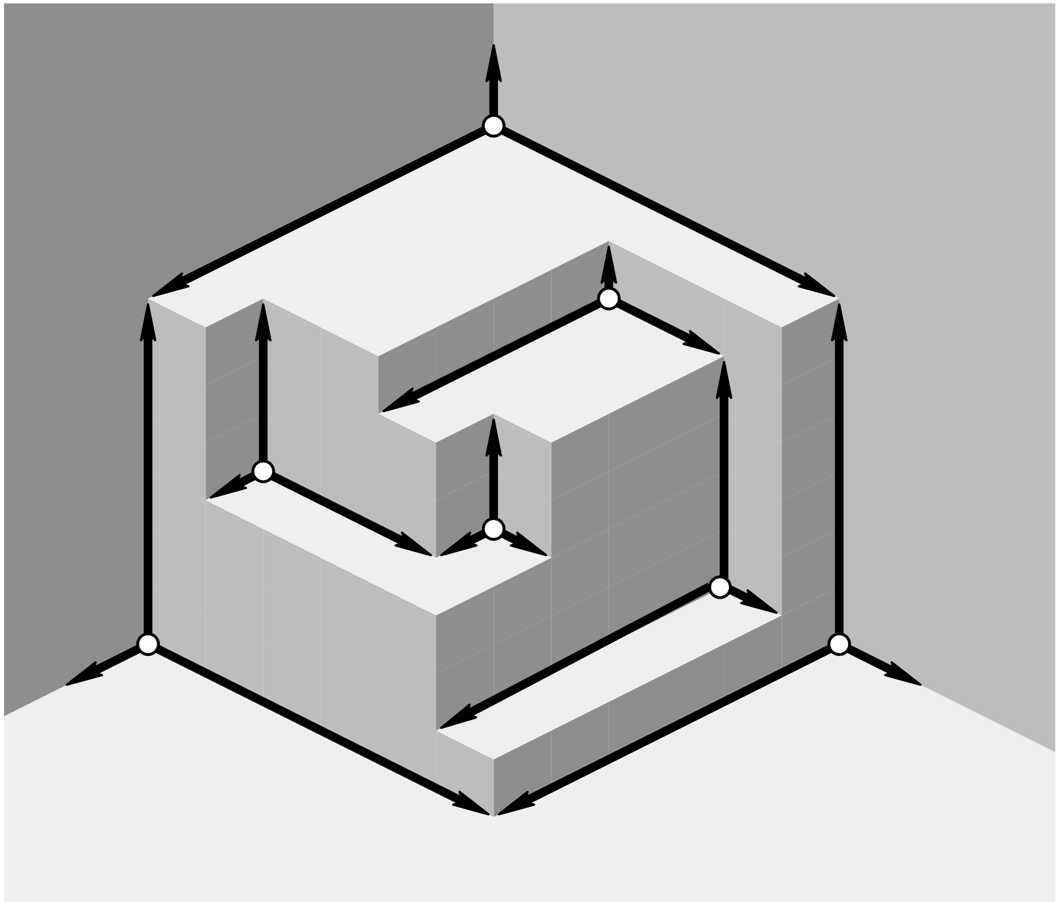
Orthogonal Surfaces

Let $X \subset \mathbb{R}^3$ be finite. The orthogonal surface $\mathbf{S}_X$ generated by X is the boundary of the filter $I_X^{\geq}$ generated by X.



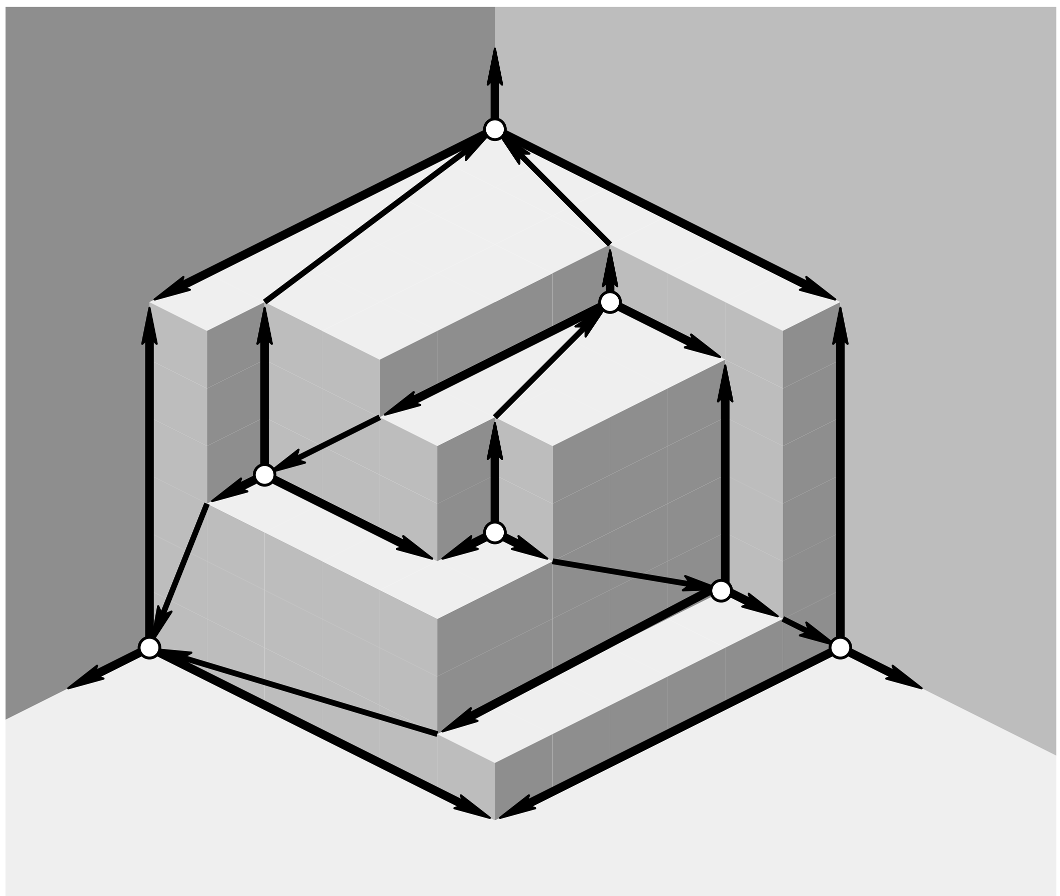
Orthogonal Surfaces

Add three orthogonal rays to each generating vertex.



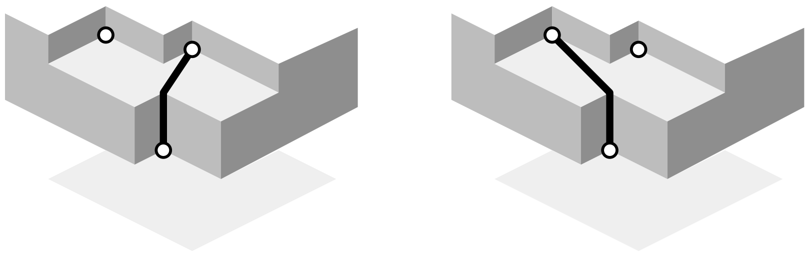
Graphs on Orthogonal Surfaces

Pending rays are completed across flats $\Longrightarrow$ a graph.

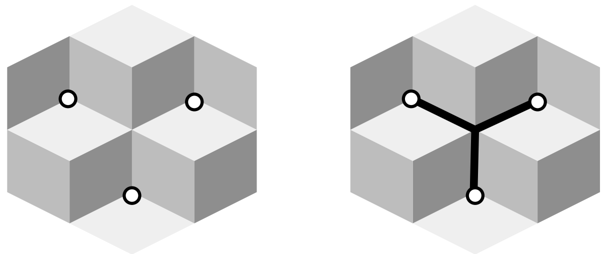


Warnings

- The graph of an orthogonal surface need not be unique

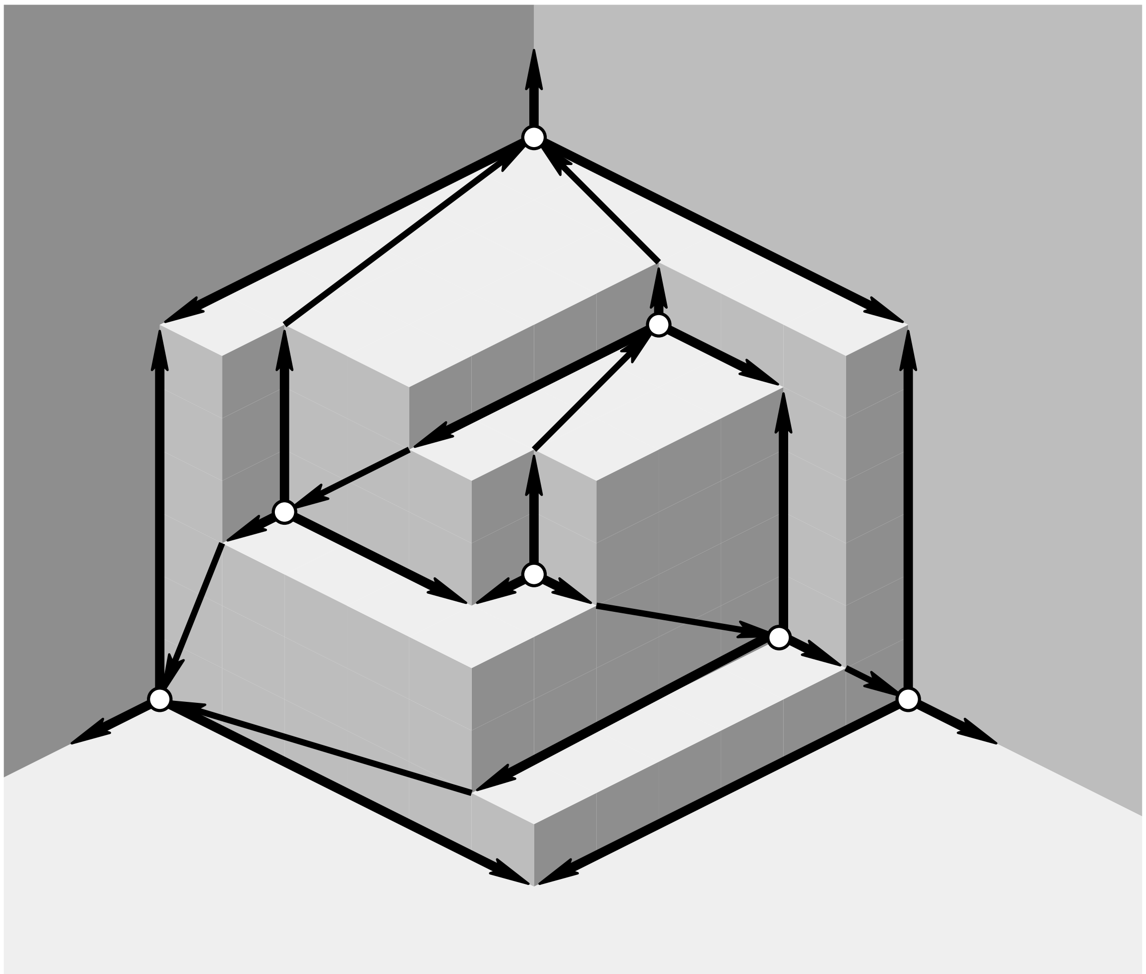


- Not every orthogonal surface carries a graph



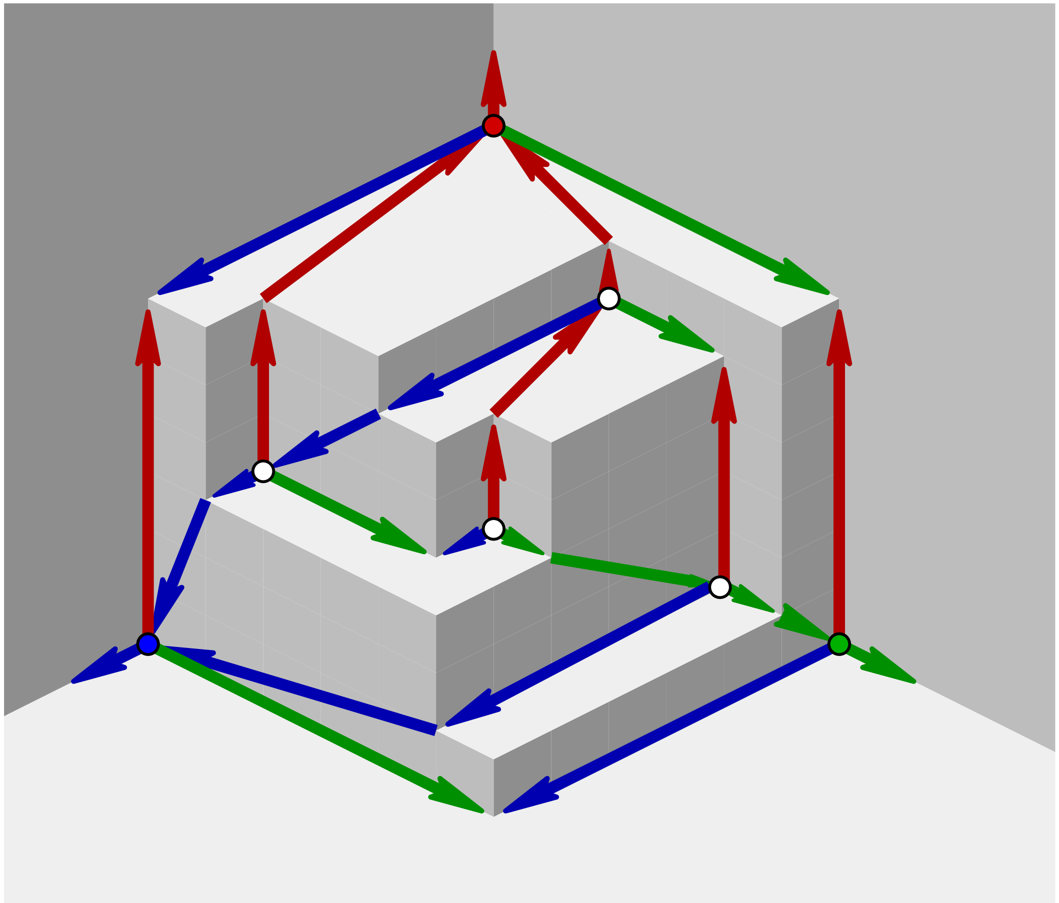
Graphs on Orthogonal Surfaces

A planar graph associated to $\mathbf{S}_{\mathsf{X}}$.



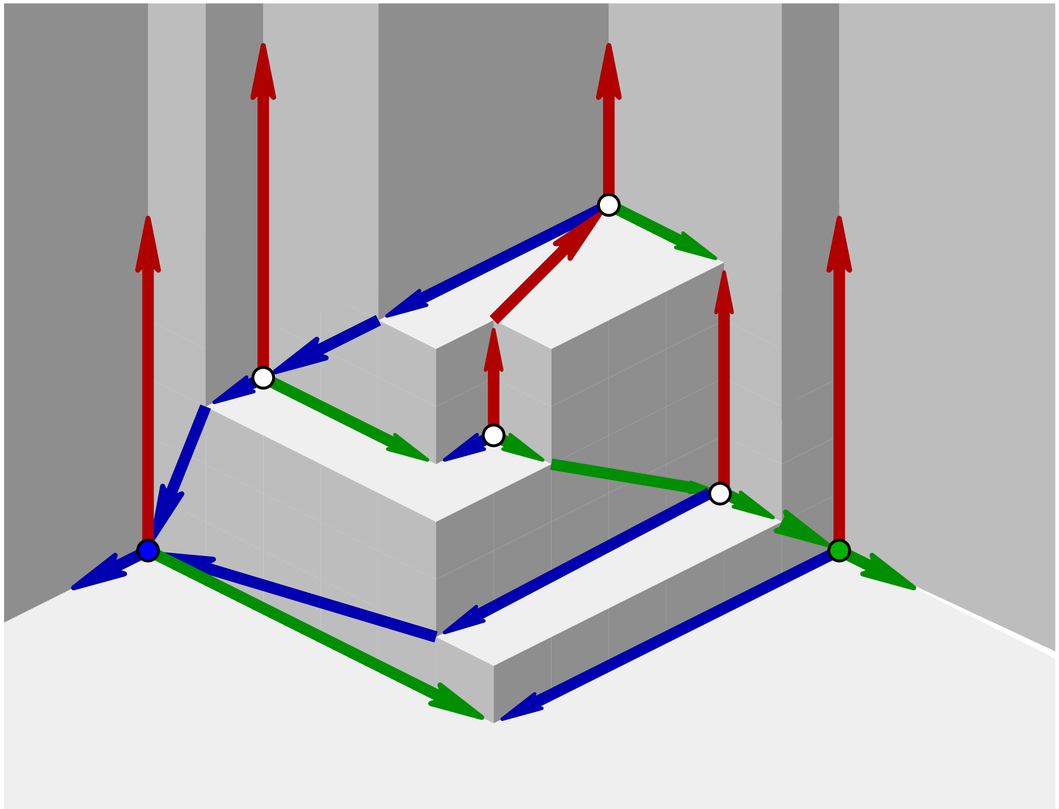
Schnyder Woods on Orthogonal Surfaces

Add color to rays and edges $\Longrightarrow$ a Schnyder wood?



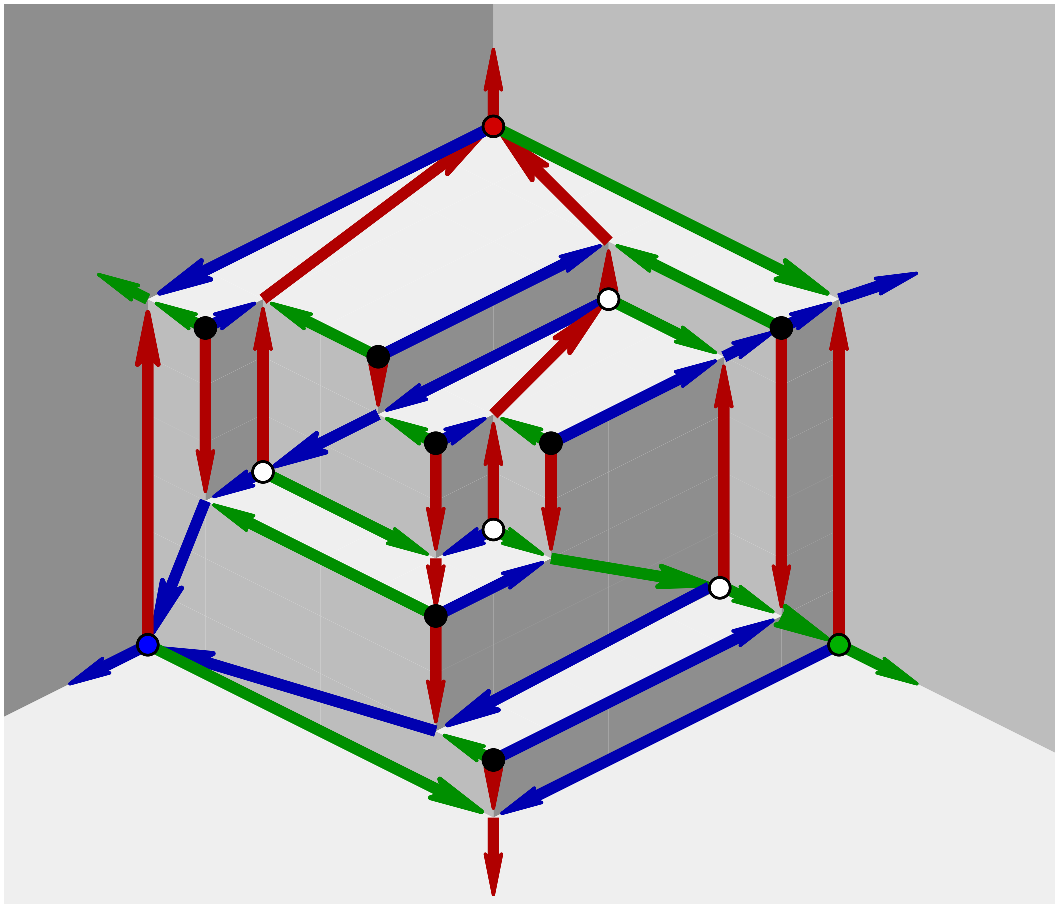
Schnyder Woods on Orthogonal Surfaces

A Schnyder wood? — Yes, if the set is suspended.

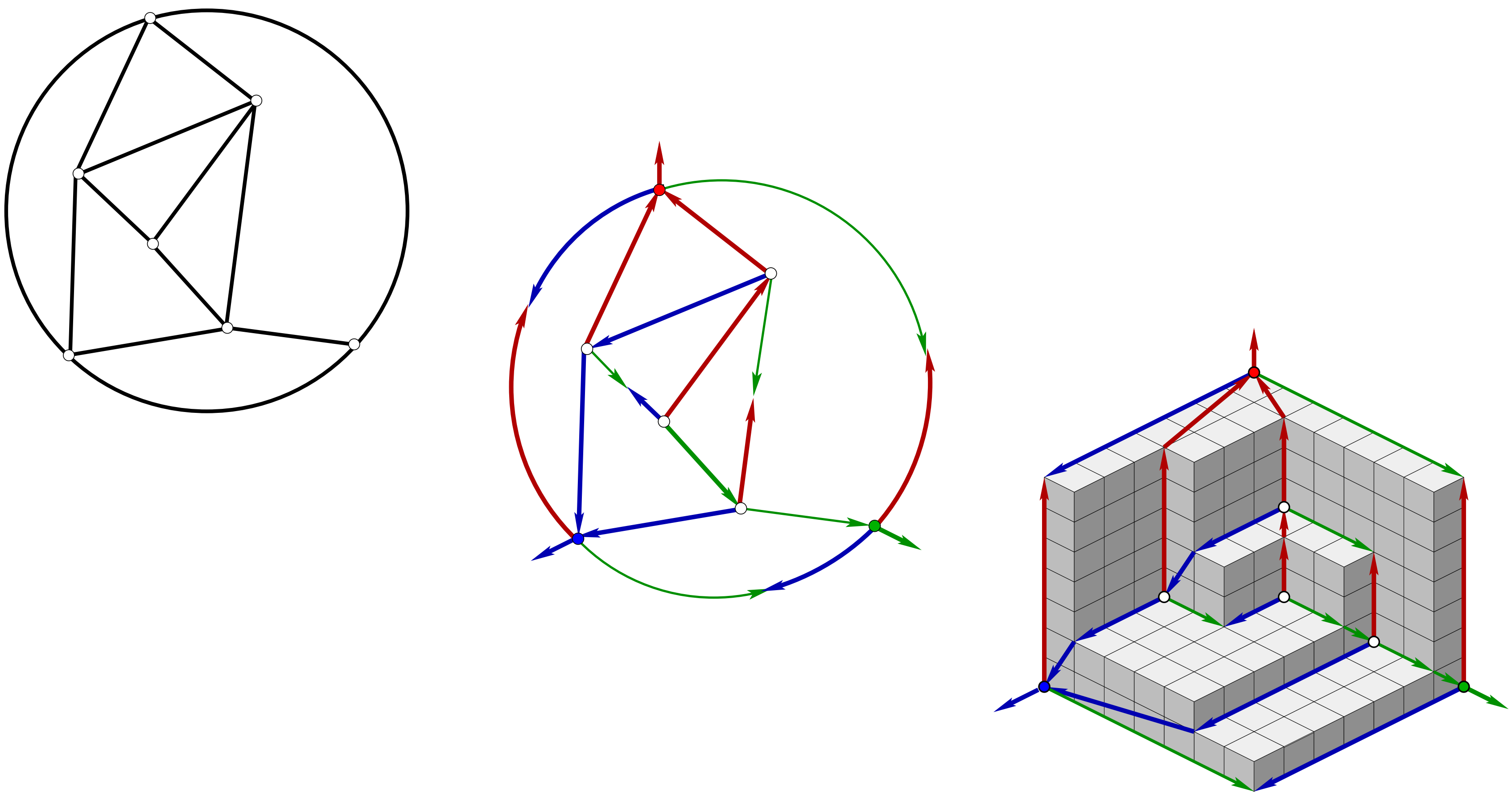


Dual Schnyder Woods on Orthogonal Surfaces

Dual pairs of Schnyder woods reappear naturally.



From Graphs to Orthogonal Surfaces



From Graphs to Orthogonal Surfaces

Let $G = (V, E)$ be a 3-connected planar graph with distinguished outer face F_∞ and vertices $a_1, a_2, a_3 \in F_\infty$.

- G has a Schnyder wood.
- $v \in G$ let $v_i = \#$ faces in $R_i(v)$.
  The orthogonal surface of $X = \{(v_1, v_2, v_3) : v \in V\}$ supports G and the defining Schnyder wood.

Modules

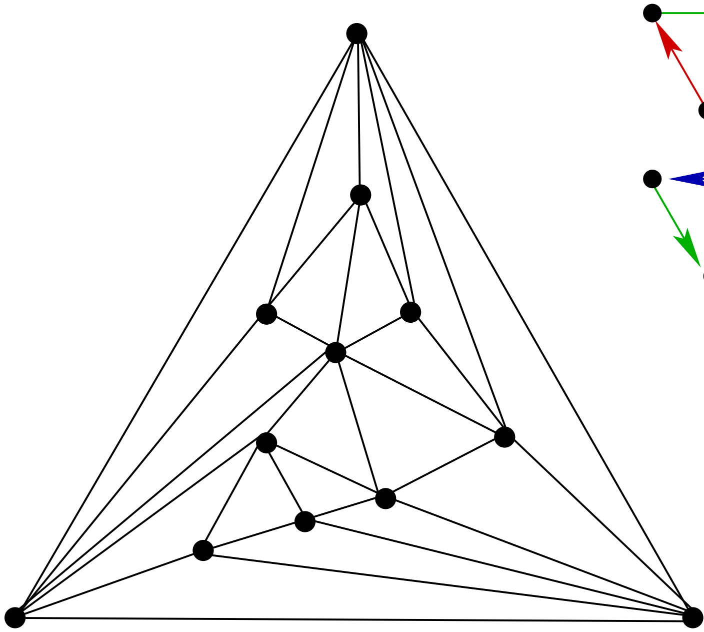
Duality and Orthogonal Surfaces

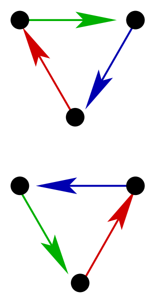
Flips and Lattices

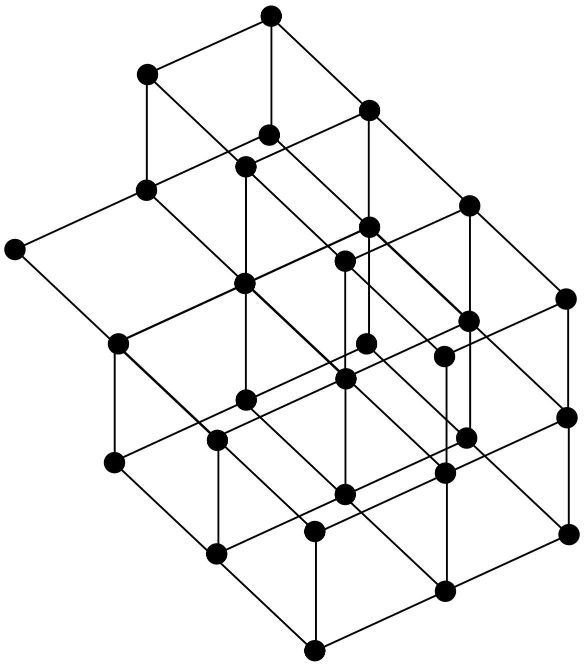
More Compact Convex Drawings

The Lattice of Schnyder Woods

Theorem [Mendez & Brehm]**.** The set of Schnyder woods of a plane triangulation G has the structure of a distributive lattice.



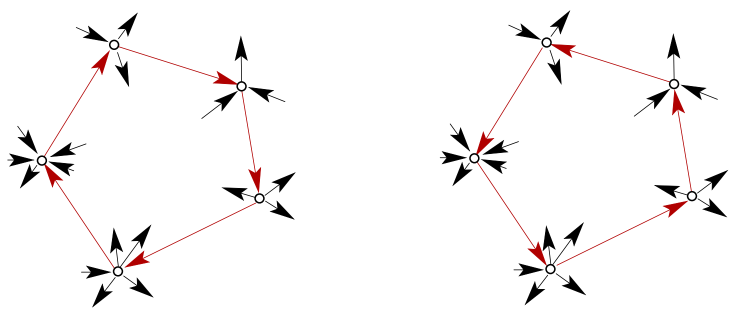




A General Theorem

Definition. Given $G = (V, E)$ and $\alpha : V \to \mathbb{N}$.
An α-orientation of G is an orientation with
$\text{outdeg}(v) = \alpha(v)$ for all v.

- Reverting directed cycles preserves α-orientations.



Theorem. The set of α-orientations of a planar graph G has the structure of a distributive lattice.

Rigidity and Essential Cycles

Definition.
rigidity of edges, cuts and cycles.

Definition.
A cycle C of G is an essential cycle if

- C is chord-free and simple,
- the interior cut of C is rigid,
- there exists an α-orientation X such that C is a directed cycle in X. ∎

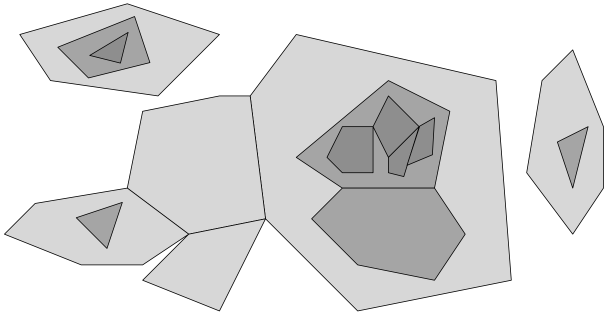
Lemma. If e is oppositionally directed in two α-orientations then there is a C with $e \in C$ which is oppositionally directed in both.

Essential Cycles

Lemma. If cycle C has no chordal path in X then it has no chordal path in Y. ∎

Lemma. C has no chordal path iff interior cut is rigid. ∎

Lemma. Essential cycles are interiorly disjoint or contained.



Flips of Essential Cycles

Lemma. If C is a directed cycle in X then we can also reach X^C by a sequence of reversals of essential cycles. ∎

Lemma. If C is a simple directed ccw-cycle in X, then X^C can also be obtained by a sequence of reversals of essential cycles from ccw to cw. Moreover, the set of essential cycles involved in such a sequence is the unique minimal set such that the interior regions of the essential cycles cover the interior region of C.

Flip Sequences

Lemma. If $(C_1, .., C_k)$ is a flip sequence (ccw $\longrightarrow$ cw) on X then for every edge e the essential cycles $C^{l(e)}$ and $C^{r(e)}$ alternate in the sequence. ▮

Lemma. For every edge e there is a $t_e \in \mathbb{N}$ such that for all α-orientations X a flip sequence (ccw $\longrightarrow$ cw) on X implies at most t_e reorientations of e. ▮

Proposition.
The length of any flip sequence (ccw $\longrightarrow$ cw) is bounded and there is a unique α-orientation $X_{\min}$ with the property that all cycles in $X_{\min}$ are cw-cycles.

Flips and Potentials

- $Y \prec X$ if a flip sequence (ccw $\rightarrow$ cw) $X \rightarrow Y$ exists.

Lemma.
Let $Y \prec X$ and C be an essential cycle. Every sequence $S = (C_1, \ldots, C_k)$ of flips that transforms X into Y contains the same number of flips at C.

Definition. An α-potential for G is a mapping $\wp : \mathsf{Ess}_\alpha \rightarrow \mathbb{N}$ such that

- $|\wp(C) - \wp(C')| \leq 1$, if C and C' share an edge e.

- $\wp(C) \leq 1$, if there is an edge $e \in C$ such that C is the only essential cycle to which e belongs.

- $\wp(C^{l(e)}) \leq \wp(C^{r(e)})$ for all e (orientation from $X_{\min}$)

More on Potentials

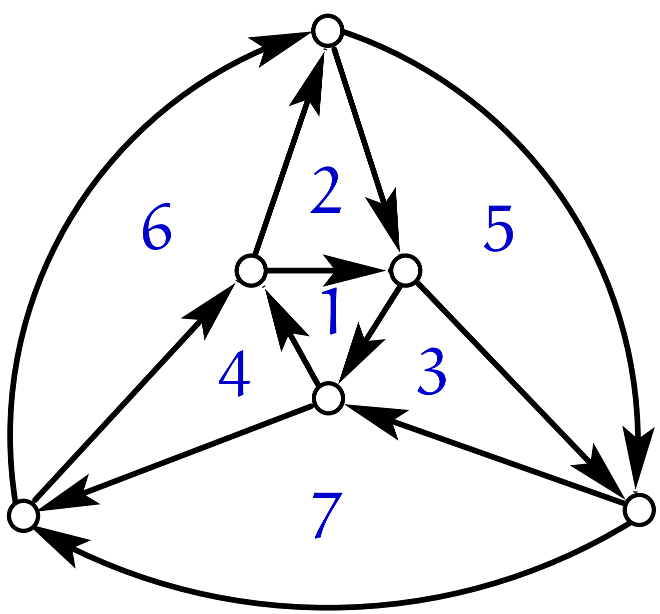
Proposition. There is a bijection between α-potentials and α-orientations. ▮

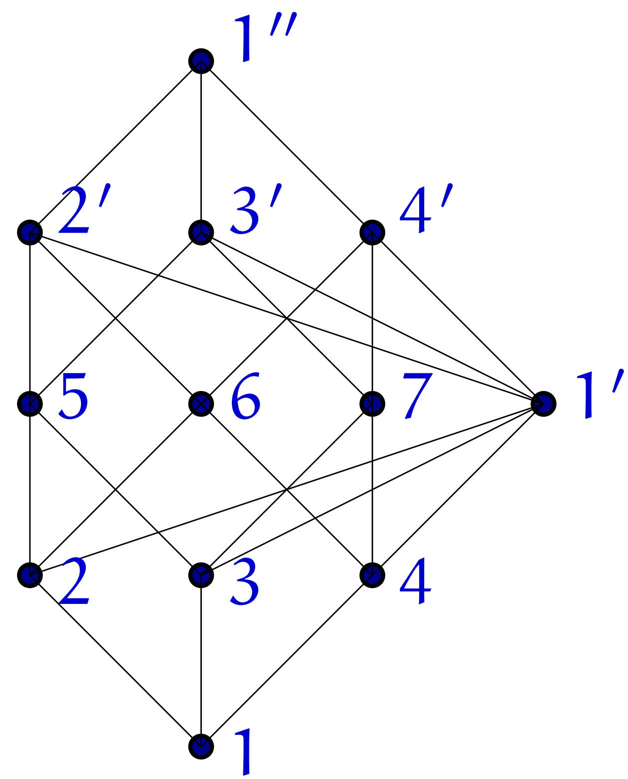
Theorem. α-potentials are a distributive lattice with

- $(\wp_1 \vee \wp_2)(C) = \max\{\wp_1(C), \wp_2(C)\}$ and
- $(\wp_1 \wedge \wp_2)(C) = \min\{\wp_1(C), \wp_2(C)\}$ for all essential C.

Application I: Eulerian Orientations

- Orientations with $\mathsf{outdeg}(v) = \mathsf{indeg}(v)$ for all v, i.e. $\alpha(v) = \frac{d(v)}{2}$

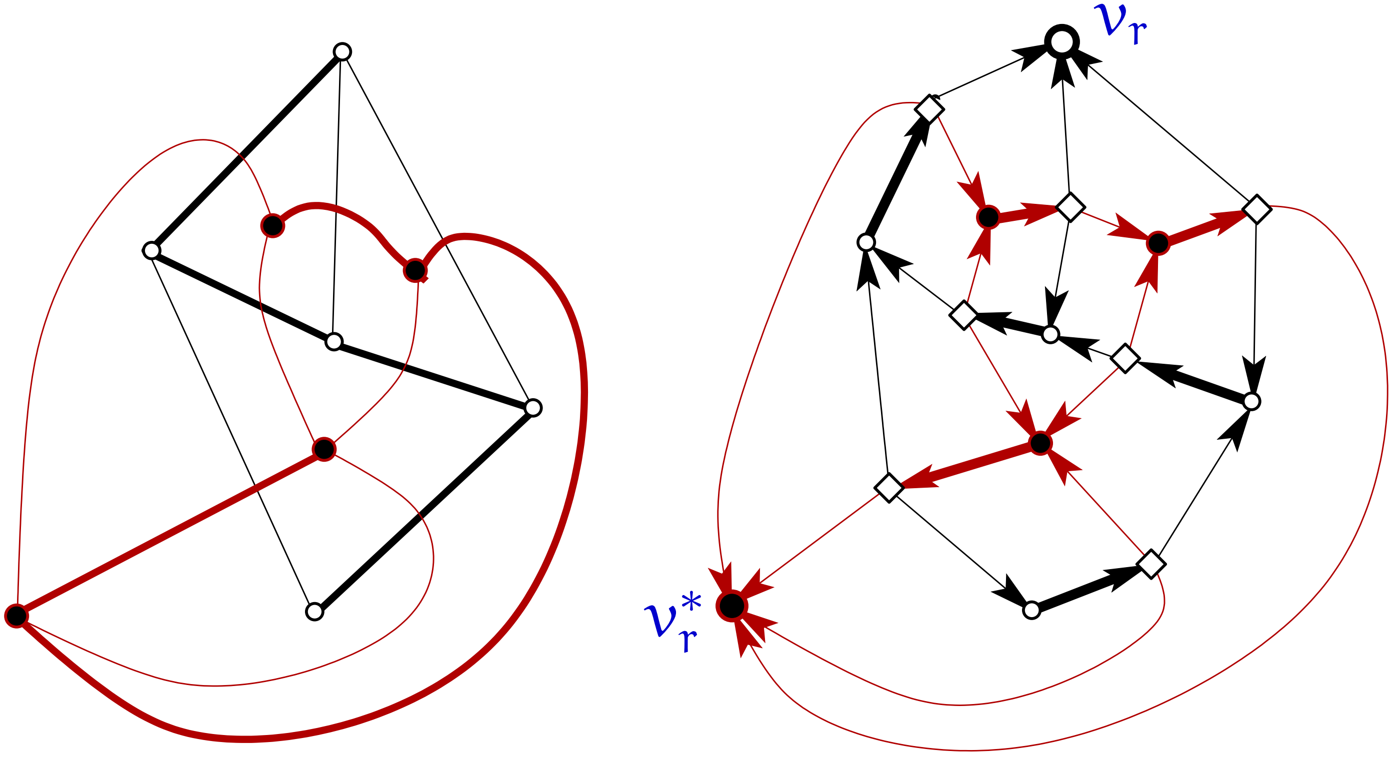




Application II: Spanning Trees

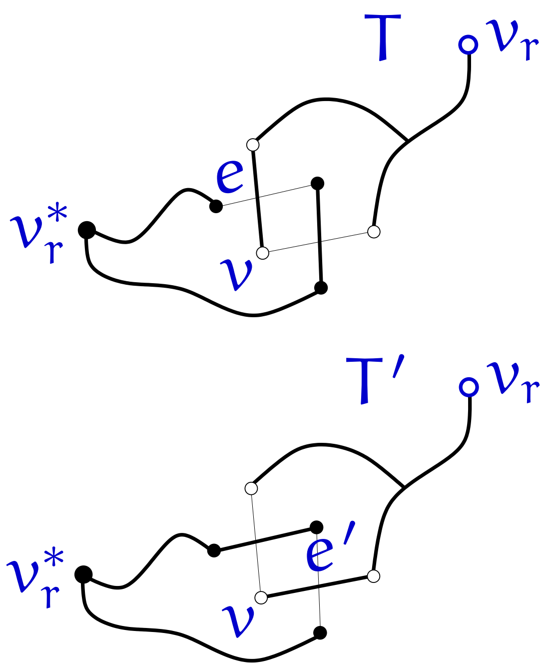
Spanning trees are in bijection with α_T orientations of a rooted primal-dual completion $\widetilde{G}$ of G

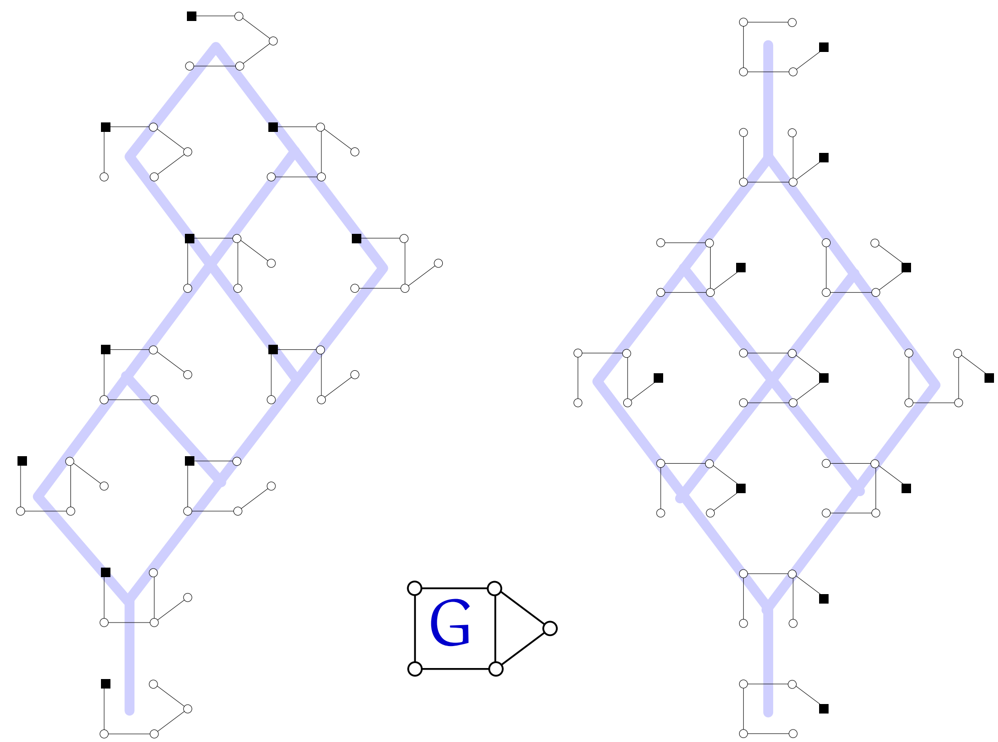
- $\alpha_T(v) = 1$ for a non-root vertex v and $\alpha_T(v_e) = 3$ for an edge-vertex v_e and $\alpha_T(v_r) = 0$ and $\alpha_T(v_r^*) = 0$.



Lattice of Spanning Trees

Gilmer and Litheland 1986, Propp 1993



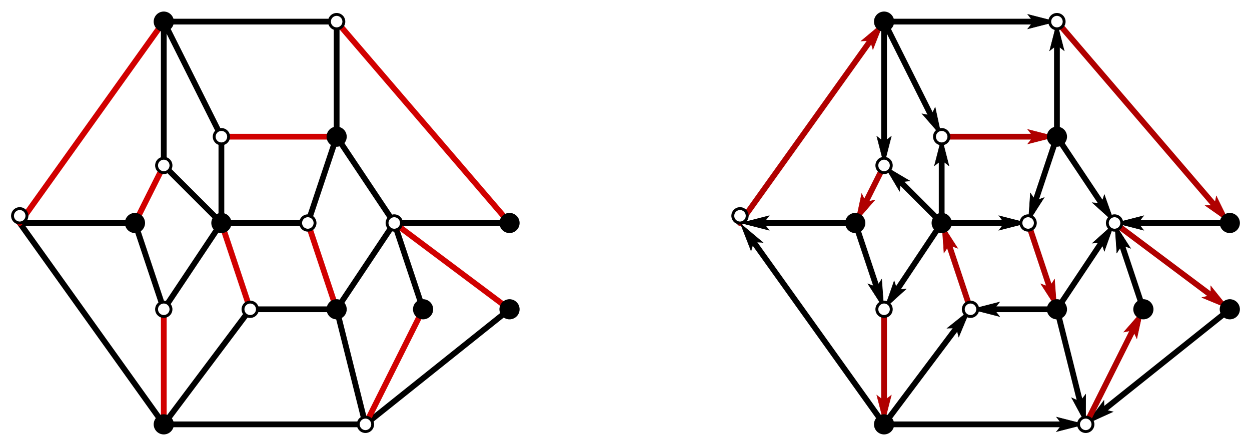


Matchings and f-Factors

Let G be planar and bipartite with parts (U, W). There is bijection between f-factors of G and α_f orientations:

- Define α_f such that $\mathsf{indeg}(u) = f(u)$ for all $u \in U$ and $\mathsf{outdeg}(w) = f(w)$ for all $w \in W$.

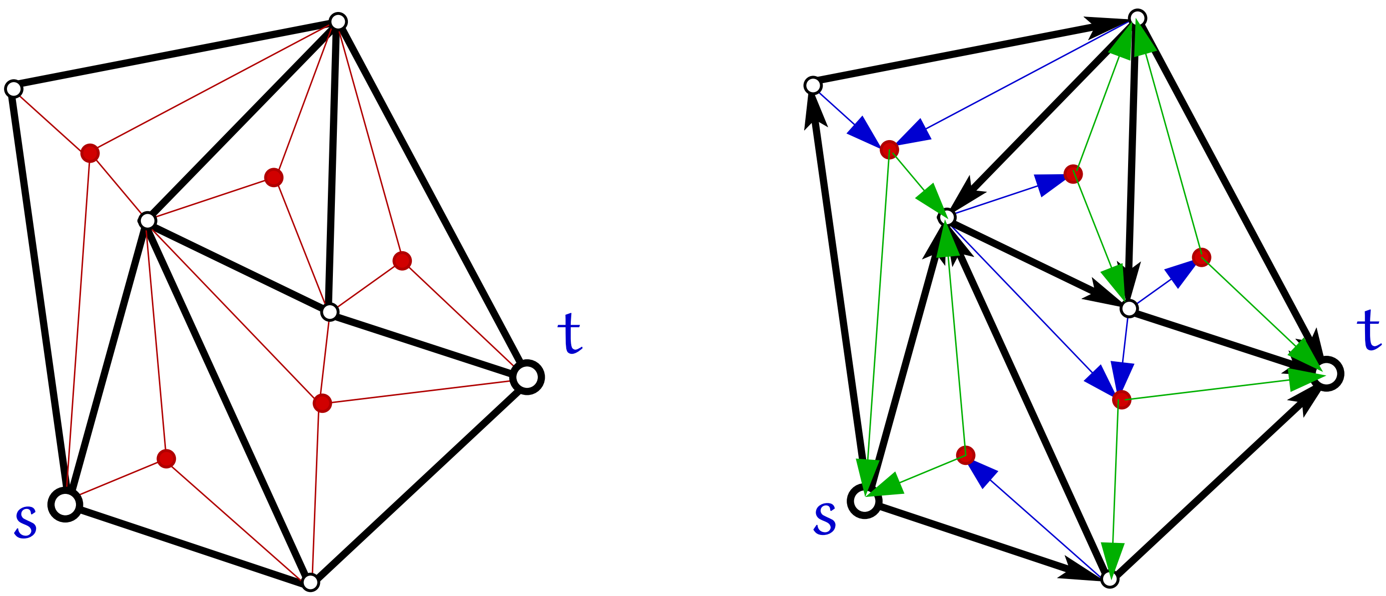
Example. A matching and the corresponding orientation.



Application III: Bipolar Orientations

Bipolar orientations of G, (s, t) are in bijection with α_B orientations of the angle graph A_G associated to G

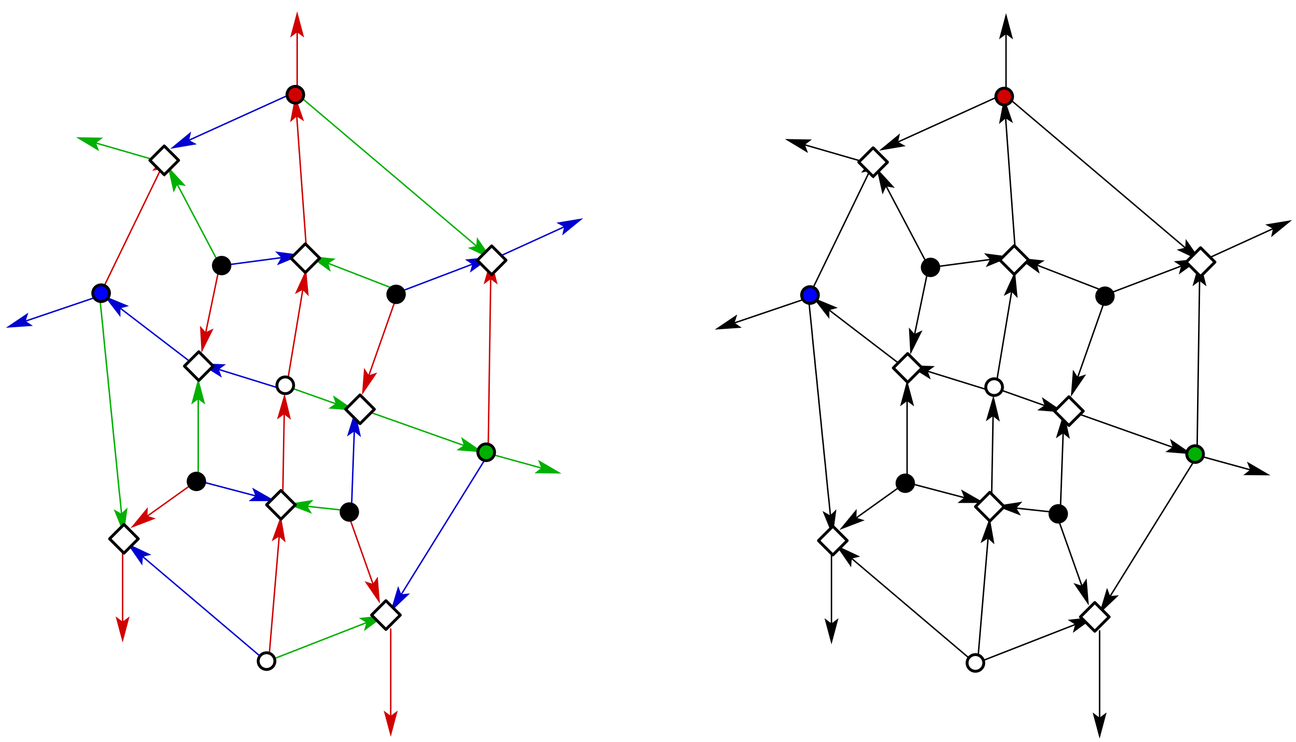
- $\alpha_B(s) = \alpha_B(t) = 0$ and $\alpha_B(v) = 1$ for vertices on the outer face $\alpha_B(x) = 2$ all other v- and f- vertices of A_G.



Application IV: Schnyder Woods

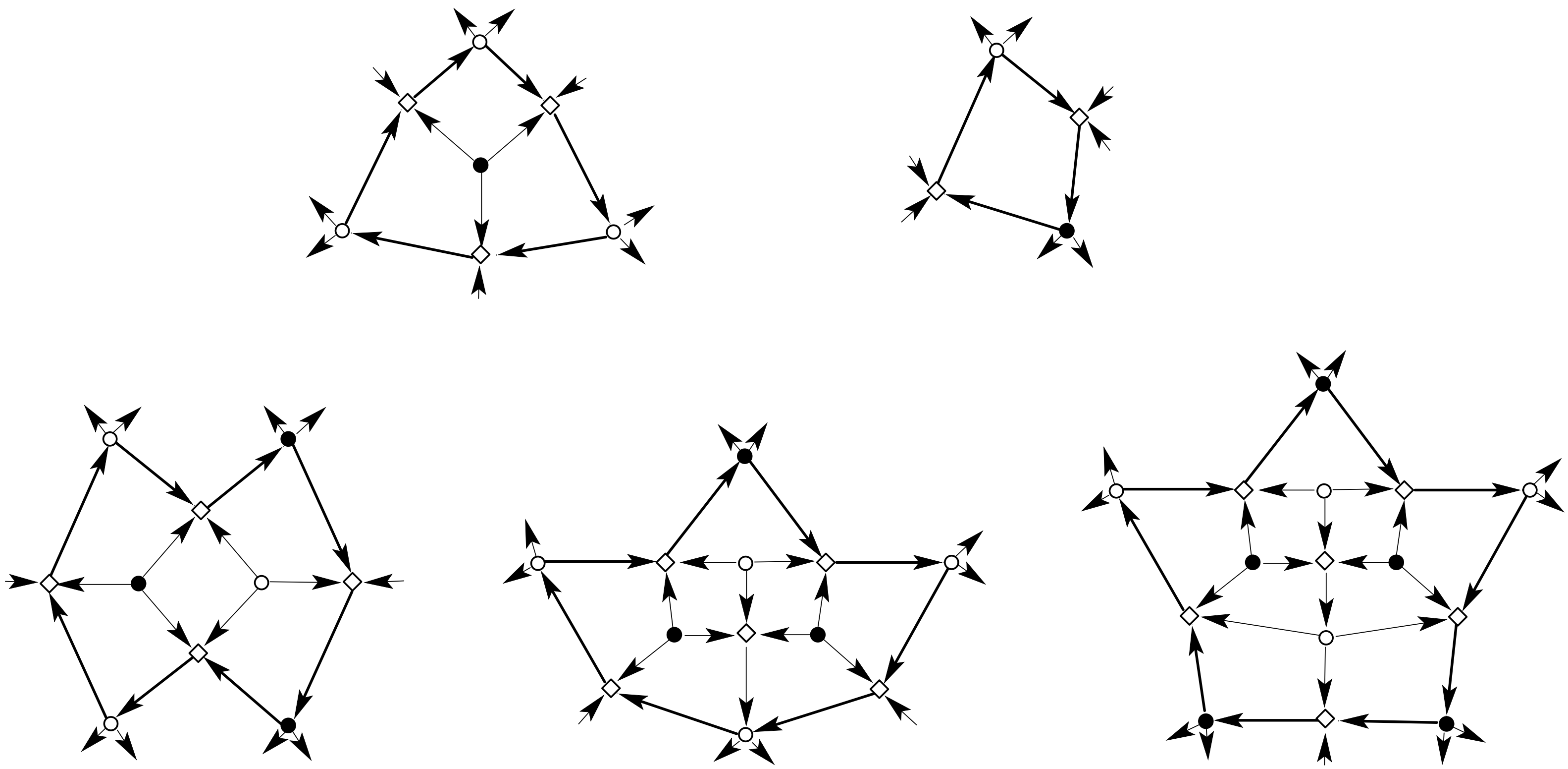
Schnyder woods are in bijection with α_S orientations of a primal-dual completion $\widetilde{G}$ of G

- $\alpha_S(v) = 3$ for a vertex v and $\alpha_S(v_e) = 1$ for an edge-vertex v_e and exterior rays.



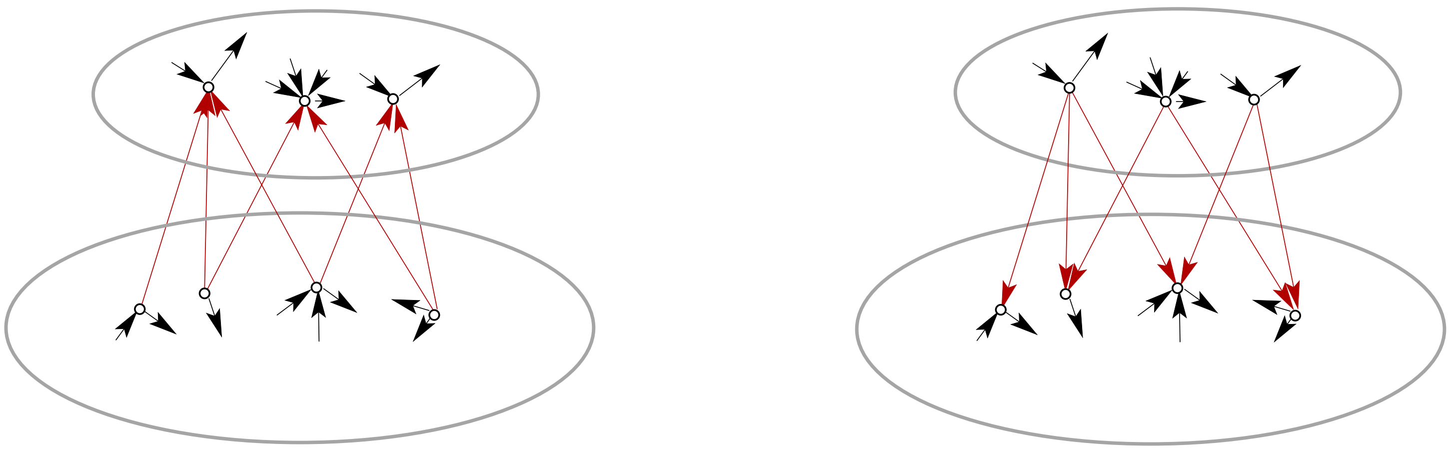
Essential Cycles

- Essential cycles for α_S.



A Dual Construction for Lattices

- Reorientations of directed cuts preserve flow-differences along cycles.



Theorem [Propp 1993].
The set of all orientations of a graph G with prescribed flow-differences for all cycles has the structure of a distributive lattice.

Modules

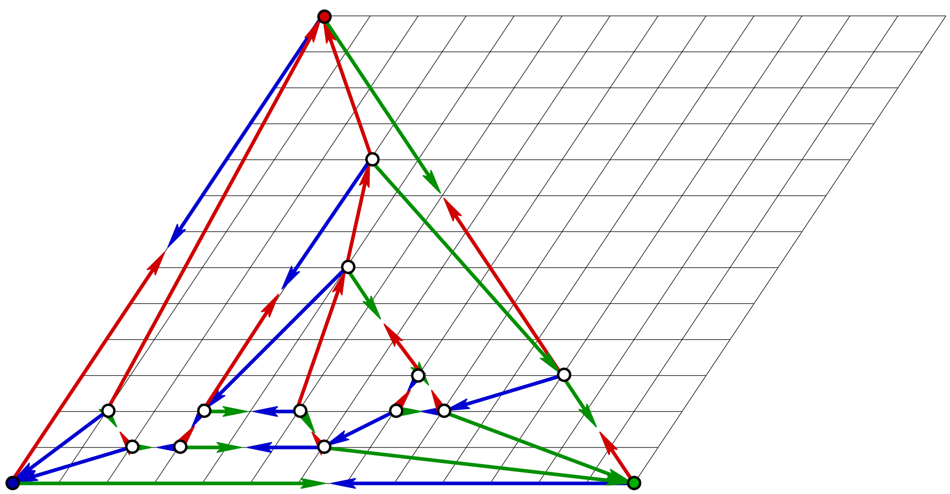
Duality and Orthogonal Surfaces

Flips and Lattices

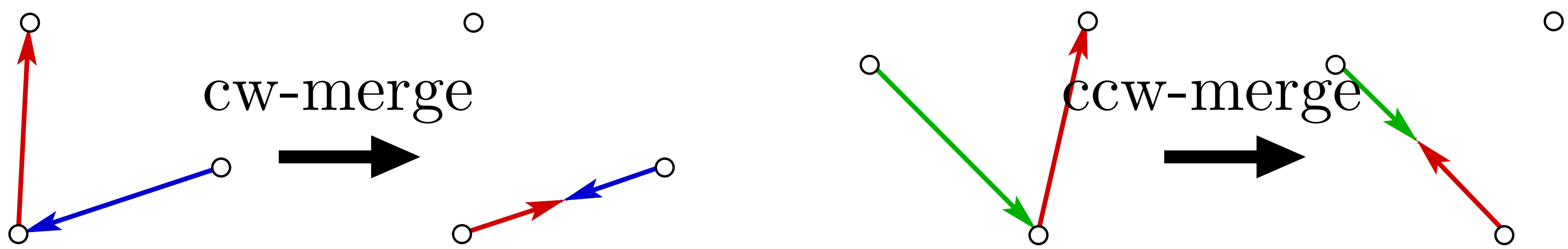
More Compact Convex Drawings

Convex Drawings

Counting faces in regions:
Convex embedding on the $(f-1) \times (f-1)$ grid.



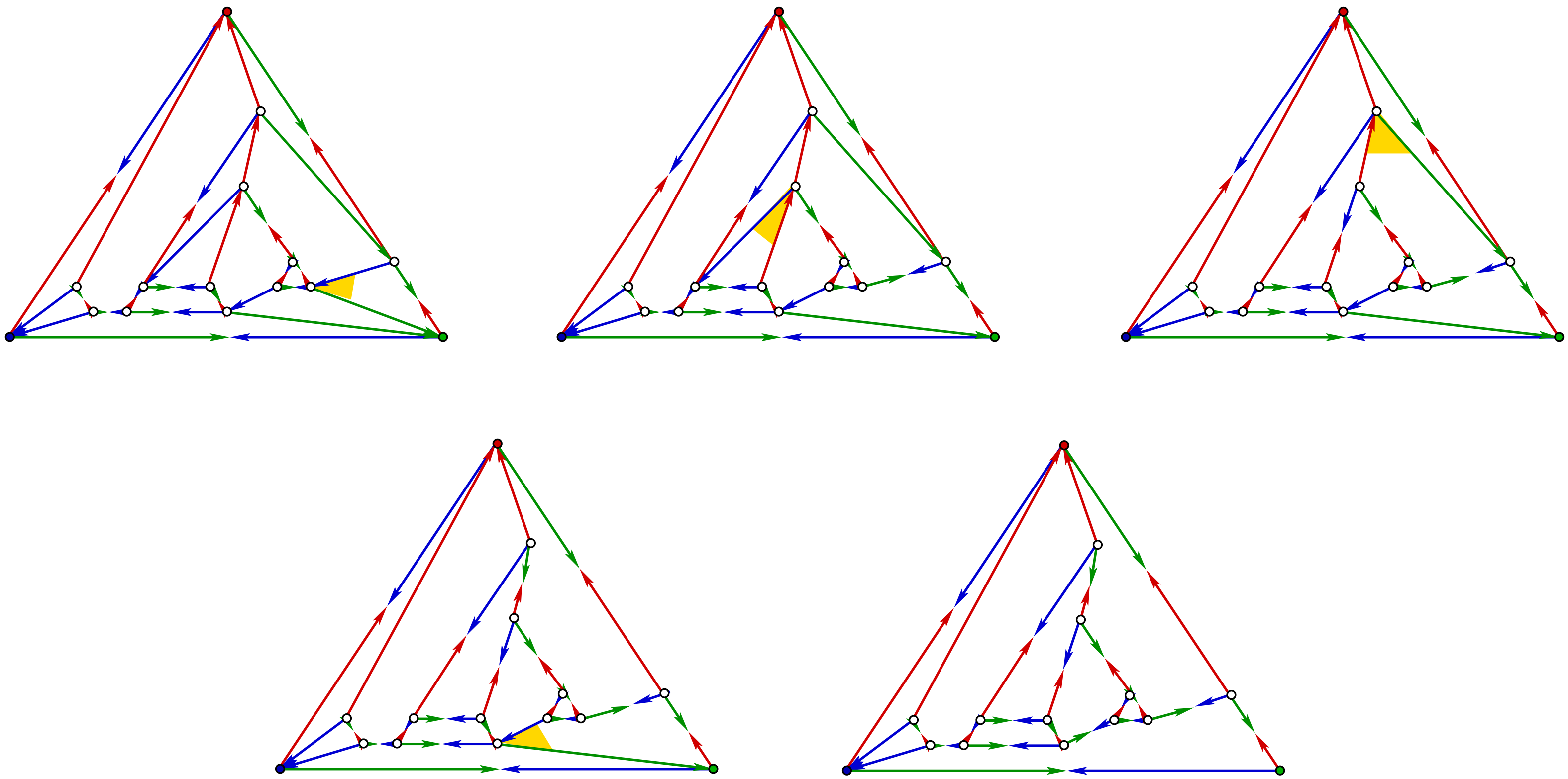
Merging Edges



Merging preserves the Schnyder wood properties.

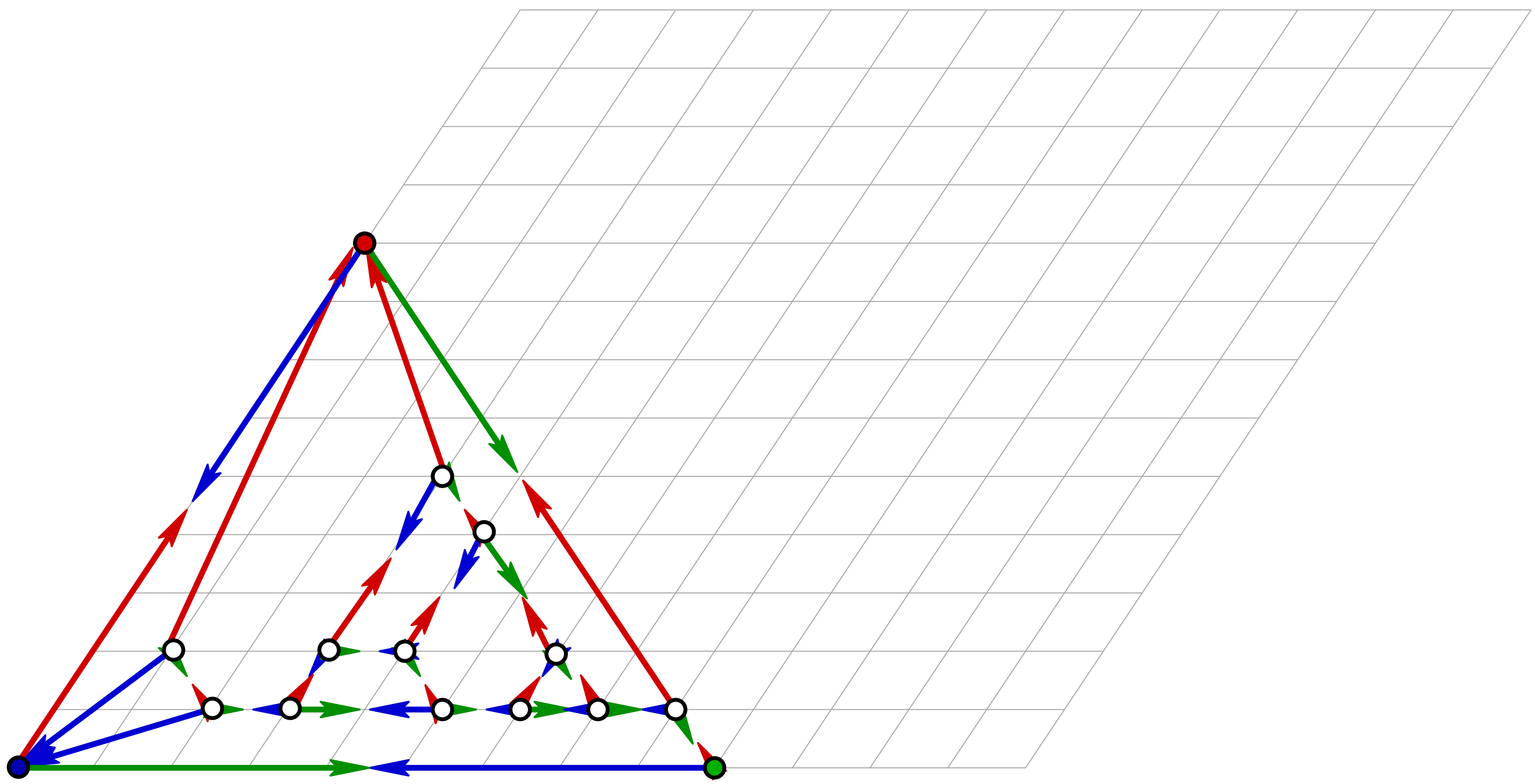
Step I: Reduction

Reduce number of faces by merging edges.



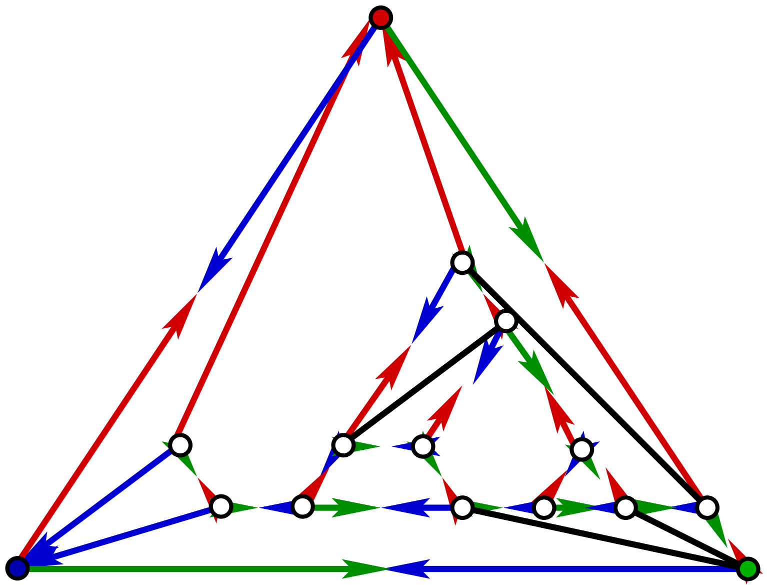
Step II: Draw

Draw the reduced graph on the $(f^{\downarrow} - 1) \times (f^{\downarrow} - 1)$ grid.



Step III: More Drawing

Reinsert the edges which have disappeared by merging.



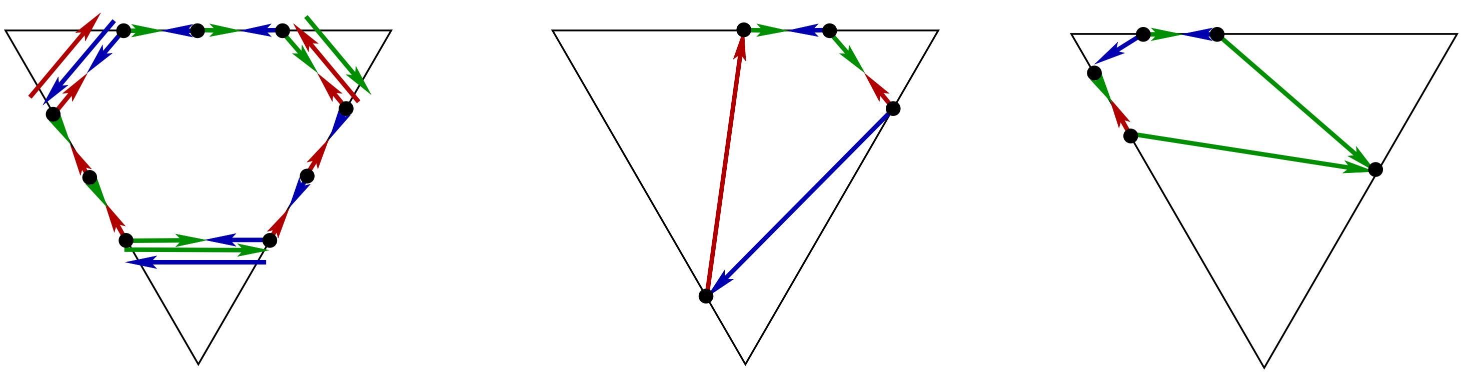
Correctness

When reinserting edges:

- The drawing remains planar.

  Here we need that there is only one type of merges.

- The drawing stays convex.



The End

Thank you.