Block Course Embedding Planar Graphs
First Lecture

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1 Planar Graphs

Definition 1 *We have a* drawing *of a graph $G = (V, E)$ if G is drawn on some space X, that means, if the vertices are mapped to X by an injective mapping $\Gamma : V \to X$, and an edge $\{u, v\}$ is mapped to a simple continuous curve with endpoints $\Gamma(u)$ and $\Gamma(v)$. In our context, we additionally assume that these curves avoid all other vertices.*
A crossing *is a point in $X \setminus \Gamma(V)$ which belongs to more than one edge.*
A graph is planar *if it has a crossing-free drawing in the plane $\mathbb{R}^2$.*
A drawn crossing-free graph is called a plane graph *or a* planar map.

Note: Every graph has a unique crossing-free drawing in $\mathbb{R}^3$ (exercise).

Theorem 2 (Jordan Curve Theorem) *A simple continuous closed curve C in the plane partitions the plane into exactly two faces, each having C as the boundary.*

Proof for the polygonal arc case. To determine whether a point p is inside or outside of C, shoot a ray from p to infinity and check the parity of the number of crossings with C. If it is odd, p is inside of C; if it is even, p is outside.
Without loss of generality we assume that there is no horizontal polygon segment, and that there are no two corners having the same y-coordinate. This can always be obtained by slightly perturbing C.
Now we sweep a horizontal line downwards through the polygon, and define a coloring of this line. It is initialized above the polygon such that all points on the line are blue. We consider the following invariant: The line is colored blue and red, and the colors change at intersections with the polygon. We claim that we can maintain this invariant.
The sweep events are the corners of the polygon. There are three basic types of events, depending on the position of the two polygon segments adjacent to the corner. If both segments reach into the upper half of the plane, we merge the two colors left of the left segment and right of the right segment, which have to

be the same since there are exactly two intersections with the polygon between them. If both segments reach into the lower half, we split the color of the line and insert the other one. Finally, if one segments reaches into the upper and one into the lower bound of the plane, we just continue the already obtained coloring of the line.
We thus get a 2-coloring of the plane. Now it suffices to show that the colored pieces are connected in $\mathbb{R}^2 \setminus C$. To see this, let p and q be two different red points of the plane. We can find a path from p to q by horizontally connecting p to C and walking along C until we can horizontally connect to q. This is possible because C is connected. Strictly speaking, this path also uses C, but from here it is easy to get a path in $\mathbb{R}^2 \setminus C$. Thus the red and the blue part of the plane are each pathwise connected. □

Proposition 3 *K_5 and $K_{3,3}$ are not planar.*

Proof. K_5 has a spanning cycle. The additional edges have to be located inside or outside of this cycle. Some pairs of edges are in conflict, i.e., they have to be mapped to different sides of C. If the vertices of C are $\{1, 2, 3, 4, 5\}$ in this order, then we get the following chain of conflicts: $(1, 4), (2, 5), (1, 3), (2, 4), (3, 5), (1, 4)$. These conflicts form an odd cycle. Therefore there is no conflict-free way of assigning the edges to two sets - and no conlict-free way of embedding the edges in the inside and outside of the cycle C.
For $K_{3,3}$ one can construct a conflict graph in an analogous way, starting with C_6: The remaining three edges are all in conflict with each other.
There is also a very straightforward way of verifying that there cannot be a planar drawing of $K_{3,3}$: If we first embed $K_{2,3}$, we see that there is basically a unique way of doing it without crossing; this drawing has three faces. Now the missing vertex has to be placed in one of these faces, but then there is always one edge which can't be inserted in a crossing-free way. □

2 Dual Graph

Definition 4 *Let G be a plane graph. The dual graph G^* is built by placing a vertex into every face of G. There is a dual edge e^* for every edge e of G: If f and g are the two faces incident to e, then $e^* = (f, g)$.*

Remarks:

- The dual graph is planar.
- Duals of simple graphs may have loops and multiple edges.
- The definition can also be applied to directed graphs, yielding a directed dual: An edge e^* is directed from the left face to the right face of e.
- Different drawings of a graph may produce different duals.

- For connected graphs, we have $G^{**} = G$.

Proposition 5 *Let G be a plane graph. $S \subseteq E$ contains a cycle iff S^* is a separating set in G^*.*

Proof. Let $C \subseteq S$ be a cycle. There is a face of G and thus a vertex v^* of G^* in the interior of C, and for the same reason there's a vertex $w*$ in the exterior of C. Every path from v^* to w^* has to intersect C. Thus, the corresponding edge set C^* separates v^* and w^*.
For the reverse direction, suppose that S is acyclic. Then $\mathbb{R}^2 \setminus C$ is connected, thus one can find a curve connecting any two dual vertices. This curve corresponds to a dual path. □

Corollary 6 *The cycle matroid of G equals the bond matroid of G^*.*

Proof. Left to those readers familiar with matroids. □

3 Euler's Formula

Theorem 7 *Let $G = (V, E)$ be a connected plane graph with face set F. Then $|V| - |E| + |F| = 2$.*

Remark: In his geometric junkyard, David Eppstein has collected nineteen proofs of Euler's Formula. The reader can find them on the webpage www.ics.uci.edu/˜eppstein/junkyard/euler.

First Proof: Duality
Let $F \subseteq E$ be an acyclic set of edges. This is equivalent to the fact that $F^* \subseteq E^*$ is non-separating, thus, to the fact that $E^* \setminus F^*$ connects all vertices of V^*. In short: Connectedness and acyclicity are dual to each other.
Now let $T \subseteq E$ be acyclic and connecting all vertices of V. Then T is a spanning tree - and the dual edges of its complement also form a spanning tree! We therefore have $|T| = |V| - 1$ as well as $|\bar{T}^*| = |V*| - 1 = |F| - 1$. Since $|T|$ and the number of edges in its complement sum up to $|E|$, we deduce $|E| = |V| - 1 + |F| - 1$, which is Euler's Formula. □

Second Proof: Induction
Let T be a spanning tree in G and color all its edges red. The edges in $E \setminus T$ are blue. Each blue edge closes a cycle in G, thus, removing a blue edge reduces the number of faces by 1. Remove all blue edge, then the number of vertices stays the same, but there's only one face left. This yields:

$$V(G) - E(G) + F(G) = V(T) - E(T) + F(T) = V(T) - (V(T) - 1) + 1 = 2$$

□

Third Proof: Local Discharging
Consider a plane drawing of G and perturb it such that no two vertex-locations have the same y-coordinate. Direct all edges upwards, such that G has exactly one source and one sink. Now assign a $+$ to each vertex and each face, and a $-$ to each edge. Shift the sign of every edge up to its (higher) end-vertex, and the sign of each face up to the highest possible location in this face, which is at a vertex since we do not have horizontal edges. This does not hold for the outer face; its sign moves up to infinity. The signs at the vertices remain at their place.
Now we consider the signs at any vertex v which is not the source. Each face f for which v is the highest incident vertex contributes a $+$, which is locally discharged by the $-$ of the left edge incident to v and bounding f. The rightmost edge with end-vertex v contributes another $-$ which cancels out the $+$ of v.
Thus, the only signs which are not canceled out are the $+$ of the outer face and the $+$ of the source - and we obtain Euler's Formula. □

4 Consequences of Euler's Formula

Corollary 8 *K_5 is not planar.*

Proof. K_5 has 5 vertices and 10 edges. Suppose that it is planar, Euler's Formula then tells us that in a planar drawing there are 7 faces. Every face is surrounded by at least 3 edges. Thus, if we sum up the degrees of all faces we can deduce that there at least 21 edges. But summing up the degrees of all faces counts every edge exactly twice, and this yields only 20 edges, a contradiction. □

Corollary 9 *A planar graph $G = (V, E)$ has at most $3|V| - 6$ edges.*

Proof. Let f_k denote the number of faces of degree k. Then:

$$2|E| = \sum_f d(f) = \sum_k k \cdot f_k \geq 3 \sum_k f_k = 3|F|$$

Thus, we have $3|F| \leq 2|E|$. Now from $|V| - 2 = |E| - |F|$ we get:

$$3|V| - 6 = 3|E| - 3|F| \geq 3|E| - 2|E| = |E|$$

□

This upper bound on the number of edges of a planar graph $G = (V, E)$ has an easy consequence that is especially useful for inductive proofs:
G has a vertex of degree at most 5, since $\sum_v d(v) = 2|E| \leq 6|V| - 12$.

Corollary 10 *A planar graph $G = (V, E)$ has at most $2|V| - 4$ faces.*

Proof. We again use the inequality $2|E| \geq 3|F|$. Then from $|V| - 2 = |E| - |F|$ we can deduce:

$$2|V| - 4 = 2|E| - 2|F| \geq 3|F| - 2|F| = |F|$$

□

5 Classical Theorems on Planar Graphs

Theorem 11 (Kuratowski 1930) *A graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.*

Corollary 12 (Wagner 1937) *A graph is planar if and only if it has no K_5- or $K_{3,3}$-minor.*

Theorem 13 (Wagner 1936, Fary 1948) *A graph is planar if and only if it has a straight line drawing without crossings.*

Theorem 14 (Whitney 1933) *A graph is planar if and only if its bond matroid is the cycle matroid of a graph.*

Theorem 15 (McLane 1937) *A graph is planar if and only there is a basis for its cycle space containing each edge at most twice.*

Theorem 16 (Tutte 1960) *A graph is planar and 3-connected if and only if it has a drawing with convex faces.*

Theorem 17 (Steinitz 1922) *A graph is planar and 3-connected if and only if it is the graph of a polytope.*

Theorem 18 (Whitney) *If a graph is planar and 3-connected, then it has a unique embedding on the sphere in the sense that there is a unique dual graph.*

Remark:
Considering different drawings of the graph $K_{2,4}$ (which has connectivity 2) we notice that the face cycles can be different; thus, this graph has different (non-isomorphic) duals.

Proof of Whitney's Theorem: Let A be a drawing of a 3-connected planar graph G, and let F be a face in A. It suffices to show that F is a face in every drawing of G.
To see this, let C be the boundary of F, and let $x, y \in V(G \setminus C)$. By Menger's Theorem there are three internally disjoint $x-y-$paths in G. Deleting C can destroy at most two of them, therefore, $G \setminus C$ is connected. Thus, if v is a vertex outside of F in a drawing, we see that every other vertex of $G \setminus C$ must be outside of F as well. It follows that F is a face in any drawing of G.
□