

## Finite Elemente: Definition und Eigenschaften

### Bezeichnung der Freiheitsgrade $\Sigma_T$

- Funktionswerte
- ⊙ Funktionswerte und erste Ableitungen
- ⊗ Funktionswerte, erste und zweite Ableitungen
- ⊙↗ Funktionswerte, erste und gemischte zweite Ableitungen
- + Normalenableitung

### Polynomräume

$$\mathcal{P}^k(\mathbb{R}^d) = \{p : \mathbb{R}^d \rightarrow \mathbb{R} : p(x_1, x_2, \dots, x_d) = \sum_{\alpha_1 + \alpha_2 + \dots + \alpha_d \leq k} a_{\alpha_1 \alpha_2 \dots \alpha_d} x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d}\}$$

$$\dim \mathcal{P}^k(\mathbb{R}^d) = \binom{d+k}{k}, \quad \mathcal{P}^2(\mathbb{R}^2) = \text{span}\{1, x, y, xy, x^2, y^2\}$$

$$\mathcal{Q}^k(\mathbb{R}^d) = \{p : \mathbb{R}^d \rightarrow \mathbb{R} : p(x_1, x_2, \dots, x_d) = p_1(x_1)p_2(x_2)\dots p_d(x_d), p_i \in \mathcal{P}^k(\mathbb{R})\}$$

$$\dim \mathcal{Q}^k(\mathbb{R}^d) = (k+1)^d, \quad \mathcal{Q}^2(\mathbb{R}^2) = \text{span}\{1, x, y, xy, x^2, y^2, x^2y, xy^2, x^2y^2\}$$

$$\mathcal{P}^k \subset \mathcal{Q}^k \subset \mathcal{P}^{dk}$$

### Baryzentrische Koordinaten

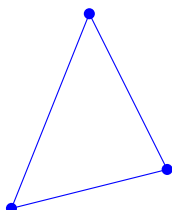
Sei  $T$  ein  $d$ -Simplex mit den Ecken  $a_1, \dots, a_{d+1} \in \mathbb{R}^d$ . Unter den baryzentrischen Koordinaten eines Punktes  $x \in \mathbb{R}^d$  versteht man die eindeutige Lösung  $\ell_1, \dots, \ell_{d+1}$  des linearen Gleichungssystems

$$\sum_{i=1}^{d+1} a_{ij} \ell_i = x_j \quad (j = 1, \dots, d), \quad \sum_{i=1}^{d+1} \ell_i = 1.$$

Ist  $\hat{T}$  das Einheitsdreieck, so gilt  $\ell_1 = 1 - \hat{x} - \hat{y}$ ,  $\ell_2 = \hat{x}$ ,  $\ell_3 = \hat{y}$ . Baryzentrische Koordinaten sind gegenüber affin-linearen Transformationen invariant.

### Konforme Dreieckelemente: Lagrange

#### Courant-Element



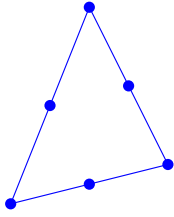
$$P_T = \mathcal{P}^1, \quad \dim P_T = 3, \quad S_h \subset C^0(\bar{\Omega}) \cap H^1(\Omega)$$

$$\Sigma_T = \{p(a_i)\}$$

$$\text{Basis: } \ell_1, \ell_2, \ell_3$$

$$\|v - \Pi_T v\|_{m,T} = \mathcal{O}(h_T^{2-m}) \quad \text{für } v \in H^2(\hat{T})$$

### Quadratisches Element



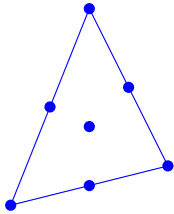
$$P_T = \mathcal{P}^2, \dim P_T = 6, S_h \subset C^0(\overline{\Omega}) \cap H^1(\Omega)$$

$$\Sigma_T = \{p(a_i), p(a_{ij})\}$$

$$\text{Basis: } \ell_i(2\ell_i - 1) \ (i = 1, 2, 3), 4\ell_i\ell_j \ (i, j = 1, 2, 3, i < j)$$

$$\|v - \Pi_T v\|_{m,T} = \mathcal{O}(h_T^{3-m}) \text{ für } v \in H^3(\overset{\circ}{T})$$

### Erweitertes quadratisches Element



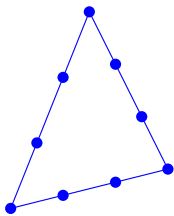
$$\mathcal{P}^2 \subset P_T \subset \mathcal{P}^3, \dim P_T = 7, S_h \subset C^0(\overline{\Omega}) \cap H^1(\Omega)$$

$$\Sigma_T = \{p(a_i), p(a_{ij}), p(a_{ijk})\}$$

$$\text{Basis: } \ell_i(2\ell_i - 1) \ (i = 1, 2, 3), 4\ell_i\ell_j \ (i, j = 1, 2, 3, i < j), 27\ell_1\ell_2\ell_3$$

$$\|v - \Pi_T v\|_{m,T} = \mathcal{O}(h_T^{3-m}) \text{ für } v \in H^3(\overset{\circ}{T})$$

### Reduziertes kubisches Element



$$\mathcal{P}^2 \subset P_T \subset \mathcal{P}^3, \dim P_T = 9, S_h \subset C^0(\overline{\Omega}) \cap H^1(\Omega)$$

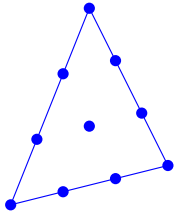
$$\Sigma_T = \{p(a_i), p(a_{ij})\}$$

$$\text{Basis: } \frac{1}{2}\ell_i(3\ell_i - 1)(3\ell_i - 2) - \frac{9}{2}\ell_1\ell_2\ell_3 \ (i = 1, 2, 3),$$

$$\frac{9}{2}\ell_i(3\ell_i - 1)\ell_j + \frac{27}{4}\ell_1\ell_2\ell_3 \ (i, j = 1, 2, 3, i \neq j)$$

$$\|v - \Pi_T v\|_{m,T} = \mathcal{O}(h_T^{3-m}) \text{ für } v \in H^3(\overset{\circ}{T})$$

### Kubisches Element



$$P_T = \mathcal{P}^3, \dim P_T = 10, S_h \subset C^0(\overline{\Omega}) \cap H^1(\Omega)$$

$$\Sigma_T = \{p(a_i), p(a_{ij}), p(a_{ijk})\}$$

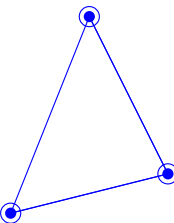
$$\text{Basis: } \frac{1}{2}\ell_i(3\ell_i - 1)(3\ell_i - 2) \ (i = 1, 2, 3), \frac{9}{2}\ell_i(3\ell_i - 1)\ell_j \ (i, j = 1, 2, 3, i \neq j),$$

$$27\ell_1\ell_2\ell_3$$

$$\|v - \Pi_T v\|_{m,T} = \mathcal{O}(h_T^{4-m}) \text{ für } v \in H^4(\overset{\circ}{T})$$

### Konforme Dreieckelemente: Hermite

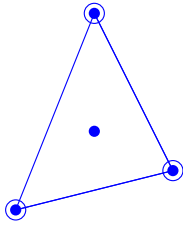
### Zienkiewicz- bzw. reduziertes kubisches Hermite-Element



$$\mathcal{P}^2 \subset P_T \subset \mathcal{P}^3, \dim P_T = 9, S_h \subset C^0(\overline{\Omega}) \cap H^1(\Omega),$$

$$\Sigma_T = \{p(a_i), \partial_k p(a_i)\}, \Sigma'_T = \{p(a_i), \nabla p(a_i) \cdot (a_j - a_i)\}$$

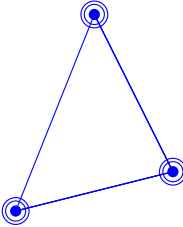
$$\|v - \Pi_T v\|_{m,T} = \mathcal{O}(h_T^{3-m}) \text{ für } v \in H^3(\overset{\circ}{T})$$

**Kubisches Hermite-Element**

$$P_T = \mathcal{P}^3, \dim P_T = 10, S_h \subset \mathcal{C}^0(\bar{\Omega}) \cap H^1(\Omega),$$

$$\Sigma_T = \{p(a_i), p(a_{ijk}), \partial_k p(a_i)\}, \Sigma'_T = \{p(a_i), p(a_{ijk}), \nabla p(a_i) \cdot (a_j - a_i)\}$$

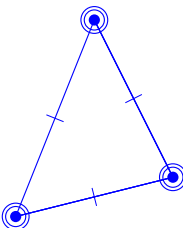
$$\|v - \Pi_T v\|_{m,T} = \mathcal{O}(h_T^{4-m}) \text{ für } v \in H^4(\mathring{T})$$

**Bell-Element**

$$P^4 \subset P_T \subset \mathcal{P}^5, \dim P_T = 18, S_h \subset \mathcal{C}^1(\bar{\Omega}) \cap H^2(\Omega)$$

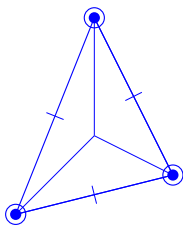
$$\Sigma'_T = \{p(a_i), \nabla p(a_i) \cdot (a_j - a_i), \nabla^2 p(a_i) \cdot (a_j - a_i) \cdot (a_k - a_i)\}$$

$$\|v - \Pi_T v\|_{m,T} = \mathcal{O}(h_T^{5-m}) \text{ für } v \in H^5(\mathring{T})$$

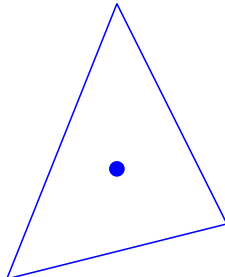
**Argyris-Element**

$$P_T = \mathcal{P}^5, \dim P_T = 21, S_h \subset \mathcal{C}^1(\bar{\Omega}) \cap H^2(\Omega)$$

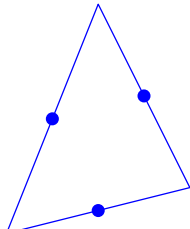
$$\|v - \Pi_T v\|_{m,T} = \mathcal{O}(h_T^{6-m}) \text{ für } v \in H^6(\mathring{T})$$

**Hsieh-Clough-Tocher-Element**  $\mathcal{P}^3 \subset P_T = \{p \in \mathcal{C}^1(T) : p|_{T_i} \in \mathcal{P}^3\}, \dim P_T = 12, S_h \subset \mathcal{C}^1(\bar{\Omega}) \cap H^2(\Omega)$ 

$$\|v - \Pi_T v\|_{m,T} = \mathcal{O}(h_T^{4-m}) \text{ für } v \in H^4(\mathring{T})$$

**Nichtkonforme Dreieckelemente** **$\mathcal{P}^0$ -Element**

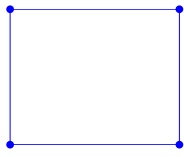
$$P_T = \mathcal{P}^0, \dim P_T = 1, S_h \not\subset \mathcal{C}^0(\bar{\Omega})$$

**Crouzeix-Raviart-Element**

$$P_T = \mathcal{P}^1, \dim P_T = 3, S_h \not\subset \mathcal{C}^0(\bar{\Omega})$$

## Konforme Rechteckelemente: Lagrange

### Bilineares Element

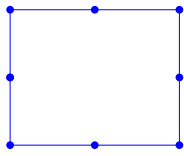


$$P_T = \mathcal{Q}^1, \dim P_T = 4, S_h \subset \mathcal{C}^0(\bar{\Omega}) \cap H^1(\Omega),$$

Basis auf dem Einheitsquadrat:  $(1-x)(1-y), x(1-y), xy, (1-x)y$

$$\|v - \Pi_T v\|_{m,T} = \mathcal{O}(h_T^{2-m}) \text{ für } v \in H^2(\mathring{T})$$

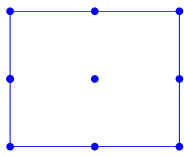
### Reduziertes biquadratisches (Serendipity-) Element



$$\mathcal{P}^2 \subset P_T \subset \mathcal{Q}^2, \dim P_T = 8, S_h \subset \mathcal{C}^0(\bar{\Omega}) \cap H^1(\Omega),$$

$$\|v - \Pi_T v\|_{m,T} = \mathcal{O}(h_T^{3-m}) \text{ für } v \in H^3(\mathring{T})$$

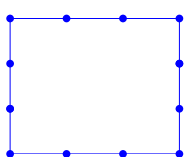
### Biquadratisches Element



$$P_T = \mathcal{Q}^2, \dim P_T = 9, S_h \subset \mathcal{C}^0(\bar{\Omega}) \cap H^1(\Omega),$$

$$\|v - \Pi_T v\|_{m,T} = \mathcal{O}(h_T^{3-m}) \text{ für } v \in H^3(\mathring{T})$$

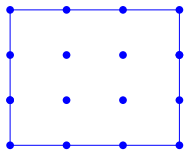
### Reduziertes bikubisches (Serendipity-) Element



$$\mathcal{P}^3 \subset P_T \subset \mathcal{Q}^3, \dim P_T = 12, S_h \subset \mathcal{C}^0(\bar{\Omega}) \cap H^1(\Omega)$$

$$\|v - \Pi_T v\|_{m,T} = \mathcal{O}(h_T^{4-m}) \text{ für } v \in H^4(\mathring{T})$$

### Bikubisches Element

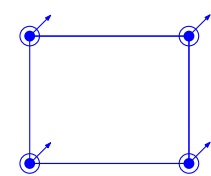


$$P_T = \mathcal{Q}^3, \dim P_T = 16, S_h \subset \mathcal{C}^0(\bar{\Omega}) \cap H^1(\Omega)$$

$$\|v - \Pi_T v\|_{m,T} = \mathcal{O}(h_T^{4-m}) \text{ für } v \in H^4(\mathring{T})$$

## Konforme Rechteckelemente: Hermite

### Bogner-Fox-Schmit-Element



$$P_T = \mathcal{Q}^3, \dim P_T = 16, S_h \subset \mathcal{C}^1(\bar{\Omega}) \cap H^2(\Omega)$$

$$\|v - \Pi_T v\|_{m,T} = \mathcal{O}(h_T^{4-m}) \text{ für } v \in H^4(\mathring{T})$$