5. Tutorium

A1 $f: \mathbb{R}_{>0} \times \mathbb{R} \longrightarrow \mathbb{R}$, $(x,y) \longmapsto \ln(x) \cdot e^{x+y-1}$ $\qquad g: \mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x,y) \longmapsto x^2 + e^{2y-x}$

a.)

$$\operatorname{grad}_{(x,y)} f = \begin{pmatrix} \ln(x) \cdot e^{x+y-1} + \frac{e^{x+y-1}}{x} \\ \ln(x) \cdot e^{x+y-1} \end{pmatrix}$$

$$\operatorname{Hess}_{(x,y)} f = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

Satz von Schwarz

$A_{12} = A_{21}$ falls f 2-mal stetig diffbar

$$A_{11} = \ln(x) \cdot e^{x+y-1} + \frac{e^{x+y-1}}{x} - \frac{1}{x^2} \cdot e^{x+y-1} + \frac{1}{x} \cdot e^{x+y-1}$$

$$A_{21} = A_{12} = \ln(x) \cdot e^{x+y-1} + \frac{e^{x+y-1}}{x}$$

$$= \ln(x)\, e^{x+y-1} + 2\,\frac{e^{x+y-1}}{x} - \frac{e^{x+y-1}}{x^2}$$

$$A_{22} = \ln(x) \cdot e^{x+y-1}$$

b.) Taylorpolynom (TP) von f im Entwicklungspunkt (EP) $(1,0)$:

Daten: $\underset{=0}{f(1,0)}$ $\quad \underset{=\begin{pmatrix}1\\0\end{pmatrix}}{\mathrm{grad}_{(1,0)} f}$ $\quad \underset{=\begin{pmatrix}1&1\\1&0\end{pmatrix}}{\mathrm{Hess}_{(1,0)} f}$

$$\left(T_1^{(1,0)} f\right)(x,y) = f(1,0) + \left(\mathrm{grad}_{(1,0)} f\right)^T \cdot \begin{pmatrix} x-1 \\ y-0 \end{pmatrix}$$

$$= 0 + \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y \end{pmatrix} = \underline{x-1}$$

$$\left(T_2^{(1,0)} f\right)(x,y) = \underbrace{f(1,0) + \left(\mathrm{grad}_{(1,0)} f\right)^T \cdot \begin{pmatrix} x-1 \\ y \end{pmatrix}}_{T_1} + \frac{1}{2} \begin{pmatrix} x-1 \\ y \end{pmatrix}^T \cdot \mathrm{Hess}_{(1,0)} f \cdot \begin{pmatrix} x-1 \\ y \end{pmatrix}$$

$$= x-1 + \frac{1}{2} \cdot \begin{pmatrix} x-1 & y \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y \end{pmatrix}$$

$$= x-1 + \frac{1}{2} (x-1)^2 + y \cdot (x-1)$$

c.) $$\mathrm{grad}_{(x,y)}\, g = \begin{pmatrix} 2x - e^{2y-x} \\ 2e^{2y-x} \end{pmatrix}; \quad \mathrm{Hess}_{(x,y)}\, g = \begin{pmatrix} 2+e^{2y-x} & -2e^{2y-x} \\ -2e^{2y-x} & 4e^{2y-x} \end{pmatrix}$$

d.) $$\left(T_1^{(2,1)} g\right)(x,y) = 5 + \underbrace{\begin{pmatrix} 3 & 2 \end{pmatrix}}_{\mathrm{grad}_{(2,1)} g} \cdot \begin{pmatrix} x-2 \\ y-1 \end{pmatrix} = -3 + 3x + 2y$$

$$\left(T_2^{(2,1)} g\right)(x,y) = \boxed{\left(T_1^{(2,1)} g\right)(x,y)} + \frac{1}{2} \cdot \begin{pmatrix} x-2 & y-1 \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 3 & -2 \\ -2 & 4 \end{pmatrix}}_{\mathrm{Hess}_{(2,1)} g} \cdot \begin{pmatrix} x-2 \\ y-1 \end{pmatrix}$$

$$= \boxed{5 + 3 \cdot (x-2) + 2 \cdot (y-1)}$$

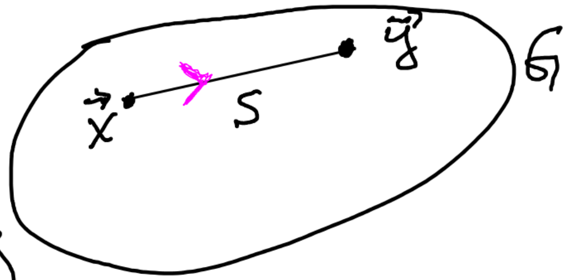
$$+ \frac{1}{2} \left(3(x-2)^2 - 4 \cdot (x-2)(y-1) + 4 \cdot (y-1)^2 \right)$$

Erinnerung: Fehlerschrankensatz

Vor.

Sei $f: \mathbb{R}^n \supset G \to \mathbb{R}$ diffbar, G offen

$\vec{x}, \vec{y} \in G$ fest,



$$S = \{ \vec{x} + t \cdot (\vec{y} - \vec{x}) : t \in [0,1] \}$$

$$\left| \frac{\partial f}{\partial x_i} (\vec{z}) \right| \leq \underset{\in \mathbb{R}_{>0}}{M_i} \quad \forall \vec{z} \in S \quad \forall i \in \{1, \dots, n\}$$

Dann: $$|f(\vec{x}) - f(\vec{y})| \leq \sum_{i=1}^{n} M_i \cdot |x_i - y_i|$$

$\boxed{A2}$ $G := \{ (x,y) \in \mathbb{R}^2 : x > 0, \ |y| < \sqrt{x} \}$

$$f: G \to \mathbb{R}$$
$$(x,y) \mapsto \ln(x - y^2)$$

$$\vec{z} = (2e^2, e) \qquad \vec{y} = (2e^2, 0)$$

① $S = \{ (2e^2, \underline{(1-t)e}) : t \in [0,1] \}$ $(*)$

② Part. Abl.

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{\ln(x-y^2)}$$

$$\boxed{\frac{\partial f}{\partial y}(x,y) = \frac{-2y}{x-y^2}}$$

③ FSS anwenden

$$|f(\vec{z})-f(\vec{y})| \leq \underbrace{\sup_{t\in[0,1]}\left(\left|\frac{\partial f}{\partial x}\big(\overbrace{\vec{z}+t\cdot(\vec{y}-\vec{z})}^{S}\big)\right|\right)}_{M_1}\cdot\underbrace{\cancel{|2e^2-2e^2|}}_{z_0-y_0}\; 0$$

$$+\sup_{t\in[0,1]}\left(\left|\frac{\partial f}{\partial y}\big(\vec{z}+t\cdot(\vec{y}-\vec{z})\big)\right|\right)\cdot|e-0|$$

1. Komp $\rightsquigarrow = 0$

2. Komp.

(*)
$$=\sup_{t\in[0,1]}\left(\frac{2\cdot(1-t)\cdot e}{2e^2-((1-t)e)^2}\right)\cdot e \underset{t=0}{=} \frac{2e}{2e^2-e^2}\cdot e = 2$$

A3

$$f:\mathbb{R}^2\longrightarrow\mathbb{R}$$
$$(x,y)\longmapsto\begin{cases}\dfrac{xy(x^2-y^2)}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & \text{sonst}\end{cases}$$

|| offen

einf. Argument

a.) Part. Abl. berechnen auf $(x,y)\neq(0,0)$

$$\frac{\partial f}{\partial x}(x,y)=\frac{x^4y+4\cdot x^2y^3-y^5}{(x^2+y^2)^2}$$

gängige Ableitungsregeln

für $(x,y) = 0$:

$\frac{\partial f}{\partial x}$ Richtungsableitung in Richtung $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(\overset{\neq (0,0)}{0+h,0}) - \cancel{f(0,0)}^{\,0}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{f(h,0)}^{\,0}}{h} = 0$$

$$\Longrightarrow \quad \frac{\partial f}{\partial x}(x,y) \begin{cases} \frac{x^4 y + 4x^2y^3 - y^5}{(x^2+y^2)^2} & \boxed{\text{für } (x,y) \neq (0,0)} \text{ offen} \\ 0 & \text{sonst} \end{cases}$$

analog: $$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2} & \text{für } (x,y) \neq (0,0) \\ 0 & \text{sonst} \end{cases}$$

b.) f total diffbar auf $\mathbb{R}^2$.

Nachweis: Die partiellen Abl. (existieren und) sind stetig.

$\frac{\partial f}{\partial x}(x,y)$ auf $(x,y) \neq (0,0)$ stetig, da Verknüpfung stetiger Fkt.

$$\frac{\partial f}{\partial x}(x,y) \text{ stetig in } (0,0$$

g stetig in x_0:
$x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$

$$\boxed{\lim_{(x,y)\to 0} \left| \frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0) \right| = 0} \; ?$$

$$\left| \frac{x^4 y + 4x^2y^3 - y^5}{(x^2+y^2)^2} \right| = |y| \cdot \left| \frac{x^4 + 4x^2y^2 - y^4}{(x^2+y^2)^2} \right|$$

$$\le |y| \cdot \left(\underbrace{\frac{x^4}{(x^2+y^2)^2}}_{\le 1} + \underbrace{\frac{2\cdot x^2y^2}{(x^2+y^2)^2}}_{\le 1} + \underbrace{\frac{2\cdot x^2y^2}{(x^2+y^2)^2}}_{\le 1} + \underbrace{\frac{y^4}{(x^2+y^2)^2}}_{\le 1} \right)$$

$\boxed{4 \cdot |y|}$

$$(x^2+y^2)^2 = x^4 + 2x^2y^2 + y^4$$

$$\boxed{\le |y| \cdot 4 \xrightarrow{(x,y)\to 0} 0} \rightsquigarrow \frac{\partial f}{\partial x} \text{ stetig in } (0,0)$$

analog $\frac{\partial f}{\partial y}$

c.) $$\underline{\underline{\frac{\partial^2 f}{\partial x \partial y}(0,0)}} = \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right)(0,0)$$

$$= \lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(0+h,0) - \frac{\partial f}{\partial y}(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^5}{h^4}}{h}$$

$$= 1$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \ldots = -1$$

Satz v. Schwarz

$\Rightarrow$ f nicht 2-mal stetig diffbar!

$\boxed{A4}$ $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x,y) \longmapsto x + x^2 + xy + y^3$

a.) Kritische Punkte: (x_0,y_0) mit $\operatorname{grad}_{(x_0,y_0)} f = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left.\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= 1 + 2x + y \\ \frac{\partial f}{\partial y}(x,y) &= x + 3y^2 \end{aligned}\right| \rightarrow \text{p,q-Formel}$$

$\boxed{y_1 = \frac{1}{2},\ x_1 = -\frac{3}{4}}$ $\boxed{y_2 = -\frac{1}{3},\ x_2 = -\frac{1}{3}}$

b.) $\operatorname{Hess}_{(x,y)} f = \begin{pmatrix} 2 & 1 \\ 1 & 6y \end{pmatrix}$

$$\operatorname{Hess}_{(x_1,y_1)} f = \begin{pmatrix} \boxed{2} & 1 \\ 1 & 3 \end{pmatrix}$$

$$\det(2) > 0$$

$$\det \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = 6 - 2 = 4 > 0$$

$\Rightarrow \operatorname{Hess}_{(x_1,y_1)} f$ positiv def.

$\Rightarrow (x_1, y_1)$ Min.

$$\operatorname{Hess}_{(x_2,y_2)} f = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

$\det \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} < 0$ gerader Hauptminor

$\Rightarrow \operatorname{Hess}_{(x_2,y_2)}$ indefinit

(x_2, y_2) kein Extremum.

c.) Keine globale Extremwert

z.B $\lim\limits_{\substack{y \to -\infty \\ x = 0}} (\cancel{x} + \cancel{x^2} + \cancel{xy} + y^3) \to -\infty.$