4. Tutorium

1. Aufgabe

$$\vec{f}: \mathbb{R}^2 \to \mathbb{R}^3, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 - y \\ x \\ y^2 \end{pmatrix}$$

$$\vec{g}: \mathbb{R}^2 \to \mathbb{R}^3, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} xy^2 \\ e^{xy} \\ x\cdot(x^2 - y) \end{pmatrix}$$

a.) $\langle \vec{f}, \vec{g} \rangle : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \left\langle \begin{pmatrix} x^2 - y \\ x \\ y^2 \end{pmatrix}, \begin{pmatrix} xy^2 \\ e^{xy} \\ x(x^2-y) \end{pmatrix} \right\rangle = (x^2 - y)\cdot xy^2$$

$$+ x\cdot e^{xy} + y^2 \cdot x(x^2 - y) = \underset{PR}{\boxed{x\cdot e^{xy}}} + 2x^3y^2 - 2xy^3$$

$$\langle \vec{f}, \vec{g} \rangle'(x,y) = \begin{pmatrix} xye^{xy} + 6x^2y^2 - 2y^3 + e^{xy} & x^2 \cdot e^{xy} + 4x^3y - 6xy^2 \end{pmatrix}$$

b.) $\langle \vec{f}, \vec{h} \rangle$: nicht definiert, da $\vec{f}$ und $\vec{h}$ unterschiedlichen Def-bereich haben.

$$\vec{h}: \mathbb{R}^3 \to \mathbb{R}^3, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} xz \\ y^2\cdot z \\ (x - y^2)^2 - z \end{pmatrix}$$

c.) $\vec{f} \times \vec{h}$:

d.) $\vec{h} \times \vec{h} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} yz - zy \\ xz - zx \\ \dots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(\vec{h} \times \vec{h})'\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Wiederholung

- **Skalarprodukt**

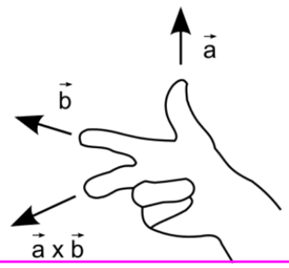
  $\vec{a}, \vec{b} \in \mathbb{R}^n$; $\langle \vec{a}, \vec{b} \rangle = \sum_{i=1}^{n} a_i \cdot b_i$

  Längen- und Winkelmessung

  - $\|\vec{a}\|_2 = \sqrt{\langle \vec{a}, \vec{a} \rangle}$
  - $\langle \vec{a}, \vec{b} \rangle = 0 \iff \vec{a} \perp \vec{b}$

- **Kreuz- bzw. Vektorprodukt**

  $\vec{a}, \vec{b} \in \mathbb{R}^3$: $\vec{a} \times \vec{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - \dots \\ \dots \end{pmatrix}$



$\langle \vec{a} \times \vec{b}, \vec{a} \rangle = 0$

$\langle \vec{a} \times \vec{b}, \vec{b} \rangle = 0$

e.) $\vec{h} \circ \vec{f} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} (x^2-y)\cdot y^2 \\ x^2 y^2 \\ \underbrace{\cancel{\left((x^2-y)-x^2\right)^2-y^2}}_{0} \end{pmatrix}$$

$$f(x,y) = \begin{pmatrix} \boxed{x^2-y}^{\,u} \\ \boxed{x}^{\,v} \\ \boxed{y^2}^{\,w} \end{pmatrix}$$

$h(u,v,w) = \ldots$

$$\left(\vec{h} \circ \vec{f}\right)'(x,y) = \begin{pmatrix} 2xy^2 & 2yx^2-3y^2 \\ 2xy^2 & 2x^2y \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{3\times 2}$$

f.) $u \cdot \vec{f} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \underbrace{x \cdot y}_{u(x,y)} \cdot \underbrace{\begin{pmatrix} x^2-y \\ x \\ y^2 \end{pmatrix}}_{f(x,y)}$$

$u : \mathbb{R}^2 \to \mathbb{R}$

$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto xy$

$$= \begin{pmatrix} x^3y - xy^2 \\ x^2y \\ xy^3 \end{pmatrix}$$

$$\left(u \cdot \vec{f}\right)'(x,y) = \begin{pmatrix} 3x^2y - y^2 & x^3 - 2xy \\ 2xy & x^2 \\ y^3 & 3xy^2 \end{pmatrix}$$

2. Aufgabe

a) Bestimme Ableitung

$\mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \sin(x)\cdot y^2$$

$$f'(x,y) = \begin{pmatrix} \cos(x)\cdot y^2 & 2\sin(x)y \end{pmatrix}$$

$\mathbb{R}^2 \to \mathbb{R}^2$

$$\vec{v}(x,y) = \begin{pmatrix} y\cdot e^x \\ e^{x+y} \end{pmatrix}$$

$$\vec{v}'(x,y) = \begin{pmatrix} y\cdot e^x & e^x \\ e^{x+y} & e^{x+y} \end{pmatrix}$$

b) Bestimme $(f\cdot\vec{v})'(0,0)$ mit Produktregel:

$$\boxed{(f\cdot g)'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)}$$

PR im $\mathbb{R}^1$

$$\frac{\partial f\vec{v}}{\boxed{\partial x}}(0,0) = f'(0,0)\cdot\vec{e}_1\cdot\vec{v}(0,0) + f(0,0)\cdot\boxed{\vec{v}'(0,0)\cdot\vec{e}_1} = \boxed{\begin{pmatrix}0\\0\end{pmatrix}}$$

erste Spalte von $\vec{v}'(0,0)$

$$\frac{\partial f\vec{v}}{\boxed{\partial y}}(0,0) = f'(0,0)\cdot\vec{e}_2\cdot\vec{v}(0,0) + f(0,0)\cdot\boxed{\vec{v}'(0,0)\cdot\vec{e}_2} = \boxed{\begin{pmatrix}0\\0\end{pmatrix}}$$

zweite Spalte von $\vec{v}'(0,0)$

$$\Rightarrow (f\cdot\vec{v})'(0,0) = \begin{pmatrix} \boxed{\begin{matrix}0\\0\end{matrix}} & \boxed{\begin{matrix}0\\0\end{matrix}} \end{pmatrix}$$

c.)

$$(f\circ\vec{v})'(0,0) = f'(\vec{v}(0,0))\circ\vec{v}\,'(0,0)$$

$$= f'\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix} =$$

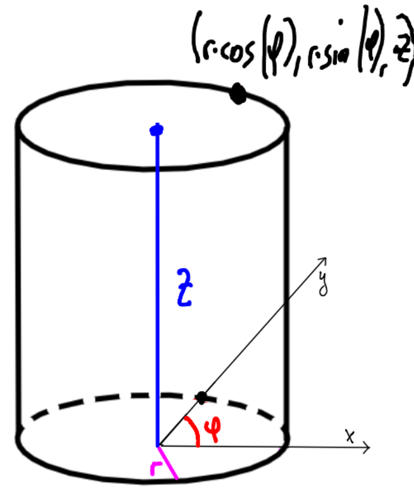
$$= \begin{pmatrix}1 & 0\end{pmatrix}\cdot\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix} = \underline{\underline{\begin{pmatrix}0 & 1\end{pmatrix}}}$$

$$\in \mathbb{R}^{1\times 2}$$

Aufgabe 3

a.) Zylinderkoord.

$$\begin{pmatrix} r\cdot\cos(\varphi)\\ r\cdot\sin(\varphi)\\ z\end{pmatrix} = \begin{pmatrix}1\\ -1\\ -\sqrt{2}\end{pmatrix}$$

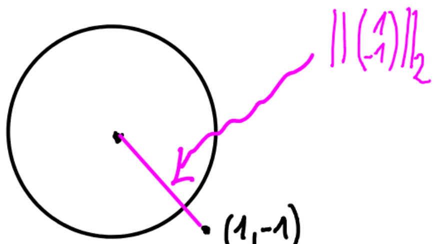


Parameterbereich: $r\in[0,\infty)$

$\varphi\in[0,2\pi]$

$z\in\mathbb{R}$

- $\underline{\underline{z = -\sqrt{2}}}$



$$r^2\cdot\cos^2(\varphi) + r^2\cdot\sin^2(\varphi) =$$

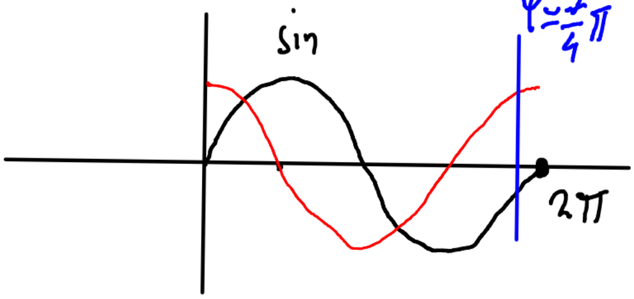
$$= \boxed{r^2}\cdot\cancel{\left(\cos^2(\varphi)+\sin^2(\varphi)\right)}^{1} = (1)^2 + (-1)^2 = 2$$

$$\underline{\underline{r = \sqrt{2}}}$$

1\. Koord.: $\sqrt{2}\cdot\cos(\varphi) = 1$

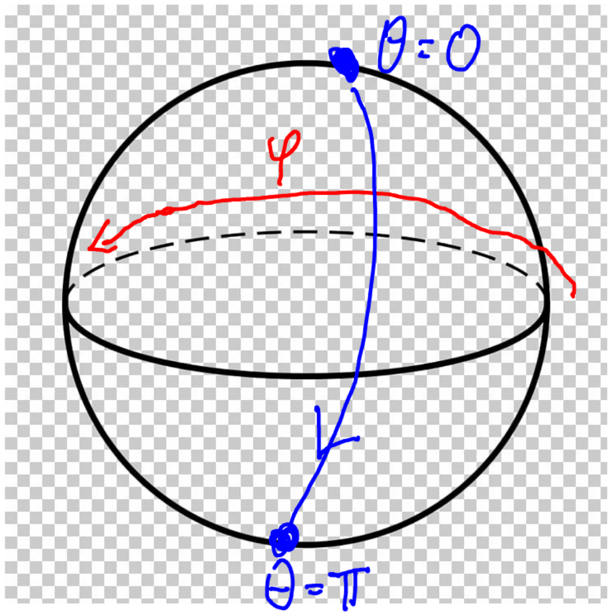
2\. Koord.: $\sqrt{2}\cdot\sin(\varphi) = -1$

$$\underline{\underline{\varphi = \frac{7}{4}\pi}}$$



Kugelkoord.:

$$\begin{pmatrix} 1 \\ -1 \\ -\sqrt{2} \end{pmatrix} = \begin{pmatrix} r\cdot\cos(\varphi)\cdot\sin(\theta) \\ r\cdot\sin(\varphi)\cdot\sin(\theta) \\ r\cdot\cos(\theta) \end{pmatrix}$$



Parameterbereich: $r \in [0, \infty)$

$\varphi \in [0, 2\pi]$

$\theta \in [0, \pi]$

Merken!

$1^2 + 1^2 + \sqrt{2}^2$

$$\underline{\underline{r^2}} = \boxed{r^2\cos^2(\varphi)\sin^2(\theta) + r^2\sin^2(\varphi)\sin^2(\theta) + r^2\cos^2(\theta)} = \underline{\underline{4}}$$

$$\underline{\underline{r = 2}}$$

dritte Koord.: $2\cdot\cos(\theta) = -\sqrt{2}$

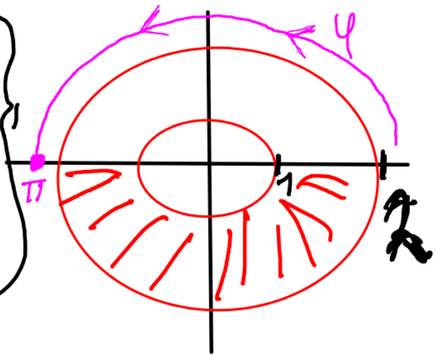
$$\underline{\underline{\theta = \frac{3}{4}\pi}}$$

Koord. 1+2: Wie vorher

$$\left.\begin{aligned} \cos(\varphi) &= \frac{1}{\sqrt{2}} \\ \sin(\varphi) &= -\frac{1}{\sqrt{2}} \end{aligned}\right\} \quad \underline{\underline{\varphi = \frac{7}{4}\pi}}$$

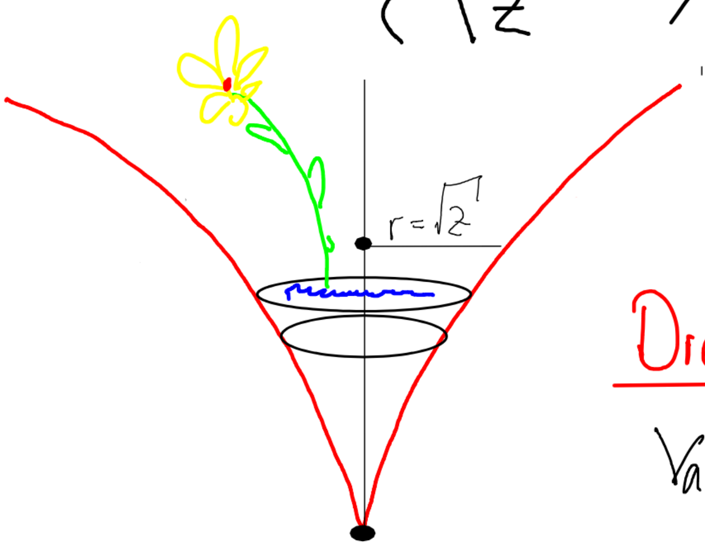
b.) $A = \{(x,y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4 \; ; \; y \leq 0\}$

$$= \left\{ \begin{pmatrix} r \cdot \cos(\varphi) \\ r \cdot \sin(\varphi) \end{pmatrix} : \; r \in [1,2], \; \varphi \in [\pi, 2\pi] \right\}$$



$$B = \left\{ (x,y,z) : \boxed{x^2 + y^2 \leq z}, \; z \geq 0 \right\}$$

$$= \left\{ \begin{pmatrix} r \cdot \cos(\varphi) \\ r \cdot \sin(\varphi) \\ z \end{pmatrix} : \; z \geq 0; \; \boxed{0 \leq r \leq \sqrt{z}}, \; \varphi \in [0, 2\pi] \right\}$$



Drehkörper

Vase

$$C := \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2\le 2,$$

$$* \; \boxed{x^2+y^2\le z^2},\; y\ge 0\}$$

ähnlich wie A

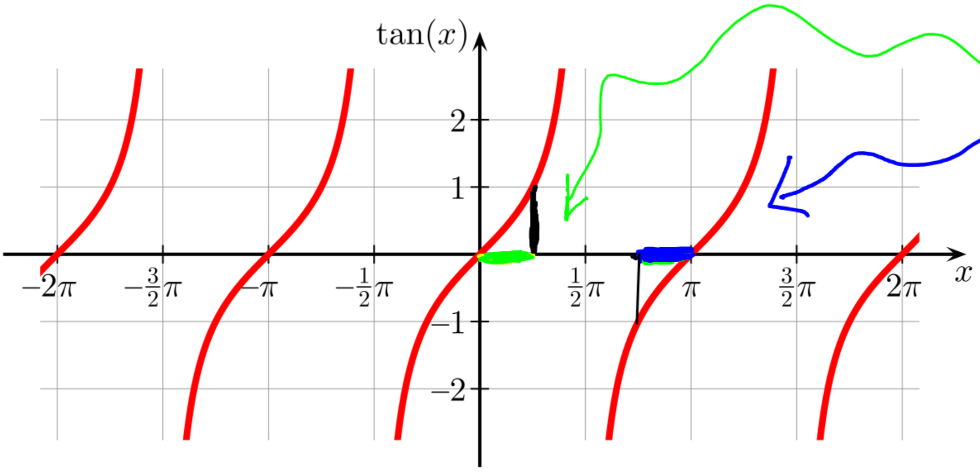
in Kugelkoord:

$$= \left\{ \begin{pmatrix} r\cos(\varphi)\sin(\theta)\\ r\sin(\varphi)\sin(\theta)\\ r\cos(\theta)\end{pmatrix} : 0\le r\le\sqrt{2};\ \varphi\in[0,\pi];\ \theta\in ??\right\}$$

$$\boxed{*}\quad r^2\cos^2(\varphi)\sin^2(\theta)+r\cdot\sin^2(\varphi)\cdot\sin^2(\theta)\le r^2\cdot\cos^2(\theta)$$

$$\iff \cancel{r^2}\sin^2(\theta) \le \cancel{r^2}\cos^2(\theta)$$

$$\iff |\tan(\theta)|\le 1 \iff \theta\in\left[0,\tfrac{1}{4}\pi\right]\cup\left[\tfrac{3}{4}\pi,\pi\right]$$



Erinnerung:
Der Parameterbereich von θ ist $[0,\pi]$.