(Local) CLT for Diffusions in Degenerate and Unbounded Random Environment

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Outline

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Consider a conductor occupying a region $\mathcal{O} \subset \mathbb{R}^d$ and suppose that the conducting material is inhomogeneous, imagine a mixture of several materials with different conductivities (heat or electrical). For example:

- composite materials (glasses, metal alloys, conductors);
- crystals, materials with impurities, materials with holes;
Consider a conductor occupying a region $\Theta \subset \mathbb{R}^d$ and suppose that the conducting material is inhomogeneous, imagine a mixture of several materials with different conductivities (heat or electrical). For example:
- composite materials (glasses, metal alloys, conductors);
- crystals, materials with impurities, materials with holes;

**Question**

Provided that the variations of the material are on a very small scale,
- can we describe the macroscopic properties of the material?
- Can we give a satisfactory approximation of the motion of a particle living in the inhomogeneous medium?
Motivation
Diffusions in heterogeneous media can frequently be described by its *effective* behavior.

This means that there is a homogeneous medium, the *effective* medium, whose diffusive properties are close to those of the real inhomogeneous medium when measured on long space-time scales.

A process of averaging or homogenization takes place so that the complicated small scale structure of the material is replaced by an asymptotically equivalent homogeneous structure.
To sample a Diffusion in Random Environment there are two steps:

- **First:** We sample the Random Environment $\omega$ from a probability space $(\Omega, \mathcal{G}, \mu)$.

- **Second:** We sample the diffusion $X^\omega$ according to a law $\mathbb{P}^\omega$ on $C([0, \infty); \mathbb{R}^d)$ depending on the environment $\omega$. The diffusion is associated to

$$L^\omega u(x) = \nabla \cdot (a(x; \omega) \nabla u(x))$$
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**Renmark (Diffusive Limit)**

Asking for the oscillations to happen on a small scale means that we are interested in studying $X_{t}^{\epsilon, \omega} := \epsilon X_{t/\epsilon^2}^\omega$ as $\epsilon \to 0$. 

(Local) CLT for Diffusions in Degenerate and Unbounded Random Environment
The Random Landscape

Definition

A **Stationary and Ergodic Random Medium** is a Probability space 
\((\Omega, \mathcal{G}, \mu)\) on which is defined a measurable group of transformations 
\(\{\tau_x\}_{x \in \mathbb{R}^d}\) acting on \(\Omega\) such that

- **(Group)** \(\tau_0 = \text{Id}_\Omega\) and \(\tau_{x+y} = \tau_x \circ \tau_y\);
- **(Stationarity)** \(\mu(\tau_x(A)) = \mu(A)\) for all \(x \in \mathbb{R}^d\) and \(A \in \mathcal{G}\);
- **(Ergodicity)** if \(\tau_x A = A\) for any \(x \in \mathbb{R}^d\) \(\implies \mu(A) \in \{0, 1\}\);
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- (Ergodicity) if \(\tau_x A = A\) for any \(x \in \mathbb{R}^d\) \(\implies\) \(\mu(A) \in \{0, 1\}\);

For \(f \in L^2(\Omega, \mu)\) we call the associated stationary field

\[
f(x; \omega) \overset{\text{def}}{=} f(\tau_x \omega)
\]

Observe that \(\{f(x)\}_{x \in \mathbb{R}^d}\) is a family of random variables
\( \tau_y \omega \) is a translation of the environment \( \omega \in \Omega \) in direction \( y \in \mathbb{R}^d \)

\[
f(y + x; \omega) = f(\tau_{x+y} \omega) = f(x; \tau_y \omega)
\]
- $\tau_y \omega$ is a translation of the environment $\omega \in \Omega$ in direction $y \in \mathbb{R}^d$

$$f(y + x; \omega) = f(\tau_{x+y} \omega) = f(x; \tau_y \omega)$$

- Stationarity $\Rightarrow \{f(x_1), \ldots, f(x_n)\} \overset{law}{=} \{f(x_1 + y), \ldots, f(x_n + y)\}$
\( \tau_y \omega \) is a translation of the environment \( \omega \in \Omega \) in direction \( y \in \mathbb{R}^d \)

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- **Stationarity** \( \Rightarrow \) \( \{f(x_1), \ldots, f(x_n)\} \overset{law}{=} \{f(x_1 + y), \ldots, f(x_n + y)\} \)
- **Ergodicity** \( \Rightarrow \) Averages with respect to \( \mu \approx \) space averages

\[
\mathbb{E}_\mu[f(x; \omega)] = \lim_{r \to \infty} \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y; \omega) \, dy
\]

(Local) CLT for Diffusions in Degenerate and Unbounded Random Environment
Periodic Media
Two realizations of a Gaussian Random Field.
A possible realization of a Poissonian field.
Random Media
Discrete Case

Random Conductances Model
Fix a random medium \((\Omega, \mathcal{G}, \mu, \{\tau_x\}_{x \in \mathbb{R}^d})\).
Consider a symmetric matrix \(a : \Omega \to \mathbb{R}^{d \times d}\) and let \(a(x; \omega) = a(\tau_x \omega)\).
The generator is formally given by

\[ L^\omega u(x) = \nabla \cdot (a(x; \omega) \nabla u(x)) \]

If we assume \(a(\cdot; \omega)\) smooth, then we look to diffusions satisfying

\[ dX_t^\omega = c(X_t^\omega; \omega) dt + \sqrt{2} \sigma(X_t^\omega; \omega) dW_t \]

where

\[ \sigma(x; \omega) = a^{1/2}(x; \omega), \quad c_i(x; \omega) = \sum_j \frac{\partial}{\partial x_j} a_{i,j}(x; \omega) \]
If $x \to a(x; \omega)$ is bounded and uniformly elliptic, uniformly in $\omega \in \Omega$, i.e.

$$c^{-1}|x|^2 \leq \sum_{i,j} a_{ij}(x; \omega)x_ix_j \leq c|x|^2$$

**Theorem (Papanicoulaou, Varadhan 1979)**

Let $X_t^{\omega}$ be the diffusion associated to $L^{\omega}$ and with $X_0^{\omega} = 0$. Then, for $\mu$-almost all $\omega$,

$$\epsilon X_t^{\omega}/\epsilon^2 \to AW_t$$

in law on $C([0, \infty); \mathbb{R}^d)$ as $\epsilon \to 0$, where $W_t$ is a $d$-dimensional BM.

**Remark**

The matrix $A$ does not depend on $\omega$, i.e. on the particular realization of the environment.
prove that there exist “correctors” $\chi^k(x; \omega)$ such that $\nabla \chi^k(x; \omega)$ is a stationary field and such that $y^k(x; \omega) := x_k - \chi^k(x; \omega)$ (harmonic coordinates) are martingales

$L^\omega y^k = 0$
Idea of the Proof

- prove that there exist “correctors” $\chi^k(x; \omega)$ such that $\nabla \chi^k(x; \omega)$ is a stationary field and such that $y^k(x; \omega) := x_k - \chi^k(x; \omega)$ (harmonic coordinates) are martingales

\[ L^\omega y^k = 0 \]

- $X^\omega_t$ can be decomposed as $X^\omega_t = y(X^\omega_t; \omega) + \chi(X^\omega_t; \omega)$;
Idea of the Proof

- prove that there exist "correctors" $\chi_k(x; \omega)$ such that $\nabla \chi_k(x; \omega)$ is a stationary field and such that $y^k(x; \omega) := x_k - \chi_k(x; \omega)$ (harmonic coordinates) are martingales

$$L^\omega y^k = 0$$

- $X_t^\omega$ can be decomposed as $X_t^\omega = y(X_t^\omega; \omega) + \chi(X_t^\omega; \omega)$;

- $M_t^{\omega, \epsilon} := \epsilon y(X_t^\omega/t; \omega)$ is a martingale (Itô formula) with quadratic variation given by

$$\langle M_h^{\omega, \epsilon}, M_k^{\omega, \epsilon} \rangle_t = 2\epsilon^2 \int_0^t \sum_{i,j} a_{ij}(X_s^\omega; \omega) \frac{\partial y^h(X_s^\omega; \omega)}{\partial x_i} \frac{\partial y^k(X_s^\omega; \omega)}{\partial x_i} \, ds$$
The environment process corresponding to the diffusion \( \{X_t^\omega, t \geq 0\} \) is an \( \Omega \) valued process defined by

\[
t \to \eta_t^\omega := \tau X_t^\omega \in \Omega, \quad t \geq 0
\]
The environment process corresponding to the diffusion \( \{X_t^\omega, t \geq 0\} \) is an \( \Omega \) valued process defined by

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\]

**Proposition**

The measure \( \mu \) is ergodic and invariant for the process \( \eta_t^\omega \).

Observe that for \( f \in L^1(\Omega, \mu) \)

\[
\frac{1}{t} \int_0^t f(X_s^\omega; \omega) \, ds = \frac{1}{t} \int_0^t f(\tau_{X_t^\omega} \omega) \, ds \to \mathbb{E}_\mu[f]
\]
By the ergodic theorem for the environment process, for \( \epsilon \to 0 \)

\[
\frac{\langle M_{i,\epsilon}^{\omega}, M_{j,\epsilon}^{\omega} \rangle_t}{t} = \frac{2\epsilon^2}{t} \int_0^{t/\epsilon^2} \sum_{i,j} a_{ij}(0; \eta_t^{\omega}) \frac{\partial y^h(0; \eta_t^{\omega})}{\partial x_i} \frac{\partial y^k(0; \eta_t^{\omega})}{\partial x_i} ds
\]
By the ergodic theorem for the environment process, for $\epsilon \to 0$

$$\frac{\langle M_i^{\omega,\epsilon}, M_j^{\omega,\epsilon} \rangle_t}{t} = \frac{2\epsilon^2}{t} \int_0^{t/\epsilon^2} \sum_{i,j} a_{ij}(0; \eta_t^{\omega}) \frac{\partial y^h(0; \eta_t^{\omega})}{\partial x_i} \frac{\partial y^k(0; \eta_t^{\omega})}{\partial x_i} ds$$

$$\to \mathbb{E}_{\mu} \left[ \sum_{i,j}^d a_{ij}(x; \omega) \left( \delta_{k,j} - \frac{\partial \chi^j(x; \omega)}{\partial x_k} \right) \left( \delta_{k,j} - \frac{\partial \chi^j(x; \omega)}{\partial x_k} \right) \right] \equiv d_{i,j}$$

CLT for Martingales $\Rightarrow$ the finite dimensional distributions of $M^{\omega,\epsilon}$ converges to that of $D^{1/2} W$ where $W$ is a BM and $D = [d_{ij}]$. 
By the ergodic theorem for the environment process, for $\epsilon \to 0$

$$\frac{\langle M_{i}^{\omega, \epsilon}, M_{j}^{\omega, \epsilon} \rangle_{t}}{t} = \frac{2\epsilon^{2}}{t} \int_{0}^{t/\epsilon^{2}} \sum_{i,j}^{d} a_{ij}(0; \eta_{t}^{\omega}) \frac{\partial y^{h}(0; \eta_{t}^{\omega})}{\partial x_{i}} \frac{\partial y^{k}(0; \eta_{t}^{\omega})}{\partial x_{i}} \, ds$$

$$\rightarrow \mathbb{E}_{\mu} \left[ \sum_{i,j}^{d} a_{ij}(x; \omega) \left( \delta_{k,j} - \frac{\partial \chi^{i}(x; \omega)}{\partial x_{k}} \right) \left( \delta_{k,j} - \frac{\partial \chi^{i}(x; \omega)}{\partial x_{k}} \right) \right] \doteq d_{i,j}$$

CLT for Martingales $\Rightarrow$ the finite dimensional distributions of $M^{\omega, \epsilon}$ converges to that of $D^{1/2}W$ where $W$ is a BM and $D = [d_{ij}]$.

Finally prove, through a priori estimates, that $\epsilon \chi(X_{t/\epsilon^{2}}^{\omega}; \omega)$ converges to 0 in law for $\mu$ almost all $\omega$;

$$\epsilon X_{t/\epsilon^{2}}^{\omega} = M_{t}^{\omega, \epsilon} + \epsilon \chi(X_{t/\epsilon^{2}}^{\omega}; \omega) \xrightarrow{law} D^{1/2}W$$

□
Remarks (Measurable coefficients)

- If $a(x; \omega)$ is assumed to be just measurable in $x \Rightarrow L^\omega$ is ill-posed;
- the general theory of Dirichlet forms is required to make sense of the problem and to associate a diffusion to that formal generator.

In some sense Dirichlet Forms approach to diffusions is comparable to the weak solution approach to PDE.

One defines a bilinear form $\mathcal{E}^\omega$ on $C_0^\infty(\mathbb{R}^d)$ "integrating by parts" $L^\omega u$ with $u \in C_0^\infty(\mathbb{R}^d)$ against a test function $v \in C_0^\infty(\mathbb{R}^d)$.

$$\mathcal{E}^\omega(u, v) := \sum_{i,j} \int_{\mathbb{R}^d} a_{i,j}(x; \omega) \partial_i u(x) \partial_j v(x) \, dx$$

and then takes the closure of $\mathcal{E}^\omega : C_0^\infty(\mathbb{R}^d) \times C_0^\infty(\mathbb{R}^d) \to \mathbb{R}$ on $L^2(\mathbb{R}^d)$. 
Degenerate and Unbounded
Periodic medium

Natural Question
What happens for degenerate and possibly unbounded matrices $a$?
Degenerate and Unbounded
Periodic medium

Natural Question

What happens for degenerate and possibly unbounded matrices $a$?

A positive result has been given by Moustapha Ba and Pierre Matthieu (2013 to appear) for the periodic case. More precisely the formal generator is given by

$$Lu(x) = e^{V(x)} \nabla \cdot (e^{-V(x)} \nabla u(x))$$

where $V$ is a periodic function satisfying

$$e^V + e^{-V} \in L^1(\mathbb{T}^d, dx)$$
Another positive result has being given in the discrete setting by the work of Deuschel, Slowik and Andres on the Random Conductances Model. Following the spirit of their technique we look at

(A.1) There exist positive random variables $\lambda$ and $\Lambda$ such that $\forall \xi \in \mathbb{R}^d$

$$\lambda(\omega)|\xi|^2 \leq \langle a(\omega)\xi, \xi \rangle \leq \Lambda(\omega)|\xi|^2, \quad \mu\text{-a.s.}$$

(A.2) There exist $p, q \in [1, \infty]$ such that $\mathbb{E}[\lambda^{-q}], \mathbb{E}[\Lambda^p] < +\infty$ and

$$\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$$

(A.3) $x \rightarrow \lambda^{-1}(x; \omega)$ and $x \rightarrow \Lambda(x; \omega)$ are in $L^\infty_{\text{loc}}(\mathbb{R}^d), \mu$-almost surely.
Clearly we look to the formal generator

\[ L^\omega u(x) = \nabla \cdot (a(x; \omega) \nabla u(x)) \]

Existence of a diffusion \((X^\omega_t, P^\omega_x, \zeta^\omega)\) for almost all points \(x \in \mathbb{R}^d\) for \(\mu\)-almost all \(\omega \in \Omega\) follows from Fukushima and \(\mathbb{E}[\Lambda], \mathbb{E}[\lambda^{-1}] < \infty\).
Clearly we look to the formal generator

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Existence of a diffusion \((X_t^\omega, P_\omega^x, \zeta^\omega)\) for almost all points \(x \in \mathbb{R}^d\) for \(\mu\)-almost all \(\omega \in \Omega\) follows from Fukushima and \(\mathbb{E}_\mu[\Lambda], \mathbb{E}_\mu[\lambda^{-1}] < \infty\).

Assumption (A.3) is given in order to construct a **minimal** diffusion process \(M^\omega = (X_t^\omega, P_\omega^x, \zeta^\omega)\) for all starting points \(x \in \mathbb{R}^d\), \(\mu\)-a.s.

The assumption of ergodicity and \(\mathbb{E}_\mu[\Lambda] < \infty\) is enough to prove that \(M^\omega\) does not explode \(\mu\)-a.s.
It can be proved that the decomposition

“corrector” + “harmonic coordinates”

is still possible for $\mathbb{E}_\mu[\Lambda], \mathbb{E}_\mu[\lambda^{-1}] < \infty$.

Question

Why do we need the $(p, q)$-condition in assumption (A.2)?
It can be proved that the decomposition

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Question

Why do we need the $(p, q)$-condition in assumption (A.2)?

To control the corrector and show that $\epsilon \chi(\frac{X^\omega_t}{\epsilon^2}; \omega) \to 0$ in law for $\mu$-a.s!

Given the convergence to zero of the corrector, the CLT follows.
Homogeneization occurs

Theorem (C. - Deuschel)

Assume (A.1), (A.2) and (A.3), \( d \geq 2 \). Let \( \mathbf{M}^{\omega} = (X_t^{\omega}, \mathbb{P}_x^{\omega}) \) be the minimal diffusion process defined above. Then the following hold

- \( \mu \)-a.s. the limits
  \[
  \lim_{t \to \infty} \frac{1}{t} \mathbb{E}_{\mu} [X_t^{\omega}(i)X_t^{\omega}(j)] = d_{ij} \quad i, j = 1, \ldots, d
  \]
  exist and are deterministic constants.

- \( \mu \)-a.s. the processes \( \epsilon X_t^{\omega} \) converges in law on \( C([0, \infty); \mathbb{R}^d) \) as \( \epsilon \to 0 \) to \( D^{1/2}W_t \), where \( D = [d_{ij}] \) is a positive definite matrix.
$(p, q)$-condition ⇒ The corrector is sublinear

To prove $\epsilon \chi(X_{t/\epsilon^2}, \omega) \to 0$ in law as $\epsilon \to 0$, it is enough to show:

**Claim:**

$$\lim_{\epsilon \to 0} \sup_{|x| \leq r} \epsilon |\chi(x/\epsilon, \omega)| = 0, \quad \forall r > 0, \mu\text{-almost surely.}$$

The *proof* goes in two steps:

- $\lim_{\epsilon \to 0} \|\epsilon \chi(x/\epsilon, \omega)\|_{2p^*, B_r} = 0$ for all $r > 0$, $\mu$-a.s. \( p^* = p/(p-1) \)
- Prove that we can control $\sup_{|x| \leq r} \epsilon |\chi(x/\epsilon, \omega)|$ by the $2p^*$ norm of $\epsilon \chi(x/\epsilon, \omega)$. 

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\((p, q)\)-condition \(\Rightarrow\) The corrector is sublinear

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- Prove that we can control \(\sup_{|x| \leq r} \epsilon |\chi(x/\epsilon, \omega)|\) by the \(2p^*\) norm of \(\epsilon \chi(x/\epsilon, \omega)\). (Moser’s iteration technique)
Recall

The corrector is constructed in such a way that solves

$$\int_B \langle a \nabla \chi^k, \nabla \phi \rangle \, dx = \int_B \langle a \nabla f_k, \nabla \phi \rangle \, dx$$

with $f_k(x) = x_k$, for all balls $B$ and $\phi \in C_0^\infty(B)$, $\phi > 0$.

We need a priori estimates for weak solutions to degenerate and unbounded elliptic PDE.
To explain the technique, set $f_k \equiv 0$, we look for a priori estimates for positive “solutions” $u$ to $L^\omega u = 0$. Start from

$$\int_B \langle a^\omega \nabla u, \nabla \phi \rangle dx = 0$$

and test it for $\phi = u^{2\alpha-1} \eta^2$ being $\eta \in C_0^\infty(B)$, $0 \leq \eta \leq 1$ a cutoff and $\alpha \geq 1$

$$\mathcal{E}_\eta^{\omega}(u^\alpha, u^\alpha) := \int_B \langle a^\omega \nabla u^\alpha, \nabla u^\alpha \rangle \eta^2 dx \leq \| \nabla \eta \|_\infty^2 \| \Lambda^\omega \|_{p, B} \| u \|_{2\alpha p^*, B}^{2\alpha}$$
On the other hand from Sobolev’s inequality with 
\( \rho = \frac{2qd}{q(d - 2) + d} \)

\[
\| u^\alpha \eta^2 \|_{\rho, B}^2 \lesssim \| (\lambda^\omega)^{-1} \|_{q, B} \left[ C_{\eta}^\omega (u^\alpha, u^\alpha) + \| \nabla \eta \|_\infty \| u \|_{2\alpha p^*, B}^{2\alpha} \right]
\]

**Iteration step**

\[
\| u^\alpha \eta^2 \|_{\rho, B}^2 \lesssim \left[ \| (\lambda^\omega)^{-1} \|_{q, B} \| \Lambda^\omega \|_{p, B} \right] \| \nabla \eta \|_\infty^2 \| u^\alpha \|_{2p^*, B}^2
\]

Observe that for this inequality to be useful we need

\[
\rho > 2p^* \iff \frac{1}{p} + \frac{1}{q} < \frac{2}{d}
\]
Setting up the iteration

For $k \in \mathbb{N}$, set

- $\alpha_k := (\rho / 2p^*)^k \geq 1$ so that $\alpha_k \rho = 2\alpha_{k+1} p^*$
- $B_k := (\frac{1+2^{-k}}{2})B$ so that $B_0 = B$, $B_\infty = B/2$.
- $\eta_k$ cutoff, $\eta_k \equiv 1$ on $B_{k+1}$, $\eta_k \equiv 0$ on $\partial B_k$ and $\|\nabla \eta_k\|_\infty \leq 2^k / |B|^{\frac{1}{d}}$
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- $\eta_k$ cutoff, $\eta_k \equiv 1$ on $B_{k+1}$, $\eta_k \equiv 0$ on $\partial B_k$ and $\|\nabla \eta_k\|_\infty \leq 2^k/|B|^\frac{1}{d}$

At step $k$ on $B_k$ we can read

$$\|u^{\alpha_k} \eta_k^2\|_{p,B_k} \lesssim \left[\|\left(\lambda^\omega\right)^{-1}\|_{q,B} \|\Lambda^\omega\|_{p,B}\right] \|\nabla \eta_k\|_\infty \|u^{\alpha_k}\|_{2p^*,B_k}^2$$

$$\downarrow$$

$$\|u\|_{2\alpha_k p^*,B_{k+1}} \lesssim \left[\frac{2^{2k} \|\left(\lambda^\omega\right)^{-1}\|_{q,B} \|\Lambda^\omega\|_{p,B}}{|B|^{2/d}}\right]^{\frac{1}{2\alpha_k}} \|u\|_{2\alpha_k p^*,B_k}$$
\[ \sup_{B/2} |u| = \limsup_{k \to \infty} \|u\|_{2\alpha_k+1, p^*/B_k+1} \lesssim \left[ \frac{\| (\lambda^\omega)^{-1} \|_{q, B} \| \Lambda^\omega \|_{p, B}}{|B|^{2/d}} \right] \sum_{i=0}^{\infty} \frac{1}{2\alpha_i} \|u\|_{2p^*/B} \]

**Final Remark**

Going back to \( \epsilon \chi(x/\epsilon; \omega) \), with similar technique we get

\[ \sup_{B/2} \epsilon |\chi(x/\epsilon; \omega)| \lesssim \left[ \frac{\| (\lambda^\omega)^{-1} \|_{q, B/\epsilon} \| \Lambda^\omega \|_{p, B/\epsilon}}{|B/\epsilon|^{1/q} |B/\epsilon|^{1/p}} \right] \gamma \|B|^{-1} \epsilon \chi(x/\epsilon; \omega) \|_{2p^*/B} \]

recall that by the ergodic theorem

\[ \lim_{\epsilon \to 0} \frac{1}{|B/\epsilon|} \int_{B/\epsilon} \lambda(x; \omega)^{-q} dx \to \mathbb{E}_\mu[\lambda^{-q}], \quad \mu\text{-a.s.} \]
How to use sublinearity?

If $\tau_{\epsilon}^\omega$ is the exit time of $\epsilon X_{t/\epsilon^2}^\omega$ from the ball of radius $R > 0$, sublinearity implies

$$\sup_{0 \leq t \leq \tau_{\epsilon}^\omega} \epsilon |\chi(X_{t/\epsilon^2}^\omega, \omega)| < 1, \quad \forall \epsilon < \epsilon_0(\omega), \mu\text{-a.s.}$$
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Hence,

$$\limsup_{\epsilon \to 0} \mathbb{P}_0^\omega \left[ \sup_{t \in [0, T]} |\epsilon X_{t/\epsilon^2}^\omega| > R \right]$$

$$\leq \limsup_{\epsilon \to 0} \mathbb{P}_0^\omega \left[ \sup_{t \in [0, T]} |\epsilon y(X_{t/\epsilon^2}^\omega, \omega)| > R - 1 \right] \leq ce^{-\gamma R}$$
How to use sublinearity?

If $\tau_\epsilon^\omega$ is the exit time of $\epsilon X_{t/\epsilon^2}^\omega$ from the ball of radius $R > 0$, sublinearity implies

$$\sup_{0 \leq t \leq \tau_\epsilon^\omega} \epsilon |\chi(X_{t/\epsilon^2}^\omega, \omega)| < 1, \quad \forall \epsilon < \epsilon_0(\omega), \mu\text{-a.s.}$$

Hence,

$$\limsup_{\epsilon \to 0} \mathbb{P}_0^\omega \left[ \sup_{t \in [0, T]} |\epsilon X_{t/\epsilon^2}^\omega| > R \right] \leq \limsup_{\epsilon \to 0} \mathbb{P}_0^\omega \left[ \sup_{t \in [0, T]} |\epsilon y(X_{t/\epsilon^2}^\omega, \omega)| > R - 1 \right] \leq ce^{-\gamma R}$$

$$\limsup_{\epsilon \to 0} \mathbb{P}_0^\omega \left[ \sup_{t \in [0, T]} |\epsilon \chi(X_{t/\epsilon^2}^\omega, \omega)| > \delta \right] \leq \limsup_{\epsilon \to 0} \mathbb{P}_0^\omega \left[ \sup_{t \in [0, \tau_\epsilon^\omega]} |\epsilon \chi(X_{t/\epsilon^2}^\omega, \omega)| > \delta \right] + \mathbb{P}_0^\omega \left[ \tau_\epsilon^\omega \leq T \right] \leq ce^{-\gamma R}$$
Moser’s iteration technique with the \((p, q)\)-condition is very robust and under (A.1), (A.2), (A.3) we can prove a local Harnack inequality which can be applied to prove a local CLT for the process formally associated to

\[
L^{\Lambda, \omega} u(x) = \frac{1}{\Lambda(x; \omega)} \nabla \cdot (a(x; \omega) \nabla u(x))
\]
Moser’s iteration technique with the \((p, q)\)-condition is very robust and under (A.1), (A.2), (A.3) we can prove a local Harnack inequality which can be applied to prove a local CLT for the process formally associated to

\[
L^{\Lambda, \omega} u(x) = \frac{1}{\Lambda(x; \omega)} \nabla \cdot (a(x; \omega) \nabla u(x))
\]

**Theorem (C. - Deuschel)**

Let \(d \geq 2\). Assume (A.1), (A.2) and (A.3). Let \(p_t^{\omega}(\cdot, \cdot)\) be the density of the process associated to \(L^{\Lambda, \omega}\). Let \(r > 0\) and \(I \subset (0, \infty)\) compact. Then for \(\mu\)-almost all \(\omega \in \Omega\)

\[
\lim_{\epsilon \to 0} \sup_{|x| \leq r} \sup_{t \in I} |e^{-d} p_t^{\omega}(0, x/\epsilon) - \mathbb{E}_{\mu}[\Lambda]^{-1} k_t^{D}(x)| = 0.
\]
Thank you for your attention!