Inflection Points of Real and Tropical Curves

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Real Algebraic Curves and Their Inflection Points

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References
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Consider a plane algebraic curve
\[ C = \{ (x : y : z) \in \mathbb{P}_k^2 : f(x, y, z) = 0 \} \], where \( k \) is any field of characteristic 0.

The **Hessian** \( H_f(x, y, z) \) is the polynomial

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H_f(x, y, z) := \text{det} \begin{pmatrix}
\frac{\partial^2 P}{\partial x^2} & \frac{\partial^2 P}{\partial x \partial y} & \frac{\partial^2 P}{\partial x \partial z} \\
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We have that \( H_f(x, y, z) = 0 \) defines the curve \( \text{Hess}_C \), known as the **Hessian** of \( C \). The set \( C \cap \text{Hess}_C \) is then defined as the set of **Inflection Points** of \( C \).
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If our field $k$ is algebraically closed: Bézout’s theorem implies that a reducible algebraic curve of degree $d \geq 2$ has exactly $3d(d - 2)$ inflection points (counting multiplicity).

If $k$ is not algebraically closed: The number of inflection points of an algebraic curve is dependent on both its degree and on the coefficients of its defining equation.
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Let us call a non-singular real algebraic curve of degree $d$ in $\mathbb{P}^2_R$ maximally inflected if it has $d(d - 2)$ distinct real inflection points.

Klein showed that maximally inflected plane algebraic curves in $\mathbb{P}^2_R$ exist.
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Tropical Geometry: A Lightning-Speed Introduction

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Tropical Shadows of Real Inflection Points

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\[ \mathbb{K} := \{ \text{field of locally convergent generalized Puiseux series} \}. \]

A locally convergent generalized Puiseux series is a formal series of the form

\[ a(t) = \sum_{r \in R} \alpha_r t^r \]

where \( R \subset \mathbb{R} \) is a well-ordered set, \( \alpha_r \in \mathbb{C} \), and the series \( a(t) \) is convergent for \( t > 0 \) small enough.
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\( \mathbb{R} \mathbb{K} := \{ f \in \mathbb{K} : \alpha_r \in \mathbb{R} \text{ for all } r \in R \} \)
The field $\mathbb{K}$ is of characteristic 0 and is algebraically closed.

$$\mathbb{K} := \{\text{field of locally convergent generalized Puiseux series}\}.$$
\( \mathbb{K} \) has a natural non-archimedean valuation

\[
val: \mathbb{K} \to \mathbb{T}
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such that

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val(x) := \begin{cases} 
-\infty & \text{if } x = 0 \\
- \min \{ r : \alpha_r \neq 0 \} & \text{if } x \neq 0
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Example: \( a(t) = t^5 + \pi t^{13} \mapsto -5 \).
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Example: \( a(t) = t^5 + \pi t^{13} \mapsto -5 \).
Consider the polynomial

\[ f = \sum_{i=1}^{k} a_i(t) \cdot x_1^{b_{i1}} \cdots x_n^{b_{in}} \in \mathbb{K}[x_1, ..., x_n], \]

Define \( f_t \) as the \textit{tropicalization} of \( f \), obtained by replacing + with \( \oplus := \max\{x, y\} \), \cdot with \( \odot := + \), and all coefficients with their valuation.

\textbf{Note:} that all tropical objects will be denoted by a subscript \( t \).
Given a tropical polynomial $f_t$, a **tropical hypersurface** $\mathcal{H}(f_t)$ is the set of all points in $\mathbb{R}^n$ at which the maximum is attained by at least two of the linear functions defining $f_t$.

**Example:** Consider $f = x + y + t^{-1}$. Its tropicalization is $f_t = x \oplus y \oplus 1 = \max\{x, y, 1\}$.
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Tropical Shadows of Real Inflection Points
Given an algebraic curve $X$ over $\mathbb{K}$, Brugallé and López de Medrano construct a method for determining where the tropicalizations of the inflection points $q$ of $X$ lie on $X_t$ by determining where the tropicalizations of the tangent lines of $q$ intersect $X_t$. 
Tropical Shadows of Real Inflection Points

Let's consider the example (on the board):
Tropical Shadows of Real Inflection Points

There are four possibilities for $E := X_t \cap T_t \cap \{q_t\}$:

**Theorem**

The vertex $v$ is a common vertex of $X_t$ and $T_t$ and is contained in $E$. Moreover, $E$ is one of the following:
Tropical Shadows of Real Inflection Points

We can further pinpoint the exact edge $\sigma$ of $E_{q_t}$ lies on:

**Theorem**

1. If $E = \{v\}$, then $\sigma = \{v\}$.
2. If $E$ is the union of two bounded edges $e_1, e_2$ and one unbounded edge $e_3$, then
   
   1. If $\ell(e_1) > \ell(e_2)$, then $\sigma = \{p_{e_1}\}$ where $p_{e_1}$ is the point of distance $\frac{\ell(e_1) - \ell(e_2)}{3}$ from $v$.
   2. If $\ell(e_1) = \ell(e_2)$, then $\sigma = e_3$
3. If $E$ is the union of 3 bounded edges $e_1, e_2, e_3$, then
   
   1. If $\ell(e_1) \geq \ell(e_2) > \ell(e_3)$, then $\sigma = \{p_{e_1}, p_{e_2}\}$, where $p_{e_i}$ is the point on $e_i$ of distance $\frac{\ell(e_i) - \ell(e_3)}{3}$ from $v$.
   2. If $\ell(e_1) > \ell(e_2) = \ell(e_3)$, then $\sigma = [v; p_{e_1}]$, where $p_{e_1}$ is the point on $e_1$ of distance $\frac{\ell(e_1) - \ell(e_2)}{3}$ from $v$.
   3. If $\ell(e_1) = \ell(e_2) = \ell(e_3)$, then $\sigma = \{v\}$. 
Tropical Shadows of Real Inflection Points

How does pinpointing where inflection points of real algebraic curves tropicalize?

**Theorem**

*(Brugallé and López de Medrano)* Let $X_t$ be a non-singular tropical curve in $\mathbb{R}^2$ defined by the tropical polynomial $f_t$. Suppose that the three edges of every vertex $v$ of $X_t$ have distinct lengths. Then the real algebraic curve defined by $f$ has exactly $d(d-2)$ inflection points in $\mathbb{P}^2_\mathbb{R}$ for $t > 0$ small enough.
Maximally inflected curves with all inflection points on a single component.
Maximally inflected quartics.
References


