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LANDAU–LIFSHITZ EQUATION, UNIAXIAL ANISOTROPY CASE: THEORY OF EXACT SOLUTIONS

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Using the inverse scattering method, we study the XXZ Landau–Lifshitz equation well-known in the theory of ferromagnetism. We construct all elementary soliton-type excitations and study their interaction. We also obtain finite-gap solutions (in terms of theta functions) and select the real solutions among them.

Keywords: multisoliton solution, ferromagnetism, finite-gap integration, theta function

1. Introduction

The Landau–Lifshitz (LL) equation [1]

$$\begin{aligned}&\vec{S}_t = [\vec{S}\times\vec{S}_{xx}] + [\vec{S}\times J\vec{S}],\\ &\vec{S} = (S_1, S_2, S_3), \qquad S_1^2+S_2^2+S_3^2 = 1, \qquad J = \operatorname{diag}(J_1, J_2, J_3),\end{aligned} \tag{1.1}$$

well-known in the theory of ferromagnetism, is an object of increased attention from specialists in theory of completely integrable nonlinear evolutionary systems. This specific interest in Eq. (1.1), compared with other models embedded in the inverse scattering method, is explained, in particular, by the following fact. The representation of the LL equation as the compatibility condition of two linear equations (a U–V pair representation) was obtained long ago: for the completely isotropic case ($J_1 = J_2 = J_3$, XXX model) by Takhtajan in 1977 [2], for the uniaxial anisotropy case ($J_1 = J_2 \neq J_3$, XXZ model) by Borovik in 1978 [3], and finally for the completely anisotropic case ($J_1 \neq J_2 \neq J_3$, XYZ model) by Sklyanin and independently by Borovik in 1979 (see [4]). It would therefore seem that for Eq. (1.1), we have a possibility of directly applying the traditional apparatus of the inverse scattering method with all its basic attributes: constructing explicit solutions, studying the Cauchy problem, and so on. But this possibility was not completely realized up to the mid-1980s. The difficulty in applying the inverse scattering method to the LL equation is explained by the fact that the LL equation (more than any other integrable system) requires reformulating the inverse scattering method into the "matrix Riemann problem method," which was not yet completely realized when the U–V pair was written for Eq. (1.1), although reformulating the inverse scattering method in this direction, strictly speaking, had already begun in 1975 in the well-known papers by Shabat, Zakharov, and Manakov [5], [6].

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Thus, starting from 1979, a quite strange situation appeared in the mathematical theory of the LL equation. This equation seemed to be embedded into the scheme of the inverse scattering method, but there was no real advantage from that. We note that Eq. (1.1) is extremely important from the physical standpoint. Therefore, the theory of the LL equation was intensively and quite successfully developing independently of the "interrelations" with the inverse scattering method. In particular, many specific solutions of it, both solitons and more complicated types, were known by 1979. In this regard, first of all, we should mention the papers by Akhiezer and Borovik [7], [8] and the research of Kosevich's group (Kosevich, Ivanov, Kovalev, Bogdan, Babich, and others) and Eleonskii's group (Eleonskii, Kirova, Kulagin, and others). We allow ourself to not present numerous citations of these authors, referring the reader to the review by Kosevich [9] for these citations as well as for a more detailed history of this problem.

Among the papers that used the inverse scattering method as much as was possible in the described period, in addition to the already mentioned papers by Takhtajan, Sklyanin, and Borovik, we note the works on the finite-gap integration of the LL equation by Bogoljubov Jr. and Prikarpatskii [10] (XXZ case), Cherednik [11], and Date, Jimbo, Koshivara, and Miwa [12] (XYZ case). We also relate the very interesting paper by Bogdan and Kovalev [13] (where the N-soliton solutions of the LL equation in the completely anisotropic case were constructed by the Hirota method) to this cycle of papers, although the authors themselves set the Hirota method in opposition to the inverse scattering method (in our view, unjustifiedly).

The fundamental contribution to the development of effective schemes for applying the inverse scattering method to the LL equation was made in 1982 after Mikhailov had rigorously defined the matrix Riemann problem with axiomatics suitable to the U–V-pair for the XYZ case of the LL equation in [14]. The result appeared immediately: the multisoliton solutions of Eq. (1.1) in the completely anisotropic case were systematically described by Mikhailov in [14] and by Rodin in [15], while Bobenko [16] and Borisov [17] proposed "dressing procedures" for this equation that allowed constructing new solutions using known ones. There was also progress in the algebraic-geometric (finite-gap) integration of Eq. (1.1): we succeeded in embedding the XXZ case of Eq. (1.1) into Krichever's scheme and reduced the finite-gap integration of this equation to explicit formulas in terms of theta functions [18], and we also implemented an analogous program for the completely anisotropic case, thus making the corresponding results in [10] and [11] more useful. Summarizing the above discussion, we can state that there is now a scheme in the theory of LL equation (1.1) that allows systematically constructing all previously known solutions, obtaining new exact solutions essentially different from those already known, and studying various effects associated with their interactions.

Our goal here is to present a detailed description of the considered scheme using the intermediate example (in complexity) of the LL equation in the uniaxial anisotropy case. This paper is intended for specialists in the ferromagnetism theory and does not assume that the reader has a deep familiarity with the ideas of the inverse scattering method. We try to give as much attention as possible to concrete results, to physically interesting particular cases, and to the effects obtainable from the approach we have developed. This purpose essentially determines the structure of this paper.

We present the foundation of the scheme in Secs. 2 and 3, formulating and discussing the "generalized matrix Riemann problem" corresponding to the XYZ case of the LL equation. Although we start with [14] mentioned above, we nevertheless use a nonstandard version of the transformation of the integrable equation into the Riemann problem, taking the version from papers by Jimbo, Miwa, and Ueno [19], [20]. It seems the most suitable foundation for developing all the basic constructions of our scheme in the subsequent sections. The mathematical apparatus in Secs. 2 and 3 is quite elementary: we use only the simplest ideas of linear algebra and complex analysis (the algebra of 2×2 matrices and the Liouville theorem).

In Secs. 4–6, staying within the framework of this apparatus, we develop the dressing procedure for the XXZ LL equation together with its use to construct, classify, and describe soliton solution interactions. We emphasize that we, apparently, do not obtain new results here. But our derived formulas have many methodological advantages compared with known representations of the exact solutions of LL equation. In particular, all the obtained answers are completely symmetric with respect to the components of the vector $\vec{S}$ from the standpoint of the degree of explicitness and compactness. Our formulas are parameterized by points of the complex plane (by the spectrum of the appropriate linear problem), which satisfy quite weak constraints (see Sec. 5). This allows studying various limits in our constructed solutions effectively.

In Sec. 6, we demonstrate the possibilities of our method by describing the effects of mutual interactions among different elementary solutions. Here, in addition to rather simple processes of two-soliton interactions, we also describe the phenomenon of a soliton passing through a domain wall.

We consider essentially new types of solutions of the XXZ LL equation in Secs. 7–10, where we present a realization of the basic conceptual theorem in Sec. 2 in the framework of the “finite-gap integration” technique based on the already less trivial mathematical apparatus of the theory of functions on compact Riemann surfaces. Unfortunately, we cannot keep the exposition closed here. But we try to minimize the need to refer to external mathematical sources. The closest source is [21]. The reader can also refer to the concluding chapters in [22] and the introductory part in [23]. We present supplementary information on reductions of Riemann surfaces (used in Sec. 9) in the appendix. A more detailed presentation of the theory of Riemann theta functions (subject to the rules for integrating nonlinear equations) can be found in [21] (also see [24]–[26]).

We now briefly describe the content of Secs. 7–10. In Sec. 7, we construct complex almost periodic (finite-gap) solutions of the LL equation in the case of uniaxial anisotropy in terms of multidimensional theta functions. In Sec. 8, we derive the reality conditions using the technique in [27]. From the standpoint of physical applications, the multidimensional theta function is a very complicated object for quantitative analysis. But there are currently program packages for calculations on Riemann surfaces [28]. In particular, they allow calculating solutions effectively in terms of theta functions. The multidimensional formulas containing theta functions hence become valuable for numerical analysis. Moreover, the general expressions in terms of theta functions are a convenient analytic basis for deriving important particular solutions.

In Sec. 9, we deduce periodic solutions described in terms of elliptic functions (cnoidal waves and their superpositions) from the general formulas in Secs. 7 and 8 using the technique for reducing theta functions of higher ranks to lower ones. In Secs. 10 and 11, we subject the general formula in Sec. 7 to a certain degeneration procedure (we pairwise merge the branch points of the original Riemann surface), resulting in convenient formulas for multisoliton solutions and solutions describing the interactions between solitons and cnoidal waves. More precisely, restricting ourself to the case of the “easy plane” anisotropy for brevity, we construct multisoliton formulas for the solutions of the “moving domain wall” type and describe the interaction effect between a single moving domain wall and a cnoidal wave.

Summarizing our description of this paper, we note the following. We hope to demonstrate that the developed scheme based on the inverse scattering method is the most appropriate technique for constructing, classifying, and studying exact solutions of the LL equation.

We dedicate this publication to our friend Ramil Faritonovich Bikbaev, untimely deceased.

2. Basic theorem

The XXZ LL equation ($J_1 = J_2$, and we take $J = \operatorname{diag}(0, 0, \varepsilon)$ without loss of generality)

$$\vec{S}_t = [\vec{S} \times \vec{S}_{xx}] + [\vec{S} \times J\vec{S}] \tag{2.1}$$

is the compatibility condition

$$U_t - V_x + [U, V] = 0 \tag{2.2}$$

for the pair of linear differential equations

$$\Psi_x = U\Psi, \qquad \Psi_t = V\Psi, \tag{2.3}$$

where U and V are given by the expressions

$$\begin{aligned} U(\lambda) &= -i\sum_{\alpha=1}^{3} S_\alpha w_\alpha \sigma_\alpha, \\ V(\lambda) &= 2i\sum_{\alpha=1}^{3} S_\alpha w_1 w_2 w_3 w_\alpha^{-1}\sigma_\alpha - i\sum_{\alpha=1}^{3}[S\times S_x]_\alpha w_\alpha \sigma_\alpha, \end{aligned} \tag{2.4}$$

$$w_1 = w_2 = \sqrt{\lambda^2 - a^2}, \qquad w_3 = \lambda, \qquad a = i\frac{\sqrt{\varepsilon}}{4}. \tag{2.5}$$

If $\varepsilon > 0$, then we have anisotropy of the easy magnetization axis type. If $\varepsilon < 0$, then we have anisotropy of the easy magnetization plane type. Both cases can be studied quite similarly. Pair (2.4) is a simple degeneration of the U–V Sklyanin–Borovik pair for the XYZ LL equation if $J_1 = J_2$. We note that the spectral parameter λ in the considered case ranges the two-sheet Riemann surface Γ of the function $\sqrt{\lambda^2 - a^2}$ instead of the complex plane. Of course, we might introduce an appropriate change of the variable λ such that U and V become rational functions of the spectral parameter. But we do not do this, because in the selected uniformization (2.5) of the relations $w_\alpha^2 - w_\beta^2 = -(J_\alpha - J_\beta)/4$, the following reduction of pair (2.4) can be taken into account very naturally:

$$\sigma_3 U(\lambda^\tau)\sigma_3 = U(\lambda), \qquad \sigma_3 V(\lambda^\tau)\sigma_3 = V(\lambda), \tag{2.6}$$

where $\lambda \to \lambda^\tau$ denotes the involution transposing the sheets of Γ and $\sqrt{(\lambda^\tau)^2 - a^2} = -\sqrt{\lambda^2 - a^2}$. This reduction (associated with the transposition of the sheets) can be easily taken into account in constructing finite-gap solutions (see Sec. 7).

The principal object in constructing exact formulas for solutions of the nonlinear equations integrable by the inverse scattering method is the function Ψ. Precisely this function is first constructed using its analytic properties following from the form of U–V-pair. The formulas for solutions of the nonlinear equations can then be constructed using this function. We formulate the so-called generalized Riemann problem corresponding to Eq. (2.1).

Reimann problem. We must find a function $\Psi(\lambda)$ defined on Γ, taking values in the set of 2×2 matrices, and having the following properties:

1. Two infinitely remote points $\infty^{1,2}$ of the surface Γ in its standard realization by the two-sheet covering of the plane of the variable λ ($\sqrt{\lambda^2 - a^2} \to \pm\lambda$ at $\lambda \to \infty^{1,2}$) are the essential singularity points of Ψ, which in the neighborhood of these points has an essential singularity differentiable in x and t of the form
$$\Psi(\lambda, x, t) = \sum_{j=0}^{\infty} \Phi_j(x,t)\lambda^{-j} e^{-i\sigma_3\lambda x + 2i\sigma_3\lambda^2 t} C\lambda^m, \tag{2.7}$$
where $\det \Phi_0(x,t) \neq 0$ and the matrix C is invertible and independent of x and t.

2. The function $\Psi(\lambda)$ also has so-called regular singularities at the points $a_1, \ldots, a_N$, i.e., $\Psi(\lambda)$ is holomorphic and invertible at all points of the set $\Gamma \setminus \{\infty^{1,2}\}$, except the points $a_1, \ldots, a_N$, independent of x and t, in a neighborhood of which we have the representation
$$\Psi(\lambda) \underset{\lambda\sim a_j}{=} \widehat{\Psi} k^{T_j} C_j, \quad j = 1, \ldots, N, \tag{2.8}$$

where T_j are diagonal constant matrices and C_j are invertible constant matrices (independent of x and t). The matrix-valued function $\widehat{\Psi}(\lambda)$ is holomorphic and invertible in a neighborhood of a_j, where k is a local parameter in the neighborhood of a_j,

$$k = \begin{cases} \lambda - a_j, & a_j \neq \pm a, \\ \sqrt{\lambda \pm a}, & a_j = \pm a. \end{cases}$$

We note that if T_j is a rational noninteger matrix, then the function $\Psi(\lambda)$ is not single-valued on Γ. In this case, the point a_j is a branch point of the covering $\widehat{\Gamma} \to \Gamma$, and the function Ψ is already single-valued on this covering surface $\widehat{\Gamma}$ and, as a function on Γ, is characterized by the property that in passing around the point a_j on Γ in the positive direction, $\Psi(\lambda)$ is multiplied on the right by a monodromy matrix,

$$\Psi(\lambda) \to \Psi(\lambda) M_j, \qquad M_j = C_j^{-1} e^{2\pi i T_j} C_j. \tag{2.9}$$

3. Let there exist contours $\mathcal{L}_i \in \Gamma$ and matrices $G_i(\lambda)$, $i = 1, \dots, M$, independent of x and t. Then the matrices $\Psi_+(\lambda)$ and $\Psi_-(\lambda)$ (the boundary values of Ψ from different sides of the contour $\mathcal{L}_i$) are coupled by the linear relations along $\mathcal{L}_i$,

$$\Psi_-(\lambda) = \Psi_+(\lambda) G_i(\lambda)\big|_{\lambda \in \mathcal{L}_i}. \tag{2.10}$$

4. The reduction constraint

$$\sigma_3 \Psi(\lambda^\tau) = \Psi(\lambda)\sigma(\lambda) \tag{2.11}$$

holds, where $\sigma(\lambda)$ is independent of x and t.

5. The normalization conditions

$$\begin{aligned} \frac{\partial}{\partial x} \log \Psi_{1i}(a) &= \frac{\partial}{\partial x} \log \Psi_{2k}(-a), \\ \frac{\partial}{\partial t} \log \Psi_{1j}(a) &= \frac{\partial}{\partial t} \log \Psi_{2l}(-a) \end{aligned} \tag{2.12}$$

hold for any i, k, j, and l.

Theorem 1. *Let a function Ψ satisfying conditions 1–5 of the Riemann problem be constructed. Then the logarithmic derivatives $\Psi_x\Psi^{-1}$ and $\Psi_t\Psi^{-1}$ have form (2.4) up to the terms proportional to the identity matrix, where*

$$\sum_{\alpha=1}^{3} S_\alpha \sigma_\alpha = \Phi_0 \sigma_3 \Phi_0^{-1}, \tag{2.13}$$

i.e., if

$$\Phi_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.14}$$

then

$$S_1 = \frac{CD - AB}{AD - BC}, \qquad S_2 = -i\frac{CD + AB}{AD - BC}, \qquad S_3 = \frac{AD + BC}{AD - BC}. \tag{2.15}$$

The functions $S_j(x,t)$ defined by relations (2.13)–(2.15) satisfy the equality

$$\sum_{\alpha=1}^{3} S_\alpha^2 = 1$$

and form a solution of XXZ LL equation (2.1).

Proof. We first note that the logarithmic derivatives $\Psi_x\Psi^{-1}$ and $\Psi_t\Psi^{-1}$ are single-valued functions on the surface Γ by virtue of properties (2.8)–(2.10), unlike the function Ψ itself. They have no singularities at the points a_j,

$$\Psi_x\Psi^{-1} \underset{\lambda\sim a_j}{=} \widehat{\Psi}_x k^{T_j} C_j C_j^{-1} k^{-T_j}\widehat{\Psi}^{-1} = \widehat{\Psi}_x\widehat{\Psi}^{-1},$$

and on the contours $\mathcal{L}_i$,

$$(\Psi_+)_x\Psi_+^{-1} = (\Psi_-)_x\Psi_-^{-1}\big|_{\lambda\in\mathcal{L}_i}.$$

Therefore, from asymptotic form (2.7) and because of the absence of singularities in the set $\Gamma\setminus\{\infty^{1,2}\}$, we have the following representations of the logarithmic derivatives $\Psi_x\Psi^{-1}$ and $\Psi_t\Psi^{-1}$:

$$\Psi_x\Psi^{-1} = \lambda A_1 + \sqrt{\lambda^2-a^2}A_2 + A_3,$$

$$\Psi_t\Psi^{-1} = \lambda^2 B_1 + \lambda\sqrt{\lambda^2-a^2}B_2 + \lambda B_3 + \sqrt{\lambda^2-a^2}B_4 + B_5,$$

where A_i and B_i are matrices depending only on x and t with

$$\operatorname{Tr}(A_1+A_2) = \operatorname{Tr}(B_1+B_2) = \operatorname{Tr}(B_3+B_4) = 0.$$

Further, it follows from reduction (2.11) that the matrices A_1, A_3, B_1, B_3, and B_5 are diagonal and the matrices A_2, B_2, and B_4 are antidiagonal. Hence,

$$\Psi_x\Psi^{-1} = -i\sum_{\alpha=1}^{3} S_\alpha w_\alpha\sigma_\alpha + A(x,t)\sigma_3 + \alpha(x,t)I,$$

$$\Psi_t\Psi^{-1} = 2i\sum_{\alpha=1}^{3} P_\alpha w_1w_2w_3w_\alpha^{-1}\sigma_\alpha + i\sum_{\alpha=1}^{3} Q_\alpha w_\alpha\sigma_\alpha + B(x,t)\sigma_3 + \beta(x,t)I.$$

It follows from normalization condition (2.12) that $A(x,t) = B(x,t) = 0$. Substituting asymptotic expansion (2.7) in (2.3) and equating terms with like degrees of λ, we finally confirm formula (2.13) and the equalities $P_\alpha = S_\alpha$ and $Q_\alpha = -[S\times S_x]_\alpha$. The theorem is proved.

Therefore, the function Ψ (and consequently the solution of the XXZ LL equation) is defined by the following data of the generalized Riemann problem (the "scattering data"):

$$\Lambda = \{a_1,\dots,a_N,\, T_j,\dots,T_N,\, C_1,\dots,C_N,\, \mathcal{L}_1,\dots,\mathcal{L}_M,\, G_1(\lambda),\dots,G_M(\lambda)\}, \tag{2.16}$$

where T_j and C_j, $j=1,\dots,N$, are defined in (2.8) and $\lambda\in\mathcal{L}_i$, $i=1,\dots,M$, is the argument of $G_i(\lambda)$. Further, we find exact expressions for Ψ with some particular data Λ and thus construct solutions of Eq. (2.1).

Remark 1. Let the function Ψ satisfy the reduction

$$\sigma_2\overline{\Psi(\bar\lambda)} = \Psi(\lambda)M(\lambda), \tag{2.17}$$

where the conjugation anti-involution on Γ is naturally given as

$$\big(\lambda, \sqrt{\lambda^2-a^2}\,\big) \to \big(\bar\lambda, \overline{\sqrt{\lambda^2-a^2}}\,\big)$$

and $M(\lambda)$ is a matrix-valued function independent of x and t. Then the solution of Eq. (2.1) (defined by (2.13)–(2.15)) is real.

Remark 2. Let the function $\Psi(\lambda)$ satisfy conditions (2.7)–(2.12) and define the solution $\vec{S}(x,t)$. Then the function

$$\Psi_\gamma(\lambda) = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix} \Psi(\lambda), \quad \gamma = \text{const} \in \mathbb{C}, \tag{2.18}$$

also satisfies a generalized Riemann problem. The corresponding solution $\vec{S}_\gamma$ of Eq. (2.1) differs from $\vec{S}$ (we consider only real solutions, $|\gamma| = 1$) by a simple rotation of axes 1 and 2 in the plane of these axes (in the plane perpendicular to the anisotropy axis 3). Obviously, the solutions $\vec{S}$ and $\vec{S}_\gamma$ are equivalent from the physical standpoint. Further, the function $\Psi_\varphi(\lambda) = \varphi(x,t)\Psi(\lambda)$, where $\varphi(x,t)$ is an arbitrary scalar function of$x$ and t, also satisfies a Riemann problem. The corresponding solutions of Eq. (2.1) coincide: $\vec{S}_\varphi(x,t) = \vec{S}(x,t)$.

Remark 3. There are some a priori constraints on the matrices $\sigma(\lambda)$ and $M(\lambda)$ involved in reduction identities (2.11) and (2.17). In particular, successively applying two transformations (2.11) to $\Psi(\lambda)$, we obtain the relation

$$\sigma(\lambda)\sigma(\lambda^\tau) \equiv I. \tag{2.19}$$

Successively applying two transformations (2.17) to $\Psi(\lambda)$, we obtain the relation

$$M(\lambda)\overline{M(\bar{\lambda})} \equiv -I. \tag{2.20}$$

Successively applying transformations (2.11) and (2.17) to $\Psi(\lambda)$, we obtain the relation

$$\sigma(\lambda)M(\lambda^\tau) + M(\lambda)\overline{\sigma(\bar{\lambda})} = 0. \tag{2.21}$$

3. External field

It is well known that a homogeneous external field directed along the anisotropy axis (axis 3 in our case) does not destroy the integrability of a system. In this section, we construct a generalization of the Riemann problem associated with Eq. (1.1) and formulated in Sec. 2. The XXZ LL equation with an external field directed along the anisotropy axis is embedded into the proposed model.

Theorem 2. *Let the function $\Psi(\lambda,x,t)$ have properties 1–4 of the Riemann problem and satisfy the normalization condition*

$$\begin{aligned} \frac{\partial}{\partial x}\log\Psi_{1i}(a) &= \frac{\partial}{\partial x}\log\Psi_{2k}(-a) + i\frac{\partial}{\partial x}f(x,t), \\ \frac{\partial}{\partial t}\log\Psi_{1j}(a) &= \frac{\partial}{\partial t}\log\Psi_{2l}(-a) + i\frac{\partial}{\partial t}f(x,t) \end{aligned} \tag{3.1}$$

for some indices i, j, k, and l. Then the logarithmic derivatives $\Psi_x\Psi^{-1}$ and $\Psi_t\Psi^{-1}$ are respectively equal (up to terms proportional to the identity matrix) to

$$U_f = U + \frac{i}{2}\sigma_3\frac{\partial f}{\partial x}, \qquad V_f = V + \frac{i}{2}\sigma_3\frac{\partial f}{\partial t},$$

where U and V are matrices (2.4). Zakharov–Shabat equation (2.2) for U_f and V_f leads to the equation

$$\vec{S}_t = [\vec{S}\times\vec{S}_{xx}] + [\vec{S}\times\tilde{J}\vec{S}] + [\vec{S}\times\vec{H}] - 2\vec{S}_x S_3\frac{\partial f}{\partial x}, \tag{3.2}$$

where

$$\tilde{J} = \operatorname{diag}\left(0,0,\varepsilon + \left(\frac{\partial f}{\partial x}\right)^2\right), \qquad \vec{H} = \left(0,0,\frac{\partial f}{\partial t}\right).$$

A particular case of Eq. (3.2) is physically interesting. It describes the nonlinear dynamics of a ferromagnet in a homogeneous external magnetic field $\vec{H}(t)$ arbitrarily depending on t.

Corollary 1. *If $f(x,t) = \int^t H(s)\,ds$ depends only on t, then this Riemann problem corresponds to the equation*

$$\vec{S}_t = [\vec{S}\times\vec{S}_{xx}] + [\vec{S}\times J\vec{S}] + [\vec{S}\times\vec{H}], \qquad \vec{H} = (0,0,H(t)). \tag{3.3}$$

The elementary procedure for constructing solutions of Eq. (3.3) using solutions of Eq. (2.1) follows from Corollary 1.

Corollary 2. *If s solution $\vec{S}(x,t) = (S_1,S_2,S_3)$ of LL equation (2.1) is expressed in terms of A, B, C, and D using formulas (2.15), then the quantities*

$$A_f = A\exp\biggl(\int^t H(s)\,ds\biggr), \qquad B_f = B\exp\biggl(\int^t H(s)\,ds\biggr), \qquad C_f = C, \qquad D_f = D$$

define the function $\vec{S}_f(x,t)$ (which is a solution of Eq. (3.3)) by the same formulas (2.15).

Of course, this purely algebraic fact can be verified directly. Obviously, the solution $\vec{S}_f$ is real if $\vec{S}$ and $H(t)$ are real.

4. Dressing procedure: General scheme

It is convenient to start this section with the proof of one auxiliary statement [19].

Lemma 1. *Let $\Psi(\lambda)$ be a 2×2 matrix function holomorphic in the neighborhood of the point $\lambda = a$, which is a simple zero of $\det\Psi(\lambda)$. Then representation (2.8) holds for the matrix $\Psi(\lambda)$ in the neighborhood of the point a with $T = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$. In this case, any invertible matrix such that the first column of the matrix C^{-1} belongs to $\ker\Psi(a)$ can be taken for C.*

Proof. Let C and T be as described in the lemma. The statement in the lemma is equivalent to the holomorphicity and matrix invertibility of the function

$$\widehat{\Psi}(\lambda) = \Psi(\lambda)C^{-1}(\lambda-a)^{\left(\begin{smallmatrix}-1&0\\0&0\end{smallmatrix}\right)}$$

in some neighborhood of the point a. Let $C^{-1} = (X,Y)$. Then (because $\Psi(a)X = 0$)

$$\begin{aligned}\widehat{\Psi}(\lambda) &= \Psi(\lambda)(X,Y)\begin{pmatrix}(\lambda-a)^{-1} & 0\\ 0 & 1\end{pmatrix} = \biggl(\frac{1}{\lambda-a}\Psi(\lambda)X, \Psi(\lambda)Y\biggr) = \\ &= \biggl(\frac{1}{\lambda-a}[\Psi(a)X + \Psi'(a)X(\lambda-a)+\cdots], \Psi(a)Y + (\lambda-a)\Psi'(a)Y+\cdots\biggr) = \\ &= \sum_{k=0}^{\infty}(\lambda-a)^k\Psi_k,\end{aligned}$$

and the function $\widehat{\Psi}(\lambda)$ is consequently holomorphic in a neighborhood of a. After the holomorphicity of $\widehat{\Psi}(\lambda)$ is established, its matrix invertibility follows because the zero of $\det\Psi(\lambda)$ at a is simple. The lemma is proved.

Now let $\Psi_0(\lambda, x, t)$ be a function satisfying all conditions in Theorem 1 and thus leading to some solution $\vec{S}_0(x,t)$ of the LL equation. We let

$$\Lambda_0 = \{a_1^0, \dots, a_N^0,\ T_1^0, \dots, T_N^0,\ C_1^0, \dots, C_N^0,\ \mathcal{L}_1^0, \dots, \mathcal{L}_N^0,\ G_1^0(\lambda), \dots, G_M^0,\ m^0\}$$

denote the data of the generalized Riemann problem corresponding to Ψ_0. Using Ψ_0, we want to explicitly construct a new function $\Psi(\lambda, x, t)$ satisfying the same conditions 1–5 of the Riemann problem but with a new set of data $\Lambda = \Lambda_0 \oplus \Lambda'$. Hence, having the known solution $\vec{S}_0(x,t)$ of the LL equation, we would construct a new solution of it. We seek the function Ψ in the form

$$\Psi(\lambda, x, t) = f(\lambda, x, t)\Psi_0(\lambda, x, t), \tag{4.1}$$

where f is a 2×2 matrix function meromorphic on Γ and the simple poles at the points $\infty^{1,2}$ are the only singularities. We require that f satisfy the relation

$$\sigma_3 f(\lambda^\tau)\sigma_3 = f(\lambda). \tag{4.2}$$

This equality together with the mentioned conditions for the singularities of $f(\lambda)$ leads to its representation

$$f(\alpha) = \sum_{\alpha=1}^{3} q_\alpha w_\alpha(\alpha)\sigma_\alpha + (q_0\lambda + p_0)I + p_3\sigma_3, \tag{4.3}$$

where the scalar functions $q_\alpha(x,t)$ and $p_\alpha(x,t)$ must still be defined. To determine them, we take the points $\lambda_1, \lambda_2 \notin \{a_1^0, \dots, a_N^0\} \cup \{a, -a\}$ and two complex numbers A_1 and A_2. Let

$$\Psi(\lambda_j)\begin{pmatrix} 1 \\ A_j \end{pmatrix} = 0, \quad j = 1, 2, \qquad p_3 = -aq_0. \tag{4.4}$$

These relations form a linear homogeneous algebraic system of five equations for six unknowns (the q_α and p_α). Solving this system, we find the sought functions $q_\alpha(x,t)$ and $p_\alpha(x,t)$ up to a common functional factor. Because of Remark 2 at the end of Sec. 1, this freedom is already inessential for us.

Theorem 3. *The function $\Psi(\lambda, x, t)$ defined by formulas* (4.1), (4.3), *and* (4.4) *satisfies all conditions in Theorem* 1. *The corresponding data of the generalized Riemann problem differ from the original data Λ_0 in the number of regular singular points, which is increased by four,*

$$\{a_1^0, \dots, a_N^0\} \to \{a_1^0, \dots, a_N^0\} \oplus \{\lambda_1, \lambda_2, \lambda_1^\tau, \lambda_2^\tau\},$$

and by the shift $m^0 \to m^0 + I$ of the matrix m^0 by the identity matrix. The corresponding (*complex*) *solution $\vec{S}(x,t)$ of the LL equation*[1] *is related to the seed solution $\vec{S}_0(x,t)$,*

$$S = QS_0Q^{-1}, \qquad Q = \sum_{\alpha=1}^{3} q_\alpha\sigma_\alpha + q_0 I. \tag{4.5}$$

Before proving Theorem 3, we prove the following lemma.

[1] We use the notation $\vec{S}$ for the vectors (S_1, S_2, S_3) and S for the matrix $\sum_{\alpha=1}^{3} S_\alpha\sigma_\alpha$.

Lemma 2. *The zeros of* $\det f(\lambda)$ *as a function on* Γ *are simple and are located at the points* λ_1, λ_2, λ_1^τ, *and* λ_2^τ. *The representation of form* (2.8) *with matrices* T *and* C *independent of* x *and* t *holds for the function* $\Psi(\lambda)$ *at all these points.*

Proof. By formula (4.3), the function $\det f(\lambda)$ has two second-order poles at the points $\infty^{1,2}$ on Γ. It must therefore have four zeros. The first two vector equalities in (4.4) indicate that the points λ_1 and λ_2 must necessarily be among those zeros. Together with λ_1 and λ_2, the points λ_1^τ and λ_2^τ are necessarily zeros of $\det f(\lambda)$, which follows from reduction identity (4.2). Finally, the last statement in the lemma follows directly from Lemma 1 and from the fact that equality (4.2) yields the validity of reduction identity (2.11) for $\Psi(\lambda)$ with the same matrix $\sigma_0(\lambda)$ as for the seed function $\Psi_0(\lambda)$. The matrices T and C corresponding to the points λ_j and λ_j^τ are

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \\ -A_j & 1 \end{pmatrix}$$

for λ_j and

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \\ -A_j & 1 \end{pmatrix} \sigma_0$$

for λ_j^τ. This ends the proof of the lemma.

Proof of Theorem 3. By Lemma 2, the function $f(\lambda)$ is holomorphic and matrix invertible at all the "old" regular singular points a_j^0. Therefore, right multiplication of $\Psi_0(\lambda)$ by $f(\lambda)$ does not destroy its behavior at the points a_j^0 , i.e., relation (2.8) remains valid for $\Psi(\lambda)$ at these points with the same matrices T_j^0 and C_j^0. The same holds for the conjugation conditions on the "old" contours $\mathcal{L}_j$, while new discontinuity lines obviously cannot appear.

Further, asymptotic condition (2.7) is satisfied for $\Psi(\lambda)$ with the matrix $m = m^0 + I$, and (as noted in the proof of Lemma 2) reduction identity (2.11) holds with the matrix $\sigma(\lambda) = \sigma_0(\lambda)$. The only new singularities of $\Psi(\lambda)$ are zeros of its determinant at the points λ_j and λ_j^τ, $j = 1, 2$. But the required representation (2.8) with the matrices T and C independent of x and t is ensured by Lemma 2. Hence, all conditions in Theorem 1 for $\Psi(\lambda)$ are verified except the last one, which is normalization conditions (2.12). We show that it follows from the scalar equation $p_3 = -aq_0$ of system (4.4). In fact, because conditions (2.12) are satisfied for the seed function $\Psi_0(x, t, \lambda)$, the equalities

$$\begin{aligned} (\Psi_0)_{1k}(a) &= \alpha(\Psi_0)_{2l}(-a), \qquad \alpha_x = 0, \\ (\Psi_0)_{1j}(a) &= \beta(\Psi_0)_{2r}(-a), \qquad \beta_t = 0 \end{aligned} \tag{4.6}$$

hold for some k, l and j, r.

The matrix $f(\lambda)$ (for $\lambda = \pm a$) is diagonal. The equality $p_3 = -aq_0$ is equivalent to the relation $f_{11}(a) = f_{22}(-a)$. Therefore, for the same k, l and j, r as in (4.6), we have

$$\Psi_{1k}(a) = f_{11}(a)(\Psi_0)_{1k}(a) = \alpha f_{11}(a)(\Psi_0)_{2l}(-a) = \alpha f_{22}(-a)(\Psi_0)_{2l}(-a) = \alpha \Psi_{2l}(-a).$$

Consequently,

$$\frac{\partial}{\partial x} \log \Psi_{1k}(a) = \frac{\partial}{\partial x} \log \Psi_{2l}(-a).$$

Similarly,

$$\Psi_{1j}(a) = f_{11}(a)(\Psi_0)_{1j}(a) = \beta f_{11}(a)(\Psi_0)_{2r}(-a) = \beta f_{22}(-a)(\Psi_0)_{2r}(-a) = \beta \Psi_{2r}(-a).$$

Consequently,

$$\frac{\partial}{\partial t}\log\Psi_{1j}(a)=\frac{\partial}{\partial t}\log\Psi_{2r}(-a),$$

i.e., normalization conditions (2.12) for the "dressed" function Ψ are satisfied for the same k,l and j,r as for the seed function Ψ_0. Theorem 3 is proved.

We note that taking the relation $p_3=-aq_0$ into account, we can represent the matrix $f(\lambda)$ in the form

$$f(\lambda)=D_1(\lambda)\bigl(Q+d_0R(\lambda)\bigr)D(\lambda), \tag{4.7}$$

where $d_0=p_0+aq_3$ and

$$D_1(\lambda)=\begin{pmatrix}1 & 0\\ 0 & \sqrt{\frac{\lambda+a}{\lambda-a}}\end{pmatrix},\qquad R(\lambda)=\begin{pmatrix}\frac{1}{\lambda-a} & 0\\ 0 & \frac{1}{\lambda+a}\end{pmatrix},\qquad D(\lambda)=\begin{pmatrix}\lambda-a & 0\\ 0 & \sqrt{\lambda^2-a^2}\end{pmatrix}.$$

We can then write the vector equation of system (4.4) as

$$QX_j=-d_0R(\lambda_j)X_j,\quad j=1,2, \tag{4.8}$$

where

$$\vec{X}_j=D(\lambda_j)\Psi_0(\lambda_j)\begin{pmatrix}1\\ A_j\end{pmatrix}. \tag{4.9}$$

We assume that two matrix functions X and Y are connected by the relation $X\cong Y$ if there is a singular scalar function $d(x,t)$ such that $X=dY$. From Eqs. (4.8), we then obtain the explicit representation for the matrix Q (which appears directly in transition (4.5) from the old solution $\vec{S}_0$ to the new solution $\vec{S}$) in terms of the seed function $\Psi_0(\lambda)$ and the transformation parameters (λ_j,A_j):

$$Q\cong VW^{-1},\qquad W=(\vec{X}_1,\vec{X}_2),\qquad V=\bigl(R(\lambda_1)\vec{X}_1,\ R(\lambda_2)\vec{X}_2\bigr). \tag{4.10}$$

As mentioned in Sec. 1, the LL equation in the complex case is invariant under the gauge transformation

$$S\to\begin{pmatrix}1 & 0\\ 0 & \delta\end{pmatrix}S\begin{pmatrix}1 & 0\\ 0 & \delta^{-1}\end{pmatrix},\qquad \delta\in\mathbb{C}\setminus\{0\}.$$

Therefore, we can say that by fixing the values λ_j and A_j, we select not a single solution of the LL equation but a whole gauge class. Each representative in this class is characterized by its own value of the parameter δ and is described by the function

$$S=Q_\delta S_0Q_\delta^{-1},\qquad Q_\delta=\begin{pmatrix}1 & 0\\ 0 & \delta\end{pmatrix}Q. \tag{4.11}$$

Hereafter, we are most interested in solutions corresponding to the special parameter value

$$\delta=\delta_0\equiv\sqrt{\frac{(\lambda_1+a)(\lambda_2+a)}{(\lambda_1-a)(\lambda_2-a)}}.$$

For the correspondent matrix $Q_0\equiv Q_{\delta_0}$, the expanded version of the representation

$$Q_\delta\cong\begin{pmatrix}1 & 0\\ 0 & \delta\end{pmatrix}VW^{-1}$$

has the most symmetric form:

$$Q_0 \cong \begin{pmatrix} \alpha_1\beta_2\sqrt{\lambda_2^2-a^2}-\alpha_2\beta_1\sqrt{\lambda_1^2-a^2} & \alpha_1\alpha_2(\lambda_1-\lambda_2) \\ \beta_1\beta_2(\lambda_2-\lambda_1) & \alpha_1\beta_2\sqrt{\lambda_1^2-a^2}-\alpha_2\beta_1\sqrt{\lambda_2^2-a^2} \end{pmatrix}, \tag{4.12}$$

where we introduce the notation

$$\begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} \equiv \Psi_0(\lambda_j)\begin{pmatrix} 1 \\ A_j \end{pmatrix}, \quad j=1,2. \tag{4.13}$$

The corresponding solution of the LL equation and the associated function Ψ are

$$S = Q_0 S_0 Q_0^{-1}, \tag{4.14}$$

$$\Psi(\lambda) \cong D_1(\lambda)\begin{pmatrix} 1 & 0 \\ 0 & \delta_0 \end{pmatrix}\bigl(VW^{-1}-R(\lambda)\bigr)D(\lambda)\Psi_0(\lambda). \tag{4.15}$$

With these formulas, we end the description of the dressing procedure for the complex case and proceed to discuss conditions on the parameters (λ_j, A_j) that preserve the realness of the vector $\vec{S}$ in transformation (4.12)–(4.14).

We first note that for $\vec{S}$ to be real, it suffices to satisfy the relation

$$\sigma_2\bar{Q}_0\sigma_2 \cong Q_0. \tag{4.16}$$

In fact, the realness condition for $\vec{S}$ is equivalent to the equality $\sigma_2\bar{S}\sigma_2 = -S$. By assumption, this equality holds for $\vec{S}_0$. From (4.16), we therefore obtain

$$\sigma_2\bar{S}\sigma_2 = \sigma_2\bar{Q}_0\bar{S}_0\bar{Q}_0^{-1}\sigma_2 = Q_0\sigma_2\bar{S}_0\sigma_2\bar{Q}_0^{-1} = -QS_0Q^{-1} = -S.$$

We now consider the realness condition directly.

Theorem 4. *Let the seed solution $\vec{S}_0$ be real and the seed function Ψ_0 consequently satisfy identity* (2.17) *with some matrix $M = M_0$. Then relation* (4.16) *holds in two cases*:

a. *where* $\operatorname{Im}\lambda_j \neq 0$, $\lambda_1 = \overline{\lambda}_2$, *and*

$$\det\left(M_0(\lambda_1)\begin{pmatrix} 1 \\ \bar{A}_2 \end{pmatrix}, \begin{pmatrix} 1 \\ A_1 \end{pmatrix}\right) = 0 \tag{4.17}$$

and

b. *where $a = \bar{a}$ (easy plane)*, $\operatorname{Im}\lambda_j = 0$, $|\lambda_j| < a$, *and*

$$\det\left(\sigma_0(\lambda_j)M_0(\lambda_j^\tau)\begin{pmatrix} 1 \\ \bar{A}_j \end{pmatrix}, \begin{pmatrix} 1 \\ A_j \end{pmatrix}\right) = 0, \tag{4.18}$$

where $\sigma_0(\lambda)$ is the matrix $\sigma(\lambda)$ used in reduction identity (2.11) *for the function Ψ_0.*

Proof. We proceed with case a. Equality (4.17) means that the vectors

$$M_0(\lambda_1)\begin{pmatrix}1\\ \bar{A}_2\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}1\\ A_1\end{pmatrix}$$

are proportional to each other. Recalling the a priori identity $\overline{M}(\bar{\lambda})M(\lambda) = -I$, we verify that the vectors

$$M_0(\lambda_2)\begin{pmatrix}1\\ \bar{A}_1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}1\\ A_2\end{pmatrix}.$$

are collinear. We have

$$M_0(\lambda_1)\begin{pmatrix}1\\ \bar{A}_2\end{pmatrix} = \varkappa\begin{pmatrix}1\\ A_1\end{pmatrix}, \quad -\begin{pmatrix}1\\ \bar{A}_2\end{pmatrix} = \varkappa\overline{M}_0(\bar{\lambda}_1)\begin{pmatrix}1\\ A_1\end{pmatrix}, \quad M_0(\lambda_2)\begin{pmatrix}1\\ \bar{A}_1\end{pmatrix} = -\frac{1}{\bar{\varkappa}}\begin{pmatrix}1\\ A_2\end{pmatrix}.$$

Therefore,

$$\begin{aligned}
\sigma_2\bar{\Psi}_0(\lambda_2)\begin{pmatrix}1\\ \bar{A}_2\end{pmatrix} &= \Psi_0(\lambda_1)M_0(\lambda_1)\begin{pmatrix}1\\ \bar{A}_2\end{pmatrix} = \varkappa\Psi_0(\lambda_1)\begin{pmatrix}1\\ A_1\end{pmatrix},\\
\sigma_2\bar{\Psi}_0(\lambda_1)\begin{pmatrix}1\\ \bar{A}_1\end{pmatrix} &= \Psi_0(\lambda_2)M_0(\lambda_2)\begin{pmatrix}1\\ \bar{A}_1\end{pmatrix} = -\frac{1}{\bar{\varkappa}}\Psi_0(\lambda_2)\begin{pmatrix}1\\ A_2\end{pmatrix},
\end{aligned} \tag{4.19}$$

which we can write in terms of α_j and β_j as

$$\begin{aligned}
\bar{\alpha}_1 &= \frac{i}{\bar{\varkappa}}\beta_2, \qquad & \bar{\beta}_1 &= -\frac{i}{\bar{\varkappa}}\alpha_2,\\
\bar{\alpha}_2 &= -i\varkappa\beta_1, \qquad & \bar{\beta}_2 &= i\varkappa\alpha_1.
\end{aligned} \tag{4.20}$$

Taking

$$\overline{\sqrt{\lambda_1^2 - a^2}} = \sqrt{\lambda_2^2 - a^2}, \qquad \overline{\sqrt{\lambda_2^2 - a^2}} = \sqrt{\lambda_1^2 - a^2}$$

into account (see the rule for the action of the anti-involution $\lambda \to \bar{\lambda}$ on Γ in Sec. 1), we obtain the theorem statement in case a directly from formula (4.12) through a simple verification with relations (4.20) taken into account.

Proceeding to case b, we note that $\operatorname{Re}\sqrt{\lambda_j^2 - a^2} = 0$ under the given conditions for λ_j. This means that

$$\overline{\sqrt{\lambda_j^2 - a^2}} = -\sqrt{\lambda_j^2 - a^2} = \sqrt{(\lambda_j^\tau)^2 - a^2}, \tag{4.21}$$

i.e., the involution $\lambda \to \bar{\lambda}$ acts on λ_j (points on the surface Γ) as the involution $\tau\colon \bar{\lambda}_j = \lambda_j^\tau$. Precisely this circumstance is taken into account in condition (4.18) for the parameters A_j: taken in this form, they lead to relations analogous to relations (4.19) in the preceding case:

$$\begin{aligned}
\sigma_2\bar{\Psi}_0(\lambda_j)\begin{pmatrix}1\\ \bar{A}_j\end{pmatrix} &= \Psi_0(\lambda_j^\tau)M_0(\lambda_j^\tau)\begin{pmatrix}1\\ \bar{A}_j\end{pmatrix} =\\
&= \sigma_3\Psi_0(\lambda_j)\sigma_0(\lambda_j)M_0(\lambda_j^\tau)\begin{pmatrix}1\\ \bar{A}_j\end{pmatrix} = \varkappa\sigma_3\Psi_0(\lambda_j)\begin{pmatrix}1\\ A_j\end{pmatrix}.
\end{aligned}$$

Hence, relations (4.20) are replaced with

$$\sigma_2\begin{pmatrix}\bar{\alpha}_j\\ \bar{\beta}_j\end{pmatrix} = \varkappa\sigma_3\begin{pmatrix}\alpha_j\\ \beta_j\end{pmatrix} \qquad \Longleftrightarrow \qquad \begin{aligned}\bar{\beta}_j &= -i\varkappa\alpha_j,\\ \bar{\alpha}_j &= -i\varkappa\beta_j.\end{aligned} \tag{4.22}$$

The proof of the theorem in case b, as in case a, ends with a direct verification in formula (4.12) with the first equality in (4.21) and relation (4.22) taken into account. This ends the proof of the theorem itself.

Remark 4. Using the identity $M(\lambda)\overline{M}(\bar{\lambda}) = -I$, we can show that the realness condition forbids two situations in the considered dressing procedure: $a = \bar{a}$ (easy plane), $\operatorname{Im}\lambda_j = 0$, $|\lambda_j| > a$ and $a = -\bar{a}$ (easy axis), $\operatorname{Im}\lambda_j = 0$. In fact, the points λ_j as points of the surface Γ are fixed points of the involution $\lambda \to \bar{\lambda}$ in both these cases. Therefore, assuming that $\sigma_2\bar{\Psi}(\bar{\lambda}) = \Psi(\lambda)M(\lambda)$, we obtain the contradiction

$$\Psi(\lambda_j)\begin{pmatrix} 1 \\ A_j \end{pmatrix} = 0 \quad\Longrightarrow\quad \bar{\Psi}(\lambda_j)\begin{pmatrix} 1 \\ \bar{A}_j \end{pmatrix} = 0 \quad\Longrightarrow\quad \Psi(\lambda_j)M(\lambda_j)\begin{pmatrix} 1 \\ \bar{A}_j \end{pmatrix} = 0,$$

consequently

$$M(\lambda_j)\begin{pmatrix} 1 \\ \bar{A}_j \end{pmatrix} = \varkappa\begin{pmatrix} 1 \\ A_j \end{pmatrix} \quad\Longrightarrow\quad -\begin{pmatrix} 1 \\ \bar{A}_j \end{pmatrix} = \varkappa\overline{M}(\lambda_j)\begin{pmatrix} 1 \\ A_j \end{pmatrix},$$

and therefore

$$M(\lambda_j)\begin{pmatrix} 1 \\ \bar{A}_j \end{pmatrix} = -\frac{1}{\bar{\varkappa}}\begin{pmatrix} 1 \\ A_j \end{pmatrix} \quad\Longrightarrow\quad |\varkappa|^2 = -1.$$

Remark 5. As already mentioned, the "dressed" function Ψ given by (4.15) satisfies reduction identity (2.11) with the same matrix $\sigma_0(\lambda)$ as the seed function $\Psi_0(\lambda)$. As we now verify, a similar statement also holds for realness reduction (2.17). More precisely, let the conditions in Theorem 4 hold. Then the function Ψ corresponding to solution (4.12)–(4.14) satisfies identity (2.17) with the matrix $M(\lambda) \cong M_0(\lambda)$. In fact, considering the function $\widetilde{\Psi}(\lambda) = \sigma_2\bar{f}(\bar{\lambda})\sigma_2\Psi_0(\lambda)$ together with the function $\Psi(\lambda) = f(\lambda)\Psi_0(\lambda)$[2], we can easily verify that under conditions (4.17) or (4.18), both these functions have the same set of Riemann problem data (the function $\widetilde{\Psi}$ satisfies the same vector equalities of system (4.4)). Therefore, $\widetilde{\Psi}(\lambda) = \Phi\Psi(\lambda)$, and the matrix Φ is independent of λ. On the other hand, $\widetilde{\Psi}(\lambda) = \sigma_2\bar{\Psi}(\bar{\lambda})M_0^{-1}(\lambda)$, and therefore

$$\widetilde{\Psi} \quad\longleftrightarrow\quad \widetilde{S} = S \quad\longleftrightarrow\quad \Psi \quad\Longrightarrow\quad S = \Phi S\Phi^{-1}. \tag{4.23}$$

Further, the functions $\widetilde{\Psi}$ and Ψ satisfy reduction identity (2.11) with the same matrix $\sigma_0(\lambda)$. Therefore,

$$\sigma_3\Phi\sigma_3 = \Phi \quad\Longleftrightarrow\quad \Phi = \operatorname{diag}(c, d).$$

Comparing the last relation with (4.23), we immediately conclude that $c = d$, i.e., $\Phi = cI$, and consequently

$$\sigma_2\bar{\Psi}(\bar{\lambda}) = c\Psi(\lambda)M_0(\lambda),$$

which is equivalent to $M(\lambda) \cong M_0(\lambda)$.

5. Dressing procedure: Soliton solutions

The simplest application of the scheme proposed in the preceding section is for dressing two types of "vacuum" Ψ functions:

$$\Psi_0(\lambda, x, t) = e^{-i\sigma_3\lambda x + 2i\sigma_3(\lambda^2 - a^2)t}, \qquad \vec{S}_0 = (0, 0, 1), \tag{5.1a}$$

$$\Psi_0(\lambda, x, t) = e^{-i\sigma_1\sqrt{\lambda^2 - a^2}(x - 2\lambda t)}, \qquad \vec{S}_0 = (1, 0, 0). \tag{5.1b}$$

As already mentioned in Sec. 1, the solutions thus obtained possibly contain all previously known solutions of the LL equation expressed in terms of elementary functions. We again note that we do not pretend to obtain any new physical results in this section. Our only purpose here is methodological: to confirm the validity of the developed approach using the simplicity of description and scope of the known effects. In this regard, assuming that all the facts below are well known to the specialist in ferromagnetism theory, we omit numerous priority references, which the reader can find in [9] if necessary.

It is convenient to introduce the following "nonphysical" conventional terminology. We call solutions of the LL equation obtained as a result of applying dressing procedure (4.12)–(4.14) once to seed solution (5.1a) S_3 solitons and solutions similarly obtained from seed solution (5.1b) S_1 solitons.

[2] The factor $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \delta_0 \end{smallmatrix}\right)$ is included in $f(\lambda)$.

5.1. The S_3 solitons. The matrices $\sigma_0(\lambda)$ and $M_0(\lambda)$ for seed solution (5.1a) are

$$\sigma_0(\lambda) \equiv \sigma_3, \qquad M_0(\lambda) \equiv \sigma_2.$$

Theorem 4 then leads to two types of conditions on the parameters (λ_j, A_j):

$$\lambda_1 = \bar{\lambda}_2 \equiv \mu, \quad \operatorname{Im}\mu > 0, \qquad A_1 = -\frac{1}{\bar{A}_2} \equiv A, \quad A \in \mathbb{C} \setminus \{0\}, \tag{5.2a}$$

$$\operatorname{Im}\lambda_j = 0, \quad |\lambda_j| < a, \qquad A_j = e^{2i\varphi_j}, \quad \varphi_j \in \mathbb{R}, \quad j = 1, 2 \tag{5.2b}$$

(the second type is possible only for $a = \bar{a}$). According to conditions (5.2a) and (5.2b), for the matrix Q_0, we have (see (4.12))

$$Q_0 \cong \begin{pmatrix} \sqrt{\bar{\mu}^2 - a^2} + |E|^2\sqrt{\mu^2 - a^2} & (\bar{\mu} - \mu)\bar{E} \\ E(\bar{\mu} - \mu) & \sqrt{\mu^2 - a^2} + |E|^2\sqrt{\bar{\mu}^2 - a^2} \end{pmatrix}, \tag{5.3a}$$

$$Q_0 \cong \begin{pmatrix} i\sqrt{a^2 - \lambda_2^2}E_1 - i\bar{E}_1\sqrt{a^2 - \lambda_1^2} & \bar{E}_2(\lambda_1 - \lambda_2) \\ E_2(\lambda_2 - \lambda_1) & iE_1\sqrt{a^2 - \lambda_1^2} - i\bar{E}_1\sqrt{a^2 - \lambda_2^2} \end{pmatrix}, \tag{5.3b}$$

where

$$E(x,t) = Ae^{2i\mu x - 4i(\mu^2 - a^2)t},$$

$$E_1(x,t) = e^{i(\lambda_2 - \lambda_1)x - 2i(\lambda_2^2 - \lambda_1^2)t + \varphi_2 - \varphi_1},$$

$$E_2(x,t) = e^{i(\lambda_1 + \lambda_2)x - 2i(\lambda_2^2 + \lambda_1^2 - 2a^2)t + \varphi_2 + \varphi_1}.$$

Substituting (5.3a) and (5.3b) in (4.14), we obtain the explicit formulas for two types of S_3 solitons (the second type is possible only in the easy-plane case):

$$\begin{aligned}
&S_3(x,t) = 1 - \frac{4\eta^2}{|\mu|^2 - a^2 + |\mu^2 - a^2|\cosh 2\beta(x,t)}, \\
&S_1 - iS_2 = 4i\eta\sqrt{|\mu^2 - a^2|}\frac{e^{-i\theta(x,t)}\big(\cos\varphi\cosh\beta(x,t) - i\sin\varphi\sinh\beta(x,t)\big)}{|\mu|^2 - a^2 + |\mu^2 - a^2|\cosh 2\beta(x,t)}, \\
&\mu = \xi + i\eta, \qquad \varphi = \frac{1}{2}\arg(\mu^2 - a^2), \\
&\beta(x,t) = 2\eta(x - 4\xi t) - \log|A|, \\
&\theta(x,t) = 2\xi x - 4(\xi^2 - \eta^2 - a^2)t + \arg A;
\end{aligned} \tag{5.4a}$$

for the matrix Q_0 from (5.3a) and

$$
\begin{aligned}
&S_1(x,t) = S_1^0(x,t)\cos\theta_2(x,t) - S_2^0(x,t)\sin\theta_2(x,t),\\
&S_2(x,t) = S_1^0(x,t)\sin\theta_2(x,t) + S_2^0(x,t)\cos\theta_2(x,t),\\
&S_3(x,t) = 1 + \frac{(\lambda_1-\lambda_2)^2}{\sqrt{(a^2-\lambda_1^2)(a-\lambda_2^2)}\cos 2\theta_1(x,t) + \lambda_1\lambda_2 - a^2},\\
&\theta_1(x,t) = (\lambda_2-\lambda_1)\big(x - 2(\lambda_2+\lambda_1)t\big) + \varphi_2 - \varphi_1,\\
&\theta_2(x,t) = (\lambda_1+\lambda_2)\big(x - 2(\lambda_2+\lambda_1)t\big) + 4(a^2+\lambda_1\lambda_2)t + \varphi_2 + \varphi_1,\\
&S_1^0(x,t) = (\lambda_2-\lambda_1)\sin\theta_1(x,t)\frac{\sqrt{a^2-\lambda_1^2}+\sqrt{a^2-\lambda_2^2}}{\sqrt{(a^2-\lambda_1^2)(a^2-\lambda_2^2)}\cos 2\theta_1(x,t) + \lambda_1\lambda_2 - a^2},\\
&S_2^0(x,t) = (\lambda_2-\lambda_1)\cos\theta_1(x,t)\frac{\sqrt{a^2-\lambda_1^2}+\sqrt{a^2-\lambda_2^2}}{\sqrt{(a^2-\lambda_1^2)(a^2-\lambda_2^2)}\cos 2\theta_1(x,t) + \lambda_1\lambda_2 - a^2}
\end{aligned}
\tag{5.4b}
$$

for the matrix Q_0 from (5.3b).

Solution (5.4a) is a solitary wave moving with the velocity $v = 4\xi$. This motion is accompanied by a uniform rotation of $\vec{S}$ in the plane (S_1, S_2) with the frequency

$$
\omega = 4\xi^2 + 4\eta^2 + 4a^2 \equiv 4|\mu|^2 + 4a^2. \tag{5.5}
$$

The velocity v and the frequency ω are two independent physical parameters, which together with the initial position $x_0 = (\log|A|)/2\eta$ and the initial rotation phase $\theta_0 = \arg A$ uniquely characterize the considered soliton solution. We note that in the easy-plane case $a^2 > 0$, the rotation of $\vec{S}$ is always present, while this rotation might be absent in the easy-axis case (the solitary spin wave). The condition for the absence of the rotation has the form $\xi^2 + \eta^2 = -a^2$. Hence, the maximum velocity of solitary spin waves is $4\sqrt{-a^2}$. If this value is exceeded, then $\vec{S}$ must rotate in the plane (S_1, S_2).

Solution (5.4b) is a running periodic wave with the phase velocity $v_\Phi = 2(\lambda_1 + \lambda_2)$. Similarly to case (5.4a), there is rotation in the plane (S_1, S_2) with the frequency $\Omega = 4(a^2 + \lambda_1\lambda_2)$. Because $|\lambda_j| < a$, the rotation is unavoidable. We again note that this type of solution is possible only in the easy-plane case. We mention the interesting fact that the S_3 component of solution (5.4b) can be formally obtained from the S_3 component of solution (5.4a) by setting

$$
\eta = i\frac{\lambda_1-\lambda_2}{2}, \qquad \xi = \frac{\lambda_1+\lambda_2}{2} \tag{5.6}
$$

in the latter. Moreover, under such conditions, the frequency ω transforms into Ω. Hence, relying only on the information about the third component of $\vec{S}$ and about the rotation frequency in the plane (S_1, S_2), we might draw the wrong conclusion about the possibility that solutions of type (5.4b) periodic in x and t also exist in the easy-axis case. With the complete system of formulas (5.4a), we avoid this risk because the first two components of the vector $\vec{S}$ become imaginary under conditions (5.6).

Formulas (5.4) show that S_3 solitons also include breather-type solutions (immovable formations oscillating in time)

$$
\begin{aligned}
&S_1(x,t) = 4\eta\sqrt{|\eta^2+a^2|}\frac{\cosh 2\eta(x-x_0)\sin(4(\eta^2+a^2)t+\theta_0)}{\eta^2-a^2+|\eta^2+a^2|\cosh 4\eta(x-x_0)},\\
&S_2(x,t) = -4\eta\sqrt{|\eta^2+a^2|}\frac{\cosh 2\eta(x-x_0)\cos(4(\eta^2+a^2)t+\theta_0)}{\eta^2-a^2+|\eta^2+a^2|\cosh 4\eta(x-x_0)},\\
&S_3(x,t) = 1 - \frac{4\eta^2}{\eta^2-a^2+|\eta^2+a^2|\cosh 4\eta(x-x_0)};
\end{aligned}
\tag{5.7}
$$

at $\xi = 0$ and

$$S_1(x,t) = -4\lambda_0\sqrt{a^2-\lambda_0^2}\frac{\sin(2\lambda_0 x+\theta_1)\cos(4(a^2-\lambda_0^2)t+\theta_2)}{(a^2-\lambda_0^2)\cos(4\lambda_0 x+2\theta_1)-\lambda_0^2-a^2},$$

$$S_2(x,t) = 4\lambda_0\sqrt{a^2-\lambda_0^2}\frac{\sin(2\lambda_0 x+\theta_1)\sin(4(a^2-\lambda_0^2)t+\theta_2)}{(a^2-\lambda_0^2)\cos(4\lambda_0 x+2\theta_1)-\lambda_0^2-a^2}, \tag{5.8}$$

$$S_3(x,t) = 1+\frac{4\lambda_0^2}{(a^2-\lambda_0^2)\cos(4\lambda_0 x+2\theta_1)-\lambda_0^2-a^2}.$$

for $\lambda_1 = -\lambda_2 \equiv \lambda_0$ and $|\lambda_0| < a$.

For all parameter values $\mu \neq a$ and $A \in \mathbb{C}\setminus\{0\}$, formulas (5.4a) describe the solutions of the LL equation characterized by the large-$|x|$ behavior

$$\vec{S} \to (0,0,1), \quad |x| \to \infty. \tag{5.9}$$

But in the easy-axis case, there is a bound for the parameters μ and A such that condition (5.9) is violated. Assuming that $a = -\bar{a}$, we set

$$|A|^2 = -\frac{1}{4\gamma a^2}|\mu^2-a^2|, \quad \gamma > 0,$$

$$\arg A = -\frac{1}{2}\arg(\mu^2-a^2)+\psi, \quad \psi \in \mathbb{R},$$

and let μ tend to a. Then formulas (5.4a) result in the formulas

$$S_1(x,t) = \frac{\sin\psi}{\cosh(2|a|+\Delta)}, \qquad S_2(x,t) = \frac{-\cos\psi}{\cosh(2|a|+\Delta)}, \qquad S_3(x,t) = \tanh(2|a|x+\Delta), \tag{5.10}$$

where $\Delta = (\log\gamma)/2$. This solution is a classical "domain wall." We have $\vec{S} \to (0,0,\pm 1)$ as $x \to \pm\infty$.

In Sec. 5, we need the expressions for the matrix Q_0 and function Ψ corresponding to solution (5.10). The corresponding formulas are easily obtained by taking the limit in formulas (5.3b) and (4.15) and have the forms

$$Q_0 \cong \begin{pmatrix} -\sqrt{\gamma}c_0 e^{2|a|x+i\psi} & i \\ ic_0 e^{2i\psi} & -\sqrt{\gamma}e^{2|a|x+i\psi} \end{pmatrix}, \tag{5.11}$$

$$\Psi(\lambda) = D_1(\lambda)Q_0 D(\lambda)\Psi_0(\lambda), \qquad c_0 = \lim_{\mu\to a}\sqrt{\frac{\bar{\mu}^2-a^2}{\mu^2-a^2}}. \tag{5.12}$$

5.2. The S_1 solitons. Because

$$e^{-i\sigma_1\sqrt{\lambda^2-a^2}(x-2\lambda t)} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} e^{-i\sigma_3\sqrt{\lambda^2-a^2}(x-2\lambda t)} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

we can take the function

$$\Psi_0(\lambda,x,t) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} e^{-i\sigma_3\sqrt{\lambda^2-a^2}(x-2\lambda t)}$$

as Ψ_0 (and this turns out to be convenient). The corresponding matrices $\sigma_0(\lambda)$ and $M_0(\lambda)$ are

$$\sigma_0(\lambda) \equiv \sigma_1, \qquad M_0(\lambda) \equiv -\sigma_2,$$

and two types of realness conditions are

$$\lambda_1 = \bar{\lambda}_1 \equiv \mu, \quad \operatorname{Im}\mu > 0, \qquad A_1 = -\frac{1}{\bar{A}_2} \equiv A, \quad A \in \mathbb{C} \setminus \{0\}, \tag{5.13a}$$

$$\operatorname{Im}\lambda_j = 0, \quad |\lambda_j| < a, \qquad A_j = -\bar{A}_j, \quad j = 1,2 \tag{5.13b}$$

(the second type of conditions requires $a = \bar{a}$).

From representation (4.12) for the matrix Q_0 in cases (5.13a) and (5.13b), we respectively obtain

$$\begin{aligned}
&Q_0 \cong \begin{pmatrix} C_+\sqrt{\bar{\mu}^2 - a^2} + C_-\sqrt{\mu^2 - a^2} & S_+(\bar{\mu} - \mu) \\ S_-(\bar{\mu} - \mu) & C_+\sqrt{\mu^2 - a^2} + C_-\sqrt{\bar{\mu}^2 - a^2} \end{pmatrix}, \\
&C_\pm = \cosh\beta(x,t) \pm \cos\theta(x,t), \qquad S_\pm = -\sinh\beta(x,t) \pm i\sin\theta(x,t), \\
&\beta(x,t) = \operatorname{Re}\bigl(2i\sqrt{\mu^2 - a^2}(x - 2\mu t)\bigr) + \log|A|, \\
&\theta(x,t) = \operatorname{Im}\bigl(2i\sqrt{\mu^2 - a^2}(x - 2\mu t)\bigr) + \arg A;
\end{aligned} \tag{5.14a}$$

$$\begin{aligned}
&Q_0 \cong \begin{pmatrix} i\widetilde{E}_1^+\widetilde{E}_2^-\sqrt{a^2 - \lambda_2^2} - i\widetilde{E}_2^+\widetilde{E}_1^-\sqrt{a^2 - \lambda_1^2} & \widetilde{E}_1^+\widetilde{E}_2^+(\lambda_1 - \lambda_2) \\ \widetilde{E}_1^-\widetilde{E}_2^-(\lambda_2 - \lambda_1) & i\widetilde{E}_1^+\widetilde{E}_2^-\sqrt{a^2 - \lambda_1^2} - i\widetilde{E}_2^+\widetilde{E}_1^-\sqrt{a^2 - \lambda_2^2} \end{pmatrix}, \\
&\widetilde{E}_j^\pm = 1 \pm E_j(x,t), \qquad E_j(x,t) = i\gamma_j e^{-2\sqrt{a^2 - \lambda_j}(x - 2\lambda_j t)}, \qquad \gamma_j = \operatorname{Im}A_j, \quad j = 1,2.
\end{aligned} \tag{5.14b}$$

The solutions of the LL equation themselves can be reconstructed from formulas (5.14) as

$$S = Q_0\sigma_1 Q_0^{-1} \tag{5.15}$$

and have a rather combersome form for arbitrary μ and λ_j, which we omit here especially because all the physical information can be easily obtained directly from formulas (5.14) and (5.15).

In the general position case, solution (5.14a) describes the soliton ($\vec{S} \to (1,0,0)$, $|x| \to \infty$), localized in x and moving with a velocity determined from the linear function $\beta(x,t)$. This motion, unlike the trivial rotation of S_3 solitons, is accomplished by the time-precession of the vector $\vec{S}$. The precession frequency can be calculated using the linear function $\theta(x,t)$. The corresponding breather solution (immovable bion) is obtained from (5.14a) under the conditions $\operatorname{Re}\mu = 0$ and $|\mu| > |a|$ (in the easy-axis case). The formulas for $\vec{S}$ are significantly simplified in this case, and we can present them. At $\mu_0 = i\eta$, for $\eta > 0$ (easy plane) and $\eta > |a|$ (easy axis), we have

$$\begin{aligned}
&S_1(x,t) = \frac{\eta^2\cosh^2(kx + \beta_0) - a^2\cos^2(\omega t + \theta_0) - 2\eta^2}{\eta^2\cosh^2(kx + \beta_0) + a^2\cos^2(\omega t + \theta_0)}, \\
&S_2(x,t) = 2\eta^2\frac{\sin(\omega t + \theta_0)\sinh(kx + \beta_0)}{\eta^2\cosh^2(kx + \beta_0) + a^2\cos^2(\omega t + \theta_0)}, \\
&S_3(x,t) = -2\eta\sqrt{\eta^2 + a^2}\frac{\cos(\omega t + \theta_0)\sinh(kx + \beta_0)}{\eta^2\cosh^2(kx - \beta_0) + a^2\cos^2(\omega t + \theta_0)}, \\
&k = -2\sqrt{\eta^2 + a^2}, \qquad \omega = 4\eta\sqrt{\eta^2 + a^2}, \qquad \beta_0 = \log|A|, \qquad \theta_0 = \arg A.
\end{aligned} \tag{5.16}$$

We again emphasize the time dependence of bion (5.16), which is nontrivial compared with the S_3 breather (see relations (5.7)).

We complete the analysis of formula (5.14a) with the following observation. There is a solution decreasing in t and oscillating in x in the easy-axis case. To obtain this solution, it suffices to set $\mu = i\eta$, $0 < \eta < |a|$ in formula (5.14a). Then

$$\beta(x,t) = 4\eta\sqrt{|a|^2 - \eta^2}\,t + \beta_0, \qquad \theta(x,t) = 2\sqrt{|a|^2 - \eta^2}\,x + \theta_0.$$

The formulas for the solution itself are (for $a = -\bar{a}$)

$$\begin{aligned}
S_1(x,t) &= \frac{a^2\cosh^2(\Omega t + \beta_0) - \eta^2\cos^2(kx + \theta_0) + 2\eta^2}{a^2\cosh^2(\Omega t + \beta_0) + \eta^2\cos^2(kx + \theta_0)}, \\
S_2(x,t) &= -\frac{2\eta^2\sinh(\Omega t + \beta_0)\sin(kx + \theta_0)}{a^2\cosh^2(\Omega t + \beta_0) + \eta^2\cos^2(kx + \theta_0)}, \\
S_3(x,t) &= -\frac{2\eta\sqrt{|a|^2 - \eta^2}\cosh(\Omega t + \beta_0)\sin(kx + \theta_0)}{a^2\cosh^2(\Omega t + \beta_0) + \eta^2\cos^2(kx + \theta_0)}, \\
k &= 2\sqrt{|a|^2 - \eta^2}, \qquad \Omega = 4\eta\sqrt{|a|^2 - \eta^2}.
\end{aligned} \tag{5.17}$$

Proceeding to discuss S_1 solitons, we note that the structure of formulas (5.14b) is much simpler than that of formulas (5.14a). There are no oscillations in relations (5.14b). Essentially, these formulas describe the interaction of two simpler solutions of the one-soliton type. This one-soliton solution can be obtained from formula (5.14b) as a result of the limit transition

$$\lambda_1 \to a, \qquad A_1 \to 0. \tag{5.18}$$

The matrix Q_0 is then greatly simplified and becomes

$$Q_0 \cong \begin{pmatrix} i\widetilde{E}_2^-\sqrt{a^2 - \lambda_2^2} & \widetilde{E}_2^+(a - \lambda_2) \\ \widetilde{E}_2^-(\lambda_2 - a) & -i\widetilde{E}_2^+\sqrt{a^2 - \lambda_2^2} \end{pmatrix},$$

and we can write the solution itself as

$$\begin{aligned}
S_1(x,t) &= -\tanh\bigl(2\sqrt{a^2 - \lambda^2}(x - 2\lambda t) + \Delta\bigr), \\
S_2(x,t) &= \mp\frac{\lambda}{a}\frac{1}{\cosh(2\sqrt{a^2 - \lambda^2}(x - 2\lambda t) + \Delta)}, \\
S_3(x,t) &= \pm\frac{\sqrt{a^2 - \lambda^2}}{a}\frac{1}{\cosh(2\sqrt{a^2 - \lambda^2}(x - 2\lambda t) + \Delta)},
\end{aligned} \tag{5.19}$$

where $\lambda \equiv \lambda_2$, $\Delta = -\log|\gamma_2|$, and the upper and lower signs respectively correspond to $\gamma_2 > 0$ and $\gamma_2 < 0$ in the equalities for S_2 and S_3. Solution (5.19) is a moving domain wall ($\vec{S} \to (\mp 1, 0, 0)$, $x \to \pm\infty$).[3] Relations (5.14b) might then be treated as describing the interaction of two domain walls. But developing this standpoint further here seems to be unreasonable. We obtain the general N-soliton formulas for solutions (5.19) in Sec. 10. It is therefore natural to defer studying the mutual interaction between S_1 solitons of type (5.19) to Sec. 10.

All solutions considered above are characterized by either exponential or trigonometric behavior in x. But the solutions of the LL equation with a power-law behavior in x ("rational" solitons) are also contained in formulas (5.4) and (5.14a) as different limit cases.

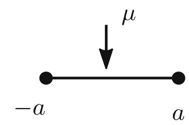

Fig. 1

5.3. Rational solitons in the easy-plane case. Let

$$\eta \to 0, \qquad |\xi| < a \quad (a^2 > 0), \qquad \theta_0 = O(1), \qquad x_0 = O(1) \tag{5.20}$$

in formula (5.4a). Then $\varphi \to \pi/2$ (see Fig. 1).[4] More precisely,

$$\varphi = \frac{\pi}{2} + \frac{1}{2}\arctan\frac{2\xi\eta}{\xi^2 - \eta^2 - a^2} = \frac{\pi}{2} - \frac{\xi}{a^2 - \xi^2}\eta + o(\eta),$$

and consequently

$$\cos\varphi = \frac{\xi}{a^2 - \xi^2}\eta + O(\eta^3), \quad \sin\varphi = 1 + O(\eta^2). \tag{5.21}$$

Estimates of the other quantities in formulas (5.4a) can also be obtained simply:

$$\begin{aligned}
&|\mu^2 - a^2| = \sqrt{(\xi^2 - \eta^2 - a^2)^2 + 4\xi^2\eta^2} = a^2 - \xi^2 + \eta^2\left(1 + \frac{2\xi^2}{a^2 - \xi^2}\right) + O(\eta^4),\\
&|\mu|^2 - a^2 = \xi^2 - a^2 + \eta^2, \qquad \theta = 2\xi x - 4(\xi^2 - a^2)t + \theta_0 + O(\eta^2),\\
&\sinh\beta = 2\eta(x - 4\xi t - x_0) + O(\eta^3), \qquad \cosh\beta = 1 + O(\eta^2),\\
&\cosh 2\beta = 1 + 8\eta^2(x - 4\xi t - x_0)^2 + O(\eta^4).
\end{aligned} \tag{5.22}$$

From relations (5.21) and (5.22), we find that solution (5.4a) in limit (5.20) transforms into a solution of the form

$$\begin{aligned}
S_3(x,t) &= 1 - \frac{2(a^2 - \xi^2)}{a^2 + 4(a^2 - \xi^2)^2(x - 4\xi t - x_0)^2},\\
S_1 + iS_2 &= -\frac{4i\sqrt{a^2 - \xi^2}\big(\xi\eta/2 + i(a^2 - \xi^2)(x - 4\xi t - x_0)\big)}{a^2 + 4(a^2 - \xi^2)^2(x - 4\xi t - x_0)^2} \times\\
&\quad \times e^{2i\xi x - 4i(\xi^2 - a^2)t + \theta_0}.
\end{aligned} \tag{5.23}$$

Again, this solution is a soliton moving uniformly and rotating uniformly in the plane (S_1, S_2). But unlike the "exponential" case, there is a restriction $|v| < 4a$ on the velocity.

5.4. Rational solitons in the easy-axis case. Now starting from S_1 soliton (5.14), (5.15) and assuming that $a = -\bar{a}$, we consider the limit

$$\begin{aligned}
&\varepsilon \downarrow 0, \qquad \mu = a + O(\varepsilon),\\
&\sqrt{\mu^2 - a^2} = \varepsilon e^{i\gamma}, \qquad \log|A| = -\varepsilon\,\mathrm{Re}(2ie^{i\gamma}x_0), \qquad \arg A = -\varepsilon\,\mathrm{Im}(4|a|e^{i\gamma}t_0),
\end{aligned}$$

[3] We again emphasize that this solution is obtained only in the easy-plane case.

[4] The choice of the cut in Fig. 1 agrees with the action of the anti-involution $\lambda \to \bar{\lambda}$ (see Sec. 2).

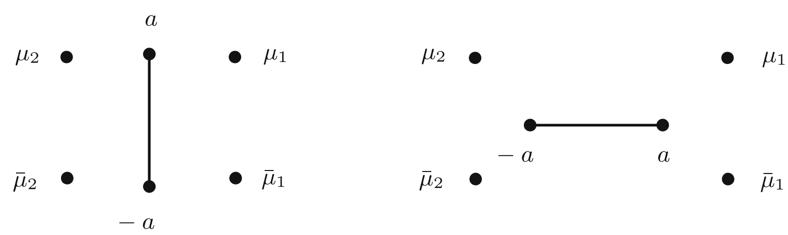

Fig. 2

where $\gamma, x_0, t_0 \in \mathbb{R}$. Obviously, the limit expression for the matrix Q_0 in (5.14a) has the form

$$Q_0 \cong \begin{pmatrix} 2e^{-i\gamma} & -4i|a|e^{-i\gamma}\big(i(x-x_0)-2|a|(t-t_0)\big) \\ 4i|a|e^{i\gamma}\big(i(x-x_0)+2|a|(t-t_0)\big) & 2e^{i\gamma} \end{pmatrix}. \tag{5.24}$$

Substituting this expression in (5.15), we obtain the formulas for the corresponding solution of the LL equation:

$$\begin{gathered} S_3(x,t) = \frac{4|a|(x-x_0)}{1+4|a|^2\big((x-x_0)^2+4|a|^2(t-t_0)^2\big)}, \\ S_1(x,t) - iS_2(x,t) = \frac{e^{-2i\gamma}\big[1-4|a|^2\big((x-x_0)+2i|a|(t-t_0)\big)^2\big]}{1+4|a|^2\big((x-x_0)^2+4|a|^2(t-t_0)^2\big)}. \end{gathered} \tag{5.25}$$

Unlike (5.23), this is a breather-type solution and is characterized by completely rational behavior in both variables. We again emphasize that a solution of type (5.25) exists only in the easy-axis case, while a solution of type (5.23) exists only in the easy-plane case.

6. Dressing procedure: Interaction of soliton solutions

Essentially, our developed dressing procedure is inductive: after applying it once, we immediately obtain a pair $(\vec{S}, \Psi)$, which can be dressed again using the same formulas (4.12)–(4.15), now considering $\Psi_0 \equiv \Psi$, and so on. It is here essential that by virtue of Remark 5, the "reduction" matrices $\sigma(\lambda)$ and $M(\lambda)$ are preserved for the whole iteration series. That is, we have the same set of conditions for the transformation parameters (λ_j, A_j) at each step. Therefore, if a single application of the dressing procedure yields the description and classification of all elementary excitations produced by the original "vacuum" (S_0, Ψ_0), then all succeeding iterations allow describing all possible interaction processes between these elementary excitations. In this section, we illustrate the power of this approach using an example of pairwise interaction between S_3 solitons.

6.1. Interaction of two S_3 solitons of type (5.4a). The two-soliton solution corresponding to the schemes in Fig. 2 can be obtained using two methods of double dressing:

$$(\Psi_0, \vec{S}_0)\big|_{S_0=(0,0,1)} \xrightarrow{(\mu_1, A_1)} (\Psi_1, \vec{S}_1) \xrightarrow{(\mu_2, A_2)} (\Psi_{12}, \vec{S}_{12}), \tag{6.1a}$$

$$(\Psi_0, \vec{S}_0)\big|_{S_0=(0,0,1)} \xrightarrow{(\mu_2, A_2)} (\Psi_2, \vec{S}_2) \xrightarrow{(\mu_2, A_1)} (\Psi_{21}, \vec{S}_{21}). \tag{6.1b}$$

It is obvious that $\vec{S}_{12} = \vec{S}_{21}$ because these two solutions have the same set of Riemann problem data. To calculate the effect of the action of the soliton (μ_1, A_1) on the soliton (μ_2, A_2), we proceed with (6.1a). We find the asymptotic behavior of Ψ_1 under the conditions $t \to \pm\infty$ and $x - 4\xi_2 t = \text{const}$. Assuming that $\xi_2 < \xi_1$ for definiteness, we have

$$\vec{X}_1 = \sqrt{\mu_1^2 - a^2} A_1 e^{-\eta_1(x-4\xi_2 t)-4\eta(\xi_2-\xi_1)t}\left[\begin{pmatrix}0\\1\end{pmatrix} + o(1)\right],$$

$$\vec{X}_2 = (\bar{\mu}_1 - a) e^{-\eta_1(x-4\xi_2 t)-4\eta(\xi_2-\xi_1)t}\left[\begin{pmatrix}1\\0\end{pmatrix} + o(1)\right]$$

in this limit (see relations (4.9), (4.10)). Consequently,

$$W = e^{-\eta_1(x-4\xi_2 t)-4\eta(\xi_2-\xi_1)t}[I + o(1)]\begin{pmatrix}0 & \bar{\mu}_1 - a\\ A_1\sqrt{\mu_1^2 - a^2} & 0\end{pmatrix},$$

$$V = e^{-\eta_1(x-4\xi_2 t)-4\eta_1(\xi_2-\xi_1)t}\left[\begin{pmatrix}\dfrac{1}{\bar{\mu}_1 - a} & 0\\ 0 & \dfrac{1}{\mu_1 + a}\end{pmatrix} + o(1)\right]\begin{pmatrix}0 & \bar{\mu}_1 - a\\ A_1\sqrt{\mu_1^2 - a^2} & 0\end{pmatrix},$$

and hence

$$VW^{-1} = \begin{pmatrix}\dfrac{1}{\bar{\mu}_1 - a} & 0\\ 0 & \dfrac{1}{\mu_1 + a}\end{pmatrix} + o(1). \tag{6.2}$$

Substituting these relations in (4.15), we obtain the sought asymptotic behavior of $\Psi_1(\lambda)$: as $t \to +\infty$ with $x - 4\xi_2 t = \text{const}$ and $\xi_2 < \xi_1$,

$$\Psi_1(\lambda) = \frac{\lambda - \bar{\mu}_1}{\bar{\mu}_1 - a}\left[\begin{pmatrix}1 & 0\\ 0 & \delta_0 \dfrac{\lambda - \mu_1}{\mu_1 + a}\dfrac{\bar{\mu}_1 - a}{\lambda - \bar{\mu}_1}\end{pmatrix} + o(1)\right]\Psi_0(\lambda). \tag{6.3}$$

Similar calculations for $x - 4\xi_2 t = \text{const}$ and $\xi_2 < \xi_1$ as $t \to -\infty$ result in

$$\Psi_1(\lambda) = \frac{\lambda - \mu_1}{\mu_1 - a}\left[\begin{pmatrix}1 & 0\\ 0 & \delta_0 \dfrac{\lambda - \bar{\mu}_1}{\lambda - \mu_1}\dfrac{\mu_1 - a}{\bar{\mu}_1 + a}\end{pmatrix} + o(1)\right]\Psi_0(\lambda). \tag{6.4}$$

Asymptotic behavior (6.3) shows that the second arrow in (6.1a) in the limit as $t \to +\infty$ with $x - 4\xi_2 t = \text{const}$ becomes the simple dressing of the "zero" seed solution $(\vec{S}_0, \Psi_0)$ characterized by the parameters

$$\mu_2, \qquad A_2^+ = A_2\delta_0 \frac{\bar{\mu}_1 - a}{\mu_1 + a}\frac{\mu_2 - \mu_1}{\mu_2 - \bar{\mu}_1}.$$

In other words, the solution $\vec{S}_{12}$ has an S_3 soliton of form (6.3) as an asymptotic form in this limit. This soliton is characterized by the velocity $v_2 = 4\xi_2$, the rotational frequency $\omega_2 = 4(\xi_2^2 + \eta_2^2 + a^2)$, the initial position

$$x_{02}^+ = \frac{1}{2\eta_2}\left(\log|A_2\delta_0| + \log\left|\frac{\bar{\mu}_1 - a}{\mu_1 + a}\frac{\mu_2 - \mu_1}{\mu_2 - \bar{\mu}_1}\right|\right),$$

and the initial rotation phase

$$\theta_{02}^+ = \arg A_2 + \arg\delta_0 + \arg\left(\frac{\bar{\mu}_1 - a}{\mu_1 + a}\frac{\mu_2 - \mu_1}{\mu_2 - \bar{\mu}_1}\right).$$

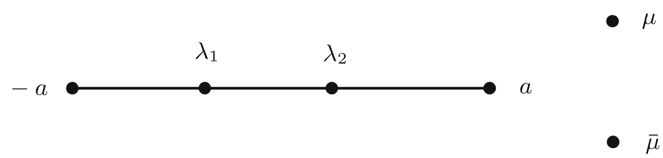

Fig. 3

Similarly, from asymptotic form (6.4), we find that in the limit $t \to -\infty$ with $x - 4\xi_2 t = \text{const}$, the solution $\vec{S}_{12}$ tends asymptotically to an S_3 soliton of form (6.3) characterized by the same velocity v_2 and the same frequency ω_2 but with different parameter values x_{02} and θ_{12}:

$$x_{02}^{-} = \frac{1}{2\eta_2}\left(\log|A_2\delta_0| + \log\left|\frac{\mu_1 - a}{\bar{\mu}_1 + a}\frac{\mu_2 - \bar{\mu}_1}{\mu_2 - \mu_1}\right|\right),$$

$$\theta_{02}^{-} = \arg A_2 + \arg\delta_0 + \arg\left(\frac{\mu_1 - a}{\bar{\mu}_1 + a}\frac{\mu_2 - \bar{\mu}_1}{\mu_2 - \mu_1}\right).$$

The action of the soliton (μ_1, A_1) on the soliton (μ_2, A_2) thus results in shifts of the mass center and initial rotation phase in the plane (S_1, S_2):

$$\begin{aligned} \Delta x_{02} &= x_{02}^{+} - x_{02}^{-} = \frac{1}{\eta_2}\log\left|\frac{\mu_2 - \mu_1}{\mu_2 - \bar{\mu}_1}\right|, \\ \Delta\theta_{02} &= \theta_{02}^{+} - \theta_{02}^{-} = -2\arg(\mu_1^2 - a^2) + 2\arg\frac{\mu_2 - \mu_1}{\mu_2 - \bar{\mu}_1}. \end{aligned} \tag{6.5}$$

To obtain the asymptotic behavior of the solution $\vec{S}_{12} = \vec{S}_{21}$ as $t \to \pm\infty$ with $x - 4\xi_1 t = \text{const}$, we must proceed with relations (6.1b). In this case, we obviously find that the asymptotic form of the solution $\vec{S}_{12}$ as $t \to \pm\infty$ is the S_3-soliton of form (5.4a), which is characterized by the velocity $v_1 = 4\xi_1$ and rotation frequency $\omega_1 = 4(\xi_1^2 + \eta_1^2 + a^2)$. The corresponding shifts of the mass center and the initial rotation phase are described by formulas analogous to (6.5):

$$\Delta x_{01} = \frac{1}{\eta_1}\log\left|\frac{\mu_1 - \bar{\mu}_2}{\mu_1 - \mu_2}\right|,$$

$$\Delta\theta_{01} = 2\arg(\mu_2^2 - a^2) + 2\arg\left(\frac{\mu_1 - \bar{\mu}_2}{\mu_1 - \mu_2}\right).$$

Concluding the analysis of this interaction case, we give the expression for the shift of the soliton mass centers in terms of the parameters v_j and ω_j:

$$\Delta x_{02} = -\frac{\eta_1}{\eta_2}\Delta x_{01} = \frac{1}{2\eta_1}\log\frac{2\omega_2 + 2\omega_1 - 16a^2 - v_1v_2 - \sqrt{4\omega_1 - 16a^2 - v_1^2}\sqrt{4\omega_2 - 16a^2 - v_2^2}}{2\omega_2 + 2\omega_1 - 16a^2 - v_1v_2 + \sqrt{4\omega_1 - 16a^2 - v_1^2}\sqrt{4\omega_2 - 16a^2 - v_2^2}}.$$

6.2. Interaction of an S_3 soliton of type (5.4a) with an S_3 soliton of type (5.4b): The easy-plane case. The scheme of the solution describing interaction of a type-(5.4a) S_3 soliton with a type-(5.4b) S_3 soliton in the easy-plane case is illustrated in Fig. 3. This solution can be realized as the double dressing:

$$(\Psi_0, \vec{S}_0)\big|_{\vec{S}_0=(0,0,1)} \xrightarrow{(\mu,A)} (\Psi_1, \vec{S}_1) \xrightarrow{(\lambda_1,\lambda_2,\varphi_1,\varphi_2)} (\Psi_{12}, \vec{S}_{12}). \tag{6.6}$$

Assuming that $\xi > (\lambda_1 + \lambda_2)/2$ and repeating the corresponding discussion in the preceding subsection, we again obtain formulas (6.3) in the limit $t \to +\infty$ with $x - 2(\lambda_1 + \lambda_2)t = \text{const}$ and formulas (6.4) as

$t \to -\infty$ with $x - 2(\lambda_1 + \lambda_2)t = \text{const}$. This allows concluding that the second arrow in (6.6) as $t \to \pm\infty$ with $x - 2(\lambda_1 + \lambda_2)t = \text{const}$ becomes the simple dressing (under conditions (5.2b)) of the "zero" seed solution $(\vec{S}_0, \Psi_0)$ characterized by the effective parameters

$$\lambda_j, \qquad \varphi_j^{\pm} = \varphi_j \mp \arg(\mu^2 - a^2) \pm 2\arg(\lambda_j - \mu), \qquad t \to \pm\infty.$$

Hence, as $t \to \pm\infty$ with $x - 2(\lambda_1 + \lambda_2)t = \text{const}$, the solution $\vec{S}_{12}$ has a periodic S_3-soliton of form (5.4b) as its asymptotic form. This soliton is characterized by the phase velocity $v_\Phi = 2(\lambda_1 + \lambda_2)$, the rotational frequency $\Omega = 4(a^2 + \lambda_1\lambda_2)$, and the initial motion and rotation phase values

$$\varphi_2^{\pm} - \varphi_1^{\pm} = \pm 2\arg\frac{\lambda_2 - \mu}{\lambda_1 - \mu},$$

$$\varphi_2^{\pm} + \varphi_1^{\pm} = \mp 2\arg(\mu^2 - a^2) \pm 2\arg(\lambda_2 - \mu)(\lambda_1 - \mu) + \varphi_2 + \varphi_1.$$

In other words, the action of an S_3 soliton of type (5.4a) on periodic wave (5.4b) is described by the formulas

$$\begin{aligned} \Delta(\varphi_2 - \varphi_1) &= 4\arg\frac{\lambda_2 - \mu}{\lambda_1 - \mu}, \\ \Delta(\varphi_2 + \varphi_1) &= -4\arg(\mu^2 - a^2) + 4\arg(\lambda_2 - \mu)(\lambda_1 - \mu). \end{aligned} \tag{6.7}$$

In terms of the physical parameters (v, ω) of the soliton and (v_Φ, Ω) of the periodic wave, we can write formulas (6.7) as

$$\Delta(\varphi_2 - \varphi_1) = 4\arg\frac{v_\Phi + 2\sqrt{v_\Phi^2/4 - \Omega + 4a^2} - \mu}{v_\Phi - 2\sqrt{v_\Phi^2/4 - \Omega + 4a^2} - \mu},$$

$$\Delta(\varphi_2 + \varphi_1) = -4\arg(\mu^2 - a^2) + 4\arg\left(\mu^2 - \frac{1}{2}v_\Phi\mu + \frac{1}{4}\Omega - a^2\right),$$

where

$$\mu = \frac{1}{4}v + i\sqrt{\frac{\omega}{4} - \frac{v^2}{16} - a^2}.$$

6.3. Interaction of an S_3 soliton of type (5.4a) with a domain wall: The easy-axis case. As shown in Sec. 5, domain wall (5.10) is the degenerate case ($\mu \to a$, $A \to 0$) of S_3 soliton (5.4a). Therefore, the case of interaction we consider here can be studied based on two interacting S_3 solitons in the limit

$$\mu_2 \to a, \qquad A_2 \equiv \frac{1}{2|a|\sqrt{\gamma}}\sqrt{\bar{\mu}_2^2 - a^2}e^{i\psi} \to 0 \tag{6.8}$$

taken in the appropriate formulas. The scheme for the solution $\vec{S}_{12}^0$ corresponding to this case is illustrated in Fig. 4.

The solution can be obtained as a result of the two sequences of dressing procedures and degenerations

$$(\Psi_0, \vec{S}_0) \xrightarrow{(\mu_1, A_1)} (\Psi_1, \vec{S}_1) \xrightarrow{(\mu_2, A_2)} (\Psi_{12}, \vec{S}_{12}) \xrightarrow[A_2 \to 0]{\mu_2 \to a} (\Psi_{12}^0, \vec{S}_{12}^0), \tag{6.9a}$$

$$(\Psi_0, \vec{S}_0) \xrightarrow{(\mu_2, A_2)} (\Psi_2, \vec{S}_2) \xrightarrow[A_2 \to 0]{\mu_2 \to a} (\Psi_2^0, \vec{S}_2^0) \xrightarrow{(\mu_1, A_1)} (\Psi_{12}^0, \vec{S}_{12}^0). \tag{6.9b}$$

Analogously to the preceding cases, to calculate the effect of the action of the soliton on the domain wall, we must compare the asymptotic forms as $t \to \pm\infty$ with $x = \text{const}$. This asymptotic behavior can be conveniently found from diagram (6.9a). Conversely, the action of the domain wall on the soliton (i.e., the asymptotic form $\vec{S}_{12}^0$ in the limit $t \to \pm\infty$ with $x - 4\xi_1 t = \text{const}$) can be more simply calculated using diagram (6.9b).

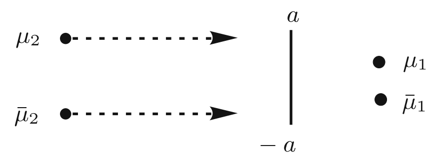

Fig. 4

6.3.1. Soliton action on a domain wall. We write the function $\Psi_1(\lambda)$ in the form (see formulas (4.1) and (4.4))

$$\Psi_1(\lambda) = f(\lambda)\Psi_0(\lambda) = \begin{pmatrix} \lambda(q_0+q_3)+p_0+p_3 & (q_1-iq_2)\sqrt{\lambda^2-a^2} \\ (q_1+iq_2)\sqrt{\lambda^2-a^2} & \lambda(q_0-q_3)+p_0-p_3 \end{pmatrix} e^{-i\sigma_3\lambda x+2i\sigma_3(\lambda^2-a^2)t}. \tag{6.10}$$

Substituting this function for Ψ_0 in formulas (4.12) and (4.13) (the second arrow in (6.9a)) and taking limit (6.8) (the third arrow in (6.9a)), we obtain explicit expressions for the elements of the matrix Q_0 corresponding to the solution $\vec{S}^0_{12}$:

$$\begin{aligned} Q^0_{11} &= -\Psi_{11}(a)\Psi_{22}(\bar{a})\sqrt{\gamma}\,2|a|e^{i\psi}c_0, \\ Q_{22} &= \frac{1}{c_0}Q^0_{11}, \\ Q^0_{12} &= -2i|a|\big(\Psi_1(a)\big)_{11}2|a|\sqrt{\gamma}e^{i\psi}c_0(q_1-iq_2)e^{|a|x}+2i|a|\Psi_{11}(a)\big(\Psi_1(\bar{a})\big)_{11}, \\ Q^0_{21} &= 2i|a|\big(\Psi_1(a)\big)_{22}2|a|\sqrt{\gamma}e^{i\psi}c_0(q_1+iq_2)e^{|a|x}+2i|a|[\Psi_1(a)]_{22}\big(\Psi_1(\bar{a})\big)_{22}c_0e^{2i\psi}, \end{aligned} \tag{6.11}$$

where

$$c_0 = \lim_{\mu\to a}\sqrt{\frac{\bar{\mu}^2-a}{\mu^2-a}} \qquad |c_0| = 1. \tag{6.12}$$

We now let $t \to +\infty$ with $x = \text{const}$. We can then use asymptotic form (6.3) for the function $\Psi_1(\lambda)$, which leads to a simplification of formulas (6.11):

$$Q^+_0 \cong \begin{pmatrix} -\left|\dfrac{\mu_1+a}{\mu_1-a}\right|e^{2|a|x}\sqrt{\gamma}e^{i\psi}c_0 & i \\ ic_0e^{2i\psi} & -\left|\dfrac{\mu_1+a}{\mu_1-a}\right|e^{2|a|x}\sqrt{\gamma}e^{i\psi} \end{pmatrix}. \tag{6.13}$$

In the considered limit, we have $\vec{S}_1 \to (0,0,1)$. Therefore, the asymptotic solution $\vec{S}^0_{12}$ has the form

$$\vec{S}^0_{12}Q^+_0\sigma_3(Q^+_0)^{-1}, \quad t\to+\infty, \quad x = \text{const}.$$

Comparing matrix (6.13) with representation (5.11) for the matrix Q_0 of the domain wall, we conclude that the asymptotic solution $\vec{S}^0_{12}$ as $t \to +\infty$ with $x = \text{const}$ is the domain wall with the parameters

$$\gamma^+ = \gamma\left|\frac{\mu_1+a}{\mu_1-a}\right|^2, \qquad \psi^+ = \psi. \tag{6.14}$$

Analogously, substituting asymptotic form (6.4) in (6.11), we conclude that as $t \to -\infty$ with $x = \text{const}$, the solution $\vec{S}^0_{12}$ is again the domain wall but with the parameters

$$\gamma^- = \gamma\left|\frac{\mu_1-a}{\mu_1+a}\right|^2, \qquad \psi^- = \psi.$$

Hence, the effect of the action of S_3 soliton (5.4a) with the velocity $v_1 = 4\,\mathrm{Re}\,\mu_1$ and frequency $\omega_1 = 4|\mu_1|^2 + 4a^2$ on domain wall (5.10) is described by the relations

$$\Delta^+ - \Delta^- = 2\log\left|\frac{\mu_1 + a}{\mu_1 - a}\right|, \qquad \psi^+ = \psi^-. \tag{6.15}$$

6.3.2. Action of a domain wall on a soliton. The result of the action of the first two arrows in (6.7) is described by formulas (5.10) and (5.12). These are a domain wall and the corresponding function Ψ, which are characterized by the parameters γ and ψ. Let $t \to +\infty$ with $x - 4\xi_1 t = \mathrm{const}$. Then $x \to +\infty$, and formulas (5.10) and (5.12) yield

$$\Psi_2^0(\lambda) = \begin{pmatrix} c_0(\lambda - a) & 0 \\ 0 & \lambda + a \end{pmatrix} e^{-i\sigma_3\lambda x + 2i\sigma_3(\lambda^2 - a^2)t}, \qquad \vec{S}_2^{\,0} = (0, 0, 1). \tag{6.16}$$

Hence, the third arrow in (6.9b) becomes the simple dressing under conditions (5.2a) with the effective parameters

$$\mu_1, \qquad A_1^+ = A_1 \frac{\mu_1 + a}{\mu_1 - a}\frac{1}{c_0}.$$

Therefore, as $t \to +\infty$ with $x - 4\xi_1 t = \mathrm{const}$, the solution $\vec{S}_{12}^{\,0}$ has an asymptotic form of S_3 soliton (5.4a) with the parameters

$$\begin{aligned} &v_1 = 4\xi_1, \qquad \omega_1 = 4|\mu_1|^2 + 4a^2, \qquad 2\chi = \arg c_0, \\ &x_0^+ = \frac{1}{2\eta_1}\log|A_1| + \frac{1}{2\eta_1}\log\left|\frac{\mu_1 + a}{\mu_1 - a}\right|, \qquad \theta_0^+ = \arg A_1 - 2\chi + \arg\frac{\mu_1 + a}{\mu_1 - a}. \end{aligned} \tag{6.17}$$

We now let $t \to -\infty$ with $x - 4\xi_1 t = \mathrm{const}$. Then $x \to -\infty$, and instead of (6.16), we have the formulas

$$\Psi_2^0(\lambda) = \sqrt{\lambda^2 - a^2}\begin{pmatrix} 0 & e^{-i\psi - i\chi} \\ e^{-i\psi + i\chi} & 0 \end{pmatrix} e^{-i\sigma_3\lambda x + 2i\sigma_3(\lambda^2 - a^2)t}, \qquad \vec{S}_2^{\,0} = (0, 0, -1). \tag{6.18}$$

We let Q_{sw}^0 denote the matrix Q_0 corresponding to the S_3 soliton $\vec{S}_{\mathrm{sw}}$ with the parameters μ_1 and A_1. It is easy to verify that the matrix Q_0^- (obtained as a result of dressing function (6.18) with (4.12)) is associated with Q_{sw}^0 by the relation

$$Q_0^- = TQ_{\mathrm{sw}}^0 T, \qquad T = \begin{pmatrix} 0 & e^{-i(\psi+\chi)} \\ e^{i(\psi+\chi)} & 0 \end{pmatrix}.$$

For the solution $\vec{S}_{12}^{\,0}$ as $t \to -\infty$ with $x - 4\xi t = \mathrm{const}$, we hence have

$$\begin{aligned} \vec{S}_{12}^{\,0} &= TQ_{\mathrm{sw}}^0 T \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} T(Q_{\mathrm{sw}}^0)^{-1} T = TQ_{\mathrm{sw}}^0 \sigma_3 Q_{\mathrm{sw}}^0 = \\ &= TS_{\mathrm{sw}}T = \begin{pmatrix} -S_{3,\mathrm{sw}} & (S_{1,\mathrm{sw}} + iS_{2,\mathrm{sw}})e^{-2i(\psi+\chi)} \\ (S_{1,\mathrm{sw}} - iS_{2,\mathrm{sw}})e^{2i(\psi+\chi)} & S_{3,\mathrm{sw}} \end{pmatrix} \end{aligned}$$

Therefore, as $t \to -\infty$ with $x - 4\xi t = \mathrm{const}$, the solution $\vec{S}_{12}^{\,0}$ becomes S_3 soliton (5.4a) rotated through 180° in the plane (S_2, S_3) (i.e., $(S_1, S_2, S_3) \to (S_1, -S_2, -S_3)$) and characterized by the parameters

$$\begin{aligned} &v_1 = 4\xi_1, \qquad \omega_1 = 4|\mu_1|^2 + 4a^2, \\ &x_0^- = \frac{1}{2\eta_1}\log|A_1|, \qquad \theta_0^- = \arg A_1 - 2\chi - 2\psi. \end{aligned}$$

Hence, the effect of S_3 soliton (5.4a) passing through domain wall (5.10) is described by the relations

$$
\begin{aligned}
&(S_1, S_2, S_3) \to (S_1, -S_2, -S_3),\\
&\Delta x_0 = x_0^+ - x_0^- = \frac{1}{2\eta_1}\log\left|\frac{\mu_1 + a}{\mu_1 - a}\right|,\\
&\Delta\theta_0 = \theta_0^+ - \theta_0^- = \arg\frac{\mu_1 + a}{\mu_1 - a} + 2\psi.
\end{aligned}
\tag{6.19}
$$

We note that unlike the interaction of two S_3 solitons, there is an obvious absence of symmetry in formulas (6.15) and (6.19) describing the interaction of the domain wall with the soliton. In particular, the soliton shift is half the domain wall shift.

With this, we complete the demonstration of applying the dressing procedure to the problem of the interaction between elementary solutions of the LL equation. We only note that by taking appropriate limits in the obtained formulas, we can simply describe the interaction processes involving rational solitons (5.21). Finally, the interaction of the S_1 solitons can be studied quite similarly. But the approach based on the degeneration of finite-gap solutions of the LL equation turned out to be more effective in this case. This approach is developed in Sec. 10.

7. Finite-gap solutions

In this section, we construct general finite-gap solutions of the XXZ LL equation. They correspond to the generalized Riemann problem data

$$
\begin{aligned}
&\Lambda_1 = \{a_1, \dots, a_{3g},\, T_1, \dots, T_{3g},\, C_1, \dots, C_{3g},\, \mathcal{L}_1, \dots, \mathcal{L}_g,\, G_1, \dots, G_g\},\\
&a_i = \begin{cases} E_i, & i = 1, \dots, 2g,\\ \mu_{i-2g}, & i = 2g+1, \dots, 3g,\end{cases}\\
&T_i = \begin{cases} \begin{pmatrix} 0 & 0\\ 0 & \frac{1}{2}\end{pmatrix}, & i = 1, \dots, 2g,\\ \begin{pmatrix} -1 & 0\\ 0 & 0\end{pmatrix}, & i = 2g+1, \dots, 3g,\end{cases} \qquad C_i = \begin{cases} \begin{pmatrix} -1 & -1\\ -1 & 1\end{pmatrix}, & i = 1, \dots, 2g,\\ I, & i = 2g+1, \dots, 3g,\end{cases}\\
&\mathcal{L}_i = [a_{2i-1}, a_{2i}], \qquad G_i = \sigma_1, \quad i = 1, \dots, g.
\end{aligned}
\tag{7.1}
$$

We construct the function Ψ satisfying reduction (2.11) with the matrix $\sigma(\lambda) = \sigma_1$:

$$
\sigma_3\Psi(\lambda^\tau) = \Psi(\lambda)\sigma_1. \tag{7.2}
$$

Hence, data (7.1) are specified on one sheet of the surface Γ. In accordance with reduction (7.2), they define the data Λ_2 on the other sheet of the surface Γ. The complete data of the generalized Riemann problem are determined as the sum $\Lambda_1 \oplus \Lambda_2$. Using the usual terminology, $\mathcal{L}_i$ are the cuts on the Riemann surface, E_i are the branch points, and μ_i are the poles of the Baker–Akhiezer function.

The function Ψ is not single-valued on Γ. According to (2.9), the monodromy matrices M_i corresponding to the points E_i are

$$
M_i = C_i^{-1}\sigma_3 C_i = \sigma_1. \tag{7.3}
$$

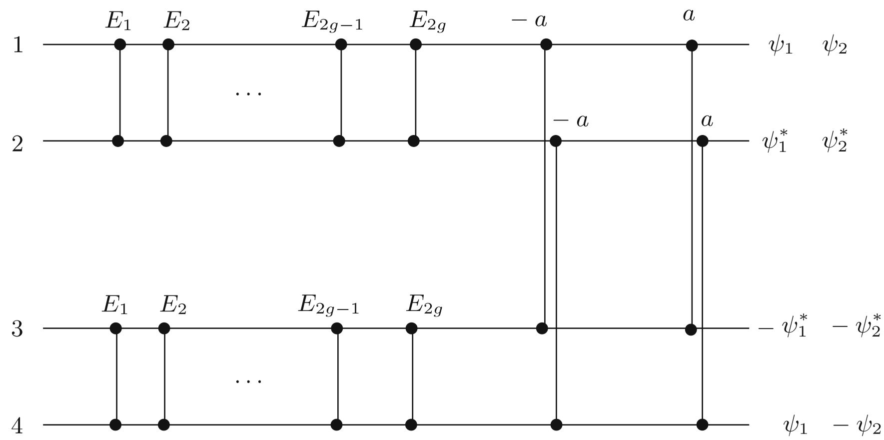

Fig. 5. The surface $\widehat{\Gamma}$.

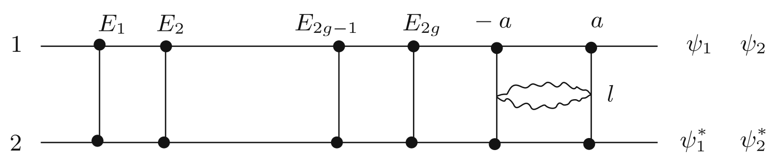

Fig. 6. The surface Γ_0.

The function Ψ becomes single-valued on the surface $\widehat{\Gamma}$ (shown schematically in Fig. 5) that is a cover of the surface Γ with the branch points E_i. In the case of the Korteweg–de Vries equation, the nonlinear Schrödinger equation, and others, the surface Γ (the surfaces where the appropriate U–V-pair is defined) is simply a complex plane. In this case, the Riemann surface $\widehat{\Gamma}$ (where the Baker–Akhiezer function is defined) is its two-sheeted cover (similarly to our case), i.e., a hyperelliptic surface.

Further, the function $\Psi(\lambda)$ can be defined by the vector Baker–Akhiezer function $\vec{\psi}(\lambda) = \big(\psi_1(\lambda), \psi_2(\lambda)\big)$ using the formula

$$\Psi(\lambda) = \begin{pmatrix} \psi_1(\lambda) & \psi_1^*(\lambda) \\ \psi_2(\lambda) & \psi_2^*(\lambda) \end{pmatrix}, \tag{7.4}$$

where $\psi_1(\lambda)$ and $\psi_2(\lambda)$ are single-valued functions on $\widehat{\Gamma}$ and $\psi^*(\lambda) = \psi(\lambda^*)$. Here, $\lambda \to \lambda^*$ is the involution of the surface $\widehat{\Gamma}$ transposing the sheets $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$. This structure of Ψ is suggested by monodromy data (7.3). It is interesting whether Ψ satisfies reduction (7.2). First, we naturally define the involution $\lambda \to \lambda^\tau$ of $\widehat{\Gamma}$ by carrying the involution of Γ (which transposes the sheets $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$) onto $\widehat{\Gamma}$. In terms of ψ_1 and ψ_2, we can write reduction (7.2) as

$$\psi_1(\lambda^\tau) = \psi_1(\lambda^*), \qquad \psi_2(\lambda^\tau) = -\psi_2(\lambda^*). \tag{7.5}$$

To construct the functions $\psi_1(\lambda)$ and $\psi_2(\lambda)$, we consider the auxiliary hyperelliptic surface Γ_0 of genus g (shown schematically in Fig. 6) and two functions on it: the single-valued function $\psi_1(\lambda)$ and the function $\psi_2(\lambda)$, $\lambda \in \Gamma_0$, changing its sign when it crosses the closed contour l passing through the points a and $-a$ (shown by the wavy line in Fig. 6),

$$\psi_2^+(\lambda) = -\psi_2^-(\lambda)\big|_{\lambda\in l}, \tag{7.6}$$

where $\psi_2^+(\lambda)$ and $\psi_2^-(\lambda)$ are the values of $\psi_2(\lambda)$ on the respective upper and lower bounds of the contour. The functions ψ_1 and ψ_2 thus constructed can be determined simply on the surface $\widehat{\Gamma}$, representing the natural domain of the analytic extension of $\psi_2(\lambda)$.

We let ψ_1 and ψ_2 denote the values of $\psi_1(\lambda)$ and $\psi_2(\lambda)$ on the first sheet of Γ_0 and ψ_1^* and ψ_2^* denote their values on the second sheet. We assume that the values of $\psi_1(\lambda)$ and $\psi_2(\lambda)$ on the first sheet of $\widehat{\Gamma}$ (see Fig. 5) coincide with their values on the first sheet of $\widehat{\Gamma}_0$ (see Fig. 6). The first and second sheets of $\widehat{\Gamma}$ are joined on the cuts $[E_{2i-1}, E_{2i}]$. Therefore, analytically extending $\psi_1(\lambda)$ and $\psi_2(\lambda)$ through these cuts, we obtain their values on the second sheet of $\widehat{\Gamma}$. These values are equal to ψ_1^* and ψ_2^*. The values on the third sheet of $\widehat{\Gamma}$ can be obtained by extending from the first sheet through the cut $[-a, a] = l$, where $\psi_2(\lambda)$ (as a function on Γ_0) changes its sign in accordance with (7.6). Therefore, the values of $\psi_1(\lambda)$ and $\psi_2(\lambda)$ on the third sheet of $\widehat{\Gamma}$ are equal to ψ_1^* and $-\psi_2^*$. Analogously, they are equal to ψ_1 and $-\psi_2$ on the fourth sheet of $\widehat{\Gamma}$.

Obviously, the functions $\psi_1(\lambda)$ and $\psi_2(\lambda)$ thus constructed on $\widehat{\Gamma}$ satisfy reduction (7.5), and the function $\Psi(\lambda)$ defined by them using (7.4) consequently satisfies the required reduction (7.2).

Theorem 5. *Let the functions $\psi_1(\lambda)$ and $\psi_2(\lambda)$ have the following properties as functions on Γ_0:*

1. *The asymptotic equalities*

$$\begin{aligned}
\psi_1(\lambda, x, t) &= (A + A_1\lambda^{-1} + \cdots)e^{-i\lambda x + 2i\lambda^2 t}, \quad \lambda \to \infty^+, \\
\psi_1(\lambda, x, t) &= (B + B_1\lambda^{-1} + \cdots)e^{i\lambda x - 2i\lambda^2 t}, \quad \lambda \to \infty^-, \\
\psi_2(\lambda, x, t) &= (C + C_1\lambda^{-1} + \cdots)e^{-i\lambda x + 2i\lambda^2 t}, \quad \lambda \to \infty^+, \\
\psi_2(\lambda, x, t) &= (D + D_1\lambda^{-1} + \cdots)e^{i\lambda x - 2i\lambda^2 t}, \quad \lambda \to \infty^-,
\end{aligned} \tag{7.7}$$

   hold, where A, B, C, and D are unknown functions of x and t and where ∞^+ and ∞^- are two infinite points on Γ_0 on the respective upper and lower sheets.

2. *The functions $\psi_1(\lambda)$ and $\psi_2(\lambda)$ are meromorphic on $\Gamma \setminus \{\infty^\pm\}$ and have the nonspecial pole divisor $\mathcal{D} = \mu_1 + \cdots + \mu_g$*

3. *The function $\psi_1(\lambda)$ is single-valued on Γ_0, and the function $\psi_2(\lambda)$ satisfies relation* (7.5).

4. *The values of $\psi_1(a)$ and $\psi_2(-a)$ are independent of x and t.*

Then the function $\Psi(\lambda)$ defined on $\widehat{\Gamma}$ by the functions $\psi_1(\lambda)$ and $\psi_2(\lambda)$ using formulas (7.4) *and the procedure described above satisfies both the generalized Riemann problem with the data $\Lambda = \Lambda_1 \oplus \Lambda_2$ (see* (7.1)*) and reduction* (7.2)*. Moreover, the first coefficient Φ_0 of expansion* (2.7) *in the neighborhood of the infinite point is given by formula* (2.14)*, where A, B, C, and D are defined by equalities* (7.7).

Proof. The proof of this theorem is simple, and we do not present it.

Hence, to construct finite-gap solutions of Eq. (2.1), it remains to construct functions $\psi_1(\lambda)$ and $\psi_2(\lambda)$, $\lambda \in \Gamma_0$, satisfying the conditions in Theorem 5.

We define canonical objects of the finite-gap integration on Γ_0 (details of this material are given in [21]). We select a canonical basis of the cycles a_i and b_i, $i = 1, \ldots, g$, such that the cycle

$$\sum a = a_1 + \cdots + a_g$$

passes around the cut $[-a, a]$, i.e., coincides with the contour l.

Let $dU_i(\lambda)$, $i = 1, \dots, g$, be the corresponding normalized basis of Abelian differentials (with the normalization $\oint_{a_i} dU_j = \delta_{ij}$) and B be the period matrix of the surface Γ_0. Let

$$\theta[\alpha, \beta](z|B) = \sum_{m \in \mathbb{Z}^g} e^{\pi i \langle B(m+\alpha), m+\alpha \rangle + 2\pi i \langle z+\beta, m+\alpha \rangle} \tag{7.8}$$

be the Riemann theta function with the characteristics $\alpha, \beta \in \mathbb{R}^g$,

$$\theta[0,0](z|B) \equiv \theta(z|B), \quad z \in \mathbb{C}^g.$$

We also define two normalized second-kind Abelian integrals (with zero a-periods) $\Omega_1(\lambda)$ and $\Omega_2(\lambda)$ by their asymptotic forms at $\infty^\pm$:

$$\Omega_1(\lambda) \to \mp(\lambda + b + \dots), \qquad \Omega_2(\lambda) \to \pm(2\lambda^2 + c + \dots), \quad \lambda \to \infty^\pm. \tag{7.9}$$

Let $V, W \in \mathbb{C}^g$ be the vectors of their b-periods.

The functions $\psi_1(\lambda)$ and $\psi_2(\lambda)$ are given by the standard formulas:

$$\begin{aligned} \psi_1(\lambda) &= \frac{\theta(U(\lambda) + \Omega + \mathcal{D})}{\theta(U(\lambda) + \mathcal{D})\,\theta(U(a) + \Omega + \mathcal{D})} e^{i(\Omega_1(\lambda) - \Omega_1(a))x + i(\Omega_2(\lambda) - \Omega_2(a))t}, \\ \psi_2(\lambda) &= \frac{\theta[0,n](U(\lambda) + \Omega + \mathcal{D})}{\theta(U(\lambda) + \mathcal{D})\,\theta[0,n](U(-a) + \Omega + \mathcal{D})} e^{i(\Omega_1(\lambda) - \Omega_1(-a))x + i(\Omega_2(\lambda) - \Omega_2(-a))t}, \end{aligned} \tag{7.10}$$

where $n = (1/2, 1/2, \dots, 1/2)$,

$$U(\lambda) = \left(\int_{p_0}^{\lambda} dU_1, \dots, \int_{p_0}^{\lambda} dU_g \right), \qquad \Omega = \frac{1}{2\pi}(Vx + Wt),$$

and $\mathcal{D} \in \mathbb{C}^g$ is an arbitrary vector in general position that is an Abelian map of the divisor $\mathcal{D}$ up to the vector of Riemann constants. It is easy to see that the function $\psi_2(\lambda)$ is not single-valued: passing the cycles $b_1, \dots, b_g$, i.e., crossing the contour l, it changes sign as prescribed by the conditions in Theorem 5. The function $\psi_2(\lambda)$ can be brought to a more convenient form if we take into account that the β-characteristics reduces to only a shift of the argument of the theta function and that $U(a) - U(-a) = n$ because

$$dU(\lambda^*) = -dU(\lambda), \qquad \int_{\sum a} dU = 2n.$$

We have

$$\psi_2(\lambda) = \frac{\theta(U(\lambda) + \Omega + \mathcal{D} + n)}{\theta(U(\lambda) + \mathcal{D})\,\theta(U(a) + \Omega + \mathcal{D})} e^{i(\Omega_1(\lambda) - \Omega_1(-a))x + i(\Omega_2(\lambda) - \Omega_2(-a))t}. \tag{7.11}$$

Further, if we take the branch point of the surface Γ_0 as the initial integration point, then we obtain the equality

$$\Omega_j(a) = \Omega_j(-a), \tag{7.12}$$

because

$$\oint_{\sum a} d\Omega_j = 0, \qquad d\Omega_j(\lambda^*) = -d\Omega_j(\lambda), \quad j = 1, 2.$$

We thus construct the function $\Psi(\lambda)$. The quantities A, B, C, and D in (7.7) are given by

$$
\begin{aligned}
A &= \frac{1}{f(\infty^+)}\theta(U(\infty^+)+\Omega+\mathcal{D})e^{i(-b-\Omega_1(a))x+i(c-\Omega_2(a))t},\\
B &= \frac{1}{f(\infty^-)}\theta(U(\infty^+)+\Omega+\mathcal{D}+r)e^{i(b-\Omega_1(a))x+i(-c-\Omega_2(a))t},\\
C &= \frac{1}{f(\infty^+)}\theta(U(\infty^+)+\Omega+\mathcal{D}+n)e^{i(-b-\Omega_1(-a))x+i(c-\Omega_2(-a))t},\\
D &= -\frac{1}{f(\infty^-)}\theta(U(\infty^+)+\Omega+\mathcal{D}+n+r)e^{i(b-\Omega_1(-a))x+i(-c-\Omega_2(-a))t},
\end{aligned}
$$

where $f(s)=\theta(U(s)+\mathcal{D})\,\theta(U(a)+\Omega+\mathcal{D})$, $r=\int_{\infty^+}^{\infty^-}dU$, the integration is over a path on Γ_0 crossing the contour l, and the parameters b and c are defined by relations (7.9).

We clarify the negative sign in the expression for D. The point is that if the value of ψ_2 on the first sheet of $\widehat{\Gamma}$ is equal to B, then D is the value of this function on the second sheet of $\widehat{\Gamma}$ (see (7.4)); adding the integral r along a path crossing the contour l to the arguments of the theta function yields the value of $\psi_2(\lambda)$ at the infinite point of the third sheet of $\widehat{\Gamma}$ (because this path relates sheets 1 and 3 rather than 1 and 2). The values of $\psi_2(\lambda)$ on sheets 2 and 3 differ in sign.

We note that expressions (2.15) for the spins are invariant under the transformation $A\to\alpha A$, $B\to\beta B$, $C\to\alpha C$, $D\to\beta D$ (or, equivalently, Ψ can be multiplied from the right by an arbitrary diagonal matrix). After appropriate simplifications using equality (7.12), we obtain the following theorem.

Theorem 6. *The general finite-gap solutions for XXZ LL equation* (2.1) *are defined by formulas* (2.15), *where*

$$
A=\theta(\Omega+\mathcal{D}),\qquad B=\theta(\Omega+\mathcal{D}+r),\qquad C=\theta(\Omega+\mathcal{D}+n),\qquad D=-\theta(\Omega+\mathcal{D}+r+n),\tag{7.13}
$$

$n=(1/2,\dots,1/2)$, $\mathcal{D}\in\mathbb{C}^g$, *and* $\Omega=(Vx+Wt)/2\pi$. *Here, the theta function is defined by the Riemann surface* Γ_0 *given by*

$$
\omega^2=(\lambda^2-a^2)\prod_{i=1}^{2g}(\lambda-E_i),
$$

whose cycle $\sum a=a_1+\dots+a_g$ *encircles the cut* $[-a,a]$, *and*

$$
r=\int_{\infty^+}^{\infty^-}dU,
$$

where the integration path crosses the cycle $\sum a$.

Remark 6. It is easy to show that if we select a canonical basis such that the contour l is $l=\sum_{i=1}^{g}(\beta_i a_i+\alpha_i b_i)$, $\alpha_i,\beta_i\in\mathbb{Z}$, then

$$
\begin{aligned}
A&=\theta(\Omega+\mathcal{D}), & B&=\theta(\Omega+\mathcal{D}+r),\\
C&=\theta\left[\frac{\alpha}{2},\frac{\beta}{2}\right](\Omega+\mathcal{D}), & D&=-\theta\left[\frac{\alpha}{2},\frac{\beta}{2}\right](\Omega+\mathcal{D}+r),
\end{aligned}\tag{7.14}
$$

where the integration path crosses the contour l.

Remark 7. We see that the formulas for solutions depend on the choice of the basis of cycles. It may therefore seem that the solution is defined not only by the Riemann problem data Λ given by (7.1), i.e., by the branch points of the divisors of the poles of ψ, but also by the canonical basis of cycles on Γ_0 on which it depends. We give a simple argument showing that this is not the case. In fact, although the formal realization of the function Ψ depends on the choice of the basis of cycles, it is nevertheless obvious that Ψ can be uniquely constructed using the data Λ. Indeed, let Ψ and $\widetilde{\Psi}$ correspond to Λ. Then

$$\Psi\widetilde{\Psi}^{-1} = \varphi(x,t)\begin{pmatrix}\gamma & 0\\ 0 & \gamma^{-1}\end{pmatrix}, \qquad \gamma = \text{const},$$

which follows from the holomorphicity of the function on $\mathbb{C}$, normalization (2.12), and reduction (2.11). We do not distinguish such functions (see Remark 2 in Sec. 1). Therefore, the solution of the LL equation defined by Ψ using formulas (2.14) and (2.15) is unique. This simple argument also shows that the solution defined by a hyperelliptic surface Γ_0 depends only on the branch points of Γ_0 and is independent of the method for constructing the cuts between them.

Different choices of cuts and the basis of cycles on the surface Γ_0 are convenient in different cases (see Secs. 8–11 below). In such cases, we obtain the same solutions of the LL equation.

Remark 8. To find the vector of b-periods of Ω, we can conveniently use the following simple fact (see, e.g., [25]). Let z be a local coordinate in a neighborhood of the point $P_0 \in \Gamma$ ($z \to 0$ as $P \to P_0$) and the basic holomorphic differentials be written in the form $dU = f(z)\,dz$ in the neighborhood of P_0. Then the normalized second-kind Abelian integral with a single singularity at the point P_0 of the form $\Omega(P) = z^{-n} + O(1)$ as $z \to 0$ has a vector of b-periods equal to

$$\Omega = -\frac{2\pi i}{(n-1)!}\,\frac{d^{n-1}}{dz^{n-1}} f(z)\bigg|_{z=0}. \tag{7.15}$$

For the hyperelliptic surface Γ given by

$$\omega^2 = \prod_{i=1}^{2g+2}(\lambda - E_i)$$

with the normalized basis

$$dU_i = \frac{1}{\omega}\sum_{j=1}^{g} c_{ij}\lambda^{g-j}, \quad i = 1,\dots,g,$$

we find that the second-kind Abelian integrals $\Omega_1(P)$ and $\Omega_2(P)$ with singularities (7.9) have the vectors of b-periods V and W given by

$$V_k = -4\pi c_{k,1}, \qquad W_k = 8\pi i\biggl(c_{k,2} + \frac{c_{k,1}}{2}\sum_{i=1}^{2g+2} E_i\biggr), \quad k = 1,\dots,g. \tag{7.16}$$

8. Selection of real finite-gap solutions

We select the real solutions among the general finite-gap solutions obtained in Sec. 7. We do not present the rather cumbersome rigorous proof that we have found all real solutions (this proof is based on the requirement to satisfy reduction (2.17)). Instead, we use a technique originally proposed in [27] for the sine-Gordon equation and based on analyzing only the final formulas for the solutions in terms of Riemann theta functions.

We first note a simple algebraic fact, which can be easily verified by direct calculation.

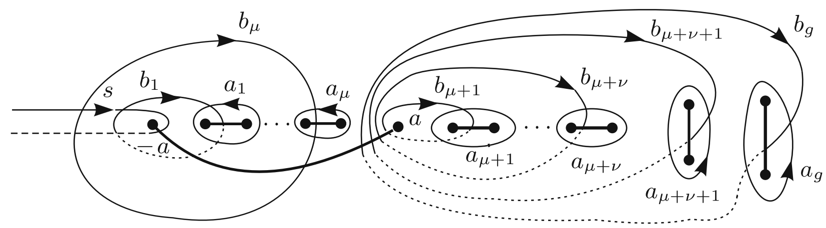

Fig. 7. The basis of cycles s on the Riemann surface Γ_{ep}: the parts of cycles on the lower sheet of the Riemann surface are depicted by dashed lines.

Lemma 3. *Formulas* (2.15) *define the real solution of equation* (2.1) *if and only if the relation*

$$D\bar{C} = -B\bar{A} \tag{8.1}$$

is satisfied.

The function Ψ corresponding to real solutions of Eq. (2.1) satisfies reduction (2.17). Consequently, the surface Γ_0 has a conjugation anti-involution. The second restriction on the solution parameters (on the vector $\mathcal{D}$ in this case) follows from Remark 2 in Sec. 1, Theorem 6, and Lemma 3:

$$|\gamma|^4 = \frac{\theta(\Omega+\mathcal{D}+n+r)\overline{\theta(\Omega+\mathcal{D}+n)}}{\theta(\Omega+\mathcal{D}+r)\overline{\theta(\Omega+\mathcal{D}+n)}} = \text{const} > 0, \tag{8.2}$$

i.e., $|\gamma|^4$ is a positive constant independent of x and t.

We present a detailed analysis of these two constraints first in the case of an anisotropy of the easy-plane type ($\varepsilon < 0$). We consider the surface Γ_{ep} (shown together with the basis of cycles in Fig. 7). Let there be $\mu+\nu$ more pairs of the branch points on the real axis in addition to the pair of the branch points a and $-a$, let μ pairs of them belong to the segment $[-a, a]$, and let Γ_{ep} have $g-\mu-\nu$ more pairs of conjugate branch points.

The conjugation anti-involution $\tau(\lambda,\omega) = (\bar{\lambda},\bar{\omega})$, which does not transpose the sheets of Γ_{ep}, acts on these cycles as

$$\begin{aligned} &a_i = \tau a_i, && b_i = -\tau b_i, && i = 1,\ldots,\mu,\\ &a_i = -\tau a_i, && b_i = \tau b_i, && i = \mu+1,\ldots\mu+\nu,\\ &a_i = -\tau a_i, && b_i = \tau b_i + \tau a_i, && i = \mu+\nu+1,\ldots,g \end{aligned} \tag{8.3}$$

(the equality holds in the homology group $H_1(\Gamma,\mathbb{Z})$), the normalized holomorphic differentials are consequently transformed under the anti-involution τ according to the rule

$$\begin{aligned} &\tau^* dU_i(\lambda) = dU_i(\tau(\lambda)) = dU_i(\bar{\lambda}) = \overline{dU_i(\lambda)}, && i = 1,\ldots,\mu,\\ &\tau^* dU_i(\lambda) = dU_i(\tau(\lambda)) = dU_i(\bar{\lambda}) = -\overline{dU_i(\lambda)}, && i = \mu+1,\ldots,g. \end{aligned} \tag{8.4}$$

The period matrix therefore has the following structure:

$\operatorname{Re} B_{ij} = 0$ for $i,j = 1,\ldots,\mu$, for $i = \mu+1,\ldots,\mu+\nu$ and $j = \mu+1,\ldots,g$, and for $i,j = \mu+\nu+1,\ldots,g$, $i \neq j$,

$\operatorname{Re} B_{ij} = 0$ for $i = 1,\ldots,\mu$ and $j = \mu+1,\ldots,g$, and

$\operatorname{Re} B_{ii} = -1/2$ for $i = \mu + \nu + 1, \ldots, g$.

We let $z = \binom{z'}{z''}$, $z' \in \mathbb{C}^{\mu}$, $z' \in \mathbb{C}^{g-\mu}$, denote a g-dimensional vector, where z' are the first μ coordinates of z and z'' are the last $g-\mu$ coordinates. The theta function defined by the matrix B given above has the symmetry

$$\overline{\theta(z)} = \overline{\theta\begin{pmatrix} z' \\ z'' \end{pmatrix}} = \overline{\theta\begin{pmatrix} -\bar{z}' \\ \bar{z}'' + \lambda \end{pmatrix}}, \qquad \lambda = \frac{1}{2}(\underbrace{0, \ldots, 0}_{\nu}, 1, \ldots, 1), \quad \lambda \in \mathbb{R}^{g-\mu}. \tag{8.5}$$

It is easy to verify that the components of the vector $\Omega = \binom{\Omega'}{\Omega''}$ satisfy the conditions $\operatorname{Re}\Omega' = 0$ and $\operatorname{Im}\Omega'' = 0$ and and the integral r calculated along the path s, $\tau s = s$ (see Fig. 7), is equal to

$$r = \begin{pmatrix} r' \\ r'' \end{pmatrix} = \int_s dU(\lambda) = \int_{\tau s = s} dU(\bar{\lambda}) = \begin{pmatrix} \bar{r}' \\ -\bar{r}'' \end{pmatrix}, \quad \operatorname{Im} r' = 0, \quad \operatorname{Re} r'' = 0. \tag{8.6}$$

Using relation (8.5), we rewrite condition (8.2) as

$$|\gamma|^4 = \frac{\theta\binom{\Omega'+\mathcal{D}'+n'+r'}{\Omega''+\mathcal{D}''+n''+r''}\theta\binom{\Omega'-\overline{\mathcal{D}}'+n'}{\Omega''+\overline{\mathcal{D}}''+n''+\lambda}}{\theta\binom{\Omega'+\mathcal{D}'+r'}{\Omega''+\mathcal{D}''+r''}\theta\binom{\Omega'-\overline{\mathcal{D}}''}{\Omega''+\overline{\mathcal{D}}''+\lambda}} = \text{const} > 0, \tag{8.7}$$

where $\mathcal{D} = \binom{\mathcal{D}'}{\mathcal{D}''}$ and $n = \binom{n'}{n''}$. This can be satisfied only if

$$\begin{pmatrix} \mathcal{D}' + n' + r' \\ \mathcal{D}'' + n'' + r'' \end{pmatrix} = \begin{pmatrix} -\overline{\mathcal{D}}' \\ \overline{\mathcal{D}}'' + \lambda \end{pmatrix} + M + BN, \quad M, N \in \mathbb{Z}^g, \tag{8.8}$$

which implies $N = 0$, $\nu = 0$, $\lambda = n''$,

$$\begin{aligned} \mathcal{D}' &= \mathcal{D}'_0 - \frac{1}{2}(r' + n' + \delta), \qquad && \operatorname{Re}\mathcal{D}'_0 = 0, \qquad \delta \in \mathbb{Z}^{\mu}/2\mathbb{Z}^{\mu}, \\ \mathcal{D}'' &= \mathcal{D}''_0 - \frac{1}{2}r'', && \operatorname{Im}\mathcal{D}''_0 = 0, \end{aligned} \tag{8.9}$$

where the vectors $\mathcal{D}'_0$ and $\mathcal{D}''_0$ are arbitrary. In this case, $|\gamma|^4 = 1$.

Therefore, the surface Γ_{ep} is described by the equation

$$\begin{gathered} \omega^2 = (\lambda^2 - a^2)\prod_{j=1}^{2}(\lambda - e_j)\prod_{i=1}^{g-\mu}(-c_i + \lambda)(\lambda - \bar{c}_i), \\ \operatorname{Im} c_i \neq 0, \quad e_j \in \mathbb{R}, \quad |e_j| < a. \end{gathered} \tag{8.10}$$

Moreover, we have 2^{μ} topologically different solution components (which cannot be transformed into each other during the dynamics with respect to the dynamical variables). These components correspond to the 2^{μ} different possible choices of the vector δ of zeros and ones in (8.9).

Remark 9. Using the addition theorem for theta functions (see, e.g., [25]),

$$\theta(z_1|B)\,\theta(z_2|B) = \sum_{2\alpha \in \mathbb{Z}^g/2\mathbb{Z}^g} \theta[\alpha, 0](z_1 + z_2|2B)\,\theta[\alpha, 0](z_1 - z_2|2B), \tag{8.11}$$

and formulas (2.15) and (7.13), we can easily show that solutions defined by the vectors $\delta_1 \in \mathbb{Z}^g$ and $\delta_2 = 2n' - \delta_1$ differ only by the transformation $(S_1, S_2, S_3) \to (S_1, -S_2, -S_3)$ reducible to the choice of axes. We do not distinguish these solutions (although they evolve differently in the presence of an external field; see Sec. 3). Consequently, the number of components of solutions is in fact halved and is equal to $2^{\mu} - 1$.

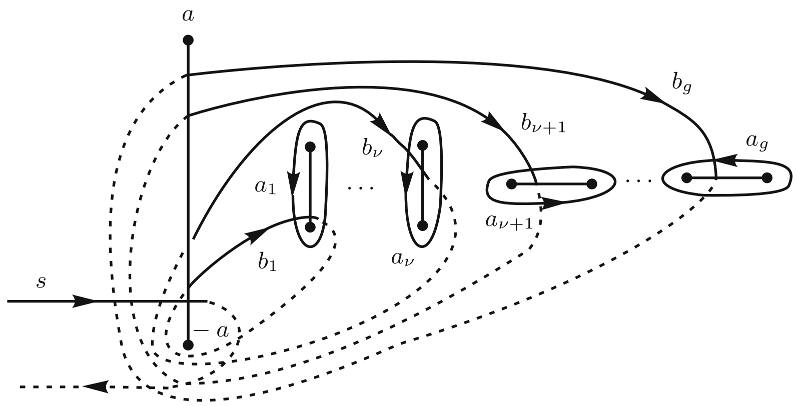

Fig. 8. The Riemann surface Γ_{ea}.

The easy-axis case ($\varepsilon > 0$) can be considered similarly. The corresponding Riemann surface Γ_{ea} is shown in Fig. 8. In addition to the points $-a$ and a, this surface has ν more pairs of conjugate branch points and $g-\nu$ pairs of real branch points. The equalities

$$a_i = -\tau a_i, \quad i = 1, \ldots, g, \qquad b_i = \begin{cases} \tau b_i - \tau a_i + \sum_{k=1}^{g} \tau a_k, & i = 1, \ldots, \nu, \\ \tau b_i + \sum_{k=1}^{g} \tau a_k, & i = \nu+1, \ldots, g, \end{cases}$$

$$B = -\bar{B} + J, \qquad J_{ij} = -1, \quad i \neq j, \qquad J_{ii} = \begin{cases} 0, & i = 1, \ldots, \nu, \\ -1, & i = \nu+1, \ldots, g, \end{cases}$$

$$\overline{\theta(z)} = \theta(\bar{z} + \lambda), \qquad \lambda = \frac{1}{2}(\underbrace{0, \ldots, 0}_{\nu}, 1, \ldots, 1);$$

$$r_0 = -\bar{r}_0 - 2n, \quad n = \frac{1}{2}(1, 1, \ldots, 1), \qquad \operatorname{Re} r_0 = -n,$$

where r_0 denotes the integral r along the path s in Fig. 8 and $\Omega \in \mathbb{R}^g$. Similarly to the easy-plane case, condition (8.2) necessarily leads to the equality

$$\mathcal{D} + n + r = \bar{\mathcal{D}} + \lambda + M + BN, \quad M, N \in \mathbb{Z}^g, \tag{8.12}$$

and in this case,

$$|\gamma|^4 = e^{2\pi i \langle N, n\rangle}, \qquad \langle N, n\rangle = \sum_{i=1}^{g} N_i n_i = \frac{1}{2}\sum_{i=1}^{g} N_i. \tag{8.13}$$

Two cases are possible.

1. If $\nu = g$, $\lambda = 0$, $N = 0$, then $|\gamma|^4 = 1$. The vector $\mathcal{D}$ is determined by

$$\mathcal{D} = \mathcal{D}_0 - \frac{1}{2} r, \quad \mathcal{D}_0 \in \mathbb{R}. \tag{8.14}$$

2. Let $\nu < g$. We take the real part of (8.12) and obtain

$$0 = \lambda + M + \operatorname{Re} BN = \lambda + M + \frac{1}{2} JN, \qquad -M = \lambda - 2\langle N, n\rangle n + \frac{1}{2} LN, \tag{8.15}$$

where

$$L = \operatorname{diag}(\underbrace{1, \ldots, 1}_{\nu}, 0, \ldots, 0).$$

In the second equality in (8.15), a vector with integer-valued coordinates is in the left-hand side, and we have $(LN)_i = 0$ and $\lambda_i = 1/2$ for $i = \nu + 1, \ldots, g$ in the right-hand side. Consequently, the number $\langle N, n\rangle$ is half-integer. As a result, we obtain $|\gamma|^4 = -1$ from expression (8.13), and there is no real solution.

Hence, the surface Γ_{ea} defining the real solutions is given by

$$\omega^2 = (\lambda^2 - a^2) \prod_{i=1}^{g} (\lambda - c_i)(\lambda - \bar{c}_i), \quad \operatorname{Im} c_i \neq 0, \tag{8.16}$$

and the following theorem is thus proved.

Theorem 7. *For the finite-gap solutions of XXZ LL equation* (2.1) *given in Theorem* 6 *to be real, it is necessary and sufficient that the Riemann surfaces* Γ_{ep} *and* Γ_{ea} *be given by the respective Eqs.* (8.10) *and* (8.16)*, the vector* $\mathcal{D}$ *be defined by the respective conditions* (8.9) *and* (8.14)*, where* $r = \int_s dU$*, and the integration paths* s *be as shown in Figs.* 7 *and* 8 *for* $\varepsilon < 0$ *and* $\varepsilon > 0$*.*

9. Simplest finite-gap solutions in terms of elliptic functions

The simplest nondegenerate finite-gap solution is the solution in the case $g = 1$. We consider the surfaces Γ_{ep} (Figs. 9a and 9b) and Γ_{ea} (Fig. 10).

As already mentioned in Remark 7 in Sec. 7, although the formal representations of the finite-gap solutions of the LL equation do depend on the choice of cuts and a basis of cycles on the surface Γ_0, the solutions themselves are defined only by the branch points of Γ_0. In what follows, it is convenient to use different representations of Γ_0. In this regard, we note that the surfaces Γ_{ep} with the basis of cycles shown in Figs. 9a and 9b, for instance, are equivalent to the surfaces with the basis shown in the respective Figs. 11a and 11b.

In all cases, there is the only holomorphic differential

$$du(\lambda) = N \frac{d\lambda}{\sqrt{(\lambda^2 - a^2)(\lambda - E_1)(\lambda - E_2)}},$$

where $(E_1, E_2) = (c, \bar{c})$ or $(E_1, E_2) = (e_1, e_2)$. The constant N can be found from the normalization condition $\oint_{a_1} dU = 1$. The period of the curve Γ_0 equals $B = \oint_{b_1} dU$. The periods V and W of the second-kind integrals are defined by equalities (7.16), whence

$$V = -4\pi i N, \qquad W = 4\pi i N(c + \bar{c}).$$

The integral $r = 2s_0$ can be calculated directly. Hence, the solution $\vec{S}(x,t) = \vec{Q}_X(x - vt)$ of genus $g = 1$, as usual, has the form of a cnoidal wave (a periodic running wave). For the surfaces shown in Figs. 9a and 10, we obtain the wave

$$\begin{aligned}
A &= \theta[0,0](2iN(-x+vt) + d - s_0|B),\\
B &= \theta[0,0](2iN(-x+vt) + d_0 + s_0|B),\\
C &= \theta\left[0, \frac{1}{2}\right](2iN(-x+vt) + d - s_0|B),\\
D &= -\theta\left[0, \frac{1}{2}\right](2iN(-x+vt) + d + s_0|B), \quad d \in \mathbb{R}
\end{aligned} \tag{9.1}$$

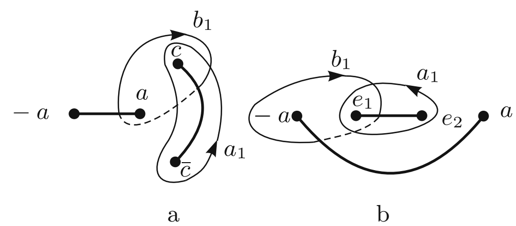

Fig. 9. The Riemann surfaces $\Gamma_{\rm ep}$ of genus 1 with the basis of cycles.

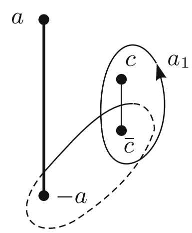

Fig. 10. The Riemann surface $\Gamma_{\rm ea}$ of genus 1 with the basis of cycles.

with the velocity

$$v = c + \bar{c}, \tag{9.2}$$

and the real period in x

$$X = \frac{i}{2N}. \tag{9.3}$$

Using $\Gamma_{\rm ep}$ (Fig. 9b), we construct solutions $\vec{Q}^{+}_{X}(x - vt)$ and $\vec{Q}^{-}_{X}(x - vt)$ corresponding to different choices of $\delta \in \mathbb{Z}/2\mathbb{R}$ in formula (8.9). It follows from Remark 9 in Sec. 8 that these solutions differ by the trivial transformation $(S_1, S_2, S_3) \to (S_2, -S_2, -S_3)$, and $\vec{Q}^{+}_{X}(x - v^t)$ is also given by expressions (9.1) with $d = d_0 + 1/4$, where d_0 is an arbitrary imaginary number. This is a cnoidal wave with the velocity

$$v = e_1 + e_2 \tag{9.4}$$

and the real period

$$X = i\frac{B}{N}, \quad N \in \mathbb{R}. \tag{9.5}$$

Using formula (8.11), we easily obtain the representation for $\vec{Q}$:

$$\begin{aligned}
Q_1 &= \frac{\theta[0,0](z|2B)\,\theta[0,0](2s_0|2B)}{\theta[0,1/2](z|2B)\theta[0,1/2](2s_0|2B)} = \frac{\theta_3(z)\,\theta_3(2s_0)}{\theta_4(z)\,\theta_4(2s_0)},\\
Q_2 &= i\frac{\theta[1/2,0](z|2B)\theta[1/2,0](2s_0|2B)}{\theta[0,1/2](z|2B)\theta[0,1/2](2s_0|2B)} = i\frac{\theta_2(z)\,\theta_2(2s_0)}{\theta_4(z)\,\theta_4(2s_0)},\\
Q_3 &= \frac{\theta[1/2,1/2](z|2B)\theta[1/2,1/2](2s_0|2B)}{\theta[0,1/2](z|2B)\theta[0,1/2](2s_0|2B)} = \frac{\theta_1(z)\,\theta_1(2s_0)}{\theta_4(z)\,\theta_4(2s_0)},
\end{aligned} \tag{9.6}$$

where $z = 4iN(-x + vt) + 2d$ and

$$\theta_1(z) = \theta\left[\frac{1}{2}, \frac{1}{2}\right](z), \qquad \theta_2(z) = \theta\left[\frac{1}{2}, 0\right](z), \qquad \theta_3(z) = \theta[0,0](z), \qquad \theta_4(z) = \theta\left[0, \frac{1}{2}\right](z)$$

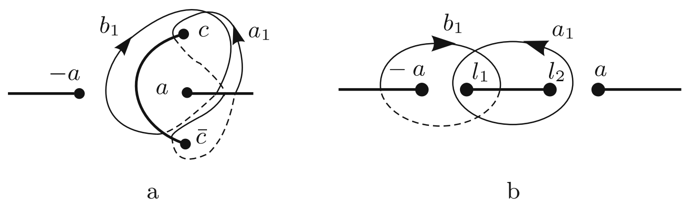

Fig. 11. The Riemann surfaces Γ_{ep} equivalent to the surfaces shown in Fig. 9 with the basis of cycles.

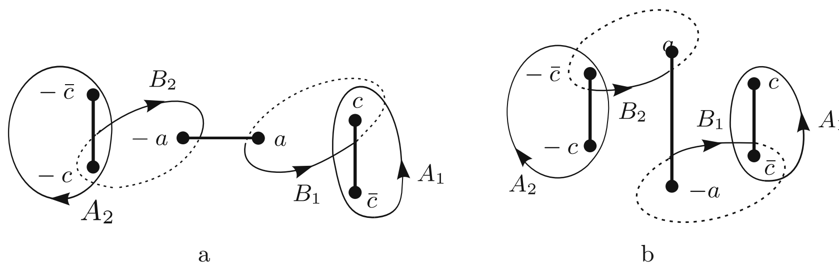

Fig. 12. (a) Symmetric surfaces Γ_{ep} and (b) symmetric surfaces Γ_{ea}.

are Jacobi theta functions [29].

We note that the solutions $\vec{Q}_X(x-vt)=(Q_1,Q_2,Q_3)$ and $\vec{Q}^{\pm}_X(x-vt)=(Q^{\pm}_1,Q^{\pm}_2,Q^{\pm}_3)$ satisfy the conditions

$$\vec{Q}_X\left(x+\frac{X}{2}-vt\right)=(Q_1,-Q_2,-Q_3),\qquad \vec{Q}^{\pm}_X\left(x+\frac{X}{2}-vt\right)=(-Q^{\pm}_1,-Q^{\pm}_2,Q^{\pm}_3).$$

The solution $\vec{Q}_X(x-vt)$ is constructed using the Riemann surface defined by the "free" boundaries of the domains c and $\bar{c}$, and $\vec{Q}^{\pm}_X(x-vt)$ is similarly defined by the boundaries of the domains e_1 and e_2. Using the curve Γ_0 with the boundaries of the domains $-c$ and $-\bar{c}$, we construct the solution $\vec{Q}_X(-x-vt)$, and using Γ_0 with the boundaries of the domains $-e_1$ and $-e_2$, we construct the solution $\vec{Q}^{\pm}_X(-x-vt))$ describing exactly the same wave moving toward the wave $\vec{Q}_X(x-vt)$.

Formulas (9.2) and (9.4) demonstrate that if $c=-\bar{c}$ (if $e_1=-e_2$), then we obtain stationary periodic solutions with periods (9.3) and (9.5), denoted by $\vec{Q}_X(x)$ (as $\vec{Q}^{\pm}_X(x)$).

Certainly, solutions of type (9.1) and (9.6) can be easily found by direct substitution of $\vec{S}(x-vt)$ in formula (2.1). But we note that in addition to solutions (9.1) and (9.6), we have found the corresponding Ψ functions, which allow applying the "dressing" procedure (see Sec. 4), i.e., constructing solutions describing the interaction of the cnoidal waves $\vec{Q}$ with solitons, breathers, and domain walls.

The solution constructed using the curve Γ_0 of genus $g=2$ is a two-phase solution describing the interaction of two cnoidal waves $\vec{Q}_{X_1}(x-v_1t)$ and $\vec{Q}_{X_2}(x-v_2t)$. In the general case, it is expressed in terms of two-dimensional Riemann theta functions. It was shown in [30], [31] that in some cases, multiphase solutions that are not degenerate (i.e., not reducible to the interaction of the cnoidal waves and solitons) can also be expressed in terms of elliptic functions. Several different methods for selecting solutions written in terms of lower-dimensional elliptic and theta functions among general finite-gap solutions were proposed in [27], [30], [32], [33]. A scheme based on reducing the multidimensional theta functions corresponding to Riemann surfaces with rich automorphism groups was given in [32].

We consider the simplest such surfaces of genera $g=2$ and $g=3$. The curves Γ_{ep} and Γ_{ea} (see Fig. 12) given by

$$\omega^2=(\lambda^2-a^2)(\lambda^2-c^2)(\lambda^2-\bar{c}^2) \tag{9.7}$$

have the involution $\phi\colon (\lambda,\omega)\to(-\lambda,-\omega)$,[5] i.e., the surface Γ_0 is a cover of the curve Γ_0/ϕ of genus $g=1$. This is a branched two-sheet cover. The general scheme for reducing the theta function of such covers is given in the appendix. In the notation in the appendix, $\widehat{\Gamma}=\widehat{C}$, the surface $C=\widehat{C}/\varphi$ is given by

$$\omega_1^2 = z(z-a^2)(z-c^2)(z-\bar{c}^2), \tag{9.8}$$

and the fixed points of the involution ϕ are $\lambda=\infty$ on both sheets ($n=1$). The cycles are $A_1=a_1$, $B_1=b_1$, $A_2=a_{1'}$, and $B_2=b_{1'}$. The normalized holomorphic differentials of $\widehat{C}$ are

$$dU_1 = u_1 = \frac{-\alpha\lambda+\beta}{\omega}\,d\lambda \qquad dU_2 = u_{1'} = \frac{-\alpha\lambda-\beta}{\omega}\,d\lambda.$$

The normalized holomorphic differential v of the curve C given by (9.8) is

$$v = u_1 - u_{1'} = \frac{2\beta}{\omega}\,d\lambda = \frac{\beta}{\omega_1}\,dz, \qquad z=\lambda^2. \tag{9.9}$$

The normalized Prym differential

$$w = u_1 + u_{1'} = -\frac{2\alpha\lambda}{\omega}\,d\lambda = -\frac{\alpha}{\omega_2}\,dz \tag{9.10}$$

is also an elliptic differential. It is defined on the curve C_π given by

$$\omega_2^2 = (z-a^2)(z-c^2)(z-\bar{c}^2). \tag{9.11}$$

The curves C and C_π (see Fig. 13) in the easy-plane case are shown together with the basis corresponding to the differentials v and w (the figures for the curve Γ_{ea} are quite similar).

The constants α and β can be found from the normalization conditions

$$\oint_A v = 1, \qquad \oint_A w = 1, \tag{9.12}$$

where we integrate over the A cycles of the respective surfaces C given by (9.8) and C_π given by (9.11). The period matrix of the curve $\widehat{C}$ is (see formula (A.4) in the appendix)

$$B = \frac{1}{2}\begin{pmatrix} \Pi+T & \Pi-T \\ \Pi-T & \Pi+T \end{pmatrix}, \qquad T=\oint_B v, \qquad \Pi=\oint_B w. \tag{9.13}$$

It defines the theta function represented according to relation (A.8) in terms of the one-dimensional theta functions:

$$\begin{aligned} \theta((z_1,z_2)|B) &= \theta[0,0](z_1+z_2|2\Pi)\,\theta[0,0](z_1-z_2|2T) + \\ &\quad + \theta\left[\frac{1}{2},0\right](z_1+z_2|2\Pi)\theta\left[\frac{1}{2},0\right](z_1-z_2|2T) = \\ &= \theta_3(z_1+z_2)|2\Pi)\,\theta_3(z_1-z_2)|2T) + \theta_2(z_1+z_2|2\Pi)\,\theta_2(z_1-z_2)|2T), \end{aligned} \tag{9.14}$$

[5] We note that ϕ does not transpose the sheets of Γ_0.

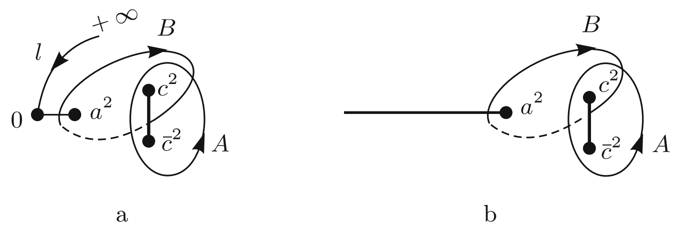

Fig. 13. (a) The curve C and (b) the curve C_π.

where θ_3 and θ_2 are the Jacobi theta functions. The quantities V, W, and r_0 (see Theorem 6 in Sec. 7) also have a certain symmetry. It follows from identity (7.16) that

$$V = 4\pi i\alpha\begin{pmatrix}1\\1\end{pmatrix}, \qquad W = 8\pi i\beta\begin{pmatrix}1\\-1\end{pmatrix}. \tag{9.15}$$

For the integral r_0, we have

$$r_0 = \int_s d\vec{U} = \int_s \begin{pmatrix}u_1\\u_{1'}\end{pmatrix} = \int_{\phi s}\begin{pmatrix}\phi(u_1)\\\phi(u_{1'})\end{pmatrix} = -\int_{\phi s}\begin{pmatrix}u_{1'}\\u_1\end{pmatrix} = -\int_s\begin{pmatrix}u_{1'}\\u_1\end{pmatrix},$$

whence

$$r_0 = s_0\begin{pmatrix}1\\-1\end{pmatrix}, \qquad s_0 = \int_l w, \tag{9.16}$$

where the contour l is shown in Fig. 13a. Substituting relations (9.14)–(9.16) in (7.13), we obtain the final expressions for the solution $\vec{S}(x,t) = \vec{Q}_{X,\tau}(x,t)$ (see (2.15)):

$$\begin{aligned}
A &= \theta_3(z_1|2\Pi)\,\theta_3(z_2 - s_0|2T) + \theta_2(z_1|2\Pi)\,\theta_2(z_2 - s_0|2\Pi),\\
B &= \theta_3(z_1|2\Pi)\,\theta_3(z_2 + s_0|2T) + \theta_2(z_1|2\Pi)\,\theta_2(z_2 + s_0|2\Pi),\\
C &= \theta_3(z_1|2\Pi)\,\theta_3(z_2 - s_0|2T) - \theta_2(z_1|2\Pi)\,\theta_2(z_2 - s_0|2\Pi),\\
D &= -\theta_3(z_1|2\Pi)\,\theta_3(z_2 + s_0|2T) + \theta_2(z_1|2\Pi)\,\theta_2(z_2 + s_0|2\Pi),
\end{aligned} \tag{9.17}$$

where $z_1 = 4i\alpha x + d_1$, $z_2 = 8i\beta t + d_2$, s_0 is integral (9.16), and the quantities $d_1, d_2 \in \mathbb{R}$ are arbitrary. In deriving this formula, we take condition (8.9) ($\mu = 0$) and the fact that $n = (1/2, 1/2)$ into account. The quantities α and β are defined from formulas (9.9), (9.10), and (9.12), where v and w are the differentials of the curves C given by (9.8) and C_π given by (9.11), whose periods are given by (9.13).

The solution $\vec{Q}_{X,\tau}(x,t)$ given by (9.15) is a standing wave periodic in x with the period $X = 1/4i\alpha$ and periodic in t with the period $\tau = 1/8i\beta$. It describes the nontrivial interaction of two waves $\vec{Q}(x - vt)$ and $\vec{Q}(-x - vt)$ (see (9.2)), running towards each other with equal velocities. Each of the quantities $A \pm C$ and $B \pm D$ is just the product of two functions with one of them depending only on x and the other depending only on t. In that sense, solution (9.17) is an analogue of the known Lamb ansatz [31] for the sine-Gordon equation.

Another interesting solution in the easy-plane case corresponds to Γ_{ep} (see Fig. 14). This solution can be analyzed quite similarly to the case considered above. As a result, we obtain four ($2^\mu = 4$) standing waves describing the interactions of various pairs of waves $\vec{Q}^{\pm}_X(x - vt)$ and $\vec{Q}^{\pm}_X(-x - vt)$ (we obviously have four combinations) given by expressions (9.17) with

$$z_1 = 4i\alpha x + d_1 + \frac{\delta_1}{2}, \qquad z_2 = 8i\beta t + d_2 + \frac{\delta_2}{2},$$

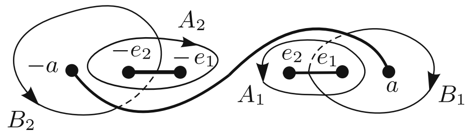

Fig. 14. The symmetric surface Γ_{ep} of genus 2.

where $d_{1,2}$ are arbitrary numbers, $\mathrm{Re}\, d_{1,2} = 0$, and $\delta_{1,2}$ can take the values 0 and 1.

It follows from Remark 9 in Sec. 8 that only two solutions are essentially different, for instance, those corresponding to the choices $\delta = (0, 0)$ or $\delta = (1, 0)$, i.e., solutions describing the interaction of the waves Q^+ and Q^+ or Q^+ and Q^-. Two other standing waves differ from them trivially.

We also note that the solution describing the interaction of the stationary wave $\vec{Q}(x)$ with the cnoidal wave $\vec{Q}(x - vt)$ corresponds to the genus-2 surface given by the equation

$$\omega^2 = (\lambda^2 - a^2)(\lambda - c)(\lambda - \bar{c})(\lambda^2 + d^2), \quad d \in \mathbb{R},$$

which does not have the additional symmetry. This interaction is therefore more complicated and is described by two-dimensional theta functions.

Finally, we consider the special solution of genus $g = 3$ corresponding to the surface $\widehat{C}$ given by

$$\omega^2 = (\lambda^2 - a^2)(\lambda^2 - c^2)(\lambda^2 - \bar{c}^2)(\lambda^2 + d^2), \quad d \in \mathbb{R}. \tag{9.18}$$

As can be seen from Fig. 15 (everything is quite similar in the easy-axis case), the solution corresponding to this surface describes the interaction of the three waves $\vec{Q}(x)$, $\vec{Q}(x - vt)$, and $\vec{Q}(-x - vt)$. Surface (9.18) has the involution ϕ: $(\lambda, \omega) \to (-\lambda, \omega)$, which does not transpose the infinite points ∞^+ and ∞^- and acts on the basis of the homology group $H_1(\widehat{C}, \mathbb{Z})$ (see Fig. 15) as shown in the appendix. Here, $C = \widehat{C}/\phi$, i.e., the surface C is given by

$$\omega_1^2 = (z - a^2)(z - c^2)(z - \bar{c}^2)(z + d^2), \tag{9.19}$$

where $\hat{g} = 3$, $g = 1$, $n = 2$, and the fixed points are 0 and ∞. The normalized differential of the surface C is

$$v = u_1 - u_{1'} = \frac{\alpha\lambda}{\omega}\, d\lambda = \frac{\alpha}{2\omega_1}\, dz, \qquad z = \lambda^2,$$

and the normalized Prym differentials are given by

$$\begin{aligned} w_1 &= u_1 + u_{1'} = \frac{\beta_1\lambda^2 + \gamma_1}{\omega}\, d\lambda = \frac{\beta_1 z + \gamma_1}{2\omega_2}\, dz, \\ w_2 &= u_2 = \frac{\beta_2\lambda^2 + \gamma_2}{\omega}\, d\lambda = \frac{\beta_2 z + \gamma_2}{2\omega_2}\, dz, \end{aligned} \tag{9.20}$$

where $\omega_2^2 = z(z - a^2)(z - c^2)(z - \bar{c}^2)(z + d^2)$. It hence follows with equality (7.16) taken into account that the vectors V and W are

$$V = -2\pi i \begin{pmatrix} \beta_1 \\ 2\beta_2 \\ \beta_1 \end{pmatrix}, \qquad W = 4\pi i \begin{pmatrix} \alpha \\ 0 \\ \alpha \end{pmatrix}. \tag{9.21}$$

According to (A.4), for the b-period matrix, we obtain

$$B = \begin{pmatrix} (\Pi_{11} + T)/2 & \Pi_{12} & (\Pi_{11} - T)/2 \\ \Pi_{12} & 2\Pi_{22} & \Pi_{12} \\ (\Pi_{11} - T)/2 & \Pi_{12} & (\Pi_{11} + T)/2 \end{pmatrix}. \tag{9.22}$$

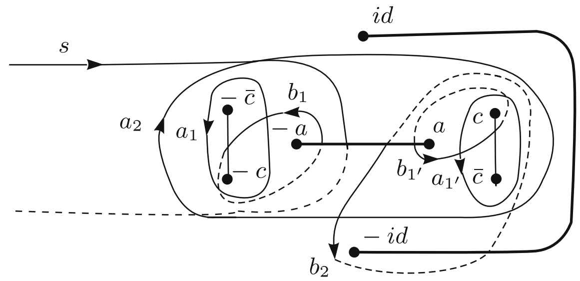

Fig. 15

An analysis similar to (9.16) shows that the integral

$$r_0 = \int_s d\vec{U} = \int_s \begin{pmatrix} u_1 \\ u_2 \\ u_{1'} \end{pmatrix} = -\int_{\phi s} \begin{pmatrix} u_{1'} \\ u_2 \\ u_1 \end{pmatrix} = -\int_s \begin{pmatrix} u_{1'} \\ u_2 \\ u_1 \end{pmatrix} - \int_{-2b_2+a_2} \begin{pmatrix} u_{1'} \\ u_2 \\ u_1 \end{pmatrix}$$

(the path s is shown in Fig. 15) is equal to

$$r_0 = \begin{pmatrix} s_0 \\ -1/2 \\ s_0 \end{pmatrix} + \begin{pmatrix} \Pi_{12} \\ -2\Pi_{22} \\ \Pi_{12} \end{pmatrix}, \qquad s_0 = \int_\infty^0 v. \tag{9.23}$$

Performing the necessary calculations (see (A.8)), we finally find that for any real d_1, d_2, and d_3, the solution of Eq. (2.1) given by formulas (2.15) is determined by the quantities

$$\begin{aligned}
A &= \theta\left[\left(0, \frac{1}{2}\right), (0,0)\right]\left(\begin{matrix} z_1 \\ z_2 \end{matrix}\middle| 2\Pi\right)\theta[0,0](z_3 - s_0|2T) + \\
&\quad + \theta\left[\left(\frac{1}{2}, \frac{1}{2}\right), (0,0)\right]\left(\begin{matrix} z_1 \\ z_2 \end{matrix}\middle| 2\Pi\right)\theta\left[\frac{1}{2}, 0\right](z_3 - s_0|2T), \\
B &= \theta\left[\left(0, \frac{1}{2}\right), (0,0)\right]\left(\begin{matrix} z_1 \\ z_2 \end{matrix}\middle| 2\Pi\right)\theta[0,0](z_3 + s_0|2T) + \\
&\quad + \theta\left[\left(\frac{1}{2}, \frac{1}{2}\right), (0,0)\right]\left(\begin{matrix} z_1 \\ z_2 \end{matrix}\middle| 2\Pi\right)\theta\left[\frac{1}{2}, 0\right](z_3 + s_0|2T), \\
C &= \theta\left[\left(0, \frac{1}{2}\right), \left(0, \frac{1}{2}\right)\right]\left(\begin{matrix} z_1 \\ z_2 \end{matrix}\middle| 2\Pi\right)\theta[0,0](z_3 - s_0|2T) + \\
&\quad + \left[\left(\frac{1}{2}, \frac{1}{2}\right), \left(0, \frac{1}{2}\right)\right]\left(\begin{matrix} z_1 \\ z_2 \end{matrix}\middle| 2\Pi\right)\theta\left[\frac{1}{2}, 0\right](z_3 - s_0|2T), \\
D &= \theta\left[\left(0, \frac{1}{2}\right), \left(0, \frac{1}{2}\right)\right]\left(\begin{matrix} z_1 \\ z_2 \end{matrix}\middle| 2\Pi\right)\theta[0,0](z_3 + s_0|2T) + \\
&\quad + \theta\left[\left(\frac{1}{2}, \frac{1}{2}\right), \left(0, \frac{1}{2}\right)\right]\left(\begin{matrix} z_1 \\ z_2 \end{matrix}\middle| 2\Pi\right)\theta\left[\frac{1}{2}, 0\right](z_3 + s_0|2T).
\end{aligned} \tag{9.24}$$

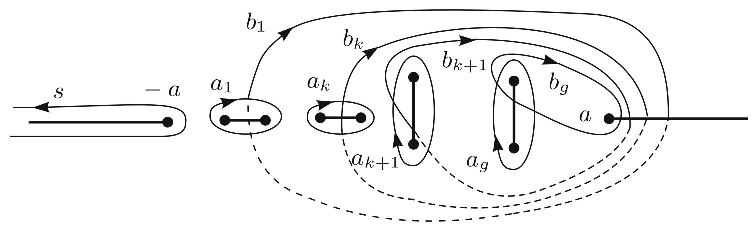

Fig. 16. Degeneration into the multisoliton solution: the surface $\Gamma_{\rm ep}$.

Here,

$$z_1 = -2i\beta_1 x + d_1, \qquad z_2 = -2i\beta_2 x + d_2, \qquad z_3 = 4i\alpha t + d_3.$$

This solution is periodic with the period $\tau = 1/4i\alpha$ in t, and the dependence on x is complicated and nonperiodic.

Similarly, for any curve of genus $\hat{g} = 2g+n-1$ having an involution with the fixed points ∞^+ and ∞^-, the dynamics in x is confined to the $(g+n-1)$-dimensional Prymian Π, and the dynamics in t is confined to the g-dimensional Jacobian T.

10. The *N*-soliton solutions in the "easy plane" case

We consider the curve $\Gamma_{\rm ep}$ shown in Fig. 16 together with the basis of cycles and subject the branch points E_j, $j = 1, \dots, 2g$, to the limit transition

$$E_{2k-1}, E_{2k} \to \lambda_k \in (-a, a), \quad k = 1, \dots, g, \quad \lambda_1 < \lambda_2 < \cdots \lambda_g. \tag{10.1}$$

In this case, the curve $\Gamma_{\rm ep}$ degenerates into a curve of genus zero (the Riemann surface of the function $\sqrt{\lambda^2 - a^2}$), and the holomorphic differentials $dU_\nu(\lambda)$ degenerate into differentials with singularities at the points λ_k:

$$dU_\nu(\lambda) \to dU_\nu^0(\lambda) = \frac{\varphi_\nu^0(\lambda)}{\sqrt{\lambda^2 - a^2}\prod_{k=1}^g(\lambda - \lambda_k)}\,d\lambda,$$

where

$$\varphi_\nu^0(\lambda) = \sum_{k=1}^g c_\nu^{0,k}\lambda^{g-k}.$$

The polynomials $\varphi_\nu^0(\lambda)$ are defined by the normalization conditions

$$\delta_\mu^\nu = \int_{a_\mu} dU_\nu^0(\lambda) = -2\pi i \operatorname{res}\big(dU_\nu^0(\lambda); \lambda_\mu\big) =$$

$$= -\frac{2\pi i}{\sqrt{\lambda_\mu^2 - a^2}}\varphi_\nu^0(\lambda_\mu)\frac{1}{\prod_{k\neq\mu}(\lambda_\mu - \lambda_k)},$$

and consequently

$$\varphi_\nu^0(\lambda) = c_\nu^{0,1}\prod_{k\neq\nu}(\lambda - \lambda_k), \qquad c_\nu^{0,1} = \frac{i\sqrt{\lambda_\nu^2 - a^2}}{2\pi} = -\frac{\varkappa_\nu}{2\pi}, \quad \varkappa_\nu = \sqrt{a^2 - \lambda_\nu^2} > 0. \tag{10.2}$$

Therefore, the differentials $dU_\nu^0(\lambda)$ can be written in the form

$$dU_\nu^0(\lambda) = -\frac{1}{2\pi}\frac{\varkappa_\nu}{\sqrt{\lambda^2 - a^2}(\lambda - \lambda_\nu)}\,d\lambda. \tag{10.3}$$

By virtue of (10.3), the limit values for the coefficients $c_\nu^{0,2}$ are

$$c_\nu^{0,2} = -c_\nu^{0,1} \sum_{k\neq\nu} \lambda_k = \frac{1}{2\pi}\varkappa_\nu \sum_{k\neq\nu} \lambda_k, \quad \nu = 1,\dots,g.$$

Hence, for the components of the vectors V and W, we have

$$\begin{aligned} V_\nu &\to V_\nu^0 = -4\pi i c_\nu^{0,1} = 2i\varkappa_\nu, \\ W_\nu &\to W_\nu^0 = 8\pi i\Big(c_\nu^{0,1}\sum_{k=1}^{g}\lambda_k + c_\nu^{0,2}\Big) = -4i\varkappa_\nu\lambda_\nu. \end{aligned} \tag{10.4}$$

We now proceed to calculate the limit values for the b-period matrix of the basis $d\vec{U}$. Let $\nu > \mu$. Then

$$B_{\nu\mu} \to B_{\nu\mu}^0 = 2\int_{\lambda_\nu}^{a} dU_\mu^0 = -\frac{i}{\pi}\log\frac{\gamma_\mu - \gamma_\nu}{\gamma_\mu + \gamma_\nu}, \qquad \gamma_\nu = \sqrt{\frac{a-\lambda_\nu}{a+\lambda_\nu}} > 0. \tag{10.5}$$

By the symmetry of the matrix B for $\nu < \mu$, from (10.5), we obtain

$$B_{\nu\mu}^0 = -\frac{i}{\pi}\log\frac{\gamma_\nu - \gamma_\mu}{\gamma_\nu + \gamma_\mu}. \tag{10.6}$$

for $B_{\nu\mu}^0$. The diagonal elements of B do not have finite limits. Simple calculations show that

$$\mathrm{Re}(iB_{\nu\nu}) = \frac{1}{\pi}\log|E_{2\nu+1} - E_{2\nu}| + O(1),$$

i.e.,

$$\mathrm{Re}(iB_{\nu\nu}) \to -\infty \tag{10.7}$$

for the considered limit.

It remains to discuss the behavior of the vectors r and $\mathcal{D}$. For the vector r, we have

$$r_\nu \to \frac{1}{2\pi i} r_\nu^0 + 1, \qquad r_\nu^0 = -4\pi i\int_a^{\infty^+} dU_\nu(\lambda) = -2\log\frac{i\gamma_\nu + 1}{i\gamma_\nu - 1}. \tag{10.8}$$

We note that $\mathrm{Re}\, r_\nu^0 = 0$. The vector $\mathcal{D}$ is a free parameter, and we can prescribe its behavior at limit (10.1) however we like. Let

$$\mathcal{D}_\nu = \frac{1}{2}B_{\nu\nu} + \frac{1}{2\pi i}\eta_\nu^0 + o(1), \tag{10.9}$$

where η_ν^0 are still arbitrary complex numbers. This completes the calculations of the limit values for all parameters in formulas (7.13), and we can now write the limit expressions for the corresponding solutions of the LL equation.

We represent the exponent in the definition of the series of the function $\theta(\Omega + \mathcal{D} + lr + kn)$ at $l = 0, 1$ and $k = 0, 1$ in the form

$$\begin{aligned} &\pi i\sum_{\nu=1}^{g} B_{\nu\nu} m_\nu(m_\nu + 1) + 2\pi i\sum_{\nu>\mu} B_{\nu\mu} m_\nu m_\mu + \\ &\qquad + \sum_{\nu=1}^{g} m_\nu\big(V_\nu x + W_\nu t + \eta_\nu^0 + l r_\nu^0 + k\pi i + o(1)\big). \end{aligned}$$

At limit (10.1) by virtue of (10.7), only terms corresponding to the vectors m from the set of vertices $\{0,-1\}^g$ of the cube $[0,-1]^g$ remain in the total infinite sum included in the definition of $\theta(\Omega+\mathcal{D}+lr+kn)$. Therefore, taking formulas (10.4)–(10.8) into account, we conclude that in limit (10.1), (10.9), we have $\theta(\Omega+\mathcal{D}+lr+kn)\to\theta_k^l(x.t)$, where

$$\begin{aligned}\theta_k^l(x,t) = \sum_{m\in\{0,-1\}^g} &\exp\biggl\{\sum_{\nu>\mu}\log\left|\frac{\gamma_\nu-\gamma_\mu}{\gamma_\nu+\gamma_\mu}\right|^2 m_\nu m_\mu\biggr\} + \\ &+\sum_{\nu=1}^{g} m_\nu(-2\varkappa_\nu x+4\varkappa_\nu\lambda_\nu t+\eta_\nu^0+lr_\nu^0+k\pi i).\end{aligned} \tag{10.10}$$

The solutions of the LL equation obtained as a result of taking the considered limit are described by the formulas

$$A=\theta_0^0(x,t),\qquad B=\theta_0^1(x,t),\qquad C=\theta_1^0(x,t),\qquad D=-\theta_1^1(x,t). \tag{10.11}$$

The conditions for the vector η^0 ensuring the realness of solution (10.11) are the last to be explored. Because all the quantities γ_ν, $\varkappa_\nu$, and λ_γ are real and r_ν^0 are purely imaginary, conjugating $\theta_k^l(x,t)$ means only simply replacing η_ν^0 with $\overline{\eta^0}_\nu$ and r_ν^0 with $-r_\nu^0$ in the right-hand side of (10.10). Hence, it is easy to understand that relation (8.1) in the considered case is equivalent to the requirement

$$\overline{\eta^0}_\nu-\eta_\nu^0=\pi i+r_\nu^0+2\pi iz,\quad z=0,-1. \tag{10.12}$$

Therefore, formulas (10.10) and (10.11) under condition (10.12) describe real solutions of the LL equation that can be parameterized by $2g$ real parameters $(\lambda_\gamma,\operatorname{Re}\eta_\nu)$, $\nu=1,\ldots,g$. For $g=1$, we obtain the simple soliton constructed above in Sec. 5 by the "dressing" method:

$$\theta_k^l=1+e^{-2\varkappa x+4\varkappa\lambda t-\eta^0-lr^0-\pi ik},$$

and consequently

$$\begin{aligned}S_1(x,t)&=-\tanh\bigl(2\sqrt{a^2-\lambda^2}(x-2\lambda t)+\Delta\bigr),\\ S_2(x,t)&=\mp\frac{\lambda}{a}\frac{1}{\cosh\bigl(2\sqrt{a^2-\lambda^2}(\varkappa-2\lambda t)+\Delta\bigr)},\\ S_3(x,t)&=\pm\frac{\sqrt{a^2-\lambda^2}}{a}\frac{1}{\cosh(2\varkappa x-4\varkappa\lambda t+\Delta)},\end{aligned} \tag{10.13}$$

where $\Delta=-\operatorname{Re}\eta^0$ and the signs plus in S_3 and minus in S_2 correspond to the choice $z=-1$ in (10.12). For $g>1$, formulas (10.10) and (10.11) describe the processes of interactions between g simple solitons (10.13). A simple standard analysis (see, e.g., [34]) of the sum in the right-hand side of (10.10) as $t\to\pm\infty$ with $x-2\lambda_j t=\text{const}$ shows that the jth soliton with the velocity λ_j and phase Δ_j^- as $t\to-\infty$ has the same velocity λ_j but the phase

$$\Delta_j^+=\Delta_j^-+2\sum_{\nu=j+1}^{g}\log\left|\frac{\gamma_j-\gamma_\nu}{\gamma_j+\gamma_\nu}\right|-2\sum_{\nu=1}^{j-1}\log\left|\frac{\gamma_j-\gamma_\nu}{\gamma_j+\gamma_\nu}\right|$$

as $t\to+\infty$.

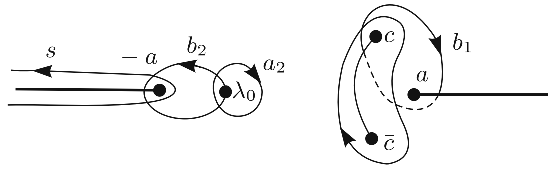

Fig. 17. The surface $\Gamma_{\rm ep}$ in the case of ptl degeneration: complex branch points.

11. Interaction of a simple soliton with a cnoidal wave: The "easy-plane" case

We consider the "partial" degeneration of the curve $\Gamma_{\rm ep}$ in the preceding section assuming $g = 2$:

$$E_1 \to E_2 \to \lambda_0 \in (-a, a), \qquad E_3 = \bar{E}_4 \equiv c, \quad \operatorname{Im} c \neq 0. \tag{11.1}$$

The "limit" curve is shown in Fig. 17. Unlike the situation in Sec. 10, the genus of the limiting curve does not reduce to zero. Also, one of the two holomorphic differentials remains holomorphic in this limit:

$$\begin{aligned} dU_1(\lambda) &\to dU_1^0(\lambda) = \frac{c_1^{0,1}}{\sqrt{(\lambda^2 - a^2)(\lambda - c)(\lambda - \bar{c})}} d\lambda, \\ dU_2(\lambda) &\to dU_2^0(\lambda) = \frac{c_2^{0,1}\lambda + c_2^{0,2}}{(\lambda - \lambda_0)\sqrt{(\lambda^2 - a^2)(\lambda - c)(\lambda - \bar{c})}} d\lambda, \end{aligned} \tag{11.2}$$

The normalization conditions in this limit are

$$\begin{aligned} c_1^{0,1} &= \left(\int_{a_1} \frac{d\lambda}{\sqrt{(\lambda^2 - a^2)(\lambda - c)(\lambda - \bar{c})}} \right)^{-1}, \\ c_2^{0,1}\lambda_0 + c_2^{0,2} &= -\frac{1}{2\pi i}\sqrt{(\lambda_0^2 - a^2)(\lambda_0 - c)(\lambda_0 - \bar{c})}, \\ \int_{a_1} & \frac{c_2^{0,1}\lambda + c_2^{0,2}}{(\lambda - \lambda_0)\sqrt{(\lambda_0^2 - a^2)(\lambda_0 - c)(\lambda_0 - \bar{c})}}\, d\lambda = 0. \end{aligned} \tag{11.3}$$

We hence obtain the representations for the components of the vectors V and W

$$\begin{aligned} V_1 &= 4\pi i N, & V_2 &= -2i\widehat{\varkappa}_0, \\ W_1 &= -4\pi i N v, & W_2 &= -4i\widehat{\varkappa}_0\hat{\lambda}_0, \end{aligned} \tag{11.4}$$

where

$$N = c_1^{0,1} = \left(\int_{a_1} \frac{d\lambda}{\sqrt{(\lambda^2 - a^2)(\lambda - c)(\lambda - \bar{c})}} \right)^{-1}, \qquad v = c + \bar{c}, \qquad \hat{\lambda}_0 = \lambda_0 + \frac{v}{2} + \Lambda_0,$$

$$\Lambda_0 = -\left(\int_{a_1} \frac{\lambda\, d\lambda}{(\lambda - \lambda_0)\sqrt{(\lambda^2 - a^2)(\lambda - c)(\lambda - \bar{c})}} \right) \times$$

$$\times \left(\int_{a_1} \frac{d\lambda}{(\lambda - \lambda_0)\sqrt{(\lambda^2 - a^2)(\lambda - c)(\lambda - \bar{c})}} \right)^{-1},$$

$$\widehat{\varkappa}_0 = -2\pi c_2^{0,1} = 2\pi \frac{\frac{1}{2\pi i}\sqrt{(\lambda_0^2 - a^2)(\lambda_0 - c)(\lambda_0 - \bar{c})}}{\lambda_0 + \Lambda_0}.$$

In the limit, the vector r and the elements B_{1j} of the b-period matrix are also expressed in terms of elliptic integrals:

$$r \to r^0 = \int_s dU^0(\lambda), \qquad B^0_{1j} = \int_{b_1} dU^0_j(\lambda), \quad j = 1, 2.$$

Unlike the situation in Sec. 10, only the element B_{22} of the b-period matrix tends to $i\infty$. Therefore, a condition of type (10.9) can be naturally imposed on the second component of the vector $\mathcal{D}$, keeping the first component bounded:

$$\mathcal{D}_1 \equiv d^0, \qquad \mathcal{D}_2 = \frac{1}{2\pi i}\eta^0 + \frac{1}{2}B_{22} + o(1), \quad d^0, \eta^0 \in \mathbb{C}. \tag{11.5}$$

As a result, in this limit, the original sum over $m \in \mathbb{Z}^2$ in the series for $\theta(\Omega + \mathcal{D} + lr + kn)$ is replaced with the sum over $m \in \mathbb{Z} \times \{0, -1\}$, and we obtain the formula for $\theta^l_k(x,t) = \lim \theta(\Omega + \mathcal{D} + lr + m)$:

$$\theta^l_k(x,t) = \theta_3\left(2iN(x - vt) + d^0 + lr^0_1 + \frac{k}{2}\middle| B^0_{11}\right) + \\ + \theta_3\left(2iN(x - vt) + d^0 + lr^0_1 + \frac{k}{2} - B^0_{12}\middle| B^0_{11}\right) e^{-2\widehat{\varkappa}_0(x - 2\hat{\lambda}_0 t) - \eta^0 - 2\pi i l r^0_2 - k\pi i}. \tag{11.6}$$

We now discuss the problem of real solutions. The conjugation anti-involution τ in the considered realization of the curve Γ_{ep} transposes the sheets and acts on the cycles a_ν and b_1 as $\tau a_\nu = a_\nu$, $\nu = 1, 2$, and $\tau b_1 = -b_1 + a_1$. Using the same arguments as in Sec. 8, we conclude that all c^j_ν are real and the quantities N, v, $\widehat{\varkappa}_0$, and $\hat{\lambda}_0$ are consequently also real. For the b-period matrix and the vector r, we have

$$\bar{B}^0_{11} = -B^0_{11} + 1, \quad \bar{B}^0_{12} = -B^0_{12}, \qquad \bar{r}_\nu = r_\nu, \quad \nu = 1, 2. \tag{11.7}$$

Therefore, considering $\bar{C} = \overline{\theta^0_1(x,t)}$, we obtain

$$\bar{C} = \theta_3(2iN(x - vt) - \bar{d}^0 | B^0_{11}) + \\ + \theta_3\big(2iN(x - vt) - \overline{d^0} - B^0_{12} | B^0_{11}\big) e^{-2\widehat{\varkappa}_0(x - 2\hat{\lambda}_0 t) - \overline{\eta^0} + \pi i}.$$

Assuming that

$$-\overline{d^0} = d^0 + r^0_1, \qquad -\overline{\eta^0} = -\eta^0 - 2\pi i r^0_2 - \pi i, \tag{11.8}$$

we easily obtain the equality $\bar{C} = B$. It is easy to verify that relations (10.8) also ensure that the equality $\bar{A} = -D$ is satisfied, i.e., the corresponding solution of the LL equation is real.

We now summarize. We set

$$d^0 = -id - \frac{1}{2}r^0_i \equiv id - s, \qquad \eta^0 = \Delta - i\pi r^0_2 - \frac{i\pi}{2}, \qquad B^0_{11} = B,$$

where Δ and d are arbitrary real numbers and $s = r^0_1/2$. A real solution of the LL equation is then given by

$$A = \theta^0_0(x,t), \qquad B = \theta^1_0(x,t), \qquad C = \theta^0_1(x,t), \qquad D = -\theta^1_1(x,t), \tag{11.9}$$

where

$$\theta^l_k(x,t) = \theta_3\left(2iN(x - vt) + id - s + 2ls + \frac{k}{2}\middle| B\right) + \\ + \theta_3\left(2iN(x - vt) + id - s + 2ls + \frac{k}{2} - B_{12}\middle| B\right) e^{-2\widehat{\varkappa}_0(x - 2\hat{\lambda}_0 t) - \Delta + i\pi r^0_2 + i\pi/2 - 2\pi i l r^0_2 - k\pi i}.$$

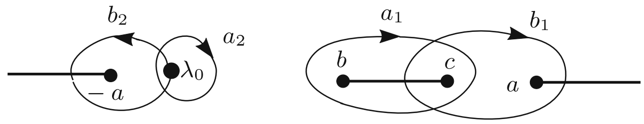

Fig. 18. The surface $\Gamma_{\rm ep}$ in the case of partial degeneration: real branch points.

Solution (11.9) can be interpreted as describing the interaction of a simple soliton characterized by the velocity $\hat{\lambda}_0$ with the cnoidal wave $\vec{Q}_X(x - vt)$ with the real period $X = i(2B - 1)/2N$ and phase velocity v. We describe this process in more detail. It follows from relations (11.9) that the presence of the soliton is significant only in the narrow strip in the plane (x, t) around the "soliton" ray $x - 2\hat{\lambda}_0 t = 0$, as expected. In the regions

$$\Omega^+ = \{(x,t)\colon \hat{\varkappa}_0(x - 2\hat{\lambda}_0 t) \ll 0\}, \qquad \Omega^- = \{(x,t)\colon \hat{\varkappa}_0(x - 2\hat{\lambda}_0 t) \gg 0\},$$

in particular, as $x \to \pm\infty$ with fixed t (as $t \to \pm\infty$ with fixed x), formulas (11.9) transform into formulas (9.1) for the unperturbed cnoidal wave:

$$\begin{aligned} A &= \theta_3(2iN(x - vt) + id - s|B), \\ B &= \theta_3(2iN(x - vt) + id + s|B), \\ C &= \theta_4(2iN(x - vt) + id - s|B), \\ D &= -\theta_4(2iN(x - vt) + id + s|B) \end{aligned} \tag{11.10}$$

in the region Ω^- and

$$\begin{aligned} A &= \theta_3(2iN(x - vt) + id - s - B_{12}|B), \\ B &= \theta_3(2iN(x - vt) + id + s - B_{12}|B), \\ C &= -\theta_4(2iN(x - vt) + id - s - B_{12}|B), \\ D &= \theta_4(2iN(x - vt) + id + s - B_{12}|B) \end{aligned} \tag{11.11}$$

in the region Ω^+ (in comparing formulas (11.10) and (11.11) with (9.1), we must take Remark 7 in Sec. 7 into account).

The action on the cnoidal wave thus has two effects: the phase shift (i.e., shift of the parameter id)

$$B_{12} = B_{21} = \int_{b_2} dU_1^0(\lambda) = -2\int_{-a}^{\lambda_0} \frac{N\,d\lambda}{\sqrt{(\lambda^2 - a^2)(\lambda - c)(\lambda - \bar{c})}}$$

and the rotation $(S_1, S_2, S_3) \to (-S_1, -S_2, -S_3)$ through $180°$ in the plane (S_1, S_2). On the other hand, as formulas (11.9) and (11.4) show, the "cnoidal" background only adds $2\Lambda_0$ to the free soliton velocity $2\lambda_0$.

Remark 10. Considering the curve $\Gamma'_{\rm ep}$ with all real branch points (see Fig. 18) instead of the curve shown in Fig. 17, we again obtain formulas (11.6) for the complex solution of the LL equation. We need only take $\bar{c} \to b$, $\operatorname{Im} c = \operatorname{Im} b = 0$, everywhere. Analyzing the realness conditions only differs from the already considered case by the fact that now $\tau b_1 = -b_1$ and consequently $\overline{B^0}_{11} = -B^0_{11}$ instead of what we have in (11.7). The last circumstance only leads to changing the conditions for the parameter $-\overline{d^0} = d^0 + r^0_1 - 1/2$, i.e.,

$$d^0 = id - \frac{1}{2}r^0_1 + \frac{1}{4} = id - s + \frac{1}{4},$$

and this in turn means that we must substitute $id \to id + 1/4$ in formulas (11.9)-(11.11).

Appendix: Branched two-sheet covers and reduction of Riemann theta functions

In this appendix, we give results concerning the reduction of the Riemann theta function contained in [35].

Let $\widehat{C} \xrightarrow{\pi} C$ be a two-sheet branched cover of genus $\hat{g} = 2g + n - 1$ of the compact Riemann surface C of genus g. Let $Q_1, \dots, Q_{2n} \in C$ be the branch points of this cover. We let $\phi\colon \widehat{C} \to \widehat{C}$ denote the involution with fixed points $Q_1, \dots, Q_{2n}$ transposing the sheets of the cover ($C = \widehat{C}/\phi$). The canonical basis of the homology group $H_1(\bar{C}, \mathbb{Z})$

$$a_1, b_1, \dots, a_g, b_g,\ a_{g+1}, b_{g+1},\ \dots,\ a_{g+n-1}, b_{g+n-1},\ a_{1'}, b_{1'},\ \dots,\ a_{g'}, b_{g'} \tag{A.1}$$

can be selected such that $\pi a_1, \pi b_1, \dots, \pi a_g, \pi b_g$ is the canonical basis of $H_1(C, \mathbb{Z})$:

$$\begin{aligned} &a_{\alpha'} + \phi a_\alpha = b_{\alpha'} + \phi b_\alpha = 0, && \alpha = 1, \dots, g, \\ &a_i + \phi a_i = b_i + \phi b_i = 0, && i = g+1, \dots, g+n-1. \end{aligned} \tag{A.2}$$

Here, ϕa_α denotes the cycles obtained from a_α under the action of ϕ. For the corresponding normalized holomorphic differentials

$$u_1, \dots, u_g, u_{g+1}, \dots, u_{g+n-1}, u_{1'}, \dots, u_{g'},$$

the equalities

$$u_\alpha(x) = -u_{\alpha'}(\phi(x)), \qquad u_i(x) = -u_i(\phi(x)), \quad x \in \widehat{C} \tag{A.3}$$

are satisfied (here $\alpha = 1, \dots, g$ and $i = g+1, \dots, g+n-1$).

The holomorphic differentials on the surface C normalized in the basis $\pi a_1, \pi b_1, \dots, \pi a_g, \pi b_g$ are $v_\alpha = u_\alpha - u_{\alpha'}$, $\alpha = 1, \dots, g$, and the expressions

$$w_\alpha = u_\alpha + u_{\alpha'}, \quad \alpha = 1, \dots, g, \qquad w_i = u_i, \quad i = g+1, \dots, g+n-1,$$

determine $g+n-1$ linearly independent normalized Prym differentials. We have

$$v_\alpha(\phi(x)) = v_\alpha(x), \qquad w_\beta(\phi(x)) = -w_\beta(x).$$

It follows from formulas (A.3) that the period matrix of $\widehat{C}$ has the form

$$B = \begin{pmatrix} (\Pi_{\alpha\beta} + T_{\alpha\beta})/2 & \Pi_{\alpha i} & (\Pi_{\alpha\beta} - T_{\alpha\beta})/2 \\ \Pi_{i\alpha} & 2\Pi_{ij} & \Pi_{i\alpha} \\ (\Pi_{\alpha\beta} - T_{\alpha\beta})/2 & \Pi_{\alpha i} & (\Pi_{\alpha\beta} + T_{\alpha\beta})/2 \end{pmatrix}, \tag{A.4}$$

where $\alpha, \beta = 1, \dots, g, i, j = g+1, \dots, g+n-1$, T is the period matrix of the surface C (composed of the differentials v_α in the basis $\pi a_1, \pi b_1, \dots, \pi a_g, \pi b_g$), and Π is the symmetric matrix of the dimension $(g+n-1) \times (g+n-1)$ given as

$$\Pi = \begin{pmatrix} \Pi_{\alpha\beta} & \Pi_{\alpha j} \\ \Pi_{i\beta} & \Pi_{ij} \end{pmatrix} = \begin{pmatrix} \displaystyle\int_{b_\beta} w_\alpha & \displaystyle\frac{1}{2}\int_{b_j} w_\alpha \\ \displaystyle\int_{b_\beta} w_i & \displaystyle\frac{1}{2}\int_{b_j} w_i \end{pmatrix}. \tag{A.5}$$

We consider the theta function with zero characteristics defined by matrix (A.4):

$$\theta(z|B) = \sum_{m\in\mathbb{Z}^{\hat{g}}} e^{\pi i\langle Bm,m\rangle+2\pi i\langle z,m\rangle}, \tag{A.6}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the usual scalar product. We let $S = (s_1|s_2|s_3)$ denote a $\hat{g}$-dimensional vector, where s_1 and s_3 are g-dimensional vectors, and s_2 is a $(n{-}1)$-dimensional vector. Let $t = (t_1|t_2)$ be a $(g{+}n{+}1)$-dimensional vector, where t_1 is a g-dimensional vector and t_2 is a $(n{-}1)$-dimensional vector.

We note that if the vector k ranges over the whole lattice $\mathbb{Z}^{\hat{g}}$ and $\hat{\delta} = (\delta|0|\delta)$, where δ ranges over all possible vectors consisting of the numbers 0 and 1/2, then the vector

$$m = N(k+\delta), \qquad N = \begin{pmatrix} I & 0 & I \\ 0 & I & 0 \\ I & 0 & -I \end{pmatrix}, \tag{A.7}$$

also ranges over $\mathbb{Z}^{\hat{g}}$ (the dimensionality of the blocks in N coincides with the dimensionality of the blocks in B in (A.4)). Therefore, the sum over m in formula (A.6) can be replaced with the sum over k and δ:

$$\langle Bm,m\rangle = \langle BN(k+\hat{\delta}), N(k+\hat{\delta})\rangle = \langle NBN(k+\hat{\delta}), (k+\hat{\delta})\rangle.$$

It is easy to see that NBN is the block matrix,

$$NBN = \begin{pmatrix} 2\Pi & 0 \\ 0 & 2T \end{pmatrix}$$

(in particular, the positive definiteness of the imaginary part of Π follows from this). Consequently,

$$\begin{aligned}\langle Bm,m\rangle + 2\langle z,m\rangle &= \langle NBN(k+\hat{\delta}), k+\hat{\delta}\rangle + 2\langle Nz, k+\hat{\delta}\rangle = \\ &= \langle 2\Pi(k_1+(\delta|0)), k_1+(\delta|0)\rangle + \langle 2T(k_2+\delta), k_2+\delta\rangle + \\ &\quad + 2\langle (z_1+z_3|z_2), k_1+(\delta|0)\rangle + 2\langle z_1-z_3, k_2+\delta\rangle,\end{aligned}$$

where $k = (k_1'|k_1''|k_2)$, $k_1 = (k_1'|k_1'')$, and $z = (z_1|z_2|z_3)$. Substituting this in expressions (A.6), we obtain the representation of the $\hat{g}$-dimensional theta function in terms of the finite sum of the products of the g-dimensional and $(g{+}n{-}1)$-dimensional theta functions:

$$\theta((z_1|z_2|z_3)|B) = \sum_{\delta\in\frac{1}{2}\mathbb{Z}^g/2\mathbb{Z}^g} \theta[(\delta|0),0]((z_1+z_3|z_2)|2\Pi)\,\theta[\delta|0](z_1-z_3|2T). \tag{A.8}$$

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