# On the Lagrangian Structure of Integrable Quad-Equations

ALEXANDER I. BOBENKO[1] and YURI B. SURIS[2]

[1]*Institut für Mathematik, MA 8-3, TU Berlin, Str. des 17. Juni 136, 10623 Berlin, Germany. e-mail: bobenko@math.tu-berlin.de*

[2]*Institut für Mathematik, MA 7-2, TU Berlin, Str. des 17. Juni 136, 10623 Berlin, Germany. e-mail: suris@math.tu-berlin.de*

**Abstract.** The new idea of flip invariance of action functionals in multidimensional lattices was recently highlighted as a key feature of discrete integrable systems. Flip invariance was proved for several particular cases of integrable quad-equations by Bazhanov, Mangazeev and Sergeev and by Lobb and Nijhoff. We provide a simple and case-independent proof for all integrable quad-equations. Moreover, we find a new relation for Lagrangians within one elementary quadrilateral which seems to be a fundamental building block of the various versions of flip invariance.

## 1. Introduction

This paper deals with some aspects of the variational (Lagrangian) structure of integrable systems on quad-graphs (planar graphs with quadrilateral faces), which serve as discretizations of integrable PDEs with a two-dimensional space–time [1,8]. We identify integrability of such systems with their multidimensional consistency [8,15]. This property was used in [1] to classify integrable systems on quad-graphs. That paper also introduced a Lagrangian formulation for them. The variational structure of discrete integrable systems is a topic which receives increasing attention in the recent years [13,16], after the pioneering work [14].

Lobb and Nijhoff [11] introduced the new idea to extend the action functional of [1] to a multidimensional lattice. The key property that makes this meaningful is the invariance of the action under elementary 3D flips of 2D quad-surfaces in $\mathbb{Z}^m$. This property was established in [11] for several particular cases of integrable equations. The proof involves computations based in particular on properties

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of the dilogarithm function. In the present paper, we prove the flip invariance for all integrable quad-equations classified in [1]; our proof is case-independent. Note that three-dimensional discrete integrable systems also possess Lagrangian formulations [6,12], and the flip invariance of action for the discrete KP equation was established in [12].

A closely related version of flip invariance of action for discrete systems of Laplace type was discussed earlier for one concrete example by Bazhanov et al. [5]. The action functional in this paper describes circle patterns and was introduced in [7]. In [5], this action was derived as a quasi-classical limit of the partition function of an integrable quantum model investigated in [10] (the Lagrangians being the quasi-classical limit of the Boltzmann weights). Invariance of the partition function under star – triangle transformations is a hallmark of integrability in the quantum context, it is usually established with the help of the quantum Yang – Baxter relation [4]. It is surprising that only now a correct classical counterpart comes to the light. Here, we extend the quasi-classical result of [5] to the whole class of integrable quad-equations. Finding the quantum version of our contribution remains an open problem.

The structure of the paper is as follows. In Section 2, we recall the definition and the classification of integrable systems on quad-graphs, the so-called ABS list [1] (see also the recent monograph [9]). In Section 3, we recall our main technical device which plays a prominent role in the subject of the present paper, namely the three-leg form of a quad-equation. Further, we recall a variational (Lagrangian) interpretation of integrable quad-equations, again following [1]. In Section 5, we prove a novel relation for Lagrangians within one elementary quadrilateral which seems to be a fundamental building block of the various versions of the flip invariance. Finally, Section 6 contains generalizations of the flip invariance results from [11] and [5] with a new case-independent proof.

The flip invariance of the action functional in multidimensional lattices is a fascinating new idea which will definitely have a serious impact on the theory of discrete integrable systems.

## 2. Integrable Systems on Quad-Graphs

We consider systems on quad-graphs, i.e., collections of equations on elementary quadrilaterals of the type

$$Q(x, u, y, v; \alpha, \beta) = 0, \tag{1}$$

where $x, u, y, v \in \mathbb{CP}^1$ are the complex variables ("fields") assigned to the four vertices of the quadrilateral, and the parameters $\alpha, \beta \in \mathbb{C}$ are assigned to its edges, as shown in Figure 1. It is required that opposite edges of any quadrilateral carry the same parameter. The function $Q$ is assumed to be multi-affine, i.e., a polynomial of degree one in each field variable. Moreover, it is supposed to possess the following property:
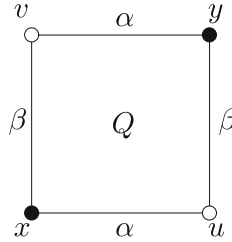
*Figure 1.* An elementary quadrilateral.

- *Symmetry*: The equation $Q=0$ is invariant under the dihedral group $D_4$ of the square symmetries:

$$Q(x, u, y, v; \alpha, \beta) = \varepsilon_1 Q(x, v, y, u; \beta, \alpha) = \varepsilon_2 Q(u, x, v, y; \alpha, \beta),$$

with $\varepsilon_1, \varepsilon_2 = \pm 1$.

As in [8,15], we consider integrability as synonymous with 3D consistency. Recall that Equation (1) is called *3D-consistent* if it may be consistently imposed on a three-dimensional lattice, so that one and the same equation hold for all six faces of any elementary cube (up to the parameter values: it is supposed that all edges of each coordinate direction carry their own parameter). More precisely, initial data $x, x_1, x_2, x_3$ determine uniquely the values $x_{12}, x_{13}, x_{23}$ by means of the equations on the faces adjacent to the vertex $x$. After that, one has three different equations for $x_{123}$, coming from the three faces of the cube adjacent to this vertex, see Figure 3. Now 3D consistency means that these three (a priori different) values for $x_{123}$ coincide for any choice of the initial data $x, x_1, x_2, x_3$.

Integrable equations on quad-graphs with multi-affine and $D_4$-symmetric functions $Q$ were classified in [1] under the following additional assumption.

- *Tetrahedron property*: The value $x_{123}$, which is well defined due to 3D consistency, depends on $x_1$, $x_2$, and $x_3$, but not on $x$.

The classification of Equations (1) up to Möbius transformation results in a list (the so-called ABS list) of nine canonical equations, named Q1–Q4, H1–H3, and A1–A2. The a priori assumption of the tetrahedron property was replaced with certain non-degeneracy conditions in [2]. This leads to the list Q1–Q4.

An important device used for the classification are the biquadratic polynomials $h$ and $g$ associated with the edges and diagonals of the elementary quadrilateral, respectively. They are obtained from $Q$ by discriminant-like operations eliminating two of the four variables. For instance,

$$QQ_{yv} - Q_y Q_v = k(\alpha, \beta) h(x, u; \alpha), \quad QQ_{yu} - Q_y Q_u = k(\alpha, \beta) h(x, v; \beta),$$
$$QQ_{uv} - Q_u Q_v = k(\alpha, \beta) g(x, y; \alpha - \beta).$$

Here, the subscripts denote partial derivatives, and $k(\alpha, \beta) = -k(\beta, \alpha)$ is a normalizing factor that makes each edge polynomial $h$ depend only on the parameter
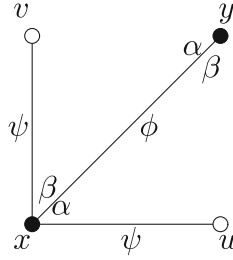
*Figure 2.* Three-leg form of a quad-equation.

assigned to the corresponding edge. The polynomials $g$ associated with the diago-
nals depend only on the difference $\alpha - \beta$ for a suitable choice of parameters, which
are naturally defined up to simultaneous re-parametrization $\alpha \mapsto \rho(\alpha)$, $\beta \mapsto \rho(\beta)$.

   The following lemma will be instrumental in the proof of our main result.

LEMMA 1. *For any quad-equation from the ABS list, the following identity is sat-
isfied for solutions of $Q = 0$:*

$$h(x, u; \alpha)h(y, v; \alpha) = h(x, v; \beta)h(y, u; \beta) = g(x, y; \alpha - \beta)g(u, v; \alpha - \beta). \qquad (2)$$

We refer the reader to [1] for further details and a proof of Lemma 1.

## 3. Three-Leg Forms

Equation (1) is said to possess a *three-leg form* centered at $x$ if it is equivalent to
the equation

$$\psi(x, u; \alpha) - \psi(x, v; \beta) = \phi(x, y; \alpha - \beta), \qquad (3)$$

for some functions $\psi$ and $\phi$, see Figure 2. It follows that the function $\phi$ must be
odd with respect to the parameter: $\phi(x, y; -\gamma) = -\phi(x, y; \gamma)$. It turns out [1] that
all equations from the ABS list possess three-leg forms. Moreover, an examination
of the list of three-leg forms leads to the following

- *Observation*: For equations Q1–Q4, the functions corresponding to the "short"
  and to the "long" legs coincide: $\psi(x, u; \alpha) = \phi(x, u; \alpha)$. Each equation H1–H3
  and A1–A2 shares the "long" leg function $\phi(x, y; \alpha - \beta)$ with some of the equa-
  tions Q1–Q3, but has a different "short" legs function $\psi(x, u; \alpha)$.

There are many applications of the three-leg form.

   First, let $\mathcal{B}$ be the "black" subgraph of the bipartite quad-graph $\mathcal{D}$ on which the
system of integrable quad-equations is considered. The edges of $\mathcal{B}$ are the diago-
nals of the quadrilateral faces of $\mathcal{D}$ connecting the "black" pairs of vertices. Let
the pairs of labels be assigned to the edges of $\mathcal{B}$ according to Figure 2, so that
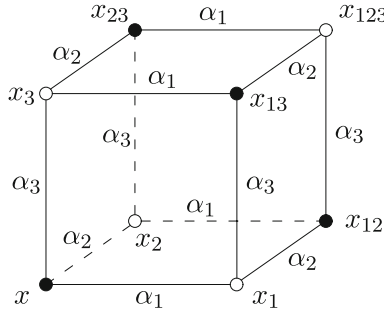
*Figure 3.* Three-dimensional consistency.

$(\alpha, \beta)$ is assigned to the edge $(x, y)$. Then the restriction of any solution of the system of quad-equations to the set of "black" vertices satisfies the so called Laplace type equations. For $x \in V(\mathcal{B})$ such an equation reads:

$$\sum_{(x, y_k) \in E(\mathcal{B})} \phi(x, y_k; \alpha_k - \alpha_{k+1}) = 0. \tag{4}$$

Here, the sum is taken over all edges $(x, y_k)$ of $\mathcal{B}$ incident with $x$ in counterclockwise order, and $(\alpha_k, \alpha_{k+1})$ are the corresponding pairs of parameters. Equation (4) is derived by adding the three-leg forms of the quad-equations for all quadrilaterals of $\mathcal{D}$ adjacent to $x$, where the contributions from the "short" legs cancel out. Of course, similar Laplace type equations hold also for the "white" subgraph of $\mathcal{D}$.

Another application of the three-leg form is the derivation of the tetrahedron property. Adding the tree-leg equations centered at $x_{123}$ on the three faces of the 3D cube adjacent to $x_{123}$ leads to the equation

$$\phi(x_{123}, x_1; \alpha_2 - \alpha_3) + \phi(x_{123}, x_2; \alpha_3 - \alpha_1) + \phi(x_{123}, x_3; \alpha_1 - \alpha_2) = 0, \tag{5}$$

which relates the fields at the vertices of the "white" tetrahedron in Figure 3. According to the above observation, this equation is actually equivalent to

$$\widehat{Q}(x_{123}, x_1, x_2, x_3; \alpha_2 - \alpha_3, \alpha_2 - \alpha_1) = 0, \tag{6}$$

where the function $\widehat{Q}(x, u, y, v; \alpha, \beta)$ is multi-affine, and, moreover, always belongs to the list Q1–Q4 (it plainly coincides with $Q$ for any of the equations Q1–Q4).

Not only does the existence of the three-leg form yield the tetrahedron property of the 3D consistent equations. The converse is also true: it has been proved in [3] that $D_4$ symmetry and the existence of a three-leg form imply 3D consistency.

## 4. Lagrangian Structures

We use the following technical statement to establish the Lagrangian structure of 3D consistent equations [1].

LEMMA 2. *For any quad-equation from the ABS list, there exists a change of variables, $x = f(X)$, $u = f(U)$, etc., such that in the new variables the leg functions $\psi$ and $\phi$ possess antiderivatives with respect to the first argument $X$ that are symmetric with respect to the permutation $X \leftrightarrow U$ and $X \leftrightarrow Y$, respectively. In other words, there exist functions $L(X, U; \alpha) = L(U, X; \alpha)$ and $\Lambda(X, Y; \alpha - \beta) = \Lambda(Y, X; \alpha - \beta)$ such that*

$$\psi(x, u; \alpha) = \psi(f(X), f(U); \alpha) = \frac{\partial}{\partial X} L(X, U; \alpha), \tag{7}$$

$$\phi(x, y; \alpha - \beta) = \phi(f(X), f(Y); \alpha - \beta) = \frac{\partial}{\partial X} \Lambda(X, Y; \alpha - \beta). \tag{8}$$

This follows from the easily verified fact that the derivatives of the leg functions with respect to their second argument, $\partial \psi / \partial U$ and $\partial \phi / \partial Y$, are symmetric with respect to $X \leftrightarrow U$ and $X \leftrightarrow Y$, respectively. Lemma 2 has the following corollaries [1].

PROPOSITION 1. *For any quad-equation from the ABS list on a bipartite quadgraph $\mathcal{D}$, the corresponding Laplace type equations (4) on the "black" subgraph $\mathcal{B}$ are the Euler – Lagrange equations for the action functional*

$$S_{\mathcal{B}} = \sum_{(x, y) \in E(\mathcal{B})} \Lambda(X, Y; \alpha - \beta), \tag{9}$$

*where the pairs of parameters $(\alpha, \beta)$ are assigned to the "black" edges $(x, y)$ as in Figure 2.*

PROPOSITION 2. *For any quad-equation from the ABS list on the regular square lattice $\mathbb{Z}^2$, the solutions are critical points of the functional*

$$S = \sum_{(x, x_1) \in E_1} L(X, X_1; \alpha_1) - \sum_{(x, x_2) \in E_2} L(X, X_2; \alpha_2) - \sum_{(x_1, x_2) \in E_3} \Lambda(X_1, X_2; \alpha_1 - \alpha_2), \tag{10}$$

*where $E_1$ and $E_2$ denote the set of horizontal and vertical edges of the square lattice $\mathbb{Z}^2$, and $E_3$ denotes the set of diagonals of all elementary quadrilaterals from northwest to south-east.*

The proof of Proposition 1 is obvious, the proof of Proposition 2 is based on the fact that $\partial S / \partial X$ is the sum of the three-leg equations on two squares adjacent to $x$ (to the north-west and to the south-east of $x$).

## 5. Fundamental Property of Lagrangians on a Single Quad

THEOREM 1. *For any equation from the ABS list, considered on a single quadrilateral, the Lagrangians $L, \Lambda$ can be chosen so that the following relation holds if*

*Equation* (1) *is satisfied*:

$$L(X, U; \alpha) + L(Y, V; \alpha) - L(X, V; \beta) - L(Y, U; \beta)$$
$$- \Lambda(X, Y; \alpha - \beta) - \Lambda(U, V; \alpha - \beta) = 0. \tag{11}$$

*Proof.* Since the symmetric antiderivatives $L$ and $\Lambda$ are determined only up to constant terms (depending on the corresponding parameters), the theorem is actually equivalent of to the statement that for any choice of $L$, $\Lambda$ there holds (for solutions of $Q = 0$):

$$\Theta = \rho(\alpha) - \rho(\beta) - \sigma(\alpha - \beta), \tag{12}$$

where $\Theta$ stands for the left-hand side of (11), and $\rho$, $\sigma$ are some functions depending only on the parameters, as indicated by the notation.

To show that the function $\Theta = \Theta(X, U, Y, V)$ is constant on the three-dimensional manifold in $(\mathbb{CP}^1)^4$ consisting of solutions of $Q(x, u, y, v; \alpha, \beta) = 0$, it is enough to prove that the directional derivatives of $\Theta$ along all tangent vectors of this manifold vanish. We prove a stronger claim, namely that the gradient of $\Theta$ vanishes on this manifold. This claim is an immediate consequence of the existence of the three-leg equations centered at each vertex of the elementary quad. Indeed, by virtue of (7), (8), and (3), one has:

$$\frac{\partial \Theta}{\partial X} = \psi(x, u; \alpha) - \psi(x, v; \beta) - \phi(x, y; \alpha - \beta) = 0. \tag{13}$$

Similarly, one shows that $\partial \Theta / \partial Y = \partial \Theta / \partial U = \partial \Theta / \partial V = 0$ for solutions. It remains to show that the constant value of $\Theta$ is of the form (12). The proof of this fact is based on identity (2) and the following lemma. $\qquad \square$

LEMMA 3. *For any equation from the ABS list, we have:*

$$\frac{\partial L(X, U; \alpha)}{\partial \alpha} = \log h(x, u; \alpha) + \kappa(X) + \kappa(U) + c(\alpha), \tag{14}$$

$$\frac{\partial \Lambda(X, Y; \alpha - \beta)}{\partial \alpha} = \log g(x, y; \alpha - \beta) + \kappa(X) + \kappa(Y) + \gamma(\alpha - \beta), \tag{15}$$

*with certain functions $\kappa, c, \gamma$ depending only on the indicated variables.*

*Proof.* Verify the relations obtained from (14), (15) by differentiation with respect to $X$:

$$\frac{\partial \psi(x, u; \alpha)}{\partial \alpha} = \frac{\partial}{\partial X} \log h(x, u; \alpha) + \kappa'(X),$$
$$\frac{\partial \phi(x, y; \alpha - \beta)}{\partial \alpha} = \frac{\partial}{\partial X} \log g(x, y; \alpha - \beta) + \kappa'(X).$$

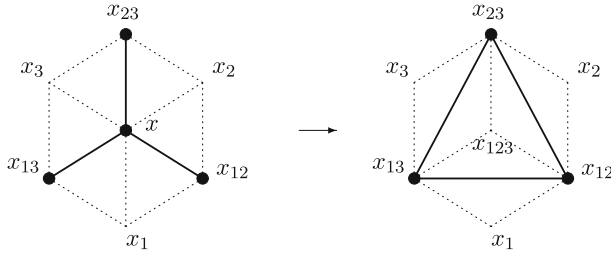*Figure 4.* Star – triangle flip.

This can be done case by case, by a direct and simple check; the leg functions $\psi$, $\phi$ and the polynomials $h$, $g$ are given for all equations of the ABS list in the Appendix. Then Equations (14) and (15) follow, since both sides of each are symmetric with respect to $x \leftrightarrow u$ and $x \leftrightarrow y$, respectively, and are defined up to an additive function of $\alpha$, resp. of $\alpha - \beta$. □

Lemma 3 and identity (2) imply

$$\frac{\partial \Theta}{\partial \alpha} = 2c(\alpha) - 2\gamma(\alpha - \beta), \quad \frac{\partial \Theta}{\partial \beta} = -2c(\beta) + 2\gamma(\alpha - \beta),$$

which yields (12). This completes the proof of Theorem 1. □

## 6. Flip Invariance of the Action Functionals

The following theorem establishes the flip invariance for the discrete Laplace type systems (with the Lagrangian structure described in Proposition 1).

THEOREM 2. *The Lagrangian* $\Lambda$ *for a discrete Laplace type system that comes from an equation of the ABS list can be chosen so that the following star – triangle relation is satisfied for solutions:*

$$\Lambda(X, X_{12}; \alpha_1 - \alpha_2) + \Lambda(X, X_{23}; \alpha_2 - \alpha_3) + \Lambda(X, X_{13}; \alpha_3 - \alpha_1) +$$
$$+ \Lambda(X_{23}, X_{13}; \alpha_1 - \alpha_2) + \Lambda(X_{13}, X_{12}; \alpha_2 - \alpha_3) + \Lambda(X_{12}, X_{23}; \alpha_3 - \alpha_1) = 0,$$

(16)

*see Figure* 4.

*Proof* Formula (16) involves the four black points $x$, $x_{12}$, $x_{23}$, $x_{13}$, which are related by a multi-affine equation

$$\widehat{Q}(x, x_{12}, x_{23}, x_{13}; \alpha_1 - \alpha_2, \alpha_1 - \alpha_3) = 0,$$

which belongs to the list Q1–Q4, compare with (6). Therefore, the claim is a particular case of Theorem 1. Indeed, combinatorially a tetrahedron is not different from a quadrilateral with diagonals, see Figure 5. □
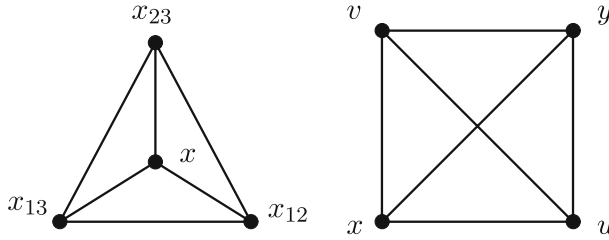
*Figure 5.* A tetrahedron versus a quadrilateral with diagonals.

Such a statement was previously established in [5] for the discrete Laplace type system which describes the radii of circle patterns with prescribed intersection angles and which comes from the so called Hirota system, a version of $(H3)_{\delta=0}$. In that paper, the action functional is derived as a classical limit of the partition function of the so called quantum Faddeeev – Volkov model. The corresponding property of the quantum model is the famous Yang – Baxter relation, the invariance of the partition function under a star – triangle transformation of the Boltzmann weights. The corresponding classical result is also established in [5], by direct computations involving the dilogarithm function.

The flips described by Theorem 2 can be considered as elementary transformations either of a planar quad-graph, or, alternatively, of its realization as a quad-surface in a multidimensional square lattice $\mathbb{Z}^m$. The Lagrangian formulation of quad-equations on $\mathbb{Z}^m$ is the main subject of [11].

The Lagrangian formulation of systems on $\mathbb{Z}^2$ used in [11] is

$$S = \sum_{\mathbb{Z}^2} \mathcal{L}(X, X_1, X_2; \alpha_1, \alpha_2), \tag{17}$$

where the 3-point Lagrangian $\mathcal{L}$ should be interpreted as a discrete 2-form, i.e., a real-valued function defined on oriented elementary squares and changing sign upon changing the orientation of the square. It is easily seen that the sum (17) is nothing but a re-arrangement of the sum (10), with

$$\mathcal{L}(X, X_1, X_2; \alpha_1, \alpha_2) = L(X, X_1; \alpha_1) - L(X, X_2; \alpha_2) - \Lambda(X_1, X_2; \alpha_1 - \alpha_2). \tag{18}$$

The main idea of the paper [11] is to extend the functional (17) to quad-surfaces $\Sigma$ in the multidimensional square lattice according to the formula

$$S = \sum_{\sigma_{ij} \in \Sigma} \mathcal{L}(\sigma_{ij}) \tag{19}$$

where for each elementary square $\sigma_{ij} = (n, n + e_i, n + e_i + e_j, n + e_j)$ there holds

$$\mathcal{L}(\sigma_{ij}) = \mathcal{L}(X, X_i, X_j; \alpha_i, \alpha_j) =$$
$$= L(X, X_i; \alpha_i) - L(X, X_j; \alpha_j) - \Lambda(X_i, X_j; \alpha_i - \alpha_j). \tag{20}$$

Let $\Delta_i$ denote the difference operator that acts on vertex functions, $\Delta_i f(x) = f(x_i) - f(x)$, so that, e.g., $\Delta_i f(x, x_j, x_k) = f(x_i, x_{ij}, x_{ik}) - f(x, x_j, x_k)$.

THEOREM 3. *For any system of quad-equations from the ABS list on $\mathbb{Z}^m$, the Lagrangian $\mathcal{L}$ given by* (20) *satisfies the following relation for solutions:*

$$\Delta_1 \mathcal{L}(X, X_2, X_3; \alpha_2, \alpha_3) + \Delta_2 \mathcal{L}(X, X_3, X_1; \alpha_3, \alpha_1) + \Delta_3 \mathcal{L}(X, X_1, X_2; \alpha_1, \alpha_2) = 0.$$
(21)

This means that the value of the action functional *for a solution* remains invariant under flips of the quad-surface. For some equations of the ABS list, namely for equations A1–A2, H1–H3, Q1, $(Q3)_{\delta=0}$, Theorem 3 was proved in [11] by long computations.

*Proof of Theorem 3.* It is enough to combine the statements of Theorem 1 for the three quadrilaterals adjacent to the vertex $x$ and the statement of Theorem 2 for the black tetrahedron.                                                              □

The following alternative proof of Theorem 3, not relying on Theorem 1, is based on the same idea as the proof of Theorem 1 but is much easier. The previous analysis of the constant value (12) is replaced by a simple and case-independent argument.

*Second proof of Theorem 3.* Let $\Delta$ denote the expression on the left-hand side of (21), considered as a function of 8 variables $x, x_i, x_{ij}, x_{123}$. We are going to show that $\Delta$ is constant on the manifold $\mathcal{S} \subset (\mathbb{CP}^1)^8$ of solutions of the system of quad-equations on the 3D cube. This manifold is four-dimensional and is parametrized, e.g., by $(x, x_1, x_2, x_3)$. We want to show that the derivatives of $\Delta$ tangent to $\mathcal{S}$ vanish. It turns out that a stronger property is easier to show, namely, that $\operatorname{grad} \Delta = 0$ on $\mathcal{S}$.

By the definition of the Lagrangian (20), we have:

$$\begin{aligned}
\Delta = \; & L(X_1, X_{12}; \alpha_2) + L(X_2, X_{23}; \alpha_3) + L(X_3, X_{13}; \alpha_1) - \\
& - L(X_1, X_{13}; \alpha_3) - L(X_2, X_{12}; \alpha_1) - L(X_3, X_{23}; \alpha_2) - \\
& - \Lambda(X_{12}, X_{13}; \alpha_2 - \alpha_3) - \Lambda(X_{23}, X_{12}; \alpha_3 - \alpha_1) - \Lambda(X_{13}, X_{23}; \alpha_1 - \alpha_2) + \\
& + \Lambda(X_2, X_3; \alpha_2 - \alpha_3) + \Lambda(X_3, X_1; \alpha_3 - \alpha_1) + \Lambda(X_1, X_2; \alpha_1 - \alpha_2).
\end{aligned}$$
(22)

Thus, $\Delta$ does not depend on either $x$ or $x_{123}$, so that its domain of definition is better visualized as an octahedron as shown in Figure 6 rather than an elementary cube, as the original definition suggests.

It remains to show that $\Delta$ does not depend on $x_i$ and $x_{ij}$ for solutions of the system of quad-equations. To show that $\Delta$ does not depend on $x_1$, say, we compute, with the help of (7) and (8):

$$\frac{\partial \Delta}{\partial X_1} = \psi(x_1, x_{12}; \alpha_2) - \psi(x_1, x_{13}; \alpha_3) + \phi(x_1, x_3; \alpha_3 - \alpha_1) + \phi(x_1, x_2; \alpha_1 - \alpha_2).$$
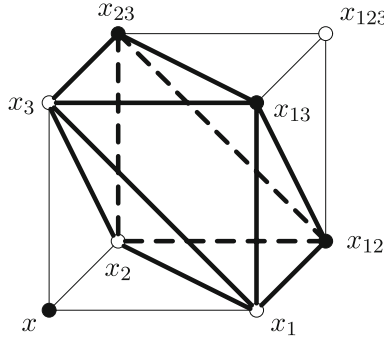
*Figure 6.* Octahedron.

But the tree-leg forms of the quad-equations on the faces $(x, x_1, x_{13}, x_3)$ and $(x, x_1, x_{12}, x_2)$, centered at $x_1$ are

$$\psi(x_1, x_{13}; \alpha_3) - \psi(x_1, x; \alpha_1) - \phi(x_1, x_3; \alpha_3 - \alpha_1) = 0,$$
$$\psi(x_1, x_{12}; \alpha_2) - \psi(x_1, x; \alpha_1) + \phi(x_1, x_2; \alpha_1 - \alpha_2) = 0.$$

Therefore, for solutions we have $\partial\Delta/\partial X_1 = 0$. That the partial derivatives of $\Delta$ with respect to all other $x_i$ and $x_{ij}$ vanish is shown similarly, because all variables enter symmetrically in $\Delta$. It is easy to understand that the manifold of solutions $\mathcal{S}$ is a connected algebraic manifold. Indeed, $\mathcal{S} = (\mathbb{CP}^1)^4 \setminus \tilde{\mathcal{S}}$, where $\tilde{\mathcal{S}}$ consists of singular curves and therefore has codimension two. Since $\operatorname{grad}\Delta = 0$ on the connected algebraic manifold $\mathcal{S}$, the function $\Delta$ is constant on $\mathcal{S}$. It remains to show that the value of this constant is 0. We need only to compute $\Delta$ on a particular solution. Consider a family of solutions defined by the following conditions:

$$x_1 = x_{23}, \quad x_2 = x_{13}, \quad x_3 = x_{12}. \tag{23}$$

(We are grateful to K. Zuev for the suggestion to consider this family.) Equations on the faces adjacent to the vertex $x$ give three different expressions for $x$. Setting them equal means imposing two (rational) conditions on the three initial values $x_1, x_2, x_3$. Thus, there is a one-parameter family of solutions satisfying (23). Thanks to the symmetry of $L$ and $\Lambda$ one sees immediately from (22) that $\Delta = 0$ on any solution from the family (23). This finishes the proof of Theorem 3.     □

## Appendix: ABS List

**List Q:**

$(Q1)_{\delta=0}$:     $Q = \alpha(xu + yv) - \beta(xv + yu) - (\alpha - \beta)(xy + uv),$

$$\psi(x, u; \alpha) = \frac{\alpha}{x - u},$$

$$h(x, u; \alpha) = \frac{1}{2\alpha}(x - u)^2;$$

$(Q1)_{\delta=1}$:  $Q = \alpha(xu + yv) - \beta(xv + yu) - (\alpha - \beta)(xy + uv) + \alpha\beta(\alpha - \beta)$,

$$\psi(x, u; \alpha) = \log \frac{x - u + \alpha}{x - u - \alpha},$$

$$h(x, u; \alpha) = \frac{1}{2\alpha}\left((x - u)^2 - \alpha^2\right) = \frac{1}{2\alpha}(x - u + \alpha)(x - u - \alpha);$$

$(Q2)$:  $Q = \alpha(xu + yv) - \beta(xv + yu) - (\alpha - \beta)(xy + uv)$
$\qquad\qquad + \alpha\beta(\alpha - \beta)(x + u + y + v) - \alpha\beta(\alpha - \beta)(\alpha^2 - \alpha\beta + \beta^2)$,

$$x = X^2,$$

$$\psi(x, u; \alpha) = \log \frac{(X + U + \alpha)(X - U + \alpha)}{(X + U - \alpha)(X - U - \alpha)},$$

$$h(x, u; \alpha) = \frac{1}{4\alpha}\left((x - u)^2 - 2\alpha^2(x + u) + \alpha^4\right) =$$
$$= \frac{1}{4\alpha}(X + U + \alpha)(X - U + \alpha)(X + U - \alpha)(X - U - \alpha);$$

$(Q3)_{\delta=0}$:  $Q = \sin(\alpha)(xu + yv) - \sin(\beta)(xv + yu) - \sin(\alpha - \beta)(xy + uv)$,

$$x = \exp(iX),$$

$$\psi(x, u; \alpha) = \log \frac{\sin\left(\dfrac{X - U + \alpha}{2}\right)}{\sin\left(\dfrac{X - U - \alpha}{2}\right)},$$

$$h(x, u; \alpha) = \frac{1}{\sin(\alpha)}\left(x^2 + u^2 - 2\cos(\alpha)xu\right) =$$
$$= \frac{\exp(iX)\exp(iU)}{\sin(\alpha)}\sin\left(\frac{X - U + \alpha}{2}\right)\sin\left(\frac{X - U - \alpha}{2}\right);$$

$(Q3)_{\delta=1}$:  $Q = \sin(\alpha)(xu + yv) - \sin(\beta)(xv + yu) - \sin(\alpha - \beta)(xy + uv) +$
$$\qquad\qquad + \sin(\alpha - \beta)\sin(\alpha)\sin(\beta),$$

$$x = \sin(X),$$

$$\psi(x, u; \alpha) = \log \frac{\cos\left(\dfrac{X + U + \alpha}{2}\right)\sin\left(\dfrac{X - U + \alpha}{2}\right)}{\cos\left(\dfrac{X + U - \alpha}{2}\right)\sin\left(\dfrac{X - U - \alpha}{2}\right)},$$

$$h(x, u; \alpha) = \frac{1}{2\sin(\alpha)}\left(x^2 + u^2 - 2\cos(\alpha)xu - \sin^2(\alpha)\right) =$$
$$= \frac{2}{\sin(\alpha)}\cos\left(\frac{X + U + \alpha}{2}\right)\cos\left(\frac{X + U - \alpha}{2}\right)\sin\left(\frac{X - U + \alpha}{2}\right)$$
$$\times \sin\left(\frac{X - U - \alpha}{2}\right);$$

$(Q4)$:  $Q = \operatorname{sn}(\alpha)(xu + yv) - \operatorname{sn}(\beta)(xv + yu) - \operatorname{sn}(\alpha - \beta)(xy + uv) +$
$$\qquad\qquad + \operatorname{sn}(\alpha - \beta)\operatorname{sn}(\alpha)\operatorname{sn}(\beta)(1 + k^2 xuyv),$$

$$x = \operatorname{sn}(X),$$

$$\psi(x, u; \alpha) =$$

$$= \log \frac{\Theta_2\left(\dfrac{X+U+\alpha}{2}\right) \Theta_3\left(\dfrac{X+U+\alpha}{2}\right) \Theta_1\left(\dfrac{X-U+\alpha}{2}\right) \Theta_4\left(\dfrac{X-U+\alpha}{2}\right)}{\Theta_2\left(\dfrac{X+U-\alpha}{2}\right) \Theta_3\left(\dfrac{X+U-\alpha}{2}\right) \Theta_1\left(\dfrac{X-U-\alpha}{2}\right) \Theta_4\left(\dfrac{X-U-\alpha}{2}\right)},$$

$$h(x, u; \alpha) = \frac{1}{2\operatorname{sn}(\alpha)} \left(x^2 + u^2 - 2\operatorname{cn}(\alpha)\operatorname{dn}(\alpha)xu - \operatorname{sn}^2(\alpha) - k^2\operatorname{sn}^2(\alpha)x^2u^2\right) =$$

$$= \frac{2\vartheta_4^2/\vartheta_2^4}{\operatorname{sn}(\alpha)} \cdot \frac{1}{\Theta_4^2(\alpha)\Theta_4^2(X)\Theta_4^2(U)} \times$$

$$\times \Theta_2\left(\dfrac{X+U+\alpha}{2}\right) \Theta_3\left(\dfrac{X+U+\alpha}{2}\right) \Theta_1\left(\dfrac{X-U+\alpha}{2}\right) \Theta_4\left(\dfrac{X-U+\alpha}{2}\right) \times$$

$$\times \Theta_2\left(\dfrac{X+U-\alpha}{2}\right) \Theta_3\left(\dfrac{X+U-\alpha}{2}\right) \Theta_1\left(\dfrac{X-U-\alpha}{2}\right) \Theta_4\left(\dfrac{X-U-\alpha}{2}\right).$$

**List H:**

(H1) $\quad Q = (x - y)(u - v) + \beta - \alpha,$

$\qquad \psi(x, u; \alpha) = x + u, \qquad \phi(x, y; \alpha - \beta) = \dfrac{\alpha - \beta}{x - y},$

$\qquad h(x, u; \alpha) = 1, \qquad g(x, y; \alpha - \beta) = \dfrac{(x - y)^2}{\alpha - \beta};$

(H2) $\quad Q = (x - y)(u - v) + (\beta - \alpha)(x + u + y + v) + \beta^2 - \alpha^2,$

$\qquad \psi(x, u; \alpha) = \log(x + u + \alpha), \qquad \phi(x, y; \alpha - \beta) = \log \dfrac{x - y + \alpha - \beta}{x - y - \alpha + \beta},$

$\qquad h(x, u; \alpha) = x + u + \alpha, \qquad g(x, y; \alpha - \beta) = \dfrac{1}{2(\alpha - \beta)} \left((x - y)^2 - (\alpha - \beta)^2\right);$

(H3) $\quad Q = e^\alpha(xu + yv) - e^\beta(xv + yu) + \delta\left(e^{2\alpha} - e^{2\beta}\right),$

$\qquad x = e^X,$

$\qquad \psi(x, u; \alpha) = -\log(xu + \delta e^\alpha) = -\log\left(e^{X+U} + \delta e^\alpha\right),$

$\qquad \phi(x, y; \alpha - \beta) = \log \dfrac{e^\alpha x - e^\beta y}{e^\beta x - e^\alpha y} = \log \dfrac{\sinh\left(\dfrac{X-Y+\alpha-\beta}{2}\right)}{\sinh\left(\dfrac{X-Y+\beta-\alpha}{2}\right)},$

$\qquad h(x, u; \alpha) = xu + \delta e^\alpha = e^{X+U} + \delta e^\alpha,$

$\qquad g(x, y; \alpha - \beta) = \dfrac{1}{e^{2\alpha} - e^{2\beta}} (e^\alpha x - e^\beta y)(e^\beta x - e^\alpha y) =$

$\qquad = \dfrac{2e^{X+Y}}{\sinh(\alpha - \beta)} \sinh\left(\dfrac{X-Y+\alpha-\beta}{2}\right) \sinh\left(\dfrac{X-Y+\beta-\alpha}{2}\right);$

**List A:**

(A1)$_{\delta=0}$ $\quad Q = \alpha(xu + yv) - \beta(xv + yu) + (\alpha - \beta)(xy + uv),$

$\qquad \psi(x, u; \alpha) = \dfrac{\alpha}{x + u}, \qquad \phi(x, y; \alpha - \beta) = \dfrac{\alpha - \beta}{x - y},$

$$h(x, u; \alpha) = \frac{1}{2\alpha}(x+u)^2, \qquad g(x, y; \alpha - \beta) = \frac{1}{2(\alpha - \beta)}(x-y)^2;$$

$(A1)_{\delta=1}$ $\quad Q = \alpha(xu + yv) - \beta(xv + yu) + (\alpha - \beta)(xy + uv) - \alpha\beta(\alpha - \beta),$

$$\psi(x, u; \alpha) = \log \frac{x+u+\alpha}{x+u-\alpha}, \qquad \phi(x, y; \alpha - \beta) = \log \frac{x+y+\alpha-\beta}{x+u-\alpha+\beta},$$

$$h(x, u; \alpha) = \frac{1}{2\alpha}\left((x+u)^2 - \alpha^2\right) = \frac{1}{2\alpha}(x+u+\alpha)(x+u-\alpha),$$

$$g(x, y; \alpha - \beta) = \frac{1}{2\alpha}\left((x-y)^2 - (\alpha-\beta)^2\right)$$

$$= \frac{1}{2\alpha}(x-y+\alpha-\beta)(x-y-\alpha+\beta);$$

(A2) $\quad Q = \sin(\alpha)(xv + yu) - \sin(\beta)(xu + yv) - \sin(\alpha - \beta)(1 + xuyv),$

$\quad x = \exp(iX),$

$$\psi(x, u; \alpha) = \log \frac{\sin\left(\dfrac{X+U+\alpha}{2}\right)}{\sin\left(\dfrac{X+U-\alpha}{2}\right)},$$

$$\phi(x, y; \alpha - \beta) = \log \frac{\sin\left(\dfrac{X-Y+\alpha-\beta}{2}\right)}{\sin\left(\dfrac{X-Y-\alpha+\beta}{2}\right)},$$

$$h(x, u; \alpha) = -\frac{1}{\sin(\alpha)}\left(x^2 u^2 + 1 - 2\cos(\alpha)xu\right) =$$

$$= \frac{\exp(iX)\exp(iU)}{\sin(\alpha)}\sin\left(\frac{X+U+\alpha}{2}\right)\sin\left(\frac{X+U-\alpha}{2}\right),$$

$$g(x, y; \alpha - \beta) = \frac{1}{\sin(\alpha - \beta)}\left(x^2 + y^2 - 2\cos(\alpha - \beta)xy\right) =$$

$$= \frac{\exp(iX)\exp(iY)}{\sin(\alpha - \beta)}\sin\left(\frac{X-Y+\alpha-\beta}{2}\right)\sin\left(\frac{X-Y-\alpha+\beta}{2}\right).$$

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