International Mathematics Research Notices Advance Access published February 7, 2009
Bobenko, A. I., and Yu. B. Suris. (2009) "Discrete Koenigs Nets and Discrete Isothermic Surfaces," International Mathematics Research Notices, Article ID rnp008, 37 pages.
doi:10.1093/imrn/rnp008

Discrete Koenigs Nets and Discrete Isothermic Surfaces

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We discuss discretization of Koenigs nets (conjugate nets with equal Laplace invariants) and of isothermic surfaces. Our discretization is based on the notion of dual quadrilaterals: two planar quadrilaterals are called dual if their corresponding sides are parallel, and their noncorresponding diagonals are parallel. Discrete Koenigs nets are defined as nets with planar quadrilaterals admitting dual nets. Several novel geometric properties of discrete Koenigs nets are found; in particular, two-dimensional discrete Koenigs nets can be characterized by coplanarity of the intersection points of diagonals of elementary quadrilaterals adjacent to any vertex; this characterization is invariant with respect to projective transformations. Discrete isothermic nets are defined as circular Koenigs nets. This is a new geometric characterization of discrete isothermic surfaces introduced previously as circular nets with factorized cross-ratios.

1 Introduction

This paper is devoted to the discretization of a geometrically important class of two-dimensional conjugate nets, very popular in the classical differential geometry under the name of *nets with equal invariants*. With a view toward discretization, we

Received November 21, 2008; Revised November 21, 2008; Accepted January 14, 2009
Communicated by Prof. Igor Krichever

prefer to call them *Koenigs nets*, for the following reason: among various geometric and analytic characterizations, the property of having equal Laplace invariants belongs to the minor part which *do not* survive by discretization, at least literally. Therefore the term “discrete nets with equal invariants” would be misleading. On the other hand, the French geometer G. Koenigs contributed a lot to the study of their properties [26, 27], see also [p. 19–47 in 12, 20]. The term “discrete Koenigs nets” will be suggestive and well justified.

Discrete Koenigs nets are closely related to *discrete Moutard nets* which recently found an important mathematical application in the problem of a geometric characterization of Prym varieties [21].

Another class of nets, whose discretizations are discussed in the present paper, are *isothermic surfaces*. Classically, their theory was considered as one of the highest achievements of the local differential geometry, see [1, 11–14, 20] and modern studies [8–10, 23, 25, 35].

Both classes of nets have already been discretized. Historically, discrete isothermic surfaces were introduced earlier [2], as circular nets with factorized cross-ratios. An approach to the discretization of Koenigs nets have been proposed in [17], based on a characterization of smooth Koenigs nets as conjugate nets possessing a so-called conic of Koenigs in each tangent plane [27] (a conic of Koenigs has a second-order contact both with the u_1 tangent line at the corresponding point of the Laplace transform f_{-1} and with the u_2 tangent line of f at the corresponding point of the Laplace transform f_1).

In the present paper, we propose a novel definition of discrete Koenigs nets and discrete isothermic surfaces. This definition is based on one of the characterizations of the Koenigs nets and isothermic surfaces, namely on the notion of *duality*. We believe that it is this definition that lies in the core of the whole theory and leads most directly to various other properties. All discretizations we consider belong to the class of Q-nets, or nets with planar elementary quadrilaterals [19], which are the fundamental objects of discrete differential geometry (see [3, 4] for a detailed presentation of the current state of discrete differential geometry as well as for historical remarks).

Two planar quadrilaterals are said to be *dual* if their corresponding sides are parallel and their noncorresponding diagonals are also parallel. In [31], this property has been identified as a characterization of pairs of quadrilaterals with parallel sides and with the vanishing mixed area, and it has been observed that the corresponding circular quadrilaterals of dual discrete isothermic surfaces possess this property. These observations stimulated the development presented here. Namely, we study here the geometric and analytic properties of nets all of whose quadrilaterals can be dualized simultaneously.

A net with planar quadrilaterals admitting a dual net is called a *discrete Koenigs net*. A discrete Koenigs net with all circular quadrilaterals is called a *discrete isothermic net*.

Discrete surfaces we arrive at are not new. The class of discrete isothermic surfaces turns out to coincide with the original class introduced in [2], so that we get just a novel characterization of the latter. In the case of discrete Koenigs nets, the history is more complicated: they first appeared in [32] (see also [33, 34]) in the context of infinitesimal deformations of surfaces, with exactly the same definition as we use (dual quadrilaterals are called *antiparallel* there); however, the geometric and analytic properties of these nets remained to a large extent unexplored. Recently, this class has been studied in [5, 18], but again some of the crucial properties passed unnoticed. The main novel results of the present paper include:

- Definition of discrete Koenigs nets as those admitting dual nets (Definition 3.4).
- A characterization of discrete Koenigs nets in terms of an exact multiplicative one-form on diagonals, defined through ratios of diagonal segments (Theorem 3.7). Integrating this exact form, we arrive at the function ν defined at the vertices of a discrete Koenigs net. This function is a novel and important ingredient of an analytic description of discrete Koenigs nets. In particular, the function ν allows us to find a discrete analog of a Laplace equation with equal invariants (Equation (30)), and defines the Moutard representatives of a discrete Koenigs net (Theorem 3.15).
- A novel projective-geometric characterization of two-dimensional Koenigs nets: intersection points of diagonals of elementary quadrilaterals of such a net form a net with planar quadrilaterals (Theorem 3.9). Interestingly, the net comprised by the intersection points of diagonals of quadrilaterals of a discrete Koenigs net in the sense of our definition turns out to satisfy the definition of discrete Koenigs nets from [17].
- A novel definition of discrete isothermic nets as circular nets admitting dual nets (Definition 4.1).
- A novel understanding of the discrete metric of a discrete isothermic net, as the function ν in the circular context (Theorem 4.8).

2 Koenigs Nets and Isothermic Surfaces

2.1 Definitions and duality

Definition 2.1 (Koenigs net). A map $f:\mathbb{R}^2\to\mathbb{R}^N$ is called a *Koenigs net* if it satisfies a differential equation

$$\partial_1\partial_2 f = (\partial_2 \log \nu)\,\partial_1 f + (\partial_1 \log \nu)\,\partial_2 f \tag{1}$$

with some scalar function $\nu:\mathbb{R}^2\to\mathbb{R}^*$. □

The following characterization of Koenigs nets will be of a fundamental importance for us.

Theorem 2.2 (dual Koenigs net). A conjugate net $f:\mathbb{R}^2\to\mathbb{R}^N$ is a Koenigs net if and only if there exists a scalar function $\nu:\mathbb{R}^2\to\mathbb{R}$ such that the differential one-form df^* defined by

$$\partial_1 f^* = \frac{\partial_1 f}{\nu^2}, \qquad \partial_2 f^* = -\frac{\partial_2 f}{\nu^2} \tag{2}$$

is exact. In this case the map $f^*:\mathbb{R}^2\to\mathbb{R}^N$, defined (up to a translation) by the integration of this one-form, is also a Koenigs net, called *dual* to f. □

This follows immediately by cross-differentiating Equation (2). A different way to formulate the latter equations is:

$$\begin{aligned} \partial_1 f^* \parallel \partial_1 f, \quad & \partial_2 f^* \parallel \partial_2 f, \\ (\partial_1+\partial_2) f^* \parallel (\partial_1-\partial_2) f, \quad & (\partial_1-\partial_2) f^* \parallel (\partial_1+\partial_2) f. \end{aligned} \tag{3}$$

Definition 2.3 (isothermic surface). A curvature line parameterized surface $f:\mathbb{R}^2\to\mathbb{R}^N$ is called an *isothermic surface* if its first fundamental form is conformal, possibly upon a re-parameterization of independent variables $u_i\mapsto\varphi_i(u_i)$ $(i=1,2)$, i.e. if at every point $u\in\mathbb{R}^2$ of the definition domain there holds $|\partial_1 f|^2/|\partial_2 f|^2=\alpha_1(u_1)/\alpha_2(u_2)$. □

In other words, isothermic surfaces are characterized by the relations $\partial_1\partial_2 f\in \operatorname{span}(\partial_1 f,\partial_2 f)$ and

$$\langle\partial_1 f,\partial_2 f\rangle = 0, \quad |\partial_1 f|^2=\alpha_1 s^2, \quad |\partial_2 f|^2 = \alpha_2 s^2, \tag{4}$$

with some $s : \mathbb{R}^2 \to \mathbb{R}_+$ (conformal metric coefficient) and with the functions α_i depending on u_i only ($i = 1, 2$). Conditions (4) may be equivalently represented as

$$\partial_1 \partial_2 f = (\partial_2 \log s)\partial_1 f + (\partial_1 \log s)\partial_2 f, \qquad \langle \partial_1 f, \partial_2 f \rangle = 0. \tag{5}$$

Comparison with Equation (1) shows that *isothermic surfaces are nothing but orthogonal Koenigs nets*, the role of the function ν being played by the metric coefficient s.

In the case of isothermic surfaces the duality is specialized as follows.

Theorem 2.4 (dual isothermic surface). Let $f : \mathbb{R}^2 \to \mathbb{R}^N$ be an isothermic surface. Then the $\mathbb{R}^N$-valued one-form df^* defined by

$$\partial_1 f^* = \alpha_1 \frac{\partial_1 f}{|\partial_1 f|^2} = \frac{\partial_1 f}{s^2}, \qquad \partial_2 f^* = -\alpha_2 \frac{\partial_2 f}{|\partial_2 f|^2} = -\frac{\partial_2 f}{s^2} \tag{6}$$

is exact. The surface $f^* : \mathbb{R}^2 \to \mathbb{R}^N$, defined (up to a translation) by the integration of this one-form, is isothermic, with

$$\langle \partial_1 f^*, \partial_2 f^* \rangle = 0, \quad |\partial_1 f^*|^2 = \alpha_1 s^{-2}, \quad |\partial_2 f^*|^2 = \alpha_2 s^{-2}. \tag{7}$$

The surface f^* is called *dual* to, or the *Christoffel transform* of the surface f. □

2.2 Moutard representatives

Remarkably, the defining property (1) turns out to be invariant under projective transformations of $\mathbb{R}^N$, so that the notion of Koenigs nets actually belongs to projective geometry. If one considers the ambient space $\mathbb{R}^N$ of a Koenigs net as an affine part of $\mathbb{RP}^N$, then there is an important choice of representatives for $f \sim (f, 1)$ in the space $\mathbb{R}^{N+1}$ of homogeneous coordinates, namely

$$y = \nu^{-1}(f, 1). \tag{8}$$

Indeed, a straightforward computation shows that the representatives (8) satisfy the following simple differential equation:

$$\partial_1 \partial_2 y = q y \tag{9}$$

with the scalar function $q = \nu\partial_1\partial_2(\nu^{-1})$. Differential equation (9) is known as the *Moutard equation*. Accordingly, we call a map $y : \mathbb{R}^2 \to \mathbb{R}^{N+1}$ a *Moutard net* if it satisfies the Moutard equation (9) with some $q : \mathbb{R}^2 \to \mathbb{R}$.

Theorem 2.5 (Koenigs nets = Moutard nets in homogeneous coordinates). For a Koenigs net $f : \mathbb{R}^2 \to \mathbb{R}^N$, the lift (8) is a Moutard net. Conversely, given a Moutard net $y : \mathbb{R}^2 \to \mathbb{R}^{N+1}$ with a nonvanishing last component $\nu^{-1} : \mathbb{R}^2 \to \mathbb{R}^*$, define $f : \mathbb{R}^2 \to \mathbb{R}^N$ by Equation (8), then f is a Koenigs net. □

More generally, for a given Moutard net y in $\mathbb{R}^{N+1}$, it is not difficult to figure out the condition for a scalar function $\nu : \mathbb{R}^2 \to \mathbb{R}^*$, under which $\tilde{f} = \nu y$ satisfies equation of the Laplace type: ν^{-1} has to be a solution of the same Moutard equation (9) (not necessarily one of the components of the vector y), and then there holds

$$\partial_1\partial_2\tilde{f} = (\partial_2 \log \nu)\partial_1\tilde{f} + (\partial_1 \log \nu)\partial_2\tilde{f}.$$

Of course, Moutard nets can be considered also on their own rights, i.e. one does not have to regard the ambient space $\mathbb{R}^{N+1}$ of a Moutard net as the space of homogeneous coordinates for $\mathbb{RP}^N$. Nevertheless, such an interpretation is useful in the most cases.

In application to isothermic surfaces, the construction of Moutard representatives can be performed within the projective model of Möbius geometry. Recall that, although conditions (4) are formulated in Euclidean terms, they are invariant not only with respect to Euclidean motions and dilations in $\mathbb{R}^N$, but also with respect to the inversion $f \to f/\langle f, f\rangle$. Therefore, the notion of isothermic surfaces belongs to Möbius differential geometry.

Recall (see, e.g. [8] or [23]) that the basic space of the projective model of Möbius geometry in $\mathbb{R}^N$ is the projectivization $\mathbb{P}(\mathbb{R}^{N+1,1})$ of the Minkowski space $\mathbb{R}^{N+1,1}$. The latter is the space spanned by $N+2$ linearly independent vectors $\mathbf{e}_1, \ldots, \mathbf{e}_{N+2}$ and equipped with the Minkowski scalar product

$$\langle \mathbf{e}_i, \mathbf{e}_j\rangle = \begin{cases} 1, & i = j \in \{1, \ldots, N+1\}, \\ -1, & i = j = N+2, \\ 0, & i \neq j. \end{cases}$$

It is convenient to introduce two isotropic vectors $\mathbf{e}_0 = \frac{1}{2}(\mathbf{e}_{N+2} - \mathbf{e}_{N+1})$, $\mathbf{e}_\infty = \frac{1}{2}(\mathbf{e}_{N+2} + \mathbf{e}_{N+1})$, satisfying $\langle \mathbf{e}_0, \mathbf{e}_\infty\rangle = -\frac{1}{2}$.

A point $f \in \mathbb{R}^N$ is modeled in the space $\mathbb{P}(\mathbb{R}^{N+1,1})$ by the element with homogeneous coordinates $\hat{f} = f + \mathbf{e}_0 + |f|^2\mathbf{e}_\infty$. Thus, points $f \in \mathbb{R}^N \cup \{\infty\}$ are in a one-to-one correspondence with points of the projectivized light cone $\mathbb{P}(\mathbb{L}^{N+1,1})$, where

$$\mathbb{L}^{N+1,1} = \left\{\xi \in \mathbb{R}^{N+1,1} : \langle \xi, \xi \rangle = 0\right\}. \tag{10}$$

A surface $f : \mathbb{R}^2 \to \mathbb{R}^N$ is curvature lines parameterized if and only if its lift $\hat{f} : \mathbb{R}^2 \to \mathbb{L}^{N+1,1}$ into the light cone is a conjugate net. In particular, Equations (5) are equivalent to

$$\partial_1\partial_2\hat{f} = (\partial_2 \log s)\partial_1\hat{f} + (\partial_1 \log s)\partial_2\hat{f}.$$

Thus, the following result by Darboux [p. 267 in 14] holds:

Theorem 2.6 (isothermic surfaces = Moutard nets in the light cone). For an isothermic surface $f : \mathbb{R}^2 \to \mathbb{R}^N$ with the conformal metric coefficient $s : \mathbb{R}^2 \to \mathbb{R}_+$, define its lift into the light cone, $y : \mathbb{R}^2 \to \mathbb{L}^{N+1,1}$, by

$$y = s^{-1}\left(f + \mathbf{e}_0 + |f|^2\mathbf{e}_\infty\right). \tag{11}$$

Then y satisfies the Moutard equation (9) with $q = s\partial_1\partial_2(s^{-1})$.

Conversely, given a Moutard net $y : \mathbb{R}^2 \to \mathbb{L}^{N+1,1}$ in the light cone, define $s : \mathbb{R}^2 \to \mathbb{R}^*$ and $f : \mathbb{R}^2 \to \mathbb{R}^N$ by Equation (11), so that s^{-1} is the $\mathbf{e}_0$-component, and $s^{-1}f$ is the $\mathbb{R}^N$-part of y in the basis $\mathbf{e}_1, \ldots, \mathbf{e}_N, \mathbf{e}_0, \mathbf{e}_\infty$. Then f is an isothermic surface, and the definition (4) holds with the functions $\alpha_i = \langle \partial_i y, \partial_i y \rangle$ depending on u_i only. □

Note that in the second part of the theorem we can always assume that $s : \mathbb{R}^2 \to \mathbb{R}_+$, changing y to $-y$ if necessary.

3 Discrete Koenigs and Moutard Nets

3.1 Notion of dual quadrilaterals

Definition 3.1 (dual quadrilaterals; see [31–34]). Two quadrilaterals (A, B, C, D) and (A^*, B^*, C^*, D^*) in a plane are called *dual* if their corresponding sides are parallel:

$$(A^*B^*) \parallel (AB), \quad (B^*C^*) \parallel (BC), \quad (C^*D^*) \parallel (CD), \quad (D^*A^*) \parallel (DA), \tag{12}$$

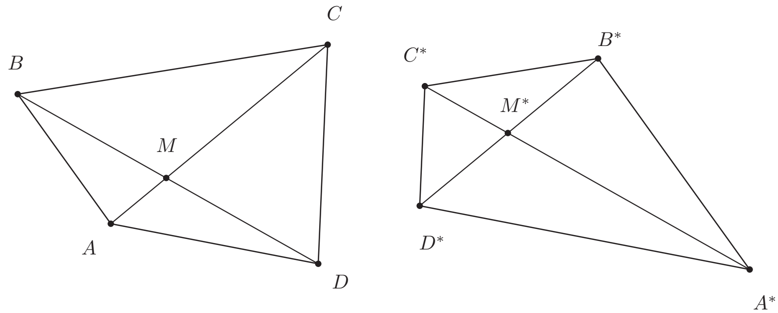

Fig. 1. Dual quadrilaterals.

and the noncorresponding diagonals are parallel:

$$(A^*C^*) \parallel (BD), \quad (B^*D^*) \parallel (AC). \tag{13}$$

See Figure 1. □

Lemma 3.2 (existence and uniqueness of dual quadrilateral). For any planar quadrilateral (A, B, C, D) a dual one exists and is unique up to scaling and translation. □

Proof. Uniqueness of the form of the dual quadrilateral can be argued as follows. Denote the intersection point of the diagonals of (A, B, C, D) by $M = (AC) \cap (BD)$. Take an arbitrary point M^* in the plane as the designated intersection point of the diagonals of the dual quadrilateral. Draw two lines ℓ_1 and ℓ_2 through M^* parallel to (AC) and (BD), respectively, and choose an arbitrary point on ℓ_2 to be A^*. Then the rest of construction is unique: draw the line through A^* parallel to (AB), its intersection point with ℓ_1 will be B^*; draw the line through B^* parallel to (BC), its intersection point with ℓ_2 will be C^*; draw the line through C^* parallel to (CD), its intersection point with ℓ_1 will be D^*. It remains to be seen that this construction closes, namely that the line through D^* parallel to (DA) intersects ℓ_2 at A^*. Clearly, this property does not depend on the initial choice of A^* on ℓ_2, since this choice only affects the scaling of the dual picture. Therefore, it is enough to demonstrate the closing property for some choice of A^*, or, in other words, to show the existence of one dual quadrilateral. This can be done as follows.

Denote by e_1 and e_2 some vectors along the diagonals, and introduce the coefficients $\alpha, \dots, \delta$ by

$$\overrightarrow{MA} = \alpha e_1, \quad \overrightarrow{MB} = \beta e_2, \quad \overrightarrow{MC} = \gamma e_1, \quad \overrightarrow{MD} = \delta e_2, \tag{14}$$

so that

$$\overrightarrow{AB} = \beta e_2 - \alpha e_1, \qquad \overrightarrow{BC} = \gamma e_1 - \beta e_2,$$
$$\overrightarrow{CD} = \delta e_2 - \gamma e_1, \qquad \overrightarrow{DA} = \alpha e_1 - \delta e_2. \tag{15}$$

Construct a quadrilateral (A^*, B^*, C^*, D^*) by setting

$$\overrightarrow{M^*A^*} = -\frac{e_2}{\alpha}, \quad \overrightarrow{M^*B^*} = -\frac{e_1}{\beta}, \quad \overrightarrow{M^*C^*} = -\frac{e_2}{\gamma}, \quad \overrightarrow{M^*D^*} = -\frac{e_1}{\delta}. \tag{16}$$

Its diagonals are parallel to the noncorresponding diagonals of the original quadrilateral, by construction. The corresponding sides are parallel as well:

$$\overrightarrow{A^*B^*} = -\frac{1}{\beta}e_1 + \frac{1}{\alpha}e_2 = \frac{1}{\alpha\beta}\overrightarrow{AB}, \qquad \overrightarrow{B^*C^*} = -\frac{1}{\gamma}e_2 + \frac{1}{\beta}e_1 = \frac{1}{\beta\gamma}\overrightarrow{BC},$$
$$\overrightarrow{C^*D^*} = -\frac{1}{\delta}e_1 + \frac{1}{\gamma}e_2 = \frac{1}{\gamma\delta}\overrightarrow{CD}, \qquad \overrightarrow{D^*A^*} = -\frac{1}{\alpha}e_2 + \frac{1}{\delta}e_1 = \frac{1}{\delta\alpha}\overrightarrow{DA}.$$

Thus, the quadrilateral (A^*, B^*, C^*, D^*) is dual to (A, B, C, D). ■

Note that the quantities $\alpha, \ldots, \delta$ in Equation (14) are not well defined by the geometry of the quadrilateral (A, B, C, D), since they depend on the choice of the vectors e_1, e_2. Well defined are their ratios, which can be viewed also as ratios of the directed lengths of the corresponding segments of diagonals, say $\gamma : \alpha = l(M, C) : l(M, A)$ and $\delta : \beta = l(M, D) : l(M, B)$. It is natural to associate these ratios with *directed* diagonals:

Definition 3.3 (ratio of diagonal segments). Given a quadrilateral (A, B, C, D), with the intersection point of diagonals $M = (AC) \cap (BD)$, we set

$$q(\overrightarrow{AC}) = \frac{l(M, C)}{l(M, A)}, \qquad q(\overrightarrow{BD}) = \frac{l(M, D)}{l(M, B)}. \tag{17}$$

Changing the direction of a diagonal corresponds to inverting the associated quantity q. □

Note that

$$(A, B, C, D) \text{ convex} \quad \Leftrightarrow \quad q(\overrightarrow{AC}) < 0 \text{ and } q(\overrightarrow{BD}) < 0. \tag{18}$$

3.2 Notion of discrete Koenigs nets

In dealing with discrete nets $f:\mathbb{Z}^m\to\mathbb{R}^N$ we will use the usual notations

$$\tau_i f(u) = f(u+e_i), \qquad \delta_i f(u) = f(u+e_i) - f(u),$$

where e_i stands for the unit vector of the ith coordinate direction. Moreover, we often abbreviate $f(u)$, $\tau_i f(u)$, $\tau_i\tau_j f(u)$ to f, f_i, f_{ij}, respectively. Recall that a Q-net is a map $f:\mathbb{Z}^m\to\mathbb{R}^N$ such that all elementary quadrilaterals (f, f_j, f_{ij}, f_j) are planar. The following definition is the fundamental one for the present paper.

Definition 3.4 (discrete Koenigs net). A Q-net $f:\mathbb{Z}^m\to\mathbb{R}^N$ is called a *discrete Koenigs net* if it admits a *dual net*, i.e. a Q-net $f^*:\mathbb{Z}^m\to\mathbb{R}^N$ such that all elementary quadrilaterals of the net f^* are dual to the corresponding quadrilaterals of f:

$$\delta_i f^* \parallel \delta_i f, \quad f^*_{ij} - f^* \parallel f_i - f_j, \quad f^*_i - f^*_j \parallel f_{ij} - f. \tag{19}$$

□

This definition can be seen as a discretization of conditions (3).

In order to understand restrictions imposed on a Q-net by this definition, we start with the following construction. Each lattice $\mathbb{Z}^m$ is bipartite: one can color its vertices black and white so that each edge connects a black vertex with a white one (for instance, one can call vertices $u=(u_1,\cdots,u_m)$ with an even value of $|u|=u_1+\cdots+u_m$ black and those with an odd value of $|u|$ white). Each elementary quadrilateral has a black diagonal (the one connecting two black vertices) and a white one. One can introduce the *black graph* $\mathbb{Z}^m_{\rm even}$ with the set of vertices consisting of the white vertices of $\mathbb{Z}^m$ and the set of edges consisting of black diagonals of all elementary quadrilaterals of $\mathbb{Z}^m$, and the analogous *white graph* $\mathbb{Z}^m_{\rm odd}$. The geometry of the elementary quadrilaterals of a Q-net $f:\mathbb{Z}^m\to\mathbb{R}^N$ induces, according to Definition 3.3, the quantities q (ratios of directed lengths of diagonal segments) on all directed diagonals, white and black.

Definition 3.5 (multiplicative one-form). Given a graph G with the set of vertices V and with the set of directed edges $\vec{E}$, the function $q:\vec{E}\to\mathbb{R}^*$ is called a *multiplicative one-form* on G if $q(-e)=1/q(e)$ for any directed edge $e\in\vec{E}$. It is called *exact* if for any cycle of directed edges the product of values of q along this cycle is equal to one. □

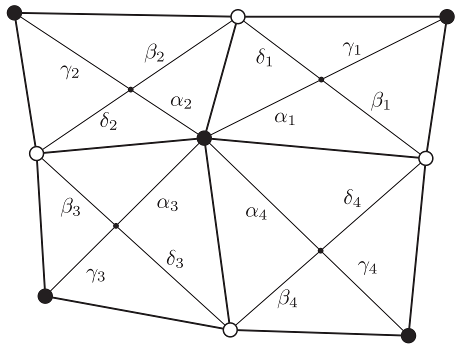

Fig. 2. Four quadrilaterals around a vertex of a two-dimensional net.

The main general statement about exact multiplicative one-forms is the following.

Theorem 3.6 (integration of an exact form). If $q : \vec{E} \to \mathbb{R}^*$ is an exact multiplicative one-form on G, then there exists a function $\nu : V \to \mathbb{R}^*$ such that for any $e = (x, y) \in \vec{E}$ there holds $q(e) = \nu(y)/\nu(x)$. Such a function ν is defined up to a multiplicative constant, which can be fixed by prescribing ν arbitrarily at one vertex. □

Any Q-net yields a multiplicative one-form q (or, better, two multiplicative one-forms) on both the black and the white graphs of $\mathbb{Z}^m$, as introduced in Definition 3.3.

Theorem 3.7 (algebraic characterization of discrete Koenigs nets). A Q-net $f : \mathbb{Z}^m \to \mathbb{R}^N$ is a Koenigs net if and only if the multiplicative one-form q is exact on both $\mathbb{Z}^m_{\text{even}}$ and $\mathbb{Z}^m_{\text{odd}}$. □

Proof. For a given Q-net, one can try to construct a dual net, applying Lemma 3.2, starting with an arbitrary quadrilateral. It is easy to realize that obstructions in extending this construction to the whole net may appear when running along closed chains of elementary quadrilaterals in which any two subsequent quadrilaterals share an edge.

$\boldsymbol{m = 2}$. The basic example of a closed chain of quadrilaterals in this case is given by four elementary quadrilaterals attached to a (black, say) vertex f. Let the diagonals of each quadrilateral be divided by their intersection point in the relations $\gamma_k : \alpha_k$ and $\delta_k : \beta_k$ $(k = 1, \dots, 4)$, as in Figure 2. The dual quadrilaterals are determined up to scaling factors λ_k $(k = 1, \dots, 4)$, say. Matching the edge shared by the dual quadrilaterals 1 and 2,

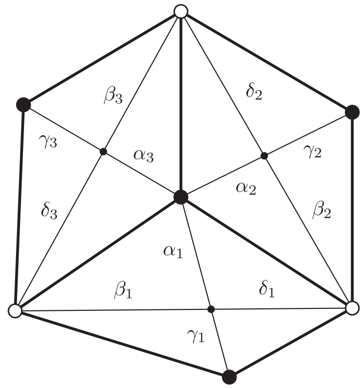

Fig. 3. Three quadrilaterals around a vertex of a three-dimensional net.

we find the relation between their scaling factors:

$$\frac{\lambda_1}{\alpha_1\delta_1} = \frac{\lambda_2}{\alpha_2\beta_2} \quad \Leftrightarrow \quad \frac{\lambda_1}{\lambda_2} = \frac{\alpha_1\delta_1}{\alpha_2\beta_2}.$$

Similarly, we find:

$$\frac{\lambda_2}{\lambda_3} = \frac{\alpha_2\delta_2}{\alpha_3\beta_3}, \qquad \frac{\lambda_3}{\lambda_4} = \frac{\alpha_3\delta_3}{\alpha_4\beta_4}, \qquad \frac{\lambda_4}{\lambda_1} = \frac{\alpha_4\delta_4}{\alpha_1\beta_1}.$$

All four edges adjacent to f can be matched if and only if the cyclic product of expressions for the quotients of scaling factors is equal to one. This condition reads:

$$\frac{\alpha_1\delta_1}{\alpha_2\beta_2}\cdot\frac{\alpha_2\delta_2}{\alpha_3\beta_3}\cdot\frac{\alpha_3\delta_3}{\alpha_4\beta_4}\cdot\frac{\alpha_4\delta_4}{\alpha_1\beta_1} = 1,$$

or

$$\frac{\delta_1}{\beta_1}\cdot\frac{\delta_2}{\beta_2}\cdot\frac{\delta_3}{\beta_3}\cdot\frac{\delta_4}{\beta_4} = 1. \tag{20}$$

This is nothing but the closeness condition of the form q for an elementary quadrilateral of the white graph. All other white and black cycles are products of elementary ones, therefore (20) for all elementary white and black cycles are necessary and sufficient for the closeness of the form q. But it is easy to see that if the closeness condition is fulfilled for all white and black cycles, then no closed chain of quadrilaterals can lead to an obstruction by the construction of the dual net.

***m* = 3**. In this case the most elementary closed chain of quadrilaterals is given by three faces of any elementary hexahedron of the net, sharing a (black, for definiteness) vertex f; see Figure 3. The further arguments are completely analogous to the two-dimensional case. Matching the edges shared by the dual quadrilaterals 1 and 2, by the dual quadrilaterals 2 and 3, and by the dual quadrilaterals 3 and 1, we find the relations between their scaling factors:

$$\frac{\lambda_1}{\lambda_2} = \frac{\alpha_1\delta_1}{\alpha_2\beta_2}, \qquad \frac{\lambda_2}{\lambda_3} = \frac{\alpha_2\delta_2}{\alpha_3\beta_3}, \qquad \frac{\lambda_3}{\lambda_1} = \frac{\alpha_3\delta_3}{\alpha_1\beta_1}.$$

All three edges adjacent to f can be matched simultaneously if and only if the cyclic product of expressions for the quotients of scaling factors is equal to one, which condition after cancellations reads:

$$\frac{\delta_1}{\beta_1} \cdot \frac{\delta_2}{\beta_2} \cdot \frac{\delta_3}{\beta_3} = 1. \tag{21}$$

This is nothing but the closeness condition for the elementary cycle of the white graph of the lattice $\mathbb{Z}^3$, which is a triangle. All cycles of the white and of the black graphs (including those encountered in the $m = 2$ case, i.e. the squares of the two-dimensional slices of the white and the black graphs of $\mathbb{Z}^3$) are products of elementary triangles. Again, closeness condition for all white and black cycles guarantees that no closed chain of quadrilaterals leads to an obstruction.

$\boldsymbol{m \geq 4}$. Also, in this case any white or black cycle is a product of elementary triangles, as for $m = 3$, therefore no additional conditions appear. ■

3.3 Geometric characterization of two-dimensional discrete Koenigs nets

The definition of discrete Koenigs nets obviously belongs to affine geometry, since it relies on the notion of parallelism. It turns out, however, that the class of discrete Koenigs nets is projectively invariant (it has been pointed out already in [32, 33]). The proof of the corresponding projectively invariant characterizations will rely on the generalized Menelaus theorem [6, 7], which has a similar flavor: its conditions are of affine-geometric nature, while its conclusions are projectively invariant.

Theorem 3.8 (generalized Menelaus theorem). Let $P_1, \dots, P_{n+1}$ be $n+1$ points in general position in $\mathbb{R}^n$, so that the affine space through the points P_i is n-dimensional. Let $P_{i,i+1}$ be some points on the lines $(P_i P_{i+1})$ (indices are read modulo $n+1$). The $n+1$ points $P_{i,i+1}$ lie in an $(n-1)$-dimensional affine subspace if and only if the following relation for the ratios of the directed lengths holds:

$$\prod_{i=1}^{n+1} \frac{l(P_i, P_{i,i+1})}{l(P_{i,i+1}, P_{i+1})} = (-1)^{n+1}.$$ □

Proof. The points $P_{i,i+1}$ lie in an $(n-1)$-dimensional affine subspace if there is a nontrivial linear dependence:

$$\sum_{i=1}^{n+1} \mu_i P_{i,i+1} = 0 \quad \text{with} \quad \sum_{i=1}^{n+1} \mu_i = 0.$$

Substituting $P_{i,i+1} = (1-\xi_i)P_i + \xi_i P_{i+1}$, and taking into account the general position condition, which can be read as linear independence of the vectors $\overrightarrow{P_1 P_i}$, we come to a homogeneous system of $n+1$ linear equations for $n+1$ coefficients μ_i:

$$\xi_i \mu_i + (1-\xi_{i+1})\mu_{i+1} = 0, \quad i = 1, \ldots, n+1$$

(where indices are understood modulo $n+1$). Clearly, it admits a nontrivial solution if and only if

$$\prod_{i=1}^{n+1} \frac{\xi_i}{1-\xi_i} = \prod_{i=1}^{n+1} \frac{l(P_i, P_{i,i+1})}{l(P_{i,i+1}, P_{i+1})} = (-1)^{n+1}.$$

(Menelaus theorem corresponds to $n = 2$.) ■

In the following considerations, we use the negative indices -1, -2 to denote the downward shifts τ_1^{-1}, τ_2^{-1}. Consider four elementary quadrilaterals of a Q-net adjacent to the point $f = f(u)$, i.e. the quadrilaterals (f, f_i, f_{ij}, f_j) with $(i, j) \in \{(\pm 1, \pm 2)\}$. We assume that the vertex f is nonplanar, i.e. that there is no plane containing these four quadrilaterals (or, what is the same, there is no plane containing f and its four neighbors f_i, $i \in \{\pm 1, \pm 2\}$). Recall that we always assume that the dimension of the ambient space is $N \geq 3$.

Theorem 3.9 (discrete 2d Koenigs nets; characterization in terms of intersection points of diagonals). A two-dimensional Q-net $f : \mathbb{Z}^2 \to \mathbb{R}^N$ with nonplanar vertices is a discrete Koenigs net if and only if for every point $f = f(u)$ the intersection points of diagonals of the four quadrilaterals adjacent to f lie in a two-dimensional plane. □

Proof. This is an immediate consequence of Equation (20) and the $n = 3$ case of the generalized Menelaus theorem (Theorem 3.8). ■

Remark. Thus, intersection points of diagonals of elementary quadrilaterals of a two-dimensional Koenigs net comprise a Q-net. Such Q-nets are not generic; it turns out that they can be characterized as discrete Koenigs nets in the sense of [17]. □

Theorem 3.10 (discrete 2d Koenigs nets; characterization in terms of vertices).

(1) Let $f : \mathbb{Z}^2 \to \mathbb{R}^N$ be the a Q-net in the space of dimension $N \geq 4$. Then f is a discrete Koenigs net if and only if for every $u \in \mathbb{Z}^2$ the five points f and $f_{\pm 1,\pm 2}$ lie in a three-dimensional subspace $V = V(u) \subset \mathbb{R}^N$, not containing some (and then any) of the four points $f_{\pm 1}$, $f_{\pm 2}$.

(2) Let $f : \mathbb{Z}^2 \to \mathbb{R}^3$ be a Q-net in the space of dimension $N = 3$. Then f is a discrete Koenigs net if and only if for every $u \in \mathbb{Z}^2$ the three planes

$$\Pi^{(\mathrm{up})} = (f f_{12} f_{-1,2}), \quad \Pi^{(\mathrm{down})} = (f f_{1,-2} f_{-1,-2}), \quad \Pi^{(1)} = (f f_1 f_{-1})$$

have a common line $\ell^{(1)}$, or, equivalently, the three planes

$$\Pi^{(\mathrm{left})} = (f f_{-1,2} f_{-1,-2}), \quad \Pi^{(\mathrm{right})} = (f f_{1,2} f_{1,-2}), \quad \Pi^{(2)} = (f f_2 f_{-2})$$

have a common line $\ell^{(2)}$. □

Proof. (1) If the net f satisfies the property of Theorem 3.9, then the space V through f and $f_{\pm 1,\pm 2}$ is clearly three-dimensional. Conversely, let this space be three-dimensional. The four quadrilaterals (f, f_i, f_{ij}, f_j) lie in a four-dimensional space through f, $f_{\pm 1}$, $f_{\pm 2}$. The intersection points of their diagonals lie in the intersection of V with the three-dimensional space through $f_{\pm 1}$, $f_{\pm 2}$. The intersection of two three-dimensional subspaces of a four-dimensional space is generically a plane.

(2) Let M_{ij} denote intersection point of diagonals of the quadrilateral (f, f_i, f_{ij}, f_j), with $(i, j) \in \{(\pm 1, \pm 2)\}$. Coplanarity of the four points M_{ij} is equivalent to the statement that the lines $(M_{1,2} M_{-1,2})$ and $(M_{1,-2} M_{-1,-2})$ intersect. These two lines lie in the planes $(f_1 f_2 f_{-1})$ and $(f_1 f_{-2} f_{-1})$, respectively, therefore their intersection point has to belong to the intersection of these planes, i.e. to the line $(f_1 f_{-1})$. Thus, coplanarity of the points M_{ij} is equivalent to the fact that three lines $(M_{1,2} M_{-1,2})$, $(M_{1,-2} M_{-1,-2})$, and $(f_1 f_{-1})$ have a common point $L^{(1)}$; see Figure 4. Now the planes $\Pi^{(\mathrm{up})}$, $\Pi^{(\mathrm{down})}$ and $\Pi^{(1)}$ can be viewed as the planes through the point f and the lines $(M_{1,2} M_{-1,2})$, $(M_{1,-2} M_{-1,-2})$, and $(f_1 f_{-1})$, respectively. Therefore their intersection is the line $\ell^{(1)}$ through f and $L^{(1)}$. ■

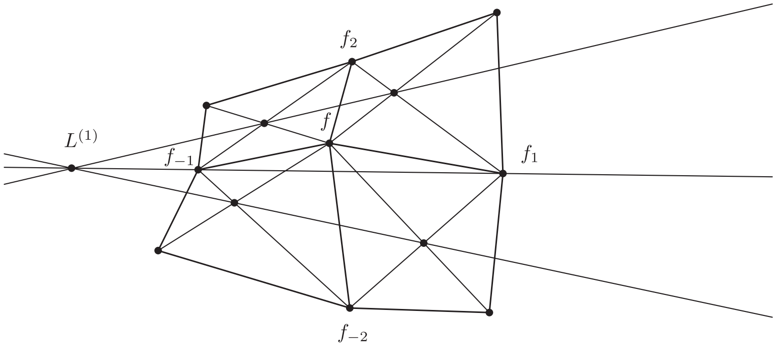

Fig. 4. Four quadrilaterals around a vertex, once more.

Remark 1. It is not difficult to see that in the dimension $N \ge 4$ the property formulated in part (1) of Theorem 3.10 automatically yields the property formulated in part (2). Indeed, for $N \ge 4$ all nine points f, $f_{\pm 1}$, $f_{\pm 2}$ and $f_{\pm 1,\pm 2}$ lie generically in a four-dimensional subspace of $\mathbb{RP}^N$. In this subspace one can consider, along with the three-dimensional subspace V, the three-dimensional subspaces $V^{(\mathrm{up})}$ containing the two quadrilaterals (f, f_1, f_{12}, f_2), $(f, f_{-1}, f_{-1,2}, f_2)$, and $V^{(\mathrm{down})}$ containing the quadrilaterals $(f, f_1, f_{1,-2}, f_{-2})$, $(f, f_{-1}, f_{-1,-2}, f_{-2})$. Obviously, one has:

$$\Pi^{(\mathrm{up})} = V^{(\mathrm{up})} \cap V, \quad \Pi^{(\mathrm{down})} = V^{(\mathrm{down})} \cap V, \quad \Pi^{(1)} = V^{(\mathrm{up})} \cap V^{(\mathrm{down})}.$$

Generically, three three-dimensional subspaces V, $V^{(\mathrm{up})}$ and $V^{(\mathrm{down})}$ of a four-dimensional space intersect along a line $\ell^{(1)}$. $\square$

Remark 2. The equivalence of two conditions in part (2) of Theorem 3.10 follows, of course, from the fact that in the notion of discrete Koenigs nets there is no asymmetry between the coordinate directions 1 and 2. However, it might be worthwhile to give an additional illustration of this equivalence. For this aim, consider a central projection of the whole picture from the point f to some plane not containing f. In this projection, the planarity of elementary quadrilaterals (f, f_i, f_{ij}, f_j) turns into collinearity of the triples of points f_i, f_j and f_{ij}. The traces of the planes $\Pi^{(\mathrm{up})}$, $\Pi^{(\mathrm{down})}$ and $\Pi^{(1)}$ on the projection plane are the lines $(f_{12}\, f_{-1,2})$, $(f_{1,-2}\, f_{-1,-2})$, and $(f_1\, f_{-1})$, respectively, and the first version of the condition of part 2) of Theorem 3.10 turns into the requirement for these three lines to meet in a point. Similarly, the traces of the planes $\Pi^{(\mathrm{left})}$, $\Pi^{(\mathrm{right})}$ and $\Pi^{(2)}$ on the projection plane are the lines $(f_{-1,2}\, f_{-1,-2})$, $(f_{1,2}\, f_{1,-2})$, and $(f_2\, f_{-2})$, respectively. The requirement for

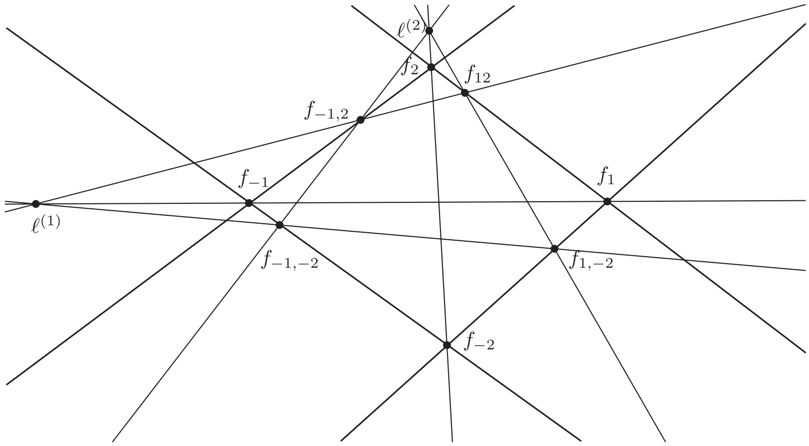

Fig. 5. Desargues theorem.

the latter three lines to meet in a point is equivalent to the previous one—this is the statement of the famous Desargues theorem; see Figure 5. □

3.4 Geometric characterization of three-dimensional discrete Koenigs nets

Theorem 3.11 (discrete 3d Koenigs nets; characterization in terms of intersection points of diagonals). A three-dimensional Q-net $f : \mathbb{Z}^3 \to \mathbb{R}^N$ is a discrete Koenigs net if and only if for every point $f = f(u)$ and for every elementary hexahedron with a vertex f, the intersection points of diagonals of the three hexahedron faces adjacent to f are collinear. □

Proof. This is nothing but the reformulation of Equation (21) in terms of Menelaus theorem ($n = 2$ case of Theorem 3.8). ■

Theorem 3.12 (discrete 3d Koenigs nets; characterization in terms of vertices). A Q-net $f : \mathbb{Z}^3 \to \mathbb{R}^N$ is a discrete Koenigs net if and only if for every elementary hexahedron of the net its four white vertices are coplanar, or its four black vertices are coplanar (each one of these conditions implies another one). □

Proof. Consider an elementary hexahedron with the vertices f, f_i, f_{ij}, f_{123}. Denote the intersection points of diagonals of the quadrilaterals (f, f_i, f_{ij}, f_j) by M_{ij}, and the intersection points of diagonals of the quadrilaterals $(f_k, f_{ik}, f_{123}, f_{jk})$ by Q_{ij}. Clearly, if the points M_{ij} are collinear, then the four points f and f_{ij} (the black ones) are coplanar.

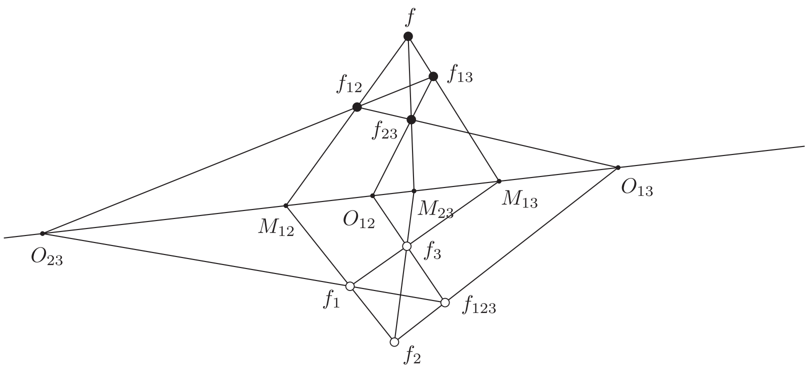

Fig. 6. Pappus theorem.

We show next that the coplanarity of the four black points yield the coplanarity of the four white points, as well.

Suppose that the four black points f, f_{ij} lie in a plane Π_0. Let Π_1 be the plane through the three points f_1, f_2, f_3. Set $\ell = \Pi_0 \cap \Pi_1$. Then the intersection points M_{ij} of diagonals of the quadrilaterals (f, f_i, f_{ij}, f_j) belong to ℓ. Denote by O_{ij} intersection points of the lines $(f_{ik} f_{jk}) \subset \Pi_0$ with ℓ. Then the three lines $(f_k O_{ij}) \subset \Pi_1$ intersect in one point, which is clearly $f_{123} \in \Pi_1$, so that the four points f_i, f_{123} are coplanar. This claim is nothing but the classical *Pappus theorem* illustrated in Figure 6. This incidence theorem of projective geometry is not to be confused with another Pappus theorem, the latter being a particular case of the Pascal hexagon theorem, when a conic section degenerates into a pair of lines. The former characterizes a *quadrilateral set* of points on a line ℓ which can be defined as consisting of intersection points of this line with the six lines connecting all pairs among four points in some plane containing ℓ. Quadrilateral sets admit several equivalent characterizations: a multi-ratio of such a set is equal to 1; in other words, the points of a quadrilateral set always build three point pairs of a projective involutive self-map of ℓ.

Now we can finish the proof of Theorem 3.12 as follows. Suppose that the black vertices of an elementary hexahedron of a Q-net are coplanar. Then also the white vertices of this hexahedron are coplanar. Then the intersection points of diagonals of all six faces of the hexahedron are collinear (they belong to the common line of the "black" and the "white" planes). According to the characterization of Theorem 3.11, the net is Koenigs. ■

Remark. The characterizations of Theorems 3.10, 3.12 coincide with the definitions of B-quadrilateral nets in [18] and of discrete Moutard nets in [5]. Thus, the point we make here is a new property of these nets, fixed as Definition 3.4 and put in the base of the whole theory. A novel derivation and understanding of the Moutard property of discrete Koenigs nets will be given below, in Section 3.6. □

3.5 Dual discrete Koenigs nets

We start with the following statement which is a direct consequence of the algebraic characterization of discrete Koenigs nets given in Theorem 3.7. Indeed, in our local setting, due to the simple-connectedness of the underlying graphs, the closeness of the multiplicative one-form q is equivalent to its exactness:

Corollary 3.13 (function ν for a discrete Koenigs net). A Q-net $f:\mathbb{Z}^m\to\mathbb{R}^N$ is a discrete Koenigs net if and only if there exists a real-valued function $\nu:\mathbb{Z}^m\to\mathbb{R}^*$ with the following property: for every elementary quadrilateral (f,f_i,f_{ij},f_j) there holds:

$$\frac{\nu_{ij}}{\nu}=q(\overrightarrow{ff_{ij}})=\frac{l(M,f_{ij})}{l(M,f)},\qquad \frac{\nu_j}{\nu_i}=q(\overrightarrow{f_if_j})=\frac{l(M,f_j)}{l(M,f_i)},\tag{22}$$

where $M=(ff_{ij})\cap(f_if_j)$ is the intersection point of diagonals. □

On both the black and the white graphs of $\mathbb{Z}^m$ such a function ν is defined up to a multiplicative constant. This freedom is fixed by prescribing values of ν arbitrarily at one black and at one white point.

Equation (22) is equivalent to

$$\frac{1}{\nu_{ij}}\,\overrightarrow{Mf_{ij}}=\frac{1}{\nu}\,\overrightarrow{Mf},\qquad \frac{1}{\nu_i}\,\overrightarrow{Mf_i}=\frac{1}{\nu_j}\,\overrightarrow{Mf_j},\tag{23}$$

which can be re-written also as

$$\frac{f_{ij}}{\nu_{ij}}-\frac{f}{\nu}=\left(\frac{1}{\nu_{ij}}-\frac{1}{\nu}\right)M,\qquad \frac{f_i}{\nu_i}-\frac{f_j}{\nu_j}=\left(\frac{1}{\nu_i}-\frac{1}{\nu_j}\right)M.\tag{24}$$

There follows:

$$\left(\frac{1}{\nu_j}-\frac{1}{\nu_i}\right)\left(\frac{f_{ij}}{\nu_{ij}}-\frac{f}{\nu}\right)=\left(\frac{1}{\nu_{ij}}-\frac{1}{\nu}\right)\left(\frac{f_j}{\nu_j}-\frac{f_i}{\nu_i}\right).\tag{25}$$

This formula can be used for an elegant representation of the dual Koenigs net for f.

Theorem 3.14 (dual Koenigs net). Let $f:\mathbb{Z}^m\to\mathbb{R}^N$ be a discrete Koenigs net, and let $\nu:\mathbb{Z}^m\to\mathbb{R}^*$ be the function defined by the property (22). Then the $\mathbb{R}^N$-valued discrete one-form δf^* defined by

$$\delta_i f^* = \frac{\delta_i f}{\nu \nu_i} \tag{26}$$

is exact. Its integration defines (up to a translation) the dual Koenigs net $f^* : \mathbb{Z}^m \to \mathbb{R}^N$. □

Proof. Equation (25) can be equivalently re-written as

$$\frac{f_{ij} - f_i}{\nu_i \nu_{ij}} + \frac{f_i - f}{\nu \nu_i} = \frac{f_{ij} - f_j}{\nu_j \nu_{ij}} + \frac{f_j - f}{\nu \nu_i}. \tag{27}$$

This is equivalent to the closeness of the discrete form δf^*. Note that Equation (26) says that the corresponding sides of elementary quadrilaterals of the nets f and f^* are parallel. It remains to show that the non-corresponding diagonals of elementary quadrilaterals of f and f^* are also parallel, so that these quadrilaterals are dual in the sense of Definitions 3.1. For this aim we demonstrate the following two formulas:

$$f^*_{ij} - f^* = a_{ij} \frac{f_j - f_i}{\nu_i \nu_j}, \quad f^*_j - f^*_i = \frac{1}{a_{ij}} \frac{f_{ij} - f}{\nu \nu_{ij}}, \tag{28}$$

where

$$a_{ij} = \left(\frac{1}{\nu_{ij}} - \frac{1}{\nu} \right) \Big/ \left(\frac{1}{\nu_j} - \frac{1}{\nu_i} \right). \tag{29}$$

Indeed, upon using Equations (25) and (29) we find:

$$\begin{aligned}
f^*_{ij} - f^* &= (f^*_{ij} - f^*_i) + (f^*_i - f^*) = \frac{f_{ij} - f_i}{\nu_i \nu_{ij}} + \frac{f_i - f}{\nu \nu_i} \\
&= \frac{1}{\nu_i} \left(\frac{f_{ij}}{\nu_{ij}} - \frac{f}{\nu} \right) - \frac{f_i}{\nu_i} \left(\frac{1}{\nu_{ij}} - \frac{1}{\nu} \right) \\
&= a_{ij} \frac{1}{\nu_i} \left(\frac{f_j}{\nu_j} - \frac{f_i}{\nu_i} \right) - a_{ij} \frac{f_i}{\nu_i} \left(\frac{1}{\nu_j} - \frac{1}{\nu_i} \right) = a_{ij} \frac{f_j - f_i}{\nu_i \nu_j},
\end{aligned}$$

and, similarly,

$$\begin{aligned}
f^*_j - f^*_i &= (f^*_{ij} - f^*_i) - (f^*_{ij} - f^*_j) = \frac{f_{ij} - f_i}{\nu_i \nu_{ij}} - \frac{f_{ij} - f_j}{\nu_j \nu_{ij}} \\
&= \frac{1}{\nu_{ij}} \left(\frac{f_j}{\nu_j} - \frac{f_i}{\nu_i} \right) - \frac{f_{ij}}{\nu_{ij}} \left(\frac{1}{\nu_j} - \frac{1}{\nu_i} \right) \\
&= \frac{1}{a_{ij} \nu_{ij}} \left(\frac{f_{ij}}{\nu_{ij}} - \frac{f}{\nu} \right) - \frac{f_{ij}}{a_{ij} \nu_{ij}} \left(\frac{1}{\nu_{ij}} - \frac{1}{\nu} \right) = \frac{1}{a_{ij}} \frac{f_{ij} - f}{\nu_{ij} \nu}.
\end{aligned}$$

Theorem 3.14 is completely proven. ■

For future reference, we note here that after some manipulations formula (25) can be transformed into

$$\delta_i\delta_j f = \frac{\nu_j\nu_{ij} - \nu\nu_i}{\nu(\nu_i - \nu_j)}\delta_i f + \frac{\nu_i\nu_{ij} - \nu\nu_j}{\nu(\nu_j - \nu_i)}\delta_j f. \tag{30}$$

This is a discrete analogue of the differential equation (1) for smooth Koenigs nets. See also Section 3.7 for more details concerning the continuous limit to smooth Koenigs nets.

3.6 Moutard representative of a discrete Koenigs net

Constructions of the previous subsection (functions ν and a_{ij} for a given Koenigs net) can be used also in a different spirit.

Theorem 3.15 (discrete Koenigs nets = discrete Moutard nets in homogeneous coordinates). A Q-net $f : \mathbb{Z}^m \to \mathbb{R}^N$ is a discrete Koenigs net if and only if there exists a function $\nu : \mathbb{Z}^m \to \mathbb{R}^*$ such that the points $y : \mathbb{Z}^m \to \mathbb{R}^{N+1}$,

$$y = \nu^{-1}(f, 1), \tag{31}$$

satisfy the *Moutard equation with minus signs*

$$\tau_i\tau_j y - y = a_{ij}(\tau_j y - \tau_i y) \tag{32}$$

with $a_{ij} \in \mathbb{R}$ given by Equation (29). The net $y = \nu^{-1}(f, 1)$, considered as a special lift of f to the space of homogeneous coordinates for $\mathbb{RP}^N$, will be called the *Moutard representative* of the discrete Koenigs net f. □

Proof. First let $f : \mathbb{Z}^m \to \mathbb{R}^N$ be a discrete Koenigs net. Define the function $\nu : \mathbb{Z}^m \to \mathbb{R}^*$, according to Corollary 3.13. Then Equation (24) holds, with M being the intersection point of diagonals of the quadrilateral (f, f_i, f_{ij}, f_j). Denoting $y = \nu^{-1}(f, 1)$, we immediately arrive at Equation (32) with the coefficients a_{ij} defined by Equation (29).

Note that the quantities a_{ij} are naturally assigned to elementary quadrilaterals of $\mathbb{Z}^m$ parallel to the coordinate plane $\mathcal{B}_{ij}$.

Conversely, given a solution $y : \mathbb{Z}^m \to \mathbb{R}^{N+1}$ of the Moutard equation (32) in $\mathbb{R}^{N+1}$, define $\nu : \mathbb{Z}^m \to \mathbb{R}$ and $f : \mathbb{Z}^m \to \mathbb{R}^N$ by $y = \nu^{-1}(f, 1)$. In other words, let ν^{-1} denote the last component of y, and let f be the vector in $\mathbb{R}^N$ obtained by multiplying the first N components of y by ν. Then, inverting the previous arguments, it is easy to show that f is a discrete Koenigs net. Indeed, one finds immediately expression (29) for the coefficient a_{ij} of the Moutard equation, then from

$$y_{ij} - y = a_{ij}(y_j - y_i)$$

there follows Equation (25). This allows to define the point M by Equation (24). The latter equation is equivalent to (23), therefore M is nothing but the intersection point of diagonals of (f, f_i, f_{ij}, f_j). There holds Equation (22), so by Corollary 3.13 f is a Koenigs net. ∎

In the context of discrete integrable systems the discrete Moutard equation (32) has been introduced in [15], its importance for discrete differential geometry has been re-iterated in [30], based on the fact that this equation expresses the permutability properties of the so called Moutard transformation for the differential Moutard equation [1, 22, 29, 30]. The role played by the discrete Moutard equation in the discrete differential geometry turns out to be manifold. In particular, the so called Lelieuvre representation of discrete asymptotic nets involves discrete Moutard nets in $\mathbb{R}^3$ [16, 28]. For the multidimensional consistency of discrete Moutard nets, which lies in the basis of the transformation theory for discrete Koenigs nets, the reader is referred to [3, 5, 18].

3.7 Continuous limit

In order for a Q-net to admit a continuous limit, all its quadrilaterals should be of a reasonable shape. Anyway, they must be convex. As mentioned in subsection 3.2, diagonals of convex quadrilaterals carry negative quantities q (ratios of segments of diagonals). Theorem 3.7 shows that a discrete Koenigs net cannot consist of convex quadrilaterals (and thus cannot admit a continuous limit) for $m \geq 3$. However, there are no obstructions in case $m = 2$. This is in a good agreement with the existence of two-dimensional smooth Koenigs nets only.

Equation (22) shows that in case $m = 2$ with all convex quadrilaterals we can assume, without losing generality, that the sign of $\nu(u)$ at $u = (u_1, u_2) \in \mathbb{Z}^2$ is either $(-1)^{u_1}$ or $(-1)^{u_2}$. Clearly, such a wildly oscillating function cannot have a well-behaved continuous limit. However, upon re-defining

$$\nu(u) \mapsto (-1)^{u_1}\nu(u), \quad \text{resp.} \quad \nu(u) \mapsto (-1)^{u_2}\nu(u) \tag{33}$$

we get a positive function, which turns out to be a proper discrete analog of the function ν for smooth Koenigs nets. Note that this re-definition is equivalent to changing Equation (22) to

$$\frac{\nu_{12}}{\nu} = \frac{l(f_{12}, M)}{l(M, f)}, \qquad \frac{\nu_2}{\nu_1} = \frac{l(f_2, M)}{l(M, f_1)}. \tag{34}$$

We mention also that Equation (30) with the re-defined ν changes its shape into

$$\delta_1\delta_2 f = \frac{\nu_2\nu_{12} - \nu\nu_1}{\nu(\nu_1 + \nu_2)}\delta_1 f + \frac{\nu_1\nu_{12} - \nu\nu_2}{\nu(\nu_1 + \nu_2)}\delta_2 f, \tag{35}$$

with Equation (1) as a continuous limit. Likewise, formulas (26) turn into

$$\delta_1 f^* = \frac{\delta_1 f}{\nu\nu_1}, \qquad \delta_2 f^* = -\frac{\delta_2 f}{\nu\nu_2}, \tag{36}$$

where the second redefinition of ν in Equation (33) has been used, for definiteness (the first one would result in changing signs of both fractions).

Thus, the change of signs (33) enables the smooth limit but breaks the symmetry of the system. In particular, we observe that the coordinate directions 1 and 2 play different roles in Equation (36), while the corresponding formulas (26) for a multidimensional Koenigs net have the same shape for all coordinate directions.

For the Moutard representative $y : \mathbb{Z}^2 \to \mathbb{R}^{N+1}$ of a two-dimensional discrete Koenigs net the change (33) leads to

$$y(u) \mapsto (-1)^{u_1} y(u), \quad \text{resp.} \quad y(u) \mapsto (-1)^{u_2} y(u). \tag{37}$$

These points satisfy the *Moutard equation with the plus signs*:

$$\tau_1\tau_2 y + y = a(\tau_1 y + \tau_2 y), \tag{38}$$

or, equivalently,

$$\delta_1\delta_2 f = \tfrac{1}{2} q(\tau_1 f + \tau_2 f), \tag{39}$$

with some $a = 1 + \frac{1}{2} q : \mathbb{Z}^2 \to \mathbb{R}$. Clearly, the latter equation has Equation (9) as continuous limit.

Again, we observe the trade between the multidimensional consistency of the Moutard equation with the minus signs (which can be posed in any dimension $m \geq 2$) and the well defined continuous limit for the Moutard equation with the plus signs (which only takes place for $m = 2$). To pass from the former to the latter, one must break the symmetry through the change of signs (37).

4 Discrete Isothermic Nets

4.1 Notion of a discrete isothermic net

Definition 4.1 (discrete isothermic net). A *discrete isothermic net* is a circular Koenigs net, i.e. a circular net $f : \mathbb{Z}^m \to \mathbb{R}^N$ admitting a dual net $f^* : \mathbb{Z}^m \to \mathbb{R}^N$ in the sense of Definition 3.4. □

We can use characterizations of Koenigs net derived in Section 3 in order to find characterizations of discrete isothermic nets. For this aim, we use the fact that for a circular net $f : \mathbb{Z}^m \to \mathbb{R}^N$ its lift $\hat{f} = f + \mathbf{e}_0 + |f|^2\mathbf{e}_\infty$ into the light cone $\mathbb{L}^{N+1,1}$ satisfies the same equation of the Laplace type as the net f itself. In particular, a circular net f in $\mathbb{R}^N$ is discrete Koenigs if and only if $\hat{f}$ is a discrete Koenigs net in $\mathbb{R}^{N+1,1}$.

Projectively invariant characterizations of Koenigs nets $\hat{f}$ in $\mathbb{R}^{N+1,1}$ immediately translate into Möbius-geometric characterizations of isothermic nets f in $\mathbb{R}^N$. Thereby conditions like "points $\hat{f}$ lie in a d-dimensional space" should be understood as "vectors $\hat{f}$ span a $(d+1)$-dimensional linear subspace", and this is translated as "points f belong to a $(d-1)$-dimensional sphere".

Translating in this fashion Theorem 3.10, applied to a two-dimensional Koenigs net $\hat{f}$ in $\mathbb{R}^{N+1,1}$, into the language of Möbius geometry in $\mathbb{R}^N$, we come to the following statement.

Theorem 4.2.

(1) **Central spheres for a discrete isothermic surface.** A two-dimensional circular net $f : \mathbb{Z}^2 \to \mathbb{R}^N$ not lying in a two-sphere is discrete isothermic if and only if for every $u \in \mathbb{Z}^2$ the five points f and $f_{\pm 1,\pm 2}$ lie on a two-sphere not containing some (and then any) of the four points $f_{\pm 1}$, $f_{\pm 2}$.

(2) **Discrete isothermic net on a sphere.** A two-dimensional circular net $f : \mathbb{Z}^2 \to S^2 \subset \mathbb{R}^N$ in a two-sphere is discrete isothermic if and only if for every $u \in \mathbb{Z}^2$ the three circles through f,

$$C^{(\mathrm{up})} = \mathrm{circle}(f, f_{12}, f_{-1,2}), \quad C^{(\mathrm{down})} = \mathrm{circle}(f, f_{1,-2}, f_{-1,-2}),$$
$$C^{(1)} = \mathrm{circle}(f, f_1, f_{-1}),$$

Fig. 7. Four circles of a generic discrete isothermic surface, with a central sphere.

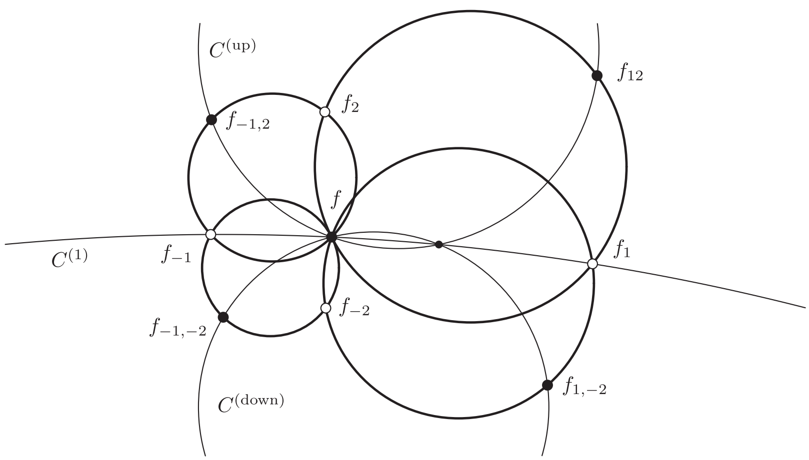

Fig. 8. Four circles of a planar (or spherical) discrete isothermic net.

have one additional point in common, or, equivalently, the three circles through f,

$$C^{(\mathrm{left})} = \mathrm{circle}(f, f_{-1,2}, f_{-1,-2}), \quad C^{(\mathrm{right})} = \mathrm{circle}(f, f_{1,2}, f_{1,-2}),$$
$$C^{(2)} = \mathrm{circle}(f, f_2, f_{-2}),$$

have one additional point in common. □

The cases (1) and (2) of Theorem 4.2 are illustrated in Figures 7 and 8, respectively.

Similarly, translating Theorem 3.12, applied to a multidimensional Koenigs net $\hat{f}$ in $\mathbb{R}^{N+1,1}$, into the language of Möbius-geometric properties of the net f in $\mathbb{R}^N$, we get the following statement.

Theorem 4.3 (multidimensional discrete isothermic nets). A circular net $f : \mathbb{Z}^m \to \mathbb{R}^N$ is discrete isothermic if and only if for any elementary hexahedron of the net its four white vertices are concircular, and its four black vertices are concircular (each one of these conditions implies another one). □

4.2 Cross-ratio characterization of discrete isothermic nets

Another characterization of discrete isothermic surfaces can be given in terms of the cross-ratios. Recall that for any four concircular points $a, b, c, d \in \mathbb{R}^N$ their (real-valued) cross-ratio is defined by

$$q(a, b, c, d) = (a - b)(b - c)^{-1}(c - d)(d - a)^{-1}, \tag{40}$$

with the Clifford multiplication in the Clifford algebra $\mathcal{C}\ell(\mathbb{R}^N)$. The Clifford product of $x, y \in \mathbb{R}^N$ satisfies $xy + yx = -2\langle x, y\rangle$, and the inverse element of $x \in \mathbb{R}^N$ in the Clifford algebra is given by $x^{-1} = -x/|x|^2$. Alternatively, one can identify the plane of the quadrilateral (a, b, c, d) with the complex plane $\mathbb{C}$, and then multiplication in Equation (40) can be interpreted as the complex multiplication. An important property of the cross-ratio is its invariance under Möbius transformations.

For discrete isothermic surfaces Theorem 4.2 yields the following characterization.

Theorem 4.4 (cross-ratios of four adjacent quadrilaterals). A two-dimensional circular net $f : \mathbb{Z}^2 \to \mathbb{R}^N$ is a discrete isothermic surface if and only if the cross-ratios $q = q(f, f_1, f_{12}, f_2)$ of its elementary quadrilaterals satisfy the following condition:

$$q \cdot q_{-1,-2} = q_{-1} \cdot q_{-2}. \tag{41}$$

Here, as usual, the negative indices $-i$ denote the backward shifts τ_i^{-1}, so that, e.g. $q_{-1} = q(f_{-1}, f, f_2, f_{-1,2})$; see Figure 9. □

Proof. Perform a Möbius transformation sending f to ∞. Under such a transformation, the four adjacent circles through f turn into four straight lines $(f_{\pm 1} f_{\pm 2})$, containing the

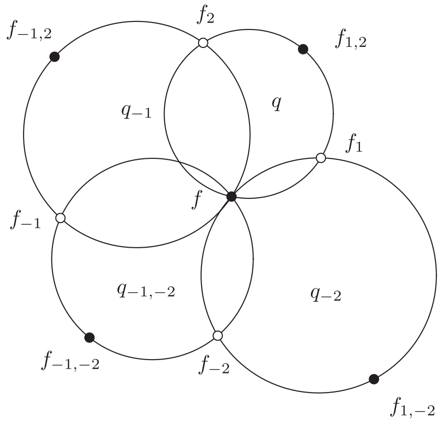

Fig. 9. Four adjacent quadrilaterals of a discrete isothermic surface: the cross-ratios satisfy $q \cdot q_{-1,-2} = q_{-1} \cdot q_{-2}$.

corresponding points $f_{\pm1,\pm2}$. The cross-ratios turn into ratios of directed lengths, e.g.

$$q(f, f_1, f_{1,2}, f_2) = -\frac{l(f_1, f_{1,2})}{l(f_{1,2}, f_2)}.$$

If the affine space through the points $f_{\pm1}$, $f_{\pm2}$ is three-dimesnional, then, according to part 1) of Theorem 4.2, the four points $f_{\pm1,\pm2}$ lie in a plane (a sphere through $f = \infty$). Generalized Menelaus theorem (Theorem 3.8) provides us with the following necessary and sufficient condition for this, which reads:

$$\frac{l(f_2, f_{1,2})}{l(f_{1,2}, f_1)} \cdot \frac{l(f_1, f_{1,-2})}{l(f_{1,-2}, f_{-2})} \cdot \frac{l(f_{-2}, f_{-1,-2})}{l(f_{-1,-2}, f_{-1})} \cdot \frac{l(f_{-1}, f_{-1,2})}{l(f_{-1,2}, f_2)} = 1. \tag{42}$$

This is equivalent to Equation (41) with $f = \infty$.

If, on the contrary, the four points $f_{\pm1}$, $f_{\pm2}$ are coplanar, then, according to part (2) of Theorem 4.2, both lines $(f_{-1,2} f_{1,2})$ and $(f_{-1,-2} f_{1,-2})$ meet the line $(f_{-1} f_1)$ at the same point $\ell^{(1)}$. Thus, we are in the situation of Figure 5, described by the Desargues theorem. Here, we apply the Menelaus theorem twice, to the triangle $\triangle(f_{-1}, f_2, f_1)$ intersected by the line $(f_{-1,2} f_{1,2})$, and to the triangle $\triangle(f_{-1}, f_{-2}, f_1)$ intersected by the line $(f_{-1,-2} f_{1,-2})$:

$$\frac{l(f_2, f_{12})}{l(f_{12}, f_1)} \cdot \frac{l(f_{-1}, f_{-1,2})}{l(f_{-1,2}, f_2)} = -\frac{l(f_{-1}, \ell^{(1)})}{l(\ell^{(1)}, f_1)} = \frac{l(f_{-2}, f_{1,-2})}{l(f_{1,-2}, f_1)} \cdot \frac{l(f_{-1}, f_{-1,-2})}{l(f_{-1,-2}, f_{-2})}.$$

This yields formula (42), again. ■

For multidimensional discrete isothermic nets Theorem 4.3 yields a similar characterization.

Theorem 4.5 (cross-ratios of three adjacent quadrilaterals). A circular net $f : \mathbb{Z}^m \to \mathbb{R}^N$ is discrete isothermic if and only if the cross-ratios of its elementary quadrilaterals satisfy the following condition:

$$q(f, f_i, f_{ij}, f_j) \cdot q(f, f_j, f_{jk}, f_k) \cdot q(f, f_k, f_{ki}, f_i) = 1 \tag{43}$$

for any triple of different indices i, j, k. □

Proof. Again, perform a Möbius transformation sending f to ∞. Under such a transformation, the three adjacent circles through f turn into three straight lines $(f_i f_j)$, $(f_j f_k)$ and $(f_k f_i)$, containing the (white) points f_{ij}, f_{jk} and f_{ki}, respectively. Concircularity of these white points with f means simply that they are collinear. The necessary and sufficient condition for this is given by the Menelaus theorem:

$$\frac{l(f_j, f_{ij})}{l(f_{ij}, f_i)} \cdot \frac{l(f_k, f_{jk})}{l(f_{jk}, f_j)} \cdot \frac{l(f_i, f_{ki})}{l(f_{ki}, f_k)} = -1. \tag{44}$$

Since the Möbius-invariant meaning of the ratios of directed lengths is given by the corresponding cross-ratios,

$$q(f, f_i, f_{ij}, f_j) = -\frac{l(f_i, f_{ij})}{l(f_{ij}, f_j)},$$

Equation (44) is equivalent to Equation (43). ■

The conclusions of Theorems 4.4 and 4.5 can be summarized with the help of the following notion:

Definition 4.6 (edge labeling). A system of real-valued functions α_i defined on the edges of $\mathbb{Z}^m$ parallel to the ith coordinate axis ($i = 1, \dots, m$) is called an *edge labeling* if they take equal values on each pair of opposite edges of any elementary quadrilateral. □

Thus, both edges $(u, u + e_i)$ and $(u + e_j, u + e_i + e_j)$ of an elementary square of $\mathbb{Z}^m$ parallel to the coordinate plane (ij) carry the label $\alpha_i = \alpha_i(u) = \alpha_i(u + e_j)$, and, similarly, both other edges $(u, u + e_j)$ and $(u + e_i, u + e_i + e_j)$ carry the label $\alpha_j = \alpha_j(u) = \alpha_j(u + e_i)$;

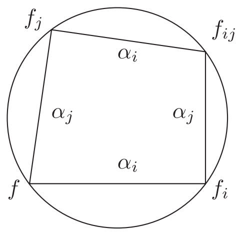

Fig. 10. Labeling of edges of a discrete isothermic net.

see Figure 10. In this notation, there holds $\tau_j\alpha_i = \alpha_i$ for $i \neq j$, so that each function $\alpha_i(u)$ depends on u_i only.

The following theorem is an immediate consequence of Theorems 4.4 and 4.5.

Theorem 4.7 (factorized cross-ratios). A circular net $f : \mathbb{Z}^m \to \mathbb{R}^N$ is discrete isothermic if and only if the cross-ratios of its elementary quadrilaterals satisfy

$$q(f, f_i, f_{ij}, f_j) = \frac{\alpha_i}{\alpha_j}, \tag{45}$$

where α_i $(i = 1, \dots, m)$ constitute a real-valued labeling of the edges of $\mathbb{Z}^m$. $\square$

Theorem 4.7 says that our definition of discrete isothermic nets coincides with the original definition from [2]. In the next subsection we will give a more concrete way of determining the labeling α_i for a given discrete isothermic net.

4.3 Metric of a discrete isothermic net

Now we turn to a characterization of discrete Koenigs nets given in Corollary 3.13. Being applied to circular nets, it says that such a net f is Koenigs if and only if there exists a function $s : \mathbb{Z}^m \to \mathbb{R}^*$ such that for any circular quadrilateral (f, f_i, f_{ij}, f_j) with the intersection point of diagonals M there holds:

$$\frac{l(M, f_{ij})}{l(M, f)} = \frac{s_{ij}}{s}, \qquad \frac{l(M, f_j)}{l(M, f_i)} = \frac{s_j}{s_i}. \tag{46}$$

(Note that the notation s comes to replace ν which we reserve for general Koenigs nets.) The function s for circular nets turns out to admit an additional property.

Theorem 4.8 (metric coefficient of discrete isothermic nets). For a discrete isothermic net f, relations (46) define a function $s : \mathbb{Z}^m \to \mathbb{R}$ uniquely, up to a black-white rescaling

($s \mapsto \lambda s$ on black vertices, $s \mapsto \mu s$ on white vertices), which can be fixed by prescribing s arbitrarily at one black and at one white point. There exists a labeling α of edges of $\mathbb{Z}^m$ such that

$$|f_i - f|^2 = \alpha_i s s_i \qquad (i = 1, \ldots, m). \tag{47}$$

A black-white rescaling of the function s results in the rescaling $\alpha \mapsto (\lambda\mu)^{-1}\alpha$ of the labeling α. □

Proof. For a circular quadrilateral (f, f_i, f_{ij}, f_j) with the intersection point of diagonals M, one has two pairs of similar triangles,

$$\triangle(f, f_i, M) \sim \triangle(f_j, f_{ij}, M), \qquad \triangle(f, f_j, M) \sim \triangle(f_i, f_{ij}, M).$$

Hence, there holds:

$$\frac{|Mf_{ij}|}{|Mf_i|} = \frac{|Mf_j|}{|Mf|} = \frac{|f_{ij} - f_j|}{|f_i - f|}, \qquad \frac{|Mf_{ij}|}{|Mf_j|} = \frac{|Mf_i|}{|Mf|} = \frac{|f_{ij} - f_i|}{|f_j - f|}. \tag{48}$$

There follows:

$$\frac{|Mf_{ij}|}{|Mf|} \cdot \frac{|Mf_j|}{|Mf_i|} = \frac{|f_{ij} - f_j|^2}{|f_i - f|^2}, \qquad \frac{|Mf_{ij}|}{|Mf|} \cdot \frac{|Mf_i|}{|Mf_j|} = \frac{|f_{ij} - f_i|^2}{|f_j - f|^2}. \tag{49}$$

This can be written as

$$\frac{l(M, f_{ij})}{l(M, f)} \cdot \frac{l(M, f_j)}{l(M, f_i)} = \frac{|f_{ij} - f_j|^2}{|f_i - f|^2}, \qquad \frac{l(M, f_{ij})}{l(M, f)} \cdot \frac{l(M, f_i)}{l(M, f_j)} = \frac{|f_{ij} - f_i|^2}{|f_j - f|^2}. \tag{50}$$

Indeed, contemplating Figure 11, it is not difficult to realize that the fractions on the left-hand side of each one of the two equations in (50) are either both negative (for an embedded quadrilateral), or both positive (for a nonembedded quadrilateral), so that the replacement of the quotients of lengths in Equation (49) by quotients of directed lengths in Equation (50) is legitimate. Substitute the defining relations (46) of the function s into Equation (50):

$$\frac{s_j s_{ij}}{s s_i} = \frac{|f_{ij} - f_j|^2}{|f_i - f|^2}, \qquad \frac{s_i s_{ij}}{s s_j} = \frac{|f_{ij} - f_i|^2}{|f_j - f|^2}. \tag{51}$$

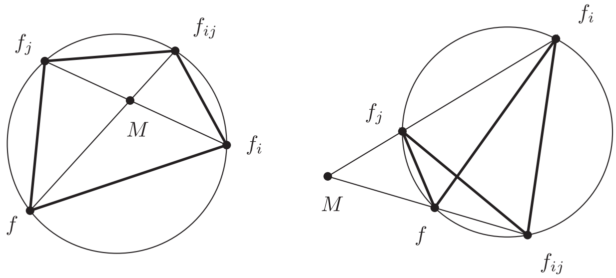

Fig. 11. Circular quadrilaterals, an embedded and a non-embedded ones.

But this is equivalent to the claim that the functions

$$\alpha_i = \frac{|f_i - f|^2}{ss_i} \tag{52}$$

possess the labeling property, $\tau_j \alpha_i = \alpha_i$. ■

The notations α_i for edge labelings in Theorems 4.7 and 4.8 coincide not without a reason.

Theorem 4.9 (origin of the edge labeling for factorized cross-ratios). If the edge labeling α_i for a discrete isothermic net $f : \mathbb{Z}^m \to \mathbb{R}^N$ is introduced according to Equation (47), then the cross-ratios of its elementary quadrilaterals are factorized as in Equation (45). □

Proof. For a circular quadrilateral (f, f_i, f_{ij}, f_j) one has:

$$q(f, f_i, f_{ij}, f_j) = \epsilon \, \frac{|f_i - f| \cdot |f_{ij} - f_j|}{|f_j - f| \cdot |f_{ij} - f_i|} \, ,$$

where $\epsilon < 0$ for an embedded quadrilateral and $\epsilon > 0$ for a nonembedded one. Thus,

$$q(f, f_i, f_{ij}, f_j) = \epsilon \, \frac{|f_i - f|^2}{|f_j - f|^2} \cdot \frac{|f_{ij} - f_j|}{|f_i - f|} \cdot \frac{|f_j - f|}{|f_{ij} - f_i|} \, .$$

Upon using Equations (47) and (48), the latter equation can be rewritten as

$$q(f, f_i, f_{ij}, f_j) = \epsilon \, \frac{\alpha_i s_i}{\alpha_j s_j} \cdot \frac{|Mf_j|}{|Mf_i|} = \frac{\alpha_i s_i}{\alpha_j s_j} \cdot \frac{l(M, f_j)}{l(M, f_i)} \, ,$$

and finally, due to Equation (51), we arrive at

$$q(f, f_i, f_{ij}, f_j) = \frac{\alpha_i s_i}{\alpha_j s_j} \cdot \frac{s_j}{s_i} = \frac{\alpha_i}{\alpha_j},$$

which proves the theorem. ■

Theorem 4.8, as it stands, cannot be reversed: existence of a function s satisfying (47) does not yield the Koenigs property. Indeed, from Equations (47) and (50) one finds:

$$\frac{l(M, f_{ij})}{l(M, f)} \cdot \frac{l(M, f_j)}{l(M, f_i)} = \frac{s_j s_{ij}}{s s_i}, \qquad \frac{l(M, f_{ij})}{l(M, f)} \cdot \frac{l(M, f_i)}{l(M, f_j)} = \frac{s_i s_{ij}}{s s_j}, \tag{53}$$

which is equivalent to

$$\frac{l(M, f_{ij})}{l(M, f)} = \pm \frac{s_{ij}}{s}, \qquad \frac{l(M, f_i)}{l(M, f_j)} = \pm \frac{s_i}{s_j} \tag{54}$$

(with the same sign $\pm$ in both equations). The latter equation is somewhat weaker than Equation (51), which is necessary and sufficient for the net f to be Koenigs. However, assuming some additional information about f, it is possible to force the plus signs in the latter formula. For instance, if it is known that all elementary quadrilaterals of a two-dimensional circular net f are embedded, then property (47) is sufficient to assure that f is Koenigs. Indeed, in this case $\alpha_2/\alpha_1 < 0$, so that Equation (47) yields $s_2/s_1 < 0$ and $s_{12}/s < 0$, and then the plus sign has to be chosen in Equation (54).

4.4 Duality of discrete isothermic nets

Specializing the notion of duality from general Koenigs nets to circular ones, the first essential observation is: the dual net for a discrete isothermic net is discrete isothermic, as well. Indeed, any quadrilateral with sides parallel to the corresponding sides of a circular quadrilateral is, obviously, also circular. A more detailed description of duality for discrete isothermic nets is contained in the following theorem.

Theorem 4.10 (dual discrete isothermic net). Let $f : \mathbb{Z}^m \to \mathbb{R}^N$ be a discrete isothermic net, with the factorized cross-ratios

$$q(f, f_i, f_{ij}, f_j) = \frac{\alpha_i}{\alpha_j} \tag{55}$$

and with the metric coefficient $s : \mathbb{Z}^m \to \mathbb{R}^*$. Then the $\mathbb{R}^N$-valued discrete one-form δf^* defined by

$$\delta_i f^* = \alpha_i \frac{\delta_i f}{|\delta_i f|^2} = \frac{\delta_i f}{s s_i}, \qquad i = 1, \dots, m, \tag{56}$$

is exact. Its integration defines (up to a translation) a net $f^* : \mathbb{Z}^2 \to \mathbb{R}^N$, called *dual* to the net f, or *Christoffel transform* of the net f. The net f^* is discrete isothermic, with the cross-ratios

$$q(f^*, f_i^*, f_{ij}^*, f_j^*) = \frac{\alpha_i}{\alpha_j} \tag{57}$$

and with the metric coefficient $s^* = s^{-1} : \mathbb{Z}^m \to \mathbb{R}^*$. Conversely, if for a given net $f : \mathbb{Z}^m \to \mathbb{R}^N$ there exists an edge labeling α_i such that the discrete one-form

$$\delta_i f^* = \alpha_i \frac{\delta_i f}{|\delta_i f|^2} \tag{58}$$

is exact, then f is a discrete isothermic net, with cross-ratios as in Equation (55). □

Proof. The first part of the theorem is a consequence of the general construction of dual Koenigs nets. To prove the converse part, observe that closeness of the one-form (58) implies that the quadrilateral (f, f_i, f_{ij}, f_j) is planar. Identifying its plane with $\mathbb{C}$, we see that the closeness condition is equivalent to (the complex conjugate of)

$$\frac{\alpha_i}{f_i - f} - \frac{\alpha_i}{f_{ij} - f_j} = \frac{\alpha_j}{f_j - f} - \frac{\alpha_j}{f_{ij} - f_i}.$$

Upon clearing denominators, the latter equation turns into the cross-ratio equation (55) (in the generic situation, when $f_{ij} - f_i - f_j + f \neq 0$). Thus, the closeness of the form (58) actually characterizes discrete isothermic nets. ■

Corollary 4.11. The noncorresponding diagonals of any elementary quadrilateral of a discrete isothermic net f and of its dual are related by

$$f_i^* - f_j^* = (\alpha_i - \alpha_j) \frac{f_{ij} - f}{|f_{ij} - f|^2}, \quad f_{ij}^* - f^* = (\alpha_i - \alpha_j) \frac{f_i - f_j}{|f_i - f_j|^2}. \tag{59}$$

□

Proof. We put Equation (45) into several equivalent forms; these computations hold not only in the Clifford algebra $\mathcal{C}\ell(\mathbb{R}^N)$, but in an arbitrary associative algebra with unit $\mathcal{A}$. Being written as

$$\alpha_i(f_{ij}-f_i)(f_i-f)^{-1}=\alpha_j(f_{ij}-f_j)(f_j-f)^{-1}, \tag{60}$$

this equation displays the symmetry with respect to the diagonal flips of an elementary quadrilateral, expressed as $f_i \leftrightarrow f_j$ and $f \leftrightarrow f_{ij}$, respectively (both have to be accompanied by the change $\alpha_i \leftrightarrow \alpha_j$). Writing Equation (60) as

$$\alpha_i(f_{ij}-f)(f_i-f)^{-1}-\alpha_i=\alpha_j(f_{ij}-f)(f_j-f)^{-1}-\alpha_j,$$

and dividing from the left by $f_{ij}-f$, we arrive at the so-called three-leg form of the cross-ratio equation

$$(\alpha_i-\alpha_j)(f_{ij}-f)^{-1}=\alpha_i(f_i-f)^{-1}-\alpha_j(f_j-f)^{-1}. \tag{61}$$

According to Equation (56), the right-hand side of Equation (61) is equal to $-(f_i^*-f^*)+(f_j^*-f^*)=f_j^*-f_i^*$. This proves the first equation in (59). The second one is analogous. ∎

4.5 Moutard representatives of discrete isothermic nets

The metric coefficient of a discrete isothermic net f can be used to produce its Moutard representative or, better, a Moutard representative of its lift $\hat{f}$ into the light cone of $\mathbb{R}^{N+1,1}$. This leads to a new characterization of discrete isothermic nets, which is manifestly Möbius invariant, since it is given entirely within the formalism of the projective model of Möbius geometry. The following statement is a discrete analog of Theorem 2.6.

Theorem 4.12 (discrete isothermic nets = discrete Moutard nets in light cone). If $f:\mathbb{Z}^m\to\mathbb{R}^N$ is a discrete isothermic net, then its lift $y=s^{-1}\hat{f}:\mathbb{Z}^m\to\mathbb{L}^{N+1,1}$ to the light cone of $\mathbb{R}^{N+1,1}$ satisfies the discrete Moutard equation (32).

Conversely, given a discrete Moutard net $y:\mathbb{Z}^m\to\mathbb{L}^{N+1,1}$ in the light cone, let the functions $s:\mathbb{Z}^m\to\mathbb{R}$ and $f:\mathbb{Z}^m\to\mathbb{R}^N$ be defined by

$$y=s^{-1}\big(f+\mathbf{e}_0+|f|^2\mathbf{e}_\infty\big) \tag{62}$$

(so that s^{-1} is the $\mathbf{e}_0$-component, and $s^{-1}f$ is the $\mathbb{R}^N$-part of y in the basis $\mathbf{e}_1, \dots, \mathbf{e}_N, \mathbf{e}_0, \mathbf{e}_\infty$). Then f is a discrete isothermic net. □

Proof. This follows from Theorem 3.15 and the fact that for a circular Koenigs net f in $\mathbb{R}^N$, the net $\hat{f} = f + \mathbf{e}_0 + |f|^2\mathbf{e}_\infty$ is also a Koenigs net in the light cone $\mathbb{L}^{N+1,1} \subset \mathbb{R}^{N+1,1}$. ■

Thus, we found an interpretation of discrete isothermic nets as an instance of discrete Moutard nets in a quadric. The edge labeling of a discrete isothermic net f (which provides the factorization (45) of its cross-ratios) is already encoded in its lift y to the light cone. Indeed,

$$\alpha_i = \frac{|f_i - f|^2}{ss_i} = -2\langle y, \tau_i y\rangle,$$

and it is easy to see that these quantities depend on u_i only.

4.6 Continuous limit

In order to enable the continuous limit to smooth isothermic surfaces, one should start with discrete isothermic surfaces (discrete isothermic nets with $m = 2$) with embedded elementary quadrilaterals. In this case the standard redefinition of the function s, namely $s(u) \mapsto (-1)^{u_2}s(u)$, assures the positivity of s. It is convenient to change the notation for the labeling, as well: $\alpha_2 \mapsto -\alpha_2$. Then formula (47) remains valid as it stands, and for the negative cross-ratios of elementary quadrilaterals, we get: $q(f, f_1, f_{12}, f_2) = -\alpha_1/\alpha_2$, with positive labels α_1 and α_2. Equation (56) turns into

$$\delta_1 f^* = \alpha_1 \frac{\delta_1 f}{|\delta_1 f|^2} = \frac{\delta_1 f}{ss_1}, \qquad \delta_2 f^* = -\alpha_2 \frac{\delta_2 f}{|\delta_2 f|^2} = -\frac{\delta_2 f}{ss_2}, \tag{63}$$

which is a direct discrete analogue of Equation (6). Once again, like in Section 3.7, we observe that a well-defined continuous limit is made possible by a break of symmetry among coordinate directions of a multidimensional net.

Acknowledgment

This work was supported by the Deutsche Forschungsgemeinschaft (Research Unit "Polyhedral Surfaces").

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