Discrete differential geometry. Integrability as consistency

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1 Introduction

The original results presented in these lectures were proved in the recent series of papers [3], [4], [6], [7]. We refer to these papers for more details, further references and complete proofs. For the geometric background in discrete differential geometry see in particular [2], [1].

2 Origin and motivation: Differential geometry

Long before the theory of solitons, geometers used integrable equations to describe various special curves, surfaces etc. At that time no relation to mathematical physics was known, and quite different geometries which appeared in this context (called integrable nowadays) were unified by their common geometric features:

- Integrable surfaces, curves etc. have nice geometric properties,
- Integrable geometries come with their interesting transformations acting within the class,
- These transformations are permutable (Bianchi permutability).

Since 'nice' and 'interesting' can hardly be treated as mathematically formulated features, let us discuss the permutability property. We shall explain it in more detail for the classical example of surfaces with constant negative Gaussian curvature (K-surface) with their Bäcklund transformations.

Let $r : \mathbb{R}^2 \to \mathbb{R}^3$ be a K-surface, and r_{10} and r_{01} two K-surfaces obtained by Bäcklund transformations of r. The classical Bianchi permutability theorem claims that there exists a unique K-surface r_{11} which is a Bäcklund transform of r_{10} and r_{01}. Moreover,

(i) the straight line connecting the points $r(x, y)$and $r_{10}(x, y)$ lies in the tangent planes of the surfaces r and r_{10} at these points,

* Partially supported by the SFB 288 "Differential geometry and quantum physics" and by the DFG research center "Mathematics for key technologies" (FZT 86) in Berlin

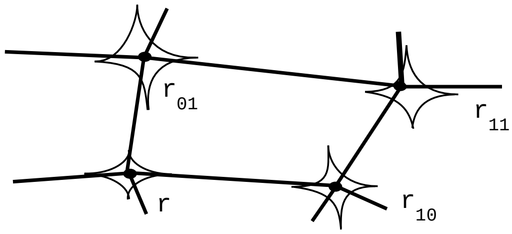

Fig. 1. Permutability of the Bäcklund transformations

(ii) the opposite edges of the quadrilateral $(r, r_{10}, r_{01}, r_{11})$ have equal lengths,

$$\|r_{10} - r\| = \|r_{11} - r_{01}\|, \qquad \|r_{01} - r\| = \|r_{11} - r_{10}\|.$$

This way a $\mathbb{Z}^2$ lattice $r_{k,\ell}$ obtained by permutable Bäcklund transformations gives rise to discrete K-surfaces. Indeed, fixing the smooth parameters (x, y) one observes that

(i) the points $r_{k,\ell}, r_{k,\ell\pm1}, r_{k\pm1,\ell}$ lie in one plane (the 'tangent' plane of the discrete K-surface at the vertex $r_{k,\ell}$),

(ii) the opposite edges of the quadrilateral $r_{k,\ell}, r_{k+1,\ell}, r_{k+1,\ell+1}, r_{k,\ell+1}$ have equal lengths.

These are exactly the characteristic properties [1] of the *discrete K-surfaces*, $r : \mathbb{Z}^2 \to \mathbb{R}^3$.

One immediately observes that the discrete K-surfaces have the same properties as their smooth counterparts. There exist deep reasons for that. The classical differential geometry of integrable surfaces may be obtained from a unifying multi-dimensional discrete theory by a refinement of the coordinate mesh-size in some of the directions.

Indeed, by refining of the coordinate mesh-size,

$$\begin{array}{ccc} r : (\epsilon\mathbb{Z})^2 \to \mathbb{R}^3 & \longrightarrow & r : \mathbb{R}^2 \to \mathbb{R}^3, \\ \text{discrete surface} & \epsilon \to 0 & \text{smooth surface,} \end{array}$$

in the limit one obtains classical smooth K-surfaces from discrete K-surfaces. This statement is visualized in Fig.2 which shows an example of a continuous Amsler surface and its discrete analogue. The subclass of Amsler surfaces is characterized by the condition that the K-surface (smooth or discrete) should contain two straight lines.

Moreover, the classical Bianchi permutability implies n-dimensional permutability of the Bäcklund transformations. This means that the set of a

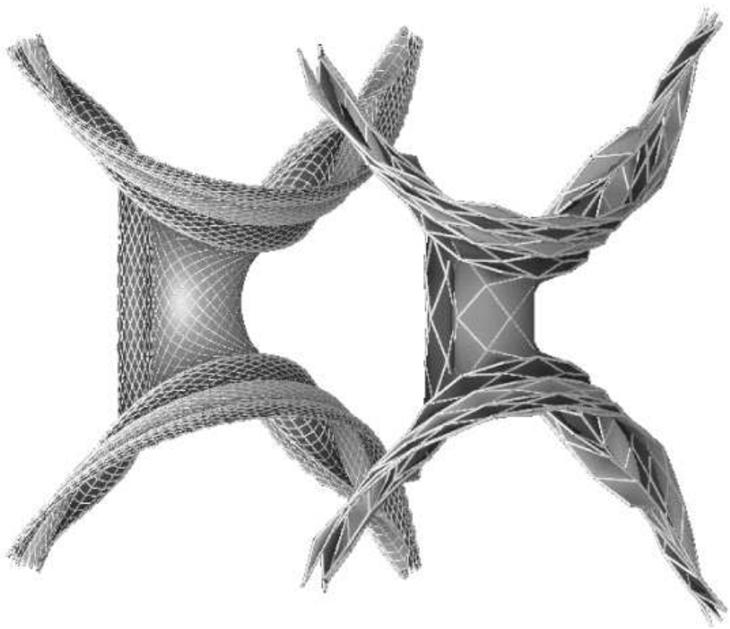

Fig. 2. A continuous and a discrete Amsler surfaces

given K-surface, $r : \mathbb{R}^2 \to \mathbb{R}^3$, with its n Bäcklund transforms $r_{10\ldots0}, r_{010\ldots0}, \ldots, r_{0\ldots01}$ can be completed to 2^n different K-surfaces $r_{i_1\ldots i_n}, i_k \in \{0,1\}$ associated to the vertices of the n-dimensional cube $C = \{0,1\}^n$. The surfaces associated to vertices of C connected by edges are Bäcklund transforms of each other.

Similar to the 2-dimensional case, this description can be extended to an n-dimensional lattice. Fixing the smooth parameters (x, y), one obtains a map,

$$r : (\epsilon_1\mathbb{Z}) \times \ldots \times (\epsilon_n\mathbb{Z}) \to \mathbb{R}^3,$$

which is an n-dimensional net obtained from one point of a K-surface by permutable Bäcklund transformations. It turns out that the whole smooth theory can be recovered from this description. Indeed, completely changing the point of view, in the limit $\epsilon_1 \to 0, \epsilon_2 \to 0, \epsilon_3 = \ldots = \epsilon_n = 1$, one arrives at a smooth K-surface with an $(n-2)$-dimensional discrete family of permutable Bäcklund transforms,

$$r : \mathbb{R}^2 \times \mathbb{Z}^{n-2} \to \mathbb{R}^3.$$

This simple idea is quite fruitful. In the discrete case all directions of the multi-dimensional lattices appear in a quite symmetric way. It leads to

– A unification of surfaces and their transformations. Discrete surfaces and their transformations are indistinguishable.

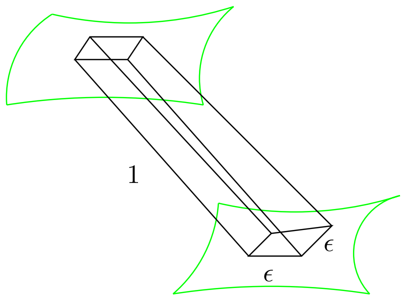

Fig. 3. Surfaces and their transformations as a limit of multidimensional lattices

- A fundamental consistency principle. Due to the symmetry of the discrete setup the same equations hold on all elementary faces of the lattice. This leads us beyond the pure differential geometry to a new understanding of the integrability, classification of integrable equations and derivation of the zero curvature (Lax) representation from the first principles.
- Interesting generalizations for $d > 2$-dimensional systems, quantum systems, discrete systems with the fields on various lattice elements (vertices, edges, faces, etc.).

As it was mentioned above, all this suggests that it might be possible to develop the classical differential geometry, including both the theory of surfaces and of their transformations, as a mesh-refining limit of the discrete constructions. On the other hand, the good quantitative properties of approximations provided by the discrete differential geometry suggest that they might be put at the basis of the practical numerical algorithms for computations in differential geometry. However until recently there were no rigorous mathematical results supporting this observation.

The first step in closing this gap was made in the paper [7] where a geometric numerical scheme for a class of nonlinear hyperbolic equations was developed and general convergence results were proved. We return to this problem in Section 6, considering in particular the sine-Gordon equation and discrete and smooth K-surfaces.

3 Equations on quad-graphs. Integrability as consistency

Traditionally, discrete integrable systems were considered for fields defined on the $\mathbb{Z}^2$ lattice. Having in mind geometric applications, it is natural to generalize this setup to include distinguished vertices with different combinatorics, and moreover to consider graphs with various global properties. A direct generalization of the Lax representation from the $\mathbb{Z}^2$ lattice to more general lattices leads to a concept of

3.1 Discrete flat connections on graphs

Integrable systems on graphs can be defined as flat connections whose values are in loop groups. More precisely, this notion includes the following component elements:

- A *cellular decomposition* $\mathcal{G}$ of an oriented surface. The set of its vertices will be denoted by $V(\mathcal{G})$, the set of its edges will be denoted by $E(\mathcal{G})$, and the set of its faces will be denoted by $F(\mathcal{G})$. For each edge, one of its possible orientations is fixed.
- A *loop group* $G[\lambda]$, whose elements are functions from $\mathbb{C}$ into some group G. The complex argument, λ, of these functions is known in the theory of integrable systems as the *spectral parameter*.
- A *wave function* $\Psi : V(\mathcal{G}) \mapsto G[\lambda]$, defined on the vertices of $\mathcal{G}$.
- A collection of *transition matrices*, $L : E(\mathcal{G}) \mapsto G[\lambda]$, defined on the edges of $\mathcal{G}$.

It is supposed that for any oriented edge, $\mathfrak{e} = (v_1, v_2) \in E(\mathcal{G})$, the values of the wave functions at its ends are connected by

$$\Psi(v_2, \lambda) = L(\mathfrak{e}, \lambda)\Psi(v_1, \lambda). \tag{1}$$

Therefore the following *discrete zero-curvature condition* is supposed to be satisfied. Consider any closed contour consisting of a finite number of edges of $\mathcal{G}$,

$$\mathfrak{e}_1 = (v_1, v_2), \quad \mathfrak{e}_2 = (v_2, v_3), \quad \ldots, \quad \mathfrak{e}_n = (v_n, v_1).$$

Then

$$L(\mathfrak{e}_n, \lambda) \cdots L(\mathfrak{e}_2, \lambda)L(\mathfrak{e}_1, \lambda) = I. \tag{2}$$

In particular, for any edge $\mathfrak{e} = (v_1, v_2)$, if $\mathfrak{e}^{-1} = (v_2, v_1)$, then

$$L(\mathfrak{e}^{-1}, \lambda) = \Big(L(\mathfrak{e}, \lambda)\Big)^{-1}. \tag{3}$$

Actually, in applications the matrices $L(\mathfrak{e}, \lambda)$ also depend on a point of some set X (the phase-space of an integrable system), so that some elements $x(\mathfrak{e}) \in X$ are attached to the edges $\mathfrak{e}$ of $\mathcal{G}$. In this case the discrete zero-curvature condition (2) becomes equivalent to the collection of equations relating the fields $x(\mathfrak{e}_1)$, …, $x(\mathfrak{e}_n)$ attached to the edges of each closed contour. We say that this collection of equations admits a *zero-curvature representation*.

3.2 Quad-graphs

Although one can, in principle, consider integrable systems in the sense of the traditional definition of Section 3.1 on very different kinds of graph, one should not go that far with the generalization.

As we have shown in [3], there is a special class of graph, called *quad-graphs*, supporting the most fundamental properties of integrability theory. This notion turns out to be a proper generalization of the $\mathbb{Z}^2$ lattice as far as integrability theory is concerned.

Definition 1. *A cellular decomposition, $\mathcal{G}$, of an oriented surface is called a* quad-graph*, if all its faces are quadrilateral.*

Here we mainly consider the local theory of integrable systems on quad-graphs. Therefore, in order to avoid the discussion of some subtle boundary and topological effects, we shall always suppose that the surface carrying the quad-graph is a topological disk; no boundary effects will be considered.

Before we proceed to integrable systems, we would like to propose a construction which, from an arbitrary cellular decomposition, produces a certain quad-graph. Towards this aim, we first recall the notion of the *dual graph*, or, more precisely, of the *dual cellular decomposition* $\mathcal{G}^*$. The vertices in $V(\mathcal{G}^*)$ are in one-to-one correspondence with the faces in $F(\mathcal{G})$ (actually, they can be chosen to be certain points inside the corresponding faces, cf. Fig. 4). Each $\mathfrak{e} \in E(\mathcal{G})$ separates two faces in $F(\mathcal{G})$, which in turn correspond to two vertices in $V(\mathcal{G}^*)$. A path between these two vertices is then declared to be an edge $\mathfrak{e}^* \in E(\mathcal{G}^*)$ dual to $\mathfrak{e}$. Finally, the faces in $F(\mathcal{G}^*)$ are in a one-to-one correspondence with the vertices in $V(\mathcal{G})$. If $v_0 \in V(\mathcal{G})$, and $v_1, \dots, v_n \in V(\mathcal{G})$ are its neighbors connected with v_0 by the edges $\mathfrak{e}_1 = (v_0, v_1), \dots, \mathfrak{e}_n = (v_0, v_n) \in E(\mathcal{G})$, then the face in $F(\mathcal{G}^*)$ corresponding to v_0 is defined by its boundary, $\mathfrak{e}_1^* \cup \ldots \cup \mathfrak{e}_n^*$ (cf. Fig. 5).

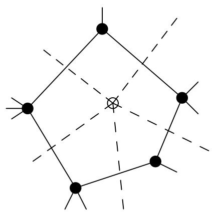

Fig. 4. The vertex in $V(\mathcal{G}^*)$ dual to the face in $F(\mathcal{G})$.

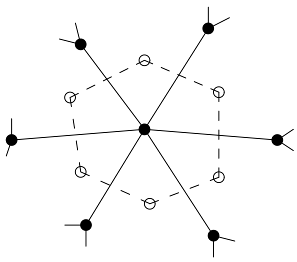

Fig. 5. The face in $F(\mathcal{G}^*)$ dual to the vertex in $V(\mathcal{G})$.

Now we introduce a new complex, the *double* $\mathcal{D}$, constructed from $\mathcal{G}$, $\mathcal{G}^*$. The set of vertices of the double ,$\mathcal{D}$, is $V(\mathcal{D}) = V(\mathcal{G}) \cup V(\mathcal{G}^*)$. Each pair of dual edges, say $\mathfrak{e} = (v_1, v_2)$ and $\mathfrak{e}^* = (f_1, f_2)$, as in Fig. 6, defines a quadrilateral (v_1, f_1, v_2, f_2), and all these quadrilaterals constitute the faces of the cell decomposition (quad-graph) $\mathcal{D}$. Let us stress that the edges of $\mathcal{D}$ belong neither to $E(\mathcal{G})$ nor to $E(\mathcal{G}^*)$. See Fig. 6.

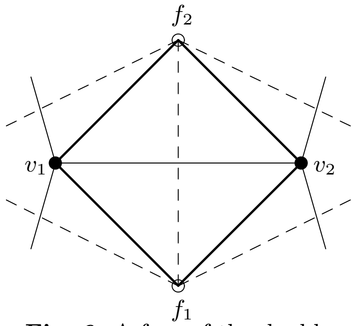

Fig. 6. A face of the double

Quad-graphs $\mathcal{D}$ arising as doubles have the following property, the set $V(\mathcal{D})$ may be decomposed into two complementary halves, $V(\mathcal{D}) = V(\mathcal{G}) \cup V(\mathcal{G}^*)$ ("black" and "white" vertices), such that the endpoints of each edge of $E(\mathcal{D})$ are of different colors. One can always color a quad-graph this way if it has no non-trivial periods, i. e., it comes from the cellular decomposition $\mathcal{G}$ of a disk.

Conversely, any such quad-graph $\mathcal{D}$ may be considered to be the double of some cellular decomposition $\mathcal{G}$. The edges in $E(\mathcal{G})$, say, are defined then as paths joining two "black" vertices of each face in $F(\mathcal{D})$. (This decomposition of $V(\mathcal{D})$ into $V(\mathcal{G})$ and $V(\mathcal{G}^*)$ is unique, up to interchanging the roles of $\mathcal{G}$ and $\mathcal{G}^*$.)

Again, since we are mainly interested in the local theory, we avoid global considerations. Therefore we always assume (without mentioning it explicitly) that our quad-graphs are cellular decompositions of a disk, thus $\mathcal{G}$ and $\mathcal{G}^*$ may be well-defined.

For the integrable systems on quad-graphs we consider here the fields z attached to the vertices of the graph[2]. They are subject to an equation

$$Q(z_1, z_2, z_3, z_4) = 0, \tag{4}$$

relating four fields residing on the four vertices of an arbitrary face in $F(\mathcal{D})$. Moreover, in all our examples it will be possible to solve equation (4) uniquely for any field $z_1, \ldots, z_4$ in terms of the other three.

The Hirota equation,

$$\frac{z_4}{z_2} = \frac{\alpha z_3 - \beta z_1}{\beta z_3 - \alpha z_1}, \tag{5}$$

[2] The systems with the fields on the edges are also very interesting, and are related to the Yang-Baxter maps (see Section 5.1).

is such an example. We observe that the equation carries parameters α and β which can be naturally associated to the edges, and the opposite edges of an elementary quadrilateral carry equal parameters (see Fig. 7). At this

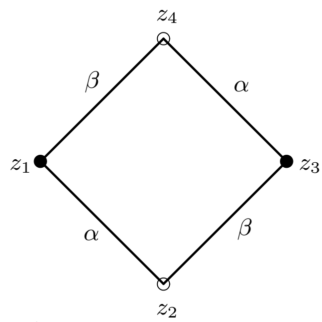

Fig. 7. A face of the labelled quad-graph

point we specify the setup further. The example illustrated in Fig. 7 can be naturally generalized. An integrable system on a quad-graph,

$$Q(z_1, z_2, z_3, z_4; \alpha, \beta) = 0 \tag{6}$$

is parametrized by a function on the set of edges, $E(\mathcal{D})$, of the quad-graph which takes equal values on the opposite edges of any elementary quadrilateral. We call such a function a *labelling* of the quad-graph. Obviously, there exist infinitely many labellings, all of which may be constructed as follows: choose some value of α for an arbitrary edge of $\mathcal{D}$, and assign consecutively the same value to all "parallel" edges along a strip of quadrilaterals, according to the definition of labelling. After that, take an arbitrary edge still without a label, choose some value of α for it, and extend the same value along the corresponding strip of quadrilaterals. Proceed similarly, till all edges of $\mathcal{D}$ are exhausted.

An elementary quadrilateral of a quad-graph can be viewed from various directions. This implies that system (6) is well defined on a general quad-graph only if it possesses the rhombic symmetry, i.e., each of the equations

$$Q(z_1, z_4, z_3, z_2; \beta, \alpha) = 0, \quad Q(z_3, z_2, z_1, z_4; \beta, \alpha) = 0$$

is equivalent to (6).

3.3 3D-consistency

Now we introduce a crucial property of discrete integrable systems which will be taken characteristic.

Let us extend the planar quad–graph $\mathcal{D}$ into the third dimension. Formally speaking, we consider a second copy $\mathcal{D}'$ of $\mathcal{D}$, and add edges connecting each vertex $v \in V(\mathcal{D})$ with its copy $v' \in V(\mathcal{D}')$. In this way we obtain a "3-dimensional quad–graph", $\mathbf{D}$, whose set of vertices is

$$V(\mathbf{D}) = V(\mathcal{D}) \cup V(\mathcal{D}'),$$

and whose set of edges is

$$E(\mathbf{D}) = E(\mathcal{D}) \cup E(\mathcal{D}') \cup \{(v, v') : v \in V(\mathcal{D})\}.$$

Elementary building blocks of $\mathbf{D}$ are "cubes" as shown in Fig. 8. Clearly, we

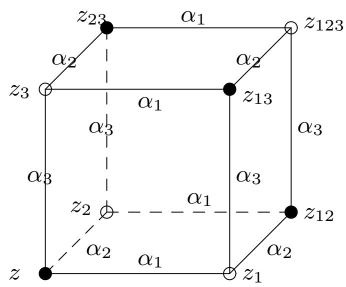

Fig. 8. Elementary cube

can still consistently subdivide the vertices of $\mathbf{D}$ into "black" and "white" vertices, so that the vertices connected by an edge have opposite colors. In the same way the labelling on $E(\mathcal{D})$ is extended to a labelling of $E(\mathbf{D})$. The opposite edges of all elementary faces (including the "vertical" ones) carry equal parameters (see Fig. 8).

Now, the fundamental property of discrete integrable system mentioned above is the *three–dimensional consistency.*

Definition 2. *Consider an elementary cube, as in Fig. 8. Suppose that the values of the field z, z_1, z_2, and z_3 are given at a vertex and at its three neighbors . Then equation (6) uniquely determines the values z_{12}, z_{23}, and z_{13}. After that the same equation (6) produces three á priori different values for the value of the field z_{123} at the eighth vertex of the cube, coming from the faces $[z_1, z_{12}, z_{123}, z_{13}]$, $[z_2, z_{12}, z_{123}, z_{23}]$ and $[z_3, z_{13}, z_{123}, z_{23}]$, respectively. Equation (6) is called* 3D-consistent *if these three values for z_{123} coincide for any choice of the initial data z, z_1, z_2, z_3.*

Proposition 1. *The Hirota equation,*

$$\frac{z_{12}}{z} = \frac{\alpha_2 z_1 - \alpha_1 z_2}{\alpha_1 z_1 - \alpha_2 z_2},$$

is 3D-consistent.

This can be verified by a straightforward computation. For the field at the eighth vertex of the cube one obtains

$$z_{123} = \frac{(l_{21} - l_{12})z_1 z_2 + (l_{32} - l_{23})z_2 z_3 + (l_{13} - l_{31})z_1 z_3}{(l_{23} - l_{32})z_1 + (l_{31} - l_{13})z_2 + (l_{12} - l_{21})z_3}, \tag{7}$$

where $l_{ij} = \dfrac{\alpha_i}{\alpha_j}$.

In [3] and [4] we suggested treating the consistency property (in the sense of Definition 2) as the characteristic one for discrete integrable systems. Thus we come to the central definition of these lectures.

Definition 3. *A discrete equation is called* integrable *if it is* consistent*.*

Note that this definition of the integrability is conceptually transparent and algorithmic: the integrability of any equation can be easily verified.

3.4 Zero curvature representation from the 3D-consistency

We show that our Definition 3 of discrete integrable systems is more fundamental then the traditional one discussed in Section 3.1. Recall that normally the problem of finding a zero-curvature representation for a given system is a difficult task whose successful solution is only possible with a large amount of luck in the guess-work. We show that finding the zero-curvature representation for a given discrete system with the consistency property becomes an algorithmically solvable problem, and we demonstrate how the corresponding flat connection in a loop group can be derived from the equation.

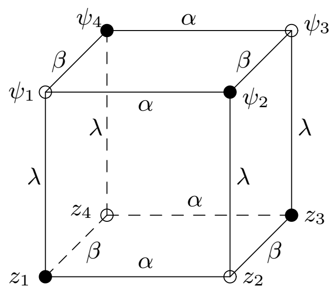

Fig. 9. Zero curvature representation from the consistency

We get rid of our symmetric notations and consider the system

$$Q(z_1, z_2, z_3, z_4; \alpha, \beta) = 0 \tag{8}$$

on the base face of the cube, and choose the vertical direction to carry an additional (spectral) parameter λ (see Fig. 9).

Assume that the left-hand-side of (8) is affine in each z_k. This gives z_4 as a fractional–linear (Möbius) transformation of z_2 with the coefficients depending on z_1 and z_3 and on α and β. One can of course freely interchange $z_1, \ldots, z_4$ in this statement. Now consider the equations on the vertical faces of the cube in Fig. 9. One obtains ψ_2 as a Möbius transformation of ψ_1

$$\psi_2 = L(z_2, z_1; \alpha, \lambda)[\psi_1],$$

with coefficients depending on the fields z_2 and z_1, on the parameter α in system (8) and on the additional parameter λ, which is to be treated as the spectral parameter. Mapping $L(z_2, z_1; \alpha, \lambda)$ is associated to the oriented edge, (z_1, z_2). For the reverse edge, (z_2, z_1), one obviously obtains the inverse transformation

$$L(z_1, z_2; \alpha, \lambda) = L(z_2, z_1; \alpha, \lambda)^{-1}.$$

Going once around the horizontal face of the cube one obtains

$$\psi_1 = L(z_1, z_4; \beta, \lambda)L(z_4, z_3; \alpha, \lambda)L(z_3, z_2; \beta, \lambda)L(z_2, z_1; \alpha, \lambda)[\psi_1].$$

The composed Möbius transformation in the right-hand-side is the identity because of the arbitrariness of ψ_1.

Using the matrix notation for the action of the Möbius transformations,

$$\frac{az+b}{cz+d} = L[z], \quad \text{where} \quad L = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and normalizing these matrices (for example by the condition $\det L = 1$), we derive the zero-curvature representation,

$$L(z_1, z_4; \beta, \lambda)L(z_4, z_3; \alpha, \lambda)L(z_3, z_2; \beta, \lambda)L(z_2, z_1; \alpha, \lambda) = I, \tag{9}$$

for (8), where the L's are elements of the corresponding loop group. Equivalently, (9) can be written as

$$L(z_3, z_2; \beta, \lambda)L(z_2, z_1; \alpha, \lambda) = L(z_3, z_4; \alpha, \lambda)L(z_4, z_1; \beta, \lambda), \tag{10}$$

where one has a little more freedom in normalizations.

Let us apply this derivation method to the Hirota equation. Equation (5) can be written as $Q = 0$ with

$$Q(z_1, z_2, z_3, z_4; \alpha, \beta) = \alpha(z_2 z_3 + z_1 z_4) - \beta(z_3 z_4 + z_1 z_2).$$

Performing the computations as above in this case we derive the zero-curvature representation for the Hirota equation (10) with the matrices

$$L(z_2, z_1, \alpha, \lambda) = \begin{pmatrix} \alpha & -\lambda z_2 \\ \dfrac{\lambda}{z_1} & -\alpha \dfrac{z_2}{z_1} \end{pmatrix}. \tag{11}$$

Another important example is the cross-ratio equation,

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)} = \frac{\alpha}{\beta}. \tag{12}$$

It is easy to show that it is 3D-consistent. Written in the form

$$\alpha(z_2 - z_3)(z_4 - z_1) - \beta(z_1 - z_2)(z_3 - z_4) = 0,$$

it obviously belongs to the class discussed in this section. By direct computation we obtain the L-matrices,

$$\tilde{L}(z_2, z_1, \alpha, \lambda) = \begin{pmatrix} 1 + \dfrac{\lambda\alpha z_2}{(z_1 - z_2)} & -\dfrac{\lambda\alpha z_1 z_2}{(z_1 - z_2)} \\ \dfrac{\lambda\alpha}{(z_1 - z_2)} & 1 - \dfrac{\lambda\alpha z_1}{(z_1 - z_2)} \end{pmatrix},$$

which are gauge-equivalent to

$$L(z_2, z_1, \alpha, \lambda) = \begin{pmatrix} 1 & z_1 - z_2 \\ \dfrac{\alpha}{\lambda(z_1 - z_2)} & 1 \end{pmatrix}. \tag{13}$$

4 Classification

We have seen that the idea of consistency is at the core of the integrability theory and may be even suggested as a definition of integrability.

Here we give a further application of the consistency approach. We show that it provides an effective tool for finding and classifying all integrable systems in certain classes of equations. In the previous section we presented two important systems which belong to our theory. Here we complete the list of the examples, classifying all integrable (in the sense of Definition 3) one-field equations on quad-graphs satisfying some natural symmetry conditions.

We consider equations

$$Q(x, u, v, y; \alpha, \beta) = 0, \tag{14}$$

on quad-graphs. Equations are associated to elementary quadrilaterals, the fields $x, u, v, y \in \mathbb{C}$ are assigned to the four vertices of the quadrilateral, and the parameters $\alpha, \beta \in \mathbb{C}$ are assigned to its edges, as shown in Fig. 10.

We now list more precisely the assumptions under which we classify the equations.

1) Consistency. Equation (14) is integrable (in the sense that it is 3D-consistent). As explained in the previous section, this property means that this equation may be consistently embedded in a three-dimensional lattice, so that the same equations hold for all six faces of any elementary cube, as in Fig. 8.

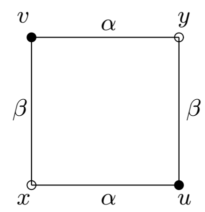

Fig. 10. An elementary quadrilateral; fields are assigned to vertices

Further, we assume that equations (14) can be uniquely solved for any one of their arguments, $x, u, v, y \in \widehat{\mathbb{CP}^1}$. Therefore, the solutions have to be fractional-linear in each of their arguments. This naturally leads to the following condition.

2) Linearity. The function $Q(x, u, v, y; \alpha, \beta)$ is linear in each argument (affine linear):

$$Q(x, u, v, y; \alpha, \beta) = a_1 xuvy + \cdots + a_{16}, \tag{15}$$

where coefficients a_i depend on α and β.

Third, we are interested in equations on quad-graphs of arbitrary combinatorics, hence it will be natural to assume that all variables involved in equations (14) are on equal footing. Therefore, our next assumption reads as follows.

3) Symmetry. Equation (14) is invariant under the group D_4 of the symmetries of the square, that is, function Q satisfies the symmetry properties

$$Q(x, u, v, y; \alpha, \beta) = \varepsilon Q(x, v, u, y; \beta, \alpha) = \sigma Q(u, x, y, v; \alpha, \beta) \tag{16}$$

with $\varepsilon, \sigma = \pm 1$. Of course, due to symmetries (16), not all coefficients a_i in (15) are independent.

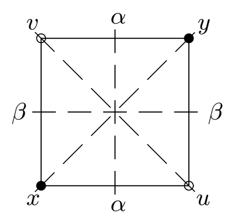

Fig. 11. D_4 symmetry

Finally, it is worth looking more attentively at expression (7) for the eighth point in the cube for the Hirota equation and at the similar formula for the cross-ratio equation

$$z_{123} = \frac{(\alpha_1-\alpha_2)z_1z_2 + (\alpha_3-\alpha_1)z_3z_1 + (\alpha_2-\alpha_3)z_2z_3}{(\alpha_3-\alpha_2)z_1 + (\alpha_1-\alpha_3)z_2 + (\alpha_2-\alpha_1)z_3}. \tag{17}$$

Looking ahead, we mention a very amazing and unexpected feature of these expressions: value z_{123} actually depends on z_1, z_2, z_3 only, and does not depend on z. In other words, four black points in Fig. 8 (the vertices of a tetrahedron) are related by a well-defined equation. This property, being rather strange at first glance, actually is valid not only in this but in all known nontrivial examples. We take it as an additional assumption in our solution of the classification problem.

4) Tetrahedron property. Function $z_{123} = f(z, z_1, z_2, z_3; \alpha_1, \alpha_2, \alpha_3)$, existing due to the 3D-consistency, actually does not depend on variable z, that is, $f_z = 0$.

Under the tetrahedron property we can paint the vertices of the cube into black and white, as in Fig. 8, and the vertices of each of two tetrahedrons satisfy an equation of the form,

$$\widehat{Q}(z_1, z_2, z_3, z_{123}; \alpha_1, \alpha_2, \alpha_3) = 0. \tag{18}$$

It is easy to see that under assumption 2) (linearity) function $\widehat{Q}$ may be also taken to be linear in each argument. (Clearly, formulas (7) and (17) may also be written in such a form.)

We identify equations related by certain natural transformations. First, acting simultaneously on all variables z by one and the same Möbius transformation does not violate our three assumptions. Second, the same holds for the simultaneous point change of all parameters, $\alpha \mapsto \varphi(\alpha)$.

Theorem 1. *[4] Up to common Möbius transformations of variables z and point transformations of the parameters α, the 3D-consistent quad-graph equations (14) with the properties 2), 3), 4) (linearity, symmetry and the tetrahedron property) are exhausted by the following three lists Q, H, and A where $x = z$, $u = z_1$, $v = z_2$, $y = z_{12}$, $\alpha = \alpha_1$, $\beta = \alpha_2$:*

List Q

(Q1) $\alpha(x-v)(u-y) - \beta(x-u)(v-y) + \delta^2\alpha\beta(\alpha-\beta) = 0,$

(Q2) $$\alpha(x-v)(u-y) - \beta(x-u)(v-y) + \alpha\beta(\alpha-\beta)(x+u+v+y) \\ -\alpha\beta(\alpha-\beta)(\alpha^2-\alpha\beta+\beta^2) = 0,$$

(Q3) $$(\beta^2-\alpha^2)(xy+uv) + \beta(\alpha^2-1)(xu+vy) - \alpha(\beta^2-1)(xv+uy) \\ -\delta^2(\alpha^2-\beta^2)(\alpha^2-1)(\beta^2-1)/(4\alpha\beta) = 0,$$

(Q4) $$a_0xuvy + a_1(xuv+uvy+vyx+yxu) + a_2(xy+uv) + \bar{a}_2(xu+vy) \\ +\tilde{a}_2(xv+uy) + a_3(x+u+v+y) + a_4 = 0,$$

where the coefficients a_i are expressed in terms of (α, a) and (β, b) with $a^2 = r(\alpha)$, $b^2 = r(\beta)$, $r(x) = 4x^3 - g_2x - g_3$, by the following formulas:

$$a_0 = a + b, \quad a_1 = -\beta a - \alpha b, \quad a_2 = \beta^2 a + \alpha^2 b,$$
$$\bar{a}_2 = \frac{ab(a+b)}{2(\alpha-\beta)} + \beta^2 a - (2\alpha^2 - \frac{g_2}{4})b,$$
$$\tilde{a}_2 = \frac{ab(a+b)}{2(\beta-\alpha)} + \alpha^2 b - (2\beta^2 - \frac{g_2}{4})a,$$
$$a_3 = \frac{g_3}{2}a_0 - \frac{g_2}{4}a_1, \quad a_4 = \frac{g_2^2}{16}a_0 - g_3 a_1.$$

List H:

(H1) $(x-y)(u-v) + \beta - \alpha = 0,$
(H2) $(x-y)(u-v) + (\beta-\alpha)(x+u+v+y) + \beta^2 - \alpha^2 = 0,$
(H3) $\alpha(xu+vy) - \beta(xv+uy) + \delta(\alpha^2 - \beta^2) = 0.$

List A:

(A1) $\alpha(x+v)(u+y) - \beta(x+u)(v+y) - \delta^2\alpha\beta(\alpha-\beta) = 0,$
(A2) $(\beta^2-\alpha^2)(xuvy+1) + \beta(\alpha^2-1)(xv+uy) - \alpha(\beta^2-1)(xu+vy) = 0.$

The proof of this theorem is rather involved and is given in [4].

Remarks

1) List A can be omitted by allowing an extended group of Möbius transformations, which act on the variables x, y differently than on u, v, white and black sublattices on Figs. 10 and 8. In this manner Eq. (A1) is related to (Q1) by the change $u \to -u$, $v \to -v$, and Eq. (A2) is related to (Q3) with $\delta = 0$ by the change $u \to 1/u$, $v \to 1/v$. So, really independent equations are given by the lists Q and H.

2) In both lists, Q and H, the last equations are the most general ones. This means that Eqs. (Q1)–(Q3) and (H1), (H2) may be obtained from (Q4) and (H3), respectively, by certain degenerations and/or limit procedures. So, one might be tempted to shorten these lists to one item each. However, on the one hand, these limit procedures are outside our group of admissible (Möbius) transformations, and, on the other, in many situations the "degenerate" equations (Q1)–(Q3) and (H1), (H2) are of interest in themselves. This resembles the situation with the six Painlevé equations and the coalescences connecting them.

3) Parameter δ in Eqs. (Q1), (Q3), (H3) can be scaled away, so that one can assume without loss of generality that $\delta = 0$ or $\delta = 1$.

4) It is natural to set in Eq. (Q4) $(\alpha, a) = (\wp(A), \wp'(A))$ and, similarly, $(\beta, b) = (\wp(B), \wp'(B))$. So, this equation is actually parametrized by two points of the elliptic curve $\mu^2 = r(\lambda)$. The appearance of an elliptic curve in our classification problem is by no means obvious from the beginning. If r has multiple roots, the elliptic curve degenerates into a rational one, and Eq. (Q4) degenerates to one of the previous equations of the list Q; for example, if $g_2 = g_3 = 0$ then inversion $x \to 1/x$ turns (Q4) into (Q2).

5) Note that the list contains the fundamental equations only. A discrete equation which is derived from an equation with the consistency property will usually lose this property.

5 Generalizations: Multidimensional and non-commutative (quantum) cases

5.1 Yang-Baxter maps

As we mentioned, however, to assign fields to the vertices is not the only possibility. Another large class of 2-dimensional systems on quad–graphs consists of those where the fields are assigned to the *edges*, see Fig. 12. In this situa-

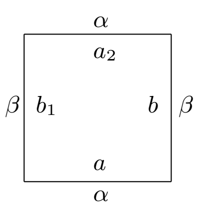

Fig. 12. An elementary quadrilateral; both fields and labels are assigned to edges

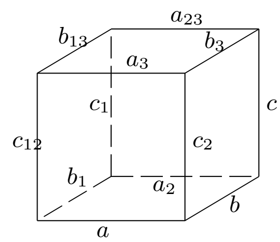

Fig. 13. Three–dimensional consistency; fields assigned to edges

tion it is natural to assume that each elementary quadrilateral carries a map $R : \mathcal{X}^2 \mapsto \mathcal{X}^2$, where $\mathcal{X}$ is the space where the fields a and b take values, so that $(a_2, b_1) = R(a, b; \alpha, \beta)$. The question of the three–dimensional consistency of such maps is also legitimate and, moreover, recently has begun to be studied. The corresponding property can be encoded in the formula

$$R_{23} \circ R_{13} \circ R_{12} = R_{12} \circ R_{13} \circ R_{23}, \tag{19}$$

where each $R_{ij} : \mathcal{X}^3 \mapsto \mathcal{X}^3$ acts as the map R on the factors i and j of the cartesian product $\mathcal{X}^3$, and acts identically on the third factor. This equation should be understood as follows. The fields a and b are supposed to be attached to the edges parallel to the 1st and the 2nd coordinate axes, respectively. Additionally, consider the fields c attached to the edges parallel to the 3rd coordinate axis. Then the left–hand side of (19) corresponds to the chain of maps along the three rear faces of the cube in Fig. 13,

$$(a, b) \mapsto (a_2, b_1), \quad (a_2, c) \mapsto (a_{23}, c_1), \quad (b_1, c_1) \mapsto (b_{13}, c_{12}),$$

while its right–hand side corresponds to the chain of maps along the three front faces of the cube,

$$(b,c) \mapsto (b_3, c_2), \quad (a, c_2) \mapsto (a_3, c_{12}), \quad (a_3, b_3) \mapsto (a_{23}, b_{13})$$

So, Eq. (19) assures that the two ways of obtaining (a_{23}, b_{13}, c_{12}) from the initial data (a, b, c) lead to the same results. The maps with this property were introduced by Drinfeld under the name of "set–theoretical solutions of the Yang–Baxter equation", an alternative name is "Yang–Baxter maps" used by Veselov. Under some circumstances, systems with fields on vertices can be regarded as systems with fields on edges or vice versa (this is the case, E.G., for systems (Q1), $(\mathrm{Q3})_{\delta=0}$, (H1), $(\mathrm{H3})_{\delta=0}$ of our list, for which the variables X enter only in combinations like $X - U$ for edges (x, u)), but in general the two classes of systems should be considered to be different. The problem of classifying of Yang–Baxter maps, like the one achieved in the previous section, has been recently solved in [5].

5.2 Four-dimensional consistency of three-dimensional systems

The consistency principle can be obviously generalized to an arbitrary dimension. We say that

> a d–dimensional discrete equation possesses the consistency property, if it may be imposed in a consistent way on all d–dimensional sublattices of a $(d+1)$–dimensional lattice.

In the three–dimensional context there are also *a priori* many kinds of systems, according to where the fields are defined: on the vertices, on the edges, or on the elementary squares of the cubic lattice.

Consider 3-dimensional systems with the fields at the vertices. In this case each elementary cube carries just one equation,

$$Q(z, z_1, z_2, z_3, z_{12}, z_{23}, z_{13}, z_{123}) = 0, \tag{20}$$

relating the fields in all its vertices. Such an equation should be solvable for any of its arguments in terms of the other seven arguments. The four–dimensional consistency of such equations is defined in the obvious way.

– Starting with initial data z, z_i $(1 \le i \le 4)$, z_{ij} $(1 \le i < j \le 4)$, equation (20) allows us to determine all fields z_{ijk} $(1 \le i < j < k \le 4)$ uniquely. Then we have *four* different ways of finding z_{1234} corresponding to four 3-dimensional cubic faces adjacent to the vertex z_{1234} of the four–dimensional hypercube, see Fig. 5.2. All four values actually coincide.

So, one can consistently impose equations (20) on all elementary cubes of the three-dimensional cubical complex, which is a three-dimensional generalization of the quad-graph. It is tempting to accept the four–dimensional

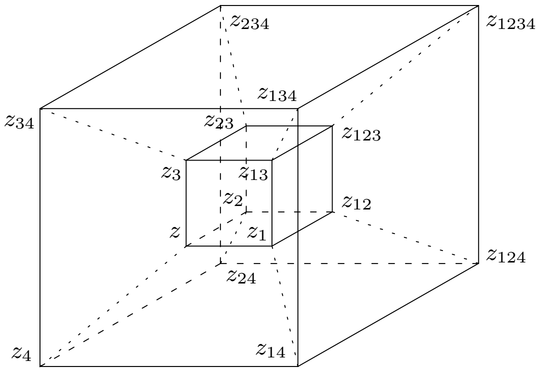

Fig. 14. Hypercube

consistency of equations of type (20) as the constructive definition of their integrability. It is important to solve the correspondent classification problem.

Let us give here some examples. Consider the equation

$$\frac{(z_1 - z_3)(z_2 - z_{123})}{(z_3 - z_2)(z_{123} - z_1)} = \frac{(z - z_{13})(z_{12} - z_{23})}{(z_{13} - z_{12})(z_{23} - z)}. \tag{21}$$

It is not difficult to see that Eq. (21) admits as its symmetry group the group D_8 of the cube. This equation can be uniquely solved for a field at an arbitrary vertex of a 3-dimensional cube, provided the fields at the seven other vertices are known.

The fundamental fact is:

Proposition 2. *Equation (21) is four–dimensionally consistent.*

A different factorization of the face variables into the vertex ones leads to another remarkable three–dimensional system known as the discrete BKP equation. For any solution $x : \mathbb{Z}^4 \mapsto \mathbb{C}$ of (21), define a function $\tau : \mathbb{Z}^4 \mapsto \mathbb{C}$ by the equations

$$\frac{\tau_i \tau_j}{\tau \tau_{ij}} = \frac{x_{ij} - x}{x_i - x_j}, \quad i < j. \tag{22}$$

Eq. (21) assures that this can be done in an essentially unique way (up to initial data on the coordinate axes, whose influence is a trivial scaling of the solution). On any 3-dimensional cube the function τ satisfies the discrete BKP equation,

$$\tau\tau_{ijk} - \tau_i\tau_{jk} + \tau_j\tau_{ik} - \tau_k\tau_{ij} = 0, \quad i < j < k. \tag{23}$$

Proposition 3. *Equation (23) is four–dimensionally consistent.*

Moreover, for the value τ_{1234} one finds a remarkable equation,

$$\tau\tau_{1234} - \tau_{12}\tau_{34} + \tau_{13}\tau_{24} - \tau_{23}\tau_{34} = 0, \tag{24}$$

which essentially reproduces the discrete BKP equation. So τ_{1234} does not actually depend on the values τ_i, $1 \le i \le 4$. This can be considered to be an analogue of the tetrahedron property of Sect. 4.

5.3 Noncommutative (quantum) cases

As it was shown in [6], the consistency approach works also in the noncommutative case, where the participating fields live in an arbitrary associative (not necessary commutative) algebra $\mathcal{A}$ (over the field $\mathcal{K}$). It turns out that finding the zero curvature representation does not hinge on the particular algebra $\mathcal{A}$ nor on prescribing some particular commutation rules for fields in the neighboring vertices. The fact that some commutation relations are preserved by the evolution is thus conceptually separate from the integrability.

As before, we deal with equations on quadrilaterals,

$$Q(x,u,v,y;\alpha,\beta) = 0.$$

Now $x,u,v,y \in \mathcal{A}$ are the fields assigned to the four vertices of the quadrilateral, and $\alpha,\beta \in \mathcal{K}$ are the parameters assigned to its edges.

We start our considerations with the following, more special equation,

$$yx^{-1} = f_{\alpha\beta}(uv^{-1}). \tag{25}$$

(Here and below, any time we encounter the inverse x^{-1} of a non-zero element, $x \in \mathcal{A}$, its existence is assumed.) We require that this equation do not depend on how we regard the elementary quadrilateral (recall that we consider equations on the quad-graphs). It is not difficult to see that this implies the following symmetries:

$$f_{\alpha\beta}(A) = f_{\beta\alpha}(A^{-1}), \tag{26}$$
$$f_{\alpha\beta}(A^{-1}) = (f_{\alpha\beta}(A))^{-1}, \tag{27}$$
$$f_{\beta\alpha}(A) = f_{\alpha\beta}^{-1}(A^{-1}). \tag{28}$$

In (28) $f_{\alpha\beta}^{-1}$ stands for the inverse function to $f_{\alpha\beta}$, which has to be distinguished from the inversion in the algebra $\mathcal{A}$ in the formula (27).

All the conditions (26)–(28) are satisfied for the function which characterizes the Hirota equation,

$$f_{\alpha\beta}(A) = \frac{1 - (\beta/\alpha)A}{(\beta/\alpha) - A}. \tag{29}$$

The 3D-consistency condition for equation (25) is

$$f_{\alpha_j\alpha_k}\Big(f_{\alpha_i\alpha_j}(z_iz_j^{-1})(f_{\alpha_i\alpha_k}(z_iz_k^{-1}))^{-1}\Big) =$$
$$f_{\alpha_i\alpha_k}\Big(f_{\alpha_i\alpha_j}(z_iz_j^{-1})(f_{\alpha_j\alpha_k}(z_jz_k^{-1}))^{-1}\Big)z_jz_i^{-1}.$$

Taking into account that $f_{\alpha\beta}$ actually depends only on β/α, we slightly abuse the notations and write $f_{\alpha\beta} = f_{\beta/\alpha}$. Setting $\lambda = \alpha_j/\alpha_i$, $\mu = \alpha_k/\alpha_j$, and $A = z_iz_j^{-1}$, $B^{-1} = z_jz_k^{-1}$, and taking into account property (27), we rewrite the above equation as

$$f_\mu\Big(f_\lambda(A)f_{\lambda\mu}(BA^{-1})\Big) = f_{\lambda\mu}\Big(f_\lambda(A)f_\mu(B)\Big)A^{-1}. \tag{30}$$

Proposition 4. *The non–commutative Hirota equation is 3D-consistent.*

To prove this theorem, one proves that function (29) satisfies this functional equation for any $\lambda, \mu \in \mathcal{K}$ and for any $A, B \in \mathcal{A}$.

Alternatively, one proves the consistency by deriving the zero-curvature representation. We show that the following two schemes for computing z_{123} lead to one and the same result:

– $(z, z_1, z_2) \mapsto z_{12}$, $(z, z_1, z_3) \mapsto z_{13}$, $(z_1, z_{12}, z_{13}) \mapsto z_{123}$.
– $(z, z_1, z_2) \mapsto z_{12}$, $(z, z_2, z_3) \mapsto z_{23}$, $(z_2, z_{12}, z_{23}) \mapsto z_{123}$.

The Hirota equation on face (z, z_1, z_{13}, z_3),

$$z_{13}z^{-1} = f_{\alpha_3\alpha_1}(z_3z_1^{-1}),$$

can be written as a formula which gives z_{13} as a fractional–linear transformation of z_3,

$$z_{13} = (\alpha_1z_3 - \alpha_3z_1)(\alpha_3z_3 - \alpha_1z_1)^{-1}z = L(z_1, z, \alpha_1, \alpha_3)[z_3], \tag{31}$$

where

$$L(z_1, z, \alpha_1, \alpha_3) = \begin{pmatrix} \alpha_1 & -\alpha_3z_1 \\ \alpha_3z^{-1} & -\alpha_1z^{-1}z_1 \end{pmatrix}. \tag{32}$$

We use here the notation which is common for Möbius transformations on $\mathbb{C}$ represented as a linear action of the group $\mathrm{GL}(2,\mathbb{C})$. In the present case we define the action of the group $\mathrm{GL}(2,\mathcal{A})$ on $\mathcal{A}$ by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}[z] = (az + b)(cz + d)^{-1}, \qquad a, b, c, d, z \in \mathcal{A}.$$

It is easy to see that this is indeed the left action of the group, provided that the multiplication in $\mathrm{GL}(2,\mathcal{A})$ is defined by the natural formula

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a'a + b'c & a'b + b'd \\ c'a + d'c & c'b + d'd \end{pmatrix}.$$

Absolutely similarly to (31), we find that

$$z_{23} = L(z_2, z, \alpha_2, \alpha_3)[z_3]. \tag{33}$$

From (33) we derive, by a shift in the direction of the first coordinate axis, the expression for z_{123} obtained by the first scheme above,

$$z_{123} = L(z_{12}, z_1, \alpha_2, \alpha_3)[z_{13}], \tag{34}$$

while from (31) we find the expression for z_{123} corresponding to the second scheme,

$$z_{123} = L(z_{12}, z_2, \alpha_1, \alpha_3)[z_{23}]. \tag{35}$$

Substituting (31) and (33) on the right-hand sides of (34) and (35), respectively, we represent the equality we want to demonstrate in the following form,

$$\begin{aligned} &L(z_{12}, z_1, \alpha_2, \alpha_3)L(z_1, z, \alpha_1, \alpha_3)[z_3] \\ &= L(z_{12}, z_2, \alpha_1, \alpha_3)L(z_2, z, \alpha_2, \alpha_3)[z_3]. \end{aligned} \tag{36}$$

It is not difficult to prove that the stronger claim holds, namely that

$$L(z_{12}, z_1, \alpha_2, \alpha_3)L(z_1, z, \alpha_1, \alpha_3) = L(z_{12}, z_2, \alpha_1, \alpha_3)L(z_2, z, \alpha_2, \alpha_3). \tag{37}$$

The last equation is nothing else but the zero-curvature condition for the noncommutative Hirota equation.

Proposition 5. *The Hirota equation admits a zero-curvature representation with matrices from the loop group* $\mathrm{GL}(2, \mathcal{A})[\lambda]$*. The transition matrix along the (oriented) edge* (x, u) *carrying the label* α *is determined by*

$$L(u, x, \alpha; \lambda) = \begin{pmatrix} \alpha & -\lambda u \\ \lambda x^{-1} & -\alpha x^{-1} u \end{pmatrix}. \tag{38}$$

Quite similar claims (3D-consistency, derivation of the zero-curvature representation) hold for the noncommutative cross-ratio equation,

$$(x - u)(u - y)^{-1}(y - v)(v - x)^{-1} = \frac{\alpha}{\beta}.$$

6 Smooth theory from the discrete one

Let us return to smooth and discrete surfaces with constant negative Gaussian curvature. The philosophy of discrete differential geometry was explained in Section 2. Surfaces and their transformations are obtained as a special limit of a discrete master-theory. The latter treats the corresponding discrete surfaces and their transformations in an absolutely symmetric way. This is possible because they are merged into multidimensional nets such that all their sublattices have the same geometric properties. The possibility of this multidimensional extension results in the permutability of the corresponding difference equations characterizing the geometry.

Let us recall the analytic description of smooth and discrete K-surfaces. Let F be a K-surface parametrized by its asymtotic lines,

$$F : \Omega(r) = [0, r] \times [0, r] \to \mathbb{R}^3.$$

This means that the vectors $\partial_x F$, $\partial_y F$, $\partial_x^2 F$ and $\partial_y^2 F$ are orthogonal to the normal vector $N : \Omega(r) \to S^2$. Reparametrizing the asymptotic lines, if necessary, we assume that $|\partial_x F| = 1$ and $|\partial_y F| = 1$. Angle $\phi = \phi(x, y)$ between the vectors $\partial_x F$, and $\partial_y F$ satisfies the sine-Gordon equation,

$$\partial_x \partial_y \phi = \sin \phi. \tag{39}$$

Moreover, a K-surface is determined by a solution to (39) essentially uniquely. The corresponding construction is as follows. Consider the matrices U and V defined by the formulas

$$U(a; \lambda) = \frac{i}{2} \begin{pmatrix} a & -\lambda \\ -\lambda & -a \end{pmatrix}, \tag{40}$$

$$V(b; \lambda) = \frac{i}{2} \begin{pmatrix} 0 & \lambda^{-1} \exp(ib) \\ \lambda^{-1} \exp(-ib) & 0 \end{pmatrix}, \tag{41}$$

taking values in the twisted loop algebra,

$$g[\lambda] = \{\xi : \mathbb{R}_* \to \mathrm{su}(2) : \ \xi(-\lambda) = \sigma_3 \xi(\lambda) \sigma_3\}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Suppose now that a and b are real-valued functions on $\Omega(r)$. Then the zero-curvature condition,

$$\partial_y U - \partial_x V + [U, V] = 0, \tag{42}$$

is satisfied identically in λ, if and only if (a, b) satisfy the system

$$\partial_y a = \sin b, \qquad \partial_x b = a, \tag{43}$$

or, in other words, if $a = \partial_x \phi$ and $b = \phi$, where ϕ is a solution of (39). Given a solution ϕ, that is, a pair of matrices (40), (41) satisfying (42), the following system of linear differential equations is uniquely solvable,

$$\partial_x \Phi = U\Phi, \quad \partial_y \Phi = V\Phi, \quad \Phi(0, 0; \lambda) = \mathbf{1}. \tag{44}$$

Here $\Phi : \Omega(r) \mapsto G[\lambda]$ takes values in the twisted loop group,

$$G[\lambda] = \{\Xi : \mathbb{R}_* \to \mathrm{SU}(2) : \ \Xi(-\lambda) = \sigma_3 \Xi(\lambda) \sigma_3\}.$$

The solution $\Phi(x, y; \lambda)$ yields the immersion $F(x, y)$ by the Sym formula,

$$F(x,y) = \big(2\lambda \Phi(x,y;\lambda)^{-1}\partial_\lambda \Phi(x,y;\lambda)\big)\Big|_{\lambda=1}. \tag{45}$$

(Here the canonical identification of su(2) with $\mathbb{R}^3$ is used.) Moreover, the right-hand side of (45) at values of λ different from $\lambda = 1$ determines a family of immersions, $F_\lambda : \Omega(r) \to \mathbb{R}^3$, all of which are K-surfaces parametrized by asymptotic lines. These surfaces F_λ constitute the so-called *associated family* of F.

Now we turn to the analytic description of discrete K-surfaces. Let ε be a discretization parameter, and we introduce discrete domains,

$$\Omega^\varepsilon(r) = [0,r]^\varepsilon \times [0,r]^\varepsilon \subset (\varepsilon\mathbb{Z})^2,$$

where $[0,r]^\varepsilon = [0,r] \cap (\varepsilon\mathbb{Z})$. Each $\Omega^\varepsilon(r)$ contains $O(\varepsilon^{-2})$ grid points. Let F^ε be a discrete surface parametrized by asymptotic lines, i.e., an immersion,

$$F^\varepsilon : \Omega^\varepsilon(r) \to \mathbb{R}^3, \tag{46}$$

such that for each $(x,y) \in \Omega^\varepsilon(r)$ the five points $F^\varepsilon(x,y)$, $F^\varepsilon(x \pm \varepsilon, y)$, and $F^\varepsilon(x, y \pm \varepsilon)$ lie in a single plane, $\mathcal{P}(x,y)$. Let us introduce the difference analogues of the partial derivatives,

$$\delta_x^\varepsilon p(x,y) = \frac{1}{\varepsilon}\Big(p(x+\varepsilon,y) - p(x,y)\Big), \quad \delta_y^\varepsilon p(x,y) = \frac{1}{\varepsilon}\Big(p(x,y+\varepsilon) - p(x,y)\Big). \tag{47}$$

It is required that all edges of the discrete surface F^ε have the same length, $\varepsilon\ell$, that is, $|\delta_x^\varepsilon F^\varepsilon| = |\delta_y^\varepsilon F^\varepsilon| = \ell$, and it turns out to be convenient to assume that $\ell = (1+\varepsilon^2/4)^{-1}$. The same relation we presented between K-surfaces and solutions to the (classical) sine–Gordon equation (39) can be found between discrete K-surfaces and solutions to the sine–Gordon equation in Hirota's discretization,

$$\begin{aligned}
&\sin\frac{1}{4}\big(\phi(x+\varepsilon,y+\varepsilon) - \phi(x+\varepsilon,y) - \phi(x,y+\varepsilon) + \phi(x,y)\big) \\
&\quad = \frac{\varepsilon^2}{4}\sin\frac{1}{4}\big(\phi(x+\varepsilon,y+\varepsilon) + \phi(x+\varepsilon,y) + \phi(x,y+\varepsilon) + \phi(x,y)\big).
\end{aligned} \tag{48}$$

Consider the matrices $\mathcal{U}^\varepsilon$, $\mathcal{V}^\varepsilon$ defined by the formulas

$$\mathcal{U}^\varepsilon(a;\lambda) = (1+\varepsilon^2\lambda^2/4)^{-1/2}\begin{pmatrix} \exp(i\varepsilon a/2) & -i\varepsilon\lambda/2 \\ -i\varepsilon\lambda/2 & \exp(-i\varepsilon a/2) \end{pmatrix}, \tag{49}$$

$$\mathcal{V}^\varepsilon(b;\lambda) = (1+\varepsilon^2\lambda^{-2}/4)^{-1/2}\begin{pmatrix} 1 & (i\varepsilon\lambda^{-1}/2)\exp(ib) \\ (i\varepsilon\lambda^{-1}/2)\exp(-ib) & 1 \end{pmatrix}. \tag{50}$$

Let a and b be real-valued functions on $\Omega^\varepsilon(r)$, and consider the discrete zero-curvature condition,

$$\mathcal{U}^\varepsilon(x, y+\varepsilon; \lambda) \cdot \mathcal{V}^\varepsilon(x, y; \lambda) = \mathcal{V}^\varepsilon(x+\varepsilon, y; \lambda) \cdot \mathcal{U}^\varepsilon(x, y; \lambda), \tag{51}$$

where $\mathcal{U}^\varepsilon$ and $\mathcal{V}^\varepsilon$ depend on $(x, y) \in \Omega^\varepsilon(r)$ by the dependence of a and b on (x, y), respectively. A direct calculation shows that (51) is equivalent to the system

$$\delta_y^\varepsilon a = \frac{2}{i\varepsilon^2} \log \frac{1-(\varepsilon^2/4)\exp(-ib-i\varepsilon a/2)}{1-(\varepsilon^2/4)\exp(ib+i\varepsilon a/2)}, \qquad \delta_x^\varepsilon b = a + \frac{\varepsilon}{2}\,\delta_y^\varepsilon a, \tag{52}$$

or, in other words, to equation (48) for the function ϕ defined by

$$a = \delta_x^\varepsilon \phi, \qquad b = \phi + \frac{\varepsilon}{2}\,\delta_y^\varepsilon \phi. \tag{53}$$

The formula (51) is the compatibility condition of the following system of linear difference equations:

$$\begin{aligned} \Psi^\varepsilon(x+\varepsilon, y; \lambda) &= \mathcal{U}^\varepsilon(x, y; \lambda)\Psi^\varepsilon(x, y; \lambda), \\ \Psi^\varepsilon(x, y+\varepsilon; \lambda) &= \mathcal{V}^\varepsilon(x, y; \lambda)\Psi^\varepsilon(x, y; \lambda), \\ \Psi^\varepsilon(0, 0; \lambda) &= \mathbf{1}. \end{aligned} \tag{54}$$

So any solution of (48) uniquely defines a matrix, $\Psi^\varepsilon : \Omega^\varepsilon(r) \to G[\lambda]$, satisfying (54). This can be used to finally construct the immersion by an analogue of the Sym formula,

$$F^\varepsilon(x, y) = \left(2\lambda \Psi^\varepsilon(x, y; \lambda)^{-1} \partial_\lambda \Psi^\varepsilon(x, y; \lambda)\right)\Big|_{\lambda=1}. \tag{55}$$

The geometric meaning of the function ϕ is the following. The angle between edges $F^\varepsilon(x+\varepsilon, y) - F^\varepsilon(x, y)$ and $F^\varepsilon(x, y+\varepsilon) - F^\varepsilon(x, y)$ is equal to $(\phi(x+\varepsilon, y) + \phi(x, y+\varepsilon))/2$; the angle between edges $F^\varepsilon(x, y+\varepsilon) - F^\varepsilon(x, y)$ and $F^\varepsilon(x-\varepsilon, y) - F^\varepsilon(x, y)$ is equal to $\pi - (\phi(x, y+\varepsilon) + \phi(x-\varepsilon, y))/2$; the angle between edges $F^\varepsilon(x-\varepsilon, y) - F^\varepsilon(x, y)$ and $F^\varepsilon(x, y-\varepsilon) - F^\varepsilon(x, y)$ is equal to $(\phi(x-\varepsilon, y) + \phi(x, y-\varepsilon))/2$; and the angle between edges $F^\varepsilon(x, y-\varepsilon) - F^\varepsilon(x, y)$ and $F^\varepsilon(x+\varepsilon, y) - F^\varepsilon(x, y)$ is equal to $\pi - (\phi(x, y-\varepsilon) + \phi(x+\varepsilon, y))/2$. In particular, the sum of these angles is 2π, so that the four neighboring vertices of $F^\varepsilon(x, y)$ lie in one plane, as they should. Again, the right-hand side of (55), at values of λ different from $\lambda = 1$ determines an associated family F_λ^ε of discrete K-surfaces parametrized by asymptotic lines.

Now we are prepared to state the approximation theorem for K-surfaces.

Theorem 2. *Let $a_0 : [0, r] \to \mathbb{R}$ and $b_0 : [0, r] \to S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ be smooth functions. Then*

– *there exists a unique K-surface parametrized by asymptotic lines, $F : \Omega(r) \to \mathbb{R}^3$ such that its characteristic angle, $\phi : \Omega(r) \to S^1$, satisfies*

$$\partial_x \phi(x, 0) = a_0(x), \quad \phi(0, y) = b_0(y), \quad x, y \in [0, r], \tag{56}$$

– *for any* $\varepsilon > 0$ *there exists a unique discrete K-surface* $F^\varepsilon : \Omega^\varepsilon(r) \to \mathbb{R}^3$ *such that its characteristic angle* $\phi^\varepsilon : \Omega^\varepsilon(r) \to S^1$ *on the coordinate axes satisfies*

$$\phi^\varepsilon(x+\varepsilon, 0) - \phi^\varepsilon(x, 0) = \varepsilon a_0(x), \quad \phi^\varepsilon(0, y+\varepsilon) + \phi^\varepsilon(0, y) = 2b_0(y), \tag{57}$$

for $x, y \in [0, r-\varepsilon]^\varepsilon$,

– *The inequality*

$$\sup_{\Omega^\varepsilon(r)} |F^\varepsilon - F| \le C\varepsilon, \tag{58}$$

where C *does not depend on* ε, *is satisfied. Moreover, for a pair* (m, n) *of nonnegative integers*

$$\sup_{\Omega^\varepsilon(r-k\varepsilon)} |(\delta_x^\varepsilon)^m (\delta_y^\varepsilon)^n F^\varepsilon - \partial_x^m \partial_y^n F| \to 0 \quad \text{as} \quad \varepsilon \to 0, \tag{59}$$

– *the estimates (58), (59) are satisfied, uniformly for* $\lambda \in [\Lambda^{-1}, \Lambda]$ *with any* $\Lambda > 1$, *if one replaces, in these estimates, the immersions* F, F^ε *by their associated families,* F_λ, F_λ^ε, *respectively.*

The complete proof of this theorem and its generalizations for nonlinear hyperbolic equations and their discretizations is presented in [7]. It is accomplished in two steps: first, the corresponding approximation results are proven for the Goursat problems for the hyperbolic systems (52) and (43), and then the approximation property is lifted to the frames Ψ^ε, Φ and finally to the surfaces F^ε, F. The proof of the C^∞-approximation goes along the same lines.

Moreover, a stronger approximation result follows from the consistency of the corresponding hyperbolic difference equations. As it was explained in Section 2, considering K-nets of higher dimensions and the corresponding consistent discrete hyperbolic systems, one obtains in the limit smooth K-surfaces with their Bäcklund transforms. The approximation results of Theorem 2 hold true also in this case. Permutability of the classical Bäcklund transformations then also easily follows.

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