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Classification of Integrable Equations on Quad-Graphs. The Consistency Approach

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Abstract: A classification of discrete integrable systems on quad–graphs, i.e. on surface cell decompositions with quadrilateral faces, is given. The notion of integrability laid in the basis of the classification is the three–dimensional consistency. This property yields, among other features, the existence of the discrete zero curvature representation with a spectral parameter. For all integrable systems of the obtained exhaustive list, the so called three–leg forms are found. This establishes Lagrangian and symplectic structures for these systems, and the connection to discrete systems of the Toda type on arbitrary graphs. Generalizations of these ideas to the three–dimensional integrable systems and to the quantum context are also discussed.

1. Introduction

The idea of consistency (or compatibility) is in the core of integrable systems theory. One is faced with it already at the very definition of the complete integrability of a Hamiltonian flow in the Liouville–Arnold sense, which means exactly that the flow may be included into a complete family of commuting (compatible) Hamiltonian flows [1]. Similarly, it is a characteristic feature of soliton (integrable) partial differential equations that they appear not separately but are always organized in hierarchies of commuting (compatible) systems. It is impossible to list all applications or reincarnations of this idea. We mention only a couple of them relevant for our present account. A condition of existence of a number of commuting systems may be taken as the basis of the *symmetry approach* which is used to single out integrable systems in some general classes and to classify them [24]. With the help of the *Miwa transformation* one can encode a hierarchy of integrable equations, like the KP one, into a single discrete (difference) equation [25]. Another way of relating continuous and discrete systems, connected with the idea of compatibility, is based on the notion of *Bäcklund transformations* and the *Bianchi permutability theorem* for them [4]. This notion, born in the classical differential geometry, found its place in the modern theory of discrete integrable systems [30]. A sort of synthesis of the analytic and the geometric approach was achieved in [5] and is being actively developed since then, see a review in [6]. These studies have revealed the fundamental importance of discrete integrable systems; it is nowadays a commonly accepted idea that they may be regarded as the cornerstone of the whole theory of integrable systems. For instance, one believes that both the differential geometry of "integrable" classes of surfaces and their transformation theory may be systematically derived from the multidimensional lattices of consistent Bäcklund transformations [6].

So, the consistency of discrete equations steps to the front stage of the integrability theater. We say that

> a d–dimensional discrete equation possesses the ***consistency*** property, if it may be imposed in a consistent way on all d–dimensional sublattices of a $(d+1)$–dimensional lattice

(a more precise definition will be formulated below). As it is seen from the above remarks, the idea that this notion is closely related to integrability, is not new. In the case $d = 1$ it was used as a possible definition of integrability of maps in [34]. A clear formulation in the case $d = 2$ was given recently in [28]. A decisive step was made in [8]: it was shown there that the integrability in the usual sense of the soliton theory (as existence of the zero curvature representation) *follows* for two–dimensional systems from the three–dimensional consistency. So, the latter property may be considered as a definition of integrability, or its criterion which may be checked in a completely algorithmic manner starting with no more information than the equation itself. Moreover, in case when this criterion gives a positive result, it delivers also the transition matrices participating in the discrete zero curvature representation. (Independently, this was found in [26].)

In the present paper we give a further application of the consistency property: we show that it provides an effective tool for finding and classifying all integrable systems in certain classes of equations. We study here integrable one–field equations on *quad–graphs*, i.e. on cellular decompositions of surfaces with all faces (2–cells) being quadrilateral [8]. In Sect. 2 we formulate our main result (Theorem 1), consisting of a complete classification of integrable systems on quad–graphs. Of course, we provide also a detailed discussion of the assumptions of Theorem 1. Sections 3, 4 are devoted to the proof of the main theorem. In Sect. 5 we discuss the so called three–leg forms [8] of all integrable equations from Theorem 1. This device, artificial from the first glance, proves to be extremely useful in several respects. First, it allows to establish a link with the discrete Toda type equations introduced in full generality in [3]. Second, it provides a mean to the most natural derivation of invariant symplectic structures for evolution problems generated by equations on quad–graphs. This is discussed in Sect. 6. Further, Sect. 7 contains a brief discussion of the relation of the equations listed in Theorem 1 with Bäcklund transformations for integrable partial differential equations. Finally, in Sect. 8 we discuss some perspectives for further work, in particular the consistency approach for three–dimensional systems, and discrete quantum systems.

2. Formulation of the Problem; Results

Basic building blocks of systems on quad–graphs are equations on quadrilaterals of the type

$$Q(x, u, v, y; \alpha, \beta) = 0, \tag{1}$$

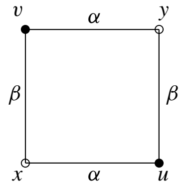

Fig. 1. An elementary quadrilateral; fields are assigned to vertices

where the fields $x, u, v, y \in \mathbb{C}$ are assigned to the four vertices of the quadrilateral, and the parameters $\alpha, \beta \in \mathbb{C}$ are assigned to its edges, as shown on Fig. 1.

A typical example is the so called "cross-ratio equation"

$$\frac{(x-u)(y-v)}{(u-y)(v-x)} = \frac{\alpha}{\beta}, \tag{2}$$

where on the left-hand side one recognizes the cross-ratio of the four complex points x, u, y, v. We shall use the cross-ratio equation to illustrate various notions and claims in this introduction.

Roughly speaking, the goal of the present paper is to classify equations (1) building integrable systems on quad-graphs. We now list more precisely the assumptions under which we solve this problem.

First of all, we assume that Eq. (1) can be uniquely solved for any one of its arguments $x, u, v, y \in \widehat{\mathbb{C}}$. Therefore, the solutions have to be fractional-linear in each of their arguments. This naturally leads to the following condition.

1) Linearity. The function $Q(x, u, v, y; \alpha, \beta)$ is linear in each argument (affine linear):

$$Q(x, u, v, y; \alpha, \beta) = a_1 xuvy + \cdots + a_{16}, \tag{3}$$

where coefficients a_i depend on α, β.

Notice that for the cross-ratio equation (2) one can take the function on the left–hand side of (1) as $Q(x, u, v, y; \alpha, \beta) = \beta(x-u)(y-v) - \alpha(u-y)(v-x)$.

Second, we are interested in equations on quad-graphs of arbitrary combinatorics, hence it will be natural to assume that all variables involved in Eqs. (1) are on equal footing. Therefore, our next assumption reads as follows.

2) Symmetry. Equation (1) is invariant under the group D_4 of the square symmetries, that is function Q satisfies the symmetry properties

$$Q(x, u, v, y; \alpha, \beta) = \varepsilon Q(x, v, u, y; \beta, \alpha) = \sigma Q(u, x, y, v; \alpha, \beta) \tag{4}$$

with $\varepsilon, \sigma = \pm 1$.

Of course, due to the symmetries (4) not all coefficients a_i in (3) are independent, cf. formulae (25), (26) below.

We are interested in ***integrable*** equations of the type (1), i.e. those admitting a ***discrete zero curvature representation***. We refer the reader to [3, 7, 8], where this notion was defined for systems on arbitrary graphs. As pointed out above, in the third of these papers it was shown that the integrability can be detected in an algorithmic manner starting with no more information than the equation itself: the criterion of integrability of an equation is its ***three-dimensional consistency***. This property means that Eq. (1)

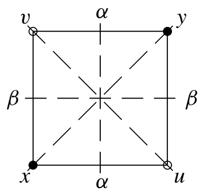

Fig. 2. D_4 symmetry

may be consistently embedded in a three-dimensional lattice, so that similar equations hold for all six faces of any elementary cube, as on Fig. 3 (it is supposed that the values of the parameters α_j assigned to the opposite edges of any face are equal to one another, so that, for instance, all edges (x_2, x_{12}), (x_3, x_{31}), and (x_{23}, x_{123}) carry the label α_1):

To describe more precisely what is meant under the three-dimensional consistency, consider the Cauchy problem with the initial data x, x_1, x_2, x_3. The equations

$$Q(x, x_i, x_j, x_{ij}; \alpha_i, \alpha_j) = 0 \tag{5}$$

allow one to determine uniquely the values x_{12}, x_{23}, x_{31}. After that one has three different equations for x_{123}, coming from the faces $(x_1, x_{12}, x_{31}, x_{123})$, $(x_2, x_{23}, x_{12}, x_{123})$, and $(x_3, x_{31}, x_{23}, x_{123})$. Consistency means that all three values thus obtained for x_{123} coincide.

For instance, consider the cross-ratio equation (2). It is not difficult to check that it possesses the property of the three-dimensional consistency, and

$$x_{123} = \frac{(\alpha_1 - \alpha_2)x_1x_2 + (\alpha_3 - \alpha_1)x_3x_1 + (\alpha_2 - \alpha_3)x_2x_3}{(\alpha_3 - \alpha_2)x_1 + (\alpha_1 - \alpha_3)x_2 + (\alpha_2 - \alpha_1)x_3}. \tag{6}$$

Looking ahead, we mention a very amazing and unexpected feature of the expression (6): the value x_{123} actually depends on x_1, x_2, x_3 only, and does not depend on x. In other words, four black points on Fig. 3 (the vertices of a tetrahedron) are related by a well-defined equation. One can rewrite Eq. (6) as

$$\frac{(x_1 - x_3)(x_2 - x_{123})}{(x_3 - x_2)(x_{123} - x_1)} = \frac{\alpha_1 - \alpha_3}{\alpha_2 - \alpha_3}, \tag{7}$$

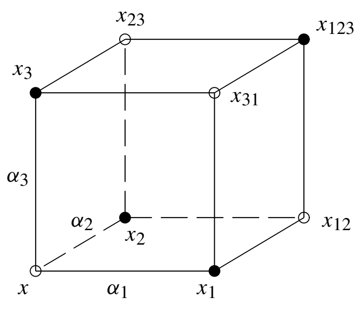

Fig. 3. Three-dimensional consistency

which also has an appearance of the cross-ratio equation for the four points (x_1, x_3, x_2, x_{123}) with the parameters $\alpha_1 - \alpha_3$ assigned to the edges (x_1, x_3), (x_2, x_{123}) and $\alpha_2 - \alpha_3$ assigned to the edges (x_2, x_3), (x_1, x_{123}).

This property, being very strange from first glance, holds actually not only in this but in all known nontrivial examples. We take it as an additional assumption in our solution of the classification problem.

3) Tetrahedron property. The function $x_{123} = z(x, x_1, x_2, x_3; \alpha_1, \alpha_2, \alpha_3)$, existing due to the three-dimensional consistency, actually does not depend on the variable x, that is, $z_x = 0$.

Under the tetrahedron condition we can paint the vertices of the cube into black and white ones, as on Fig. 3, and the vertices of each of two tetrahedrons satisfy an equation of the form

$$\widehat{Q}(x_1, x_2, x_3, x_{123}; \alpha_1, \alpha_2, \alpha_3) = 0; \tag{8}$$

it is easy to see that under the assumption 2) (linearity) the function $\widehat{Q}$ may be also taken linear in each argument. (Clearly, formulas (6), (7) may be also written in such a form.)

Actually, the tetrahedron condition is closely related to another property of Eq. (1), namely to the existence of a ***three-leg form*** of this equation [8]:

$$\psi(x, u; \alpha) - \psi(x, v; \beta) = \phi(x, y; \alpha, \beta). \tag{9}$$

The three terms in this equation correspond to three "legs": two short ones, (x, u) and (x, v), and a long one, (x, y). The short legs are the edges of the original quad-graph, while the long one is not. (We say that the three-leg form (9) is centered at the vertex x; of course, due to the symmetries of the function Q there have to exist also similar formulas centered at the other three vertices involved.) For instance, the cross-ratio equation (2) is equivalent to the following one:

$$\frac{\alpha}{u - x} - \frac{\beta}{v - x} = \frac{\alpha - \beta}{y - x}. \tag{10}$$

The three-leg form gives an explanation for the equation for the "black" tetrahedron from Fig. 3. Consider three faces adjacent to the vertex x_{123} on this figure, namely the quadrilaterals $(x_1, x_{12}, x_{31}, x_{123})$, $(x_2, x_{23}, x_{12}, x_{123})$, and $(x_3, x_{31}, x_{23}, x_{123})$. A summation of three-leg forms (centered at x_{123}) of equations corresponding to these three faces leads to the equation

$$\phi(x_{123}, x_1; \alpha_2, \alpha_3) + \phi(x_{123}, x_2; \alpha_3, \alpha_1) + \phi(x_{123}, x_3; \alpha_1, \alpha_2) = 0.$$

This equation, in any event, relates the fields in the "black" vertices of the cube only, i.e. has the form of (8). For example, for the cross-ratio equation this formula reads:

$$\frac{\alpha_2 - \alpha_3}{x_1 - x_{123}} + \frac{\alpha_3 - \alpha_1}{x_2 - x_{123}} + \frac{\alpha_1 - \alpha_2}{x_3 - x_{123}} = 0,$$

and a simple calculation convinces that this is equivalent to (6).

So, the tetrahedron property is a necessary condition for the existence of a three-leg form. On the other hand, a verification of the tetrahedron property is much more straightforward than finding the three-leg form, since the latter contains two *á priori* unknown functions ψ, ϕ.

It remains to mention that, as demonstrated in [8], the existence of the three-leg form allows one to immediately establish a relation to ***discrete systems of the Toda type*** [3].

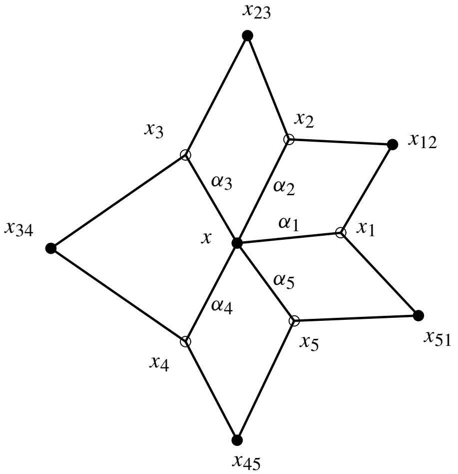

Fig. 4. Faces adjacent to the vertex x

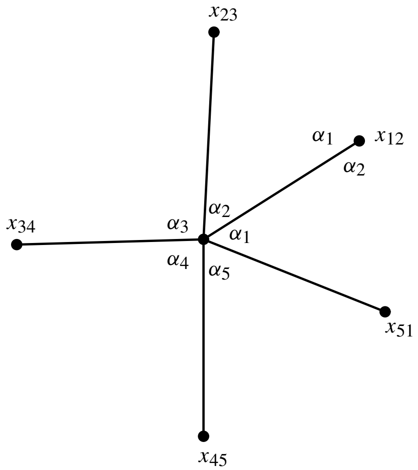

Fig. 5. The star of the vertex x in the black subgraph

Indeed, if x is a common vertex of n adjacent quadrilateral faces $(x, x_k, x_{k,k+1}, x_{k+1})$, $k = 1, 2, \dots, n$, with the parameters α_k assigned to the edges (x, x_k) (cf. Fig. 4), then the fields in the point x and in the "black" vertices of the adjacent faces satisfy the following equation:

$$\sum_{k=1}^{n} \phi(x, x_{k,k+1}, \alpha_k, \alpha_{k+1}) = 0. \tag{11}$$

This is a discrete Toda type equation (equation on stars) on the graph whose vertices are the "black" vertices of the original quad-graph, and whose edges are the diagonals of the faces of the original quad-graph connecting the "black" vertices. The parameters α are naturally assigned to the corners of the faces of the "black" subgraph. See Fig. 5. (Of course, a similar Toda type equation holds also for the "white" subgraph.)

For instance, in the case of the cross-ratio equation, the discrete Toda type system (11) reads as

$$\sum_{k=1}^{n} \frac{\alpha_k - \alpha_{k+1}}{x_{k,k+1} - x} = 0. \tag{12}$$

In the next section we will show that the tetrahedron condition naturally separates one of two subcases of the general problem of classification of integrable equations on quad-graphs. The second subcase will not be considered in this paper; the corresponding subclass of equations is certainly not empty, but we are aware only of trivial (linearizable) examples.

By solving the classification problem we identify equations related by certain natural transformations. First, acting simultaneously on all variables x by one and the same Möbius transformation does not violate our three assumptions. Second, the same holds for the simultaneous point change of all parameters $\alpha \mapsto \varphi(\alpha)$. Our results on the classification of integrable equations on quad-graphs are given by the following statement.

Theorem 1. *Up to common Möbius transformations of the variables x and point transformations of the parameters α, the three-dimensionally consistent quad-graph equations (5) with the properties 1), 2), 3) (linearity, symmetry, tetrahedron property) are exhausted by the following three lists Q, H, A* ($u = x_1$, $v = x_2$, $y = x_{12}$, $\alpha = \alpha_1$, $\beta = \alpha_2$).

List Q:

$$\text{(Q1)}\quad \alpha(x-v)(u-y) - \beta(x-u)(v-y) + \delta^2\alpha\beta(\alpha-\beta) = 0,$$

$$\text{(Q2)}\quad \alpha(x-v)(u-y) - \beta(x-u)(v-y) + \alpha\beta(\alpha-\beta)(x+u+v+y) - \alpha\beta(\alpha-\beta)(\alpha^2-\alpha\beta+\beta^2) = 0,$$

$$\text{(Q3)}\quad (\beta^2-\alpha^2)(xy+uv) + \beta(\alpha^2-1)(xu+vy) - \alpha(\beta^2-1)(xv+uy) - \delta^2(\alpha^2-\beta^2)(\alpha^2-1)(\beta^2-1)/(4\alpha\beta) = 0,$$

$$\text{(Q4)}\quad a_0xuvy + a_1(xuv+uvy+vyx+yxu) + a_2(xy+uv) + \bar{a}_2(xu+vy) + \tilde{a}_2(xv+uy) + a_3(x+u+v+y) + a_4 = 0,$$

where the coefficients a_i *are expressed through* (α, a) *and* (β, b) *with* $a^2 = r(\alpha)$, $b^2 = r(\beta)$, $r(x) = 4x^3 - g_2x - g_3$, *by the following formulae:*

$$a_0 = a + b, \quad a_1 = -\beta a - \alpha b, \quad a_2 = \beta^2 a + \alpha^2 b,$$
$$\bar{a}_2 = \frac{ab(a+b)}{2(\alpha-\beta)} + \beta^2 a - (2\alpha^2 - \frac{g_2}{4})b,$$
$$\tilde{a}_2 = \frac{ab(a+b)}{2(\beta-\alpha)} + \alpha^2 b - (2\beta^2 - \frac{g_2}{4})a,$$
$$a_3 = \frac{g_3}{2}a_0 - \frac{g_2}{4}a_1, \quad a_4 = \frac{g_2^2}{16}a_0 - g_3a_1.$$

List H:

$$\text{(H1)}\quad (x-y)(u-v) + \beta - \alpha = 0,$$
$$\text{(H2)}\quad (x-y)(u-v) + (\beta-\alpha)(x+u+v+y) + \beta^2 - \alpha^2 = 0,$$
$$\text{(H3)}\quad \alpha(xu+vy) - \beta(xv+uy) + \delta(\alpha^2-\beta^2) = 0.$$

List A:

$$\text{(A1)}\quad \alpha(x+v)(u+y) - \beta(x+u)(v+y) - \delta^2\alpha\beta(\alpha-\beta) = 0,$$
$$\text{(A2)}\quad (\beta^2-\alpha^2)(xuvy+1) + \beta(\alpha^2-1)(xv+uy) - \alpha(\beta^2-1)(xu+vy) = 0.$$

Remark. 1) The list A can be dropped down by allowing an extended group of Möbius transformations, which act on the variables x, y differently than on u, v (white and black sublattices on Figs. 1,3). In this manner Eq. (A1) is related to (Q1) (by the change $u \to -u, v \to -v$), and Eq. (A2) is related to (Q3) with $\delta = 0$ (by the change $u \to 1/u$, $v \to 1/v$). So, really independent equations are given by the lists Q and H.

2) In both lists Q, H the last equations are the most general ones. This means that Eqs. (Q1)–(Q3) and (H1), (H2) may be obtained from (Q4) and (H3), respectively, by certain degenerations and/or limit procedures. So, one could be tempted to shorten down these lists to one item each. However, on the one hand, these limit procedures are outside of our group of admissible (Möbius) transformations, and on the other hand, in many situations the "degenerate" equations (Q1)–(Q3) and (H1), (H2) are of interest for

themselves. This resembles the situation with the list of six Painlevé equations and the coalescences connecting them, cf. [15].

3) Parameter δ in Eqs. (Q1), (Q3), (H3) can be scaled away, so that one can assume without loss of generality that $\delta = 0$ or $\delta = 1$.

4) It is natural to set in Eq. (Q4) $(\alpha, a) = (\wp(A), \wp'(A))$ and, similarly, $(\beta, b) = (\wp(B), \wp'(B))$. So, this equation is actually parametrized by two points of the elliptic curve $\mu^2 = r(\lambda)$. The appearance of an elliptic curve in our classification problem is by no means obvious from the beginning, its origin will become clear later, in the course of the proof. For the cases of r with multiple roots, when the elliptic curve degenerates into a rational one, Eq. (Q4) degenerates to one of the previous equations of the list Q; for example, if $g_2 = g_3 = 0$ then the inversion $x \to 1/x$ turns (Q4) into (Q2).

Bibliographical remarks. It is difficult to track down the origin of the equations listed in Theorem 1. Probably, the oldest ones are (H1) and $(\mathrm{H3})_{\delta=0}$, which can be found in the work of Hirota [14] (of course not on general quad–graphs but only on the standard square lattice with the labels α constant in each of the two lattice directions; similar remarks apply also to other references in this paragraph). Equations (Q1) and $(\mathrm{Q3})_{\delta=0}$ go back to [30], see also a review in [27]. Equation (Q4) was found in [2]. A Lax representation for (Q4) was found in [26] with the help of the method based on the three–dimensional consistency, identical with the method introduced in [8]. Equations (Q2) and $(\mathrm{Q3})_{\delta=1}$ are particular cases of (Q4), but seem to have not appeared explicitly in the literature. The same holds for (H2) and $(\mathrm{H3})_{\delta=1}$.

3. Classification: Analysis

In principle, the three-dimensional consistency turns, under the assumptions 1), 2), into some system of functional equations for the coefficients a_i of the functions Q (see (3)). However, this system is difficult to analyze and we will take a different route.

For the first step, we consider the problem of the three-dimensional consistency in the following general setting: find triples of functions f_1, f_2, f_3 of three arguments such that if

$$x_{23} = f_1(x, x_2, x_3), \quad x_{31} = f_2(x, x_3, x_1), \quad x_{12} = f_3(x, x_1, x_2) \tag{13}$$

then

$$x_{123} := f_1(x_1, x_{12}, x_{31}) \equiv f_2(x_2, x_{23}, x_{12}) \equiv f_3(x_3, x_{31}, x_{23}) \tag{14}$$

identically in x, x_1, x_2, x_3. In other words, we ignore for a moment the conditions 1) and 2) and look to what consequences the *tetrahedron* condition leads. The proof of the following statement demonstrates that this condition separates just one of two possible subcases in the general problem.

Proposition 2. *For the functions f_1, f_2, f_3 compatible in the sense (13), (14) and satisfying the tetrahedron condition, the following relation holds:*

$$f_{3,x_2} f_{2,x_1} f_{1,x_3} = -f_{3,x_1} f_{2,x_3} f_{1,x_2} \tag{15}$$

identically in x, x_1, x_2, x_3.

Proof. Denote $x_{123} = z(x, x_1, x_2, x_3)$, and also denote for short the functions with shifted arguments (14) by capitals: $F_1 = f_1(x_1, x_{12}, x_{31})$, etc. Then differentiating (14) with respect to x_2, x_3 and x yields the following system which is linear with respect to the derivatives of F_1:

$$\begin{aligned} f_{3,x_2} F_{1,x_{12}} &= z_{x_2}, \\ f_{2,x_3} F_{1,x_{31}} &= z_{x_3}, \\ f_{3,x} F_{1,x_{12}} + f_{2,x} F_{1,x_{31}} &= z_x, \end{aligned}$$

and two analogous systems for F_2, F_3 obtained by the cyclic shift of indices. The above system is overdetermined, and, excluding the derivatives of F_1, we come to the first equation of the following system (the other two come from the similar considerations with F_2, F_3):

$$\begin{aligned} f_{2,x_3} f_{3,x} z_{x_2} + f_{3,x_2} f_{2,x} z_{x_3} &= f_{3,x_2} f_{2,x_3} z_x, \\ f_{1,x_3} f_{3,x} z_{x_1} + f_{3,x_1} f_{1,x} z_{x_3} &= f_{1,x_3} f_{3,x_1} z_x, \\ f_{1,x_2} f_{2,x} z_{x_1} + f_{2,x_1} f_{1,x} z_{x_2} &= f_{2,x_1} f_{1,x_2} z_x. \end{aligned}$$

Now, the tetrahedron condition $z_x = 0$ implies that the r.h.s. vanishes, and therefore the determinant has to vanish as well; this is equivalent to (15). □

The second possibility, which we do not consider here, would be that $z_x \neq 0$ and Eq. (15) does not hold. In this case the above system can be solved to give $z_{x_i} = \Phi_i z_x$, $i = 1, 2, 3$, where Φ_i are expressed through $f_{j,x}$, f_{j,x_k}. The compatibility of the latter three equations can be expressed as a system of differential equations for Φ_i, and therefore for f_j, which are much more complicated than (15). It certainly deserves a further investigation, see discussion in Sect. 8.

The necessary condition (15) will provide us with a finite list of candidates for the three-dimensional consistency. First of all, we rewrite the relation (15) in terms of the polynomial Q using both the *linearity* and the *symmetry* assumptions.

Proposition 3. *Relation (15) is equivalent to*

$$\begin{aligned} & g(x, x_1; \alpha_1, \alpha_2) g(x, x_2; \alpha_2, \alpha_3) g(x, x_3; \alpha_3, \alpha_1) \\ & \quad = -g(x, x_1; \alpha_1, \alpha_3) g(x, x_2; \alpha_2, \alpha_1) g(x, x_3; \alpha_3, \alpha_2), \end{aligned} \tag{16}$$

where $g(x, u; \alpha, \beta)$ is a biquadratic polynomial in x, u defined by either of the formulas

$$g(x, u; \alpha, \beta) = Q Q_{yv} - Q_y Q_v, \tag{17}$$

$$g(x, v; \beta, \alpha) = Q Q_{yu} - Q_y Q_u, \tag{18}$$

where $Q = Q(x, u, v, y; \alpha, \beta)$. The polynomial g is symmetric:

$$g(x, u; \alpha, \beta) = g(u, x; \alpha, \beta). \tag{19}$$

Proof. The equivalence of the definitions (17), (18) follows from the first symmetry property in (4) ($\alpha \leftrightarrow \beta$, $u \leftrightarrow v$), while the second one ($x \leftrightarrow u$, $y \leftrightarrow v$) implies (19).

To prove (16), let $Q = p(x, u, v)y + q(x, u, v)$, so that $y = f(x, u, v) = -q/p$. Then $f_v/f_u = (q_v p - q p_v)/(q_u p - q p_u)$, and substituting $p = Q_y$, $q = Q - y Q_y$, we obtain

$$\frac{f_v}{f_u} = \frac{Q Q_{yv} - Q_y Q_v}{Q Q_{yu} - Q_y Q_u},$$

which yields:

$$\frac{f_{k,x_j}}{f_{k,x_i}} = \frac{g(\alpha_i, \alpha_j; x, x_i)}{g(\alpha_j, \alpha_i; x, x_j)}, \quad (i, j, k) = \text{c.p.}(1, 2, 3).$$

Now (16) follows from (15). □

The biquadratic polynomials (17) and (18) are associated to the edges of the basic square. One can consider also the polynomial

$$G(x, y; \alpha, \beta) = QQ_{uv} - Q_u Q_v, \tag{20}$$

associated to the diagonal. They have the following important property.

Lemma 4. *The discriminants of the polynomials $g = g(x, u; \alpha, \beta)$, $\bar{g} = g(x, v; \beta, \alpha)$, and $G = G(x, y; \alpha, \beta)$, considered as quadratic polynomials in u, v, and y, respectively, coincide:*

$$g_u^2 - 2gg_{uu} = \bar{g}_v^2 - 2\bar{g}\bar{g}_{vv} = G_y^2 - 2GG_{yy}. \tag{21}$$

Proof. This follows solely from the fact that the function Q is linear in each argument. Indeed, calculate

$$g_u^2 - 2gg_{uu} = ((QQ_{yv} - Q_y Q_v)_u)^2 - 2(QQ_{yv} - Q_y Q_v)(QQ_{yv} - Q_y Q_v)_{uu},$$

taking into account that $Q_{uu} = 0$. The result reads:

$$\begin{aligned} g_u^2 - 2gg_{uu} = {} & Q^2 Q_{uvy}^2 + Q_u^2 Q_{vy}^2 + Q_v^2 Q_{uy}^2 + Q_y^2 Q_{uv}^2 + 4QQ_{uv}Q_{uy}Q_{vy} \\ & -2QQ_{uvy}(Q_u Q_{vy} + Q_v Q_{uy} + Q_y Q_{uv}) - 4Q_u Q_v Q_y Q_{uvy} \\ & -2Q_u Q_v Q_{uy} Q_{vy} - 2Q_u Q_y Q_{uv} Q_{vy} - 2Q_v Q_y Q_{uv} Q_{uy}. \end{aligned}$$

It remains to notice that this expression is symmetric with respect to all indices. □

In the formula (16) the variables are highly separated, and it can be effectively analyzed further on. In the next statement we demonstrate that this functional equation relating values of g with different arguments implies some properties for a *single* polynomial g.

Proposition 5. *The polynomial $g(x, u; \alpha, \beta)$ can be represented as*

$$g(x, u; \alpha, \beta) = k(\alpha, \beta)h(x, u; \alpha), \tag{22}$$

where the factor k is antisymmetric,

$$k(\beta, \alpha) = -k(\alpha, \beta), \tag{23}$$

and the coefficients of the polynomial $h(x, u; \alpha)$ depend on parameter α in such a way that its discriminant

$$r(x) = h_u^2 - 2hh_{uu} \tag{24}$$

does not depend on α.

Proof. Relation (16) implies that the fraction $g(x,x_1;\alpha_1,\alpha_2)/g(x,x_1;\alpha_1,\alpha_3)$ does not depend on x_1, and due to the symmetry (19) it does not depend on x as well. Therefore

$$\frac{g(x,x_1;\alpha_1,\alpha_2)}{g(x,x_1;\alpha_1,\alpha_3)}=\frac{\kappa(\alpha_1,\alpha_2)}{\kappa(\alpha_1,\alpha_3)},$$

where, because of (16), the function κ satisfies the equation

$$\kappa(\alpha_1,\alpha_2)\kappa(\alpha_2,\alpha_3)\kappa(\alpha_3,\alpha_1)=-\kappa(\alpha_2,\alpha_1)\kappa(\alpha_3,\alpha_2)\kappa(\alpha_1,\alpha_3).$$

This equation is equivalent to

$$\kappa(\beta,\alpha)=-\frac{\phi(\alpha)}{\phi(\beta)}\kappa(\alpha,\beta),$$

that is, the function $k(\alpha,\beta)=\phi(\alpha)\kappa(\alpha,\beta)$ is antisymmetric. We have:

$$\frac{g(x,u;\alpha,\beta)}{\kappa(\alpha,\beta)}=\frac{g(x,u;\alpha,\gamma)}{\kappa(\alpha,\gamma)}\quad\Rightarrow\quad\frac{g(x,u;\alpha,\beta)}{k(\alpha,\beta)}=\frac{g(x,u;\alpha,\gamma)}{k(\alpha,\gamma)},$$

which implies (22). To prove the last statement of the proposition, we notice that, according to (22), (23),

$$k(\alpha,\beta)h(x,u;\alpha)=g(x,u;\alpha,\beta),\quad -k(\alpha,\beta)h(x,v;\beta)=g(x,v;\beta,\alpha).$$

Due to the identity (21), we find:

$$h_u^2-2hh_{uu}=\bar h_v^2-2\bar h\bar h_{vv},\quad h=h(x,u;\alpha),\quad \bar h=h(x,v;\beta),$$

and therefore r does not depend on α. □

Thus, the three-dimensional consistency with the tetrahedron property implies the following remarkable property of the function Q which will be called property (R):

(R) the determinant $g=QQ_{vy}-Q_vQ_y$ is factorizable as in (22), (23), and the discriminant $r=h_u^2-2hh_{uu}$ of the corresponding quadratic polynomial h does not depend on parameters at all.

It remains to classify all functions Q with this property. This will be done in the next section. A finite list of functions Q with the property (R) consists, therefore, of *candidates* for the three-dimensional consistency. The final check is straightforward, and shows that the property (R) is not only necessary but also almost sufficient for the three-dimensional consistency with the tetrahedron property (the list of functions with the property (R) consists of a dozen items; only in two of them one finds functions violating the consistency).

As a preliminary step, we consider more closely the coefficients of the polynomial Q. The symmetry property (4) easily implies that two cases are possible, with $\sigma=1$ and $\sigma=-1$, respectively:

$$\begin{aligned}Q=&\,a_0xuvy+a_1(xuv+uvy+vyx+yxu)\\&+a_2(xy+uv)+\bar a_2(xu+vy)+\tilde a_2(xv+uy)\\&+a_3(x+u+v+y)+a_4,\end{aligned}\tag{25}$$

$$\begin{aligned}Q=&\,a_1(xuv+uvy-vyx-yxu)\\&+a_2(xy-uv)+a_3(x-u-v+y),\end{aligned}\tag{26}$$

where

$$a_i(\beta,\alpha)=\varepsilon a_i(\alpha,\beta),\quad \tilde{a}_2(\alpha,\beta)=\varepsilon\bar{a}_2(\beta,\alpha),\quad \varepsilon=\pm 1. \tag{27}$$

It is easy to prove that the case (26) is actually empty. Indeed, a light calculation shows that in this case

$$g(x,u;\alpha,\beta)=g(x,u;\beta,\alpha)=-P(x;\alpha,\beta)P(u;\alpha,\beta),$$

where $P(\alpha,\beta;x)=a_1x^2-a_2x-a_3$, so that the relation (16) becomes

$$P(x_1;\alpha_1,\alpha_2)P(x_2;\alpha_2,\alpha_3)P(x_3;\alpha_3,\alpha_1)=-P(x_1;\alpha_3,\alpha_1)P(x_2;\alpha_1,\alpha_2)P(x_3;\alpha_2,\alpha_3).$$

Equating to zero the coefficients at $(x_1x_2x_3)^2$, $x_1x_2x_3$, and the free term yields $a_1=a_2=a_3=0$ – a contradiction.

Turning to the case (25), we denote the coefficients of the polynomials h, r as follows:

$$h(x,u;\alpha)=b_0x^2u^2+b_1xu(x+u)+b_2(x^2+u^2)+\hat{b}_2xu+b_3(x+u)+b_4, \tag{28}$$

$$r(x)=c_0x^4+c_1x^3+c_2x^2+c_3x+c_4, \tag{29}$$

where $b_i=b_i(\alpha)$. So, we consequently descended from the polynomial Q (4 variables, 7 coefficients depending on 2 parameters) to the polynomial h (2 variables, 6 coefficients depending on 1 parameter; also take into account the factor k), and then to the polynomial r (1 variable, 5 coefficients, no parameters).

It remains to go the way back, i.e. to reconstruct k, h and Q from a given polynomial r (which does not depend on the parameter α). This will be done in the next section.

4. Classification: Synthesis

First of all, we factor out the action of the simultaneous Möbius transformations of the variables x. The action $x\mapsto(ax+b)/(cx+d)$ transforms the polynomials h, r as follows:

$$h(x,u;\alpha)\mapsto(cx+d)^2(cu+d)^2h\left(\frac{ax+b}{cx+d},\frac{au+b}{cu+d};\alpha\right),$$
$$r(x)\mapsto(cx+d)^4r\left(\frac{ax+b}{cx+d}\right).$$

Using such transformations one can bring the polynomial r into one of the following canonical forms, depending on the distribution of its roots:

- $r(x)=0$;
- $r(x)=1$ (r has one quadruple zero);
- $r(x)=x$ (r has one simple zero and one triple zero);
- $r(x)=x^2$ (r has two double zeroes);
- $r(x)=x^2-1$ (r has two simple zeroes and one double zero);
- $r(x)=4x^3-g_2x-g_3$, $\Delta=g_2^3-27g_3^2\neq 0$ (r has four simple zeroes).

Next, we find for these canonical polynomials r all possible polynomials h.

Proposition 6. *For a given polynomial $r(x)$ of the fourth degree, in one of the canonical forms above, the symmetric biquadratic polynomials $h(x,u)$ having $r(x)$ as their discriminants, $r(x) = h_u^2 - 2hh_{uu}$, are exhausted by the following list:*

$$r = 0: \qquad h = \frac{1}{\alpha}(x-u)^2; \tag{q0}$$

$$h = (\gamma_0 xu + \gamma_1(x+u) + \gamma_2)^2; \tag{h1}$$

$$r = 1: \qquad h = \frac{1}{2\alpha}(x-u)^2 - \frac{\alpha}{2}; \tag{q1}$$

$$h = \gamma_0(x+u)^2 + \gamma_1(x+u) + \gamma_2, \quad \gamma_1^2 - 4\gamma_0\gamma_2 = 1; \tag{h2}$$

$$r = x: \qquad h = \frac{1}{4\alpha}(x-u)^2 - \frac{\alpha}{2}(x+u) + \frac{\alpha^3}{4}; \tag{q2}$$

$$r = x^2: \qquad h = \gamma_0 x^2u^2 + \gamma_1 xu + \gamma_2, \quad \gamma_1^2 - 4\gamma_0\gamma_2 = 1; \tag{h3}$$

$$r = x^2 - \delta^2: \qquad h = \frac{\alpha}{1-\alpha^2}(x^2+u^2) - \frac{1+\alpha^2}{1-\alpha^2}xu + \delta^2\frac{1-\alpha^2}{4\alpha}; \tag{q3}$$

$$r = 4x^3 - g_2x - g_3: \qquad h = \frac{1}{\sqrt{r(\alpha)}}\Big[\big(xu + \alpha(x+u) + g_2/4\big)^2 - (x+u+\alpha)(4\alpha xu - g_3)\Big]. \tag{q4}$$

Proof. We have to solve the system of the form

$$\begin{aligned} b_1^2 - 4b_0b_2 &= c_0, \\ 2b_1(\hat{b}_2 - 2b_2) - 4b_0b_3 &= c_1, \\ \hat{b}_2^2 - 4b_2^2 - 2b_1b_3 - 4b_0b_4 &= c_2, \\ 2b_3(\hat{b}_2 - 2b_2) - 4b_1b_4 &= c_3, \\ b_3^2 - 4b_2b_4 &= c_4, \end{aligned}$$

where b_k are the coefficients of $h(x,u)$ and c_k are the coefficients of $r(x)$ (see (28), (29)). This is done by a straightforward analysis. For example, consider in detail the case $r = 4x^3 - g_2x - g_3$. We have: $c_0 = 0$, $c_1 = 4$, hence $b_0 \neq 0$. Set $b_0 = \rho^{-1}$, $b_1 = -2\alpha\rho^{-1}$, then $b_2 = \alpha^2\rho^{-1}$. Next, use the second and the third equations to eliminate b_3 and b_4, then the last two equations give an expression for $\hat{b}_2$ and the constraint $\rho^2 = r(\alpha)$. □

Remarks . 1) Notice that for all six canonical forms of r we have a one-parameter family of h's, denoted in the list of Proposition 6 by (q0)–(q4) (the one-parameter family for $r = x^2$ is not explicitly written down since it coincides with (q3) at $\delta = 0$). It will turn out that the polynomials Q reconstructed from these h's belong to the list Q (so that h in (qk) corresponds to Q in (Qk); h from (q0) corresponds to $(\text{Q1})_{\delta=0}$). For the polynomials $r = 0, 1, x^2$ we have additional two- or three-parameter families of h's denoted in the list of Proposition 6 by (h1)–(h3). They will lead to the correspondent Q's in (H1)–(H3), as well as to (A1), (A2).

2) The expression $\sqrt{r(\alpha)}$ in Eq. (q4) clearly shows that the elliptic curve $\mu^2 = r(\lambda)$ comes into play at this point. One can uniformize $\sqrt{r(\alpha)} = \wp'(A)$, $\alpha = \wp(A)$, where A is a point of the elliptic curve, and $\wp$ is the Weierstrass elliptic function. Actually, the polynomial $h(x,u;\alpha)$ is well–known in the theory of elliptic functions and represents the addition theorem for the $\wp$–function. Namely, the equality $h(\wp(X), \wp(U); \wp(A)) = 0$ is

equivalent to $A=\pm X\pm U$ modulo the lattice of periods of the function $\wp$. To summarize: symmetric biquadratic polynomials $h(x,u)$ with the discriminant $h_u^2-2hh_{uu}=r(x)$ are parametrized (in the non-degenerate case) by a point of the elliptic curve $\mu^2=r(\lambda)$. This is the origin of the elliptic curve in the parametrization of Eq. (q4). As a consequence, the spectral parameter in the discrete zero curvature representation of the latter equation also lives on the elliptic curve. This may be considered also as the ultimate reason for the spectral parameter of the Krichever–Novikov equation to live on an elliptic curve ([20], see also Sect. 7).

It remains to reconstruct polynomials Q for all h's from Proposition 6.

Proof of Theorem 1. For the polynomial (25) we have:

$$\begin{aligned} g(x,u;\alpha,\beta) &= (\bar a_2a_0-a_1^2)x^2u^2+(a_1(\bar a_2-\tilde a_2)+a_0a_3-a_1a_2)xu(x+u)\\ &\quad+(a_1a_3-a_2\tilde a_2)(x^2+u^2)+(\bar a_2^2-\tilde a_2^2+a_0a_4-a_2^2)xu\\ &\quad+(a_3(\bar a_2-\tilde a_2)+a_1a_4-a_2a_3)(x+u)+\bar a_2a_4-a_3^2, \end{aligned}$$

and an analogous expression for $g(x,u;\beta,\alpha)$ is obtained by the replacement $\bar a_2\leftrightarrow\tilde a_2$. Using (22) and denoting $b_i=b_i(\alpha)$, $b_i'=b_i(\beta)$, $k=k(\alpha,\beta)$, we come to the following system for the unknown quantities a_k:

$$\begin{aligned} \bar a_2a_0-a_1^2&=kb_0, & \tilde a_2a_0-a_1^2&=-kb_0',\\ a_1(\bar a_2-\tilde a_2)+a_0a_3-a_1a_2&=kb_1, & a_1(\tilde a_2-\bar a_2)+a_0a_3-a_1a_2&=-kb_1',\\ a_1a_3-a_2\tilde a_2&=kb_2 & a_1a_3-a_2\bar a_2&=-kb_2',\\ \bar a_2^2-\tilde a_2^2+a_0a_4-a_2^2&=k\hat b_2, & \tilde a_2^2-\bar a_2^2+a_0a_4-a_2^2&=-k\hat b_2',\\ a_3(\bar a_2-\tilde a_2)+a_1a_4-a_2a_3&=kb_3, & a_3(\tilde a_2-\bar a_2)+a_1a_4-a_2a_3&=-kb_3',\\ \bar a_2a_4-a_3^2&=kb_4, & \tilde a_2a_4-a_3^2&=-kb_4'. \end{aligned}$$

Of course, the quantity k is also still unknown here. Since we are looking for the function Q up to arbitrary factor, it is convenient to denote

$$a=\bar a_2-\tilde a_2,\quad A_i=\frac{a_i}{a},\quad \widehat A_2=\frac{\bar a_2}{a}-\frac12=\frac{\tilde a_2}{a}+\frac12,\quad K=\frac{k}{a^2} \tag{30}$$

(it is easy to see that $a\not\equiv0$ since otherwise $h\equiv0$). These functions are skew-symmetric:

$$A_i(\beta,\alpha)=-A_i(\alpha,\beta),\quad \widehat A_2(\beta,\alpha)=-\widehat A_2(\alpha,\beta),\quad K(\beta,\alpha)=-K(\alpha,\beta),$$

and the above system can be rewritten, after some elementary transformations, as follows:

$$\begin{aligned} K[(\hat b_2+\hat b_2')(b_0+b_0')-(b_1+b_1')^2]&=2(b_0-b_0'),\\ K[(b_0+b_0')(b_3+b_3')-(b_1+b_1')(b_2+b_2')]&=b_1-b_1',\\ K[(b_1+b_1')(b_3+b_3')-(b_2+b_2')(\hat b_2+\hat b_2')]&=2(b_2-b_2'),\\ K[(b_0+b_0')(b_4+b_4')-(b_2+b_2')^2]&=\frac12(\hat b_2-\hat b_2'),\\ K[(b_1+b_1')(b_4+b_4')-(b_2+b_2')(b_3+b_3')]&=b_3-b_3',\\ K[(\hat b_2+\hat b_2')(b_4+b_4')-(b_3+b_3')^2]&=2(b_4-b_4'),\\ A_0=K(b_0+b_0'),\quad 2A_1=K(b_1+b_1'),&\quad A_2=K(b_2+b_2'),\\ 4\widehat A_2=K(\hat b_2+\hat b_2'),\quad 2A_3=K(b_3+b_3'),&\quad A_4=K(b_4+b_4'). \end{aligned}$$

The first six lines here form a system of functional equations for b_i and K (call it the (K, b)–system), while the last two lines split away, and should be considered just as definitions of A_i. So, to any solution of the (K, b)–system there corresponds a function Q whose coefficients are given by the last two lines of the system above, and the formulas

$$\bar{A}_2 = \widehat{A}_2 + 1/2, \quad \widetilde{A}_2 = \widehat{A}_2 - 1/2,$$

which follow from (30). By construction, this function has the property (R) and is, therefore, a candidate for the three-dimensional consistency with the tetrahedron property.

Consider first the cases (q0), (q1), (q2), (q3), (q4), i.e. when we have a one-parameter family of polynomials h. A straightforward, although tedious, check proves that in these cases all six equations of the (K, b)–system lead to one and the same function K, provided the functions $b_i = b_i(\alpha)$ are defined as in (q0)–(q4). Calculating the corresponding coefficients A_i, we come to the functions Q given in Theorem 1 by the formulas $(Q1)_{\delta=0}$, $(Q1)_{\delta=1}$, (Q2), (Q3), and (Q4), respectively. A further straightforward check convinces us that all these functions indeed pass the three-dimensional consistency test with the tetrahedron propery.

It remains to consider the cases (h1), (h2), (h3). A thorough analysis of the (K, b)–system shows that in these cases we have the following solutions.

In the case (h1):

- either h does not depend on parameters at all, and then it has to be of the form $h = \big((\varepsilon_0 x + \varepsilon_1)(\varepsilon_0 u + \varepsilon_1)\big)^2$, and K is arbitrary; performing a suitable Möbius transformation, we can achieve that $h \equiv 1$; the outcome of this subcase is the following function Q which is a candidate for the three-dimensional consistency:

$$Q = (x - y)(u - v) + k(\alpha, \beta), \qquad (\widehat{\text{H1}})$$

  where k is an arbitrary skew-symmetric function, $k(\alpha, \beta) = -k(\beta, \alpha)$;
- or $h = (1/\alpha)(\varepsilon_0 xu + \varepsilon_1(x + u) + \varepsilon_2)^2$ with arbitrary constants ε_i. In this case we can use Möbius transformations to achieve $h = (1/\alpha)(x + u)^2$, and the correspondent function Q coincides with $(A1)_{\delta=0}$ from Theorem 1. This function passes the three-dimensional consistency test with the tetrahedron property.

In the case (h2):

- either $h = x + u + \alpha$, and the correspondent function Q coincides with (H2) from Theorem 1;
- or $h = (1/2\alpha)(x + u)^2 - (\alpha/2)$, and the correspondent function (Q) is given in $(A1)_{\delta=1}$ of Theorem 1.

Both functions are three-dimensionally consistent with the tetrahedron property.

Finally, in the case (h3):

- either h does not depend on parameters at all, then it has to be equal to $h = xu$, and K is arbitrary; we have in this subcase the following function Q which is a candidate for the three-dimensional consistency:

$$Q = \frac{1+k}{2}(xu + vy) - \frac{1-k}{2}(xv + uy), \qquad (\widehat{\text{H3}}_0)$$

  where k is an arbitrary skew-symmetric function, $k(\alpha, \beta) = -k(\beta, \alpha)$;
- or $h = xu + \alpha$ (possibly, upon application of the inversion $x \mapsto 1/x$, $u \mapsto 1/u$); the correspondent function Q is $(H3)_{\delta=1}$ of Theorem 1;

- or, finally, $h = \dfrac{\alpha}{1-\alpha^2}(x^2u^2+1) - \dfrac{1+\alpha^2}{1-\alpha^2}xu$; the correspondent function Q is given in (A2) of Theorem 1.

In the two last subcases the three-dimensional consistency condition is fulfilled with the tetrahedron property.

To finish the proof of Theorem 1, we have to consider the functions Q given by the formulas ($\widehat{\text{H1}}$) and ($\widehat{\text{H3}}_0$). They depend on an arbitrary skew–symmetric function k, and have the property (R) for any choice of the latter. As it turns out, these are the only situations when the property (R) does not automatically imply the three-dimensional consistency.

A direct calculation shows that the function ($\widehat{\text{H1}}$) gives a map which is three-dimensionally consistent, with

$$x_{123} = \frac{k(\alpha_1,\alpha_2)x_1x_2 + k(\alpha_2,\alpha_3)x_2x_3 + k(\alpha_3,\alpha_1)x_3x_1}{k(\alpha_3,\alpha_2)x_1 + k(\alpha_1,\alpha_3)x_2 + k(\alpha_2,\alpha_1)x_3}, \tag{31}$$

if and only if

$$k(\alpha_1,\alpha_2) + k(\alpha_2,\alpha_3) + k(\alpha_3,\alpha_1) = 0. \tag{32}$$

To solve this functional equation, differentiate it with respect to α_1 and α_2:

$$k_{\alpha_1\alpha_2}(\alpha_1,\alpha_2) = 0,$$

which together with the skew–symmetry of k yields $k(\alpha_1,\alpha_2) = f(\alpha_2) - f(\alpha_1)$. A point transformation of the parameter $f(\alpha) \mapsto \alpha$ allows us to take simply $k(\alpha,\beta) = \beta - \alpha$. Thus, we arrive at the case (H1) of Theorem 1.

Finally, consider the formula ($\widehat{\text{H3}}_0$). Denote, for brevity,

$$\ell(\alpha,\beta) = \frac{1+k(\alpha,\beta)}{1-k(\alpha,\beta)}, \quad \text{so that} \quad \ell(\beta,\alpha) = 1/\ell(\alpha,\beta).$$

We will write also ℓ_{ij} for $\ell(\alpha_i,\alpha_j)$. A straightforward inspection shows that the function ($\widehat{\text{H3}}_0$) gives a three-dimensionally consistent map with

$$x_{123} = \frac{(\ell_{21}-\ell_{12})x_1x_2 + (\ell_{32}-\ell_{23})x_2x_3 + (\ell_{13}-\ell_{31})x_3x_1}{(\ell_{23}-\ell_{32})x_1 + (\ell_{31}-\ell_{13})x_2 + (\ell_{12}-\ell_{21})x_3}, \tag{33}$$

if and only if

$$\ell(\alpha_1,\alpha_2)\ell(\alpha_2,\alpha_3)\ell(\alpha_3,\alpha_1) = 1. \tag{34}$$

Just as above, up to a point transformation of the parameter, the solution of this functional equation is given by $\ell(\alpha,\beta) = \alpha/\beta$, which leads to the equation $(\text{H3})_{\delta=0}$ of Theorem 1.

The proof of Theorem 1 is now complete. □

5. Three-Leg Forms

Theorem 1 classifies the three-dimensionally consistent equations under the tetrahedron condition. In Sect. 1 we have seen that the latter condition is necessary for the existence of a *three-leg form*

$$\psi(x, u; \alpha) - \psi(x, v; \beta) = \phi(x, y; \alpha, \beta) \tag{35}$$

of Eq. (1). Recall that the formula (35) implies the validity of equations on stars (11), or equations of the discrete Toda type, for the fields x at the vertices of the "black" sublattice; the same holds for the "white" sublattice. We prove now that a three–leg form indeed exists for all equations listed in Theorem 1. After that, in Sect. 6 we demonstrate some further applications of the three–leg forms, establishing the variational and symplectic structures for Eqs. (1) and (11).

The theorem below provides three-leg forms for all equations of the lists Q and H (the results for the list A follow from these ones). As it turns out, in almost all cases it is more convenient to write the three-leg equation (35) in the multiplicative form

$$\Psi(x, u; \alpha)/\Psi(x, v; \beta) = \Phi(x, y; \alpha, \beta). \tag{36}$$

For the list Q the functions Ψ and Φ corresponding to the "short" and to the "long" legs, respectively, essentially coincide. One has in these cases:

$$\Psi(x, u; \alpha) = F(X, U; A), \quad \Phi(x, y; \alpha, \beta) = F(X, Y; A - B), \tag{37}$$

where some suitable point transformations of the field variables and of the parameters are introduced: $x = f(X)$, $u = f(U)$, $y = f(Y)$, and $\alpha = \rho(A)$, $\beta = \rho(B)$.

Theorem 7. *The three–leg forms exist for all equations from Theorem 1. For the lists Q, H they are listed below.*

$(Q1)_{\delta=0}$: *An additive three–leg form with* $\phi(x, y; \alpha, \beta) = \psi(x, y; \alpha - \beta)$,

$$\psi(x, u; \alpha) = \frac{\alpha}{x - u}. \tag{38}$$

$(Q1)_{\delta=1}$: *A multiplicative three-leg form with* $\Phi(x, y; \alpha, \beta) = \Psi(x, y; \alpha - \beta)$,

$$\Psi(x, u; \alpha) = \frac{x - u + \alpha}{x - u - \alpha}. \tag{39}$$

(Q2): *A multiplicative three-leg form with* $\Phi(x, y; \alpha, \beta) = \Psi(x, y; \alpha - \beta)$,

$$\Psi(x, u; \alpha) = \frac{(X + U + \alpha)(X - U + \alpha)}{(X + U - \alpha)(X - U - \alpha)}, \tag{40}$$

where $x = X^2$, $u = U^2$.

$(Q3)_{\delta=0}$: *A multiplicative three-leg form with* $\Phi(x, y; \alpha, \beta) = \Psi(x, y; \alpha/\beta)$,

$$\Psi(x, u; \alpha) = \frac{\alpha x - u}{x - \alpha u}. \tag{41}$$

Under the point transformations $x = \exp(2X)$, $\alpha = \exp(2A)$, *etc., there holds (37) with*

$$\Psi(x, u; \alpha) = F(X, U; A) = \frac{\sinh(X - U + A)}{\sinh(X - U - A)}. \tag{42}$$

$(\mathrm{Q3})_{\delta=1}$: *A multiplicative three-leg form with (37),*

$$\Psi(x,u;\alpha)=F(X,U;A)=\frac{\sinh(X+U+A)\sinh(X-U+A)}{\sinh(X+U-A)\sinh(X-U-A)},\tag{43}$$

where $x=\cosh(2X)$, $\alpha=\exp(2A)$, *etc.*

(Q4): *A multiplicative three-leg form with (37),*

$$\Psi(x,u;\alpha)=F(X,U;A)=\frac{\sigma(X+U+A)\sigma(X-U+A)}{\sigma(X+U-A)\sigma(X-U-A)},\tag{44}$$

where $x=\wp(X)$, $\alpha=\wp(A)$, *etc.*

(H1): *An additive three-leg form*

$$\psi(x,u;\alpha)=x+u,\quad \phi(x,y;\alpha,\beta)=\frac{\alpha-\beta}{x-y}.\tag{45}$$

(H2): *A multiplicative three–leg form*

$$\Psi(x,u;\alpha)=x+u+\alpha,\quad \Phi(x,y;\alpha,\beta)=\frac{x-y+\alpha-\beta}{x-y-\alpha+\beta}.\tag{46}$$

(H3): *A multiplicative three–leg form*

$$\Psi(x,u;\alpha)=xu+\delta\alpha\ =\ \exp(2X+2U)+\delta\alpha,\tag{47}$$

$$\Phi(x,y;\alpha,\beta)=\frac{\beta x-\alpha y}{\alpha x-\beta y}\ =\ \frac{\sinh(X-U-A+B)}{\sinh(X-U+A-B)},\tag{48}$$

where $x=\exp(2X)$, $\alpha=\exp(2A)$, *etc.*

Proof. We start with the list H, for which the situation is somewhat simpler. Finding the three-leg forms of Eqs. (H1) and $(\mathrm{H3})_{\delta=0}$ is almost immediate: these equations are equivalent to

$$u-v=\frac{\alpha-\beta}{x-y}\qquad\text{and}\qquad\frac{u}{v}=\frac{\beta x-\alpha y}{\alpha x-\beta y},\tag{49}$$

respectively. For other equations of the list H one uses the following simple formula which will also be quoted on several occasions later on.

Lemma 8. *The relation* $Q(x,u,v,y;\alpha,\beta)=0$ *yields*

$$\frac{h(x,u;\alpha)}{h(x,v;\beta)}=-\frac{Q_v}{Q_u}.\tag{50}$$

Proof.

$$-\frac{h(x,u;\alpha)}{h(x,v;\beta)}=\frac{g(x,u;\alpha,\beta)}{g(x,v;\beta,\alpha)}=\frac{QQ_{yv}-Q_yQ_v}{QQ_{yu}-Q_yQ_u}=\frac{Q_v}{Q_u}\bigg|_{Q=0}.\quad\square$$

This lemma immediately yields the three-leg forms for the cases when $Q_{uv} = 0$: then the right–hand side of (50) does not depend on u, v and can therefore be taken as $\Phi(x, y; \alpha, \beta)$, while $\Psi(x, u; \alpha) := h(x, u; \alpha)$. This covers the cases (H2) and (H3). Observe also that under the condition $Q_{uv} = 0$ we have $G(x, y; \alpha, \beta) = QQ_{uv} - Q_u Q_v = -Q_u Q_v$. Thus, in these cases the function Ψ associated to the "short" legs is nothing but the polynomial h, while the function Φ associated to the "long" legs, is just a ratio of two linear factors of the quadratic polynomial G.

The situation with the list Q is a bit more tricky. The inspection of formulas (q1), (q2), (q3), (q4) shows that in these cases $h(x, u; \alpha)$ is a quadratic polynomial in u, and, similarly, the polynomial $G(x, y; \alpha, \beta)$ is a quadratic polynomial in y. We try a linear–fractional ansatz for Ψ, Φ:

$$\Psi(x, u; \alpha) = \frac{p_+(x, u; \alpha)}{p_-(x, u; \alpha)}, \quad \Phi(x, y; \alpha, \beta) = \frac{s_+(x, y; \alpha, \beta)}{s_-(x, y; \alpha, \beta)}, \tag{51}$$

where the functions $p_\pm(x, u, \alpha)$ are linear in u, and the functions $s_\pm(x, y; \alpha, \beta)$ are linear in y, and

$$p_+(x, u; \alpha) p_-(x, u; \alpha) = h(x, u; \alpha),$$
$$s_+(x, y; \alpha, \beta) s_-(x, y; \alpha, \beta) = G(x, y; \alpha, \beta).$$

According to (21), the coefficients of polynomials $p_\pm(\cdot, u; \cdot)$ and $s_\pm(\cdot, y; \cdot, \cdot)$ are rational functions of x and $\sqrt{r(x)}$. This justifies "uniformizing" changes of variables, namely

$x = X^2$ in the case (Q2), when $r(x) = x$,
$x = \cosh(2X)$ in the case (Q3) with $\delta = 1$, when $r(x) = x^2 - 1$,
$x = \wp(X)$ in the case (Q4), when $r(x) = 4x^3 - g_2 x - g_3$.

The ansatz (51) turns out to work. For the cases (Q1)–(Q3) identify the functions $p_\pm$ from (51) with the numerators and denominators of the expressions listed in (39)–(43) (for the case $(Q3)_{\delta=0}$ take the fraction (41), i.e. the expression through x, u). In the case (Q4) the functions $p_\pm$ linear in u are obtained by dividing the numerators and denominators of the fractions in (44) by $\sigma^2(X)\sigma^2(U)$. Make similar identifications for the functions $s_\pm$ from (51). So, under the point transformations used in the formulation of Theorem 7 we have:

$$p_\pm(x, u; \alpha) = P(X, U; \pm A), \quad s_\pm(x, y; \alpha, \beta) = P(X, U; \pm A \mp B),$$

where

$$\begin{aligned}
(Q1)_{\delta=1}:&\ P(x, u; \alpha) = x - u + \alpha,\\
(Q2):&\ P(X, U; \alpha) = (X + U + \alpha)(X - U + \alpha),\\
(Q3)_{\delta=0}:&\ P(X, U; A) = \sinh(X - U + A),\\
(Q3)_{\delta=1}:&\ P(X, U; A) = \sinh(X + U + A)\sinh(X - U + A),\\
(Q4):&\ P(X, U; A) = \frac{\sigma(X + U + A)\sigma(X - U + A)}{\sigma^2(X)\sigma^2(U)}.
\end{aligned}$$

A straightforward computation shows that

$$\begin{aligned}
&p_+(x, u; \alpha) p_-(x, v; \beta) s_-(x, y; \alpha, \beta) - p_-(x, u; \alpha) p_+(x, v; \beta) s_+(x, y; \alpha, \beta)\\
&\quad = \rho(x; \alpha, \beta) Q(x, u, v, y; \alpha, \beta),
\end{aligned} \tag{52}$$

with some factor ρ depending only on x. (Obviously, the left–hand side of the latter equation is linear in u, v, y.) Concretely, we find: $\rho = 2$ in the case $(\mathrm{Q1})_{\delta=1}$; $\rho = 4X$ in the case (Q2); $\rho = x$ in the case $(\mathrm{Q3})_{\delta=0}$; $\rho = 2\alpha\beta\sinh(2X)$ in the case $(\mathrm{Q3})_{\delta=1}$; finally,

$$\rho = \sigma^4(A)\sigma^4(B)\frac{\sigma(B-A)}{\sigma(B+A)} \cdot \frac{\sigma(2X)}{\sigma^4(X)}$$

in the case (Q4). So, in all these cases $Q = 0$ is equivalent to vanishing of the left–hand side of (52), which, in turn, is equivalent to the multiplicative three–leg formula (36) with the ansatz (51). Finally, we comment on the origin of the additive three–leg form of $(\mathrm{Q1})_{\delta=0}$. Rescaling in (39) the parameters as $\alpha \mapsto \delta\alpha$, we come to the three–leg form of Eq. (Q1) with a general $\delta \neq 0$. Sending $\delta \to 0$, we find: $\Psi = 1 + 2\delta\psi + O(\delta^2)$, $\Phi = 1 + 2\delta\phi + O(\delta^2)$, with the functions ψ, ϕ from (38), and thus we arrive at the additive formula for the case $\delta = 0$. □

Remark. The notion of the three–leg equation was formulated in [8]. The three–leg forms for Eqs. (Q1), $(\mathrm{Q3})_{\delta=0}$, (H1), $(\mathrm{H3})_{\delta=0}$ were also found there. The results for (Q2), $(\mathrm{Q3})_{\delta=1}$, (Q4), (H2), $(\mathrm{H3})_{\delta=1}$ are given here for the first time.

6. Lagrangian Structures

Recall that the three–leg form of Eq. (1) imply that the discrete Toda type equation (11) holds on the black and on the white sublattices. (Of course, for multiplicative equations (36) we set $\psi(x, u; \alpha) = \log \Psi(x, u; \alpha)$, $\phi(x, y; \alpha, \beta) = \log \Phi(x, y; \alpha, \beta)$.) We show now that all these Toda systems may be given a variational (Lagrangian) interpretation. Notice that only the functions $\phi(x, y; \alpha, \beta)$ related to the "long" legs enter the Toda equations, and that the Toda systems for the cases (H1)–(H3) are the same as in the cases (Q1), $(\mathrm{Q3})_{\delta=0}$.

Consider the point changes of variables $x = f(X)$ listed in Theorem 7; in the cases (Q1), (H1), (H2), when no such substitution is listed, set just $x = X$. For the sake of notational simplicity, we write in the present section $\psi(x, u; \alpha)$ for $\psi(f(X), f(U); \alpha)$, etc.

Lemma 9. *For all equations from Theorem 1 there exist symmetric functions $L(X, U; \alpha) = L(U, X; \alpha)$ and $\Lambda(X, Y; \alpha, \beta) = \Lambda(Y, X; \alpha, \beta)$ such that*

$$\psi(x, u; \alpha) = \psi(f(X), f(U); \alpha) = \frac{\partial}{\partial X} L(X, U; \alpha), \tag{53}$$

$$\phi(x, y; \alpha, \beta) = \phi(f(X), f(Y); \alpha, \beta) = \frac{\partial}{\partial X} \Lambda(X, Y; \alpha, \beta). \tag{54}$$

Proof. It is sufficient to notice that the functions $(\partial/\partial U)\psi(x, u; \alpha)$ are symmetric with respect to the permutation $X \leftrightarrow U$, and similarly for ϕ. □

This observation has the following immediate corollary.

Proposition 10. *Let ★ be the "black" subgraph, and let E(★) be the set of its (non-oriented) edges. Let the pairs of labels (α, β) be assigned to the edges from E(★) according to Fig. 5, so that, e.g., the pair (α_1, α_2) corresponds to the edge (x, x_{12}). Then for all*

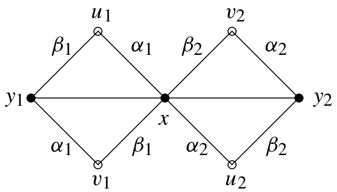

Fig. 6. Two elementary quadrilaterals of the square lattice

equations from Theorem 1 the discrete Toda type equations (11) are the Euler–Lagrange equations for the action functional

$$S = \sum_{(X,Y)\in E(\bigstar)} \Lambda(X, Y; \alpha, \beta). \tag{55}$$

This result could be anticipated, since it is quite natural to expect from a system consisting of equations on stars to have a variational origin. Our next result is, on the contrary, somewhat unexpected, since it gives a sort of a variational interpretation for the original system consisting of the equations on quadrilaterals (1). This is possible not on arbitrary quad–graphs but only on special regular lattices. We restrict ourselves here to the case of the standard square lattice. For the cases (H1), $(H3)_{\delta=0}$ (the discrete KdV equation and the Hirota equation) our results coincide with those found in [10], for all other equations from Theorem 1 they seem to be new.

For the sake of convenience orient the square lattice as in Fig. 6. Denote by E_1 (E_2) the subset of edges running from south–east to north–west (resp. from south–west to north–east). Add to the edges of the original quad–graph the set E_3 of the horizontal diagonals of all elementary quadrilaterals. Let the labels α be assigned to the edges from E_1, and let the labels β be assigned to the edges from E_2. It is natural to assign to the diagonals from E_3 the pairs (α, β) from the sides of the correspondent quadrilateral.

Proposition 11. *Solutions of all equations from Theorem 1 are critical for the following action functional:*

$$\mathbf{S} = \sum_{(X,U)\in E_1} L(X, U; \alpha) - \sum_{(X,V)\in E_2} L(X, V; \beta) - \sum_{(X,Y)\in E_3} \Lambda(X, Y; \alpha, \beta). \tag{56}$$

Proof. For any vertex x, the correspondent Euler–Lagrange equation relates the vertices of two elementary quadrilaterals, as in Fig. 6. Due to Lemma 9, this equation reads:

$$\begin{aligned} &\psi(x, u_1; \alpha_1) - \psi(x, v_1; \beta_1) - \phi(x, y_1; \alpha_1, \beta_1) + \\ &\quad \psi(x, u_2; \alpha_2) - \psi(x, v_2; \beta_2) - \phi(x, y_2; \alpha_2, \beta_2) = 0. \end{aligned}$$

This holds, since (35) is fulfilled on both elementary quadrilaterals. □

The variational interpretation allows one to find invariant symplectic structures for reasonably posed Cauchy problems for Eqs. (1). As is well–known, one way to set the Cauchy problem is to prescribe the values x on the zigzag line like in Fig. 7 and to impose periodicity in the horizontal direction (so that one is dealing with the square lattice on a cylinder) [10, 13]. Then Eq. (1) defines the evolution in the vertical direction, i.e. the map $\{x_i\}_{i\in\mathbb{Z}/2N\mathbb{Z}} \mapsto \{\widetilde{x}_i\}_{i\in\mathbb{Z}/2N\mathbb{Z}}$.

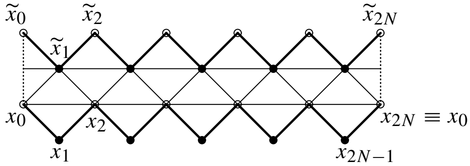

Fig. 7. The Cauchy problem on a zigzag

Proposition 12. *Let the edges* (x_i, x_{i+1}) *carry the labels* α_i*, and let the edges* $(\widetilde{x}_i, \widetilde{x}_{i+1})$ *carry the labels* $\widetilde{\alpha}_i$*, so that* $\widetilde{\alpha}_{2k} = \alpha_{2k-2}$*, and* $\widetilde{\alpha}_{2k-1} = \alpha_{2k+1}$*. Denote*

$$s(X, U; \alpha) = \frac{\partial}{\partial U}\psi(x, u; \alpha) = \frac{\partial^2}{\partial X \partial U} L(X, U; \alpha), \tag{57}$$

so that $s(X, U; \alpha) = s(U, X; \alpha)$*. Then the following relation holds for the map* $\{x_i\} \mapsto \{\widetilde{x}_i\}$*:*

$$\sum_{i \in \mathbb{Z}/2N\mathbb{Z}} s(X_i, X_{i+1}; \alpha_i) dX_i \wedge dX_{i+1} = \sum_{i \in \mathbb{Z}/2N\mathbb{Z}} s(\widetilde{X}_i, \widetilde{X}_{i+1}; \widetilde{\alpha}_i) d\widetilde{X}_i \wedge d\widetilde{X}_{i+1} \,. \tag{58}$$

Proof. We use the argument methodologically close to [23]. Consider the action functional $\mathbb{S}$ defined by the same formula as (56) but with the summations restricted to the edges depicted in Fig. 7, and restricted to such fields X which satisfy Eq. (1) on each elementary quadrilateral. Consider the differential

$$d\mathbb{S} = \sum_{\text{all vertices } x} \frac{\partial \mathbb{S}}{\partial X} dX.$$

Due to Proposition 11, the expression $\partial \mathbb{S}/\partial X$ vanishes for all vertices where six edges meet, i.e. for all vertices except x_{2k+1} and $\widetilde{x}_{2k}$. So, we find:

$$\begin{aligned} d\mathbb{S} = \sum_{k \in \mathbb{Z}/N\mathbb{Z}} & \frac{\partial L(X_{2k}, X_{2k+1}; \alpha_{2k})}{\partial X_{2k+1}} dX_{2k+1} - \frac{\partial L(X_{2k+1}, X_{2k+2}; \alpha_{2k+1})}{\partial X_{2k+1}} dX_{2k+1} \\ & - \frac{\partial L(\widetilde{X}_{2k-1}, \widetilde{X}_{2k}; \widetilde{\alpha}_{2k-1})}{\partial \widetilde{X}_{2k}} d\widetilde{X}_{2k} + \frac{\partial L(\widetilde{X}_{2k}, \widetilde{X}_{2k+1}; \widetilde{\alpha}_{2k})}{\partial \widetilde{X}_{2k}} d\widetilde{X}_{2k}. \end{aligned}$$

Differentiating this 1–form and taking into account $d^2\mathbb{S} = 0$ yields (58). □

Remarks . 1) Notice that only the functions ψ related to the "short" legs are present in formula (58).

2) One is tempted to interpret formula (58) as symplecticity of the map $\{X_i\} \mapsto \{\widetilde{X}_i\}$. However, the 2–form (58) is degenerate. One finds a genuine symplectic form, if one considers a quasi-periodic initial value problem instead of a periodic one, and extends the phase space by the correspondent monodromies (cf. [13, 21] for the case $(\text{H3})_{\delta=0}$ (the Hirota equation)). Actually, the formula (58) means the invariance of a degenerate 2–form only when $\widetilde{\alpha}_i = \alpha_i$. This is easy to achieve by setting all $\alpha_i = \alpha$, or, more generally, all $\alpha_{2k} = \alpha$ and all $\alpha_{2k+1} = \beta$.

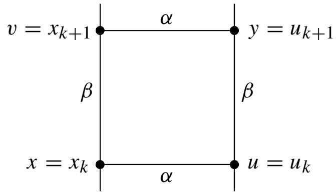

Fig. 8. To the construction of Bäcklund transformations

We close this section with a local form of Proposition 12, also based on the three–leg form of Eqs. (1). The particular case of the Hirota equation was given in [21].

Proposition 13. *Associate the two–form $\omega(e) = s(X, U; \alpha)dX \wedge dU$ to every (oriented) edge $e = (x, u)$ of the quad–graph. Then Eq. (1) with the three–leg form (35) implies that the sum of ω's along the boundary of any elementary quadrilateral, and therefore along any cycle homotopic to zero, vanishes.*

Proof. Differentiating (35) and wedging the result with dX, we have:

$$\frac{\partial}{\partial U}\psi(x, u; \alpha)dX \wedge dU + \frac{\partial}{\partial V}\psi(x, v; \beta)dV \wedge dX = \frac{\partial}{\partial Y}\phi(x, y; \alpha)dX \wedge dY.$$

On the other hand, starting with the three–leg equation centered at y (or, in other words, flipping $x \leftrightarrow y$, $u \leftrightarrow v$), we arrive at

$$\frac{\partial}{\partial V}\psi(y, v; \alpha)dY \wedge dV + \frac{\partial}{\partial U}\psi(y, u; \beta)dU \wedge dY = \frac{\partial}{\partial X}\phi(y, x; \alpha)dY \wedge dX.$$

Adding these two equations, and taking into account Lemma 9 and the notation (57), we come to the statement of the proposition. □

The statement of Proposition 12 is a particular case of Proposition 13, since the boundary of the domain in Fig. 7 (a cylindrical strip) is homotopic to zero.

7. Relation to Bäcklund Transformations

In this section we interpret the integrable quad-graph equations of Theorem 1 as non-linear superposition principles (NSP) of Bäcklund transformations for the KdV–type equations.

First of all, we show how Bäcklund transformations themselves can be derived from our equations. Towards this aim, consider Eq. (1) on one vertical strip of the standard square lattice:

$$S = \{0, 1\} \times \mathbb{Z}.$$

The fields on $S_0 = \{0\} \times \mathbb{Z}$ will be denoted by x_k, while the fields on $S_1 = \{1\} \times \mathbb{Z}$ will be denoted by u_k. So, a single square of the strip S looks as in Fig. 8. Suppose now that $x_k = x(k\epsilon)$, $k \in \mathbb{Z}$, where $x(\xi)$ is a smooth function, and similarly for u_k. In particular, this means that one has to set in Eq. (1) $v = x + \epsilon x_\xi + O(\epsilon^2)$ and $y = u + \epsilon u_\xi + O(\epsilon^2)$. If, in addition, the parameter β is chosen properly, then Eq. (1) approximates in the limit $\epsilon \to 0$ some differential equation which relates the functions $x(\xi)$ and $u(\xi)$. For the equations of the list Q the result is the most straightforward.

Proposition 14. *Set in Eq. (Q1) $\beta = \epsilon^2/2$, in Eq. (Q2) $\beta = \epsilon^2/4$, in Eq. (Q3) $\beta = 1 - \epsilon^2/2$, and in Eq. (Q4) $\beta = \wp(\epsilon^2)$. Then in the limit $\epsilon \to 0$ just described these equations tend to*

$$x_\xi u_\xi = h(x, u; \alpha), \tag{59}$$

with the correspondent polynomials h listed in the formulas (q1), (q2), (q3), (q4) of Proposition 6.

Proof. The statement is almost obvious in the cases (Q1), (Q2). To see that it holds for Eq. (Q3), the latter can be first rewritten as

$$\beta(\alpha^2 - 1)(x - v)(u - y) = \alpha(\beta^2 - 1)(xv + uy) + (1 - \beta)(\alpha^2 + \beta)(xy + uv) \\ +(\delta^2/4\alpha\beta)(\alpha^2 - \beta^2)(\alpha^2 - 1)(\beta^2 - 1).$$

If $\beta = 1 - \epsilon^2/2$, then the above equation approximates

$$(\alpha^2 - 1)x_\xi u_\xi = -\alpha(x^2 + u^2) + (\alpha^2 + 1)xu - (\delta^2/4\alpha)(\alpha^2 - 1)^2.$$

Finally, in the most intricate case (Q4) one starts by rewriting the equation as

$$\frac{\bar{a}_2 - a_2}{2a_0}(x - v)(u - y) = xuvy + \frac{a_1}{a_0}(xuv + uvy + vyx + yxu) \\ + \frac{\bar{a}_2 + a_2}{2a_0}(x + v)(u + y) \\ + \frac{\tilde{a}_2}{a_0}(xv + uy) + \frac{a_3}{a_0}(x + u + v + y) + \frac{a_4}{a_0}.$$

From expressions for the coefficients a_i given in Theorem 1, it is easy to see that if $\beta = \wp(\epsilon^2) \sim \epsilon^{-4}$, so that $b = \wp'(\epsilon^2) \sim -2\epsilon^{-6}$, then the left–hand side of the above equation tends to $a x_\xi u_\xi$, while the right–hand side tends to

$$x^2u^2 - 2\alpha xu(x + u) - \left(2\alpha^2 - \frac{g_2}{2}\right)xu + \alpha^2(x^2 + u^2) \\ + \left(g_3 + \frac{g_2}{2}\alpha\right)(x + u) + \left(\frac{g_2^2}{16} + g_3\alpha\right) = ah(x, u; \alpha).$$

This proves the proposition. □

Equation (59), read as a Riccati equation for u with the coefficients dependent on x, describes a transformation $x \mapsto u$, which turns out [2] to be a ***Bäcklund transformation*** for the Krichever–Novikov equation [20]:

$$x_t = x_{\xi\xi\xi} - \frac{3}{2x_\xi}(x_{\xi\xi}^2 - r(x)), \tag{60}$$

with the polynomial $r(x)$ being the discriminant of $h(x, u; \alpha)$. In other words, if x is a solution of (60), and u is related to x by (59), then u is also a solution of (60). It should be noticed that, in turn, the partial differential equation (60) may be derived from (59), either through a sort of continuous limit, or as a higher symmetry. In any way, it would be fair to say that the whole theory of the Eq. (60) and its Bäcklund transformations (59) is contained in the correspondent quad-graph equation (1) from the list Q. To complete the picture, we demonstrate that all Eqs. (1) listed in Theorem 1, in turn, can be interpreted as nonlinear superposition principles for the Bäcklund transformations. To this end consider the system of four differential equations of the type (59) corresponding to four sides of the quadrilateral on Fig. 1:

$$x_\xi u_\xi = h(x, u; \alpha), \qquad u_\xi y_\xi = h(u, y; \beta), \tag{61}$$

$$x_\xi v_\xi = h(x, v; \beta), \qquad v_\xi y_\xi = h(v, y; \alpha). \tag{62}$$

We will consider it for functions h corresponding to all cases listed in Theorem 1, except for those two when h actually does not depend on parameters, namely (H1) with $h(x, u) = 1$, and $(\mathrm{H3})_{\delta=0}$ with $h(x, u) = xu$. In all other cases the system (61), (62) makes perfect sense and its consistency is quite nontrivial.

Proposition 15. *The equation $Q(x, u, v, y; \alpha, \beta) = 0$ is a sufficient condition for the consistency of the system of differential equations (61), (62). Moreover, it is compatible with this system, i.e.*

$$Q_\xi|_{Q=0} = 0. \tag{63}$$

Proof. The consistency condition of the differential equations (61), (62) reads:

$$\mathcal{Q} := h(x, u; \alpha)h(v, y; \alpha) - h(x, v; \beta)h(u, y; \beta) = 0. \tag{64}$$

The left–hand side $\mathcal{Q}$ of this equation is a polynomial of degree 1 in each variable for the equations of the list H, and of degree 2 in each variable for the equations of the list Q. In the former case it is directly seen that Eq. (64) exactly coincides with $Q(x, u, v, y; \alpha, \beta) = 0$. In the latter case it can be shown that the polynomial Q divides the left–hand side of (64). More precisely, it is verified that, up to a constant factor, $\mathcal{Q} = QP$, where in the cases (Q1), (Q2) one has $P = Q|_{\beta\to-\beta}$, in the case (Q3) one has $P = Q|_{\beta\to 1/\beta}$, and in the case (Q4) one has $P = Q|_{(\beta,b)\to(\beta,-b)}$. This proves the first statement of the proposition. The second one reads

$$(Q_x x_\xi + Q_u u_\xi + Q_v v_\xi + Q_y y_\xi)|_{Q=0} = 0.$$

We will prove that actually both terms $Q_x x_\xi + Q_y y_\xi$ and $Q_u u_\xi + Q_v v_\xi$ vanish separately. For example,

$$(Q_u u_\xi + Q_v v_\xi)|_{Q=0} = \frac{1}{x_\xi}\left(Q_u h(x, u; \alpha) + Q_v h(x, v; \beta)\right)\Big|_{Q=0} = 0.$$

The last step follows from Lemma 8. □

In the two cases when the system (61), (62) becomes trivial, there still exist Bäcklund transformations of a different kind, such that the equation $Q = 0$ serves as their NSP. Namely, in the cases (H1), $(\mathrm{H3})_{\delta=0}$ the following equations come to replace (59):

$$x_\xi + u_\xi = (x - u)^2 + \alpha, \tag{65}$$

$$\frac{x_\xi}{x} + \frac{u_\xi}{u} = \frac{\alpha}{2}\left(\frac{x}{u} + \frac{u}{x}\right), \tag{66}$$

respectively. The second of these equations is probably better known in the coordinates $x = \exp(X)$, $u = \exp(U)$:

$$X_\xi + U_\xi = \alpha\cosh(X - U). \tag{67}$$

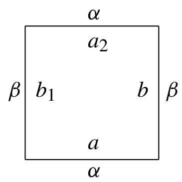

Fig. 9. An elementary quadrilateral; both fields and labels are assigned to edges

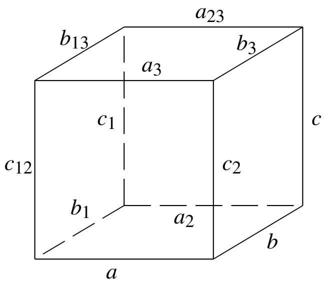

Fig. 10. Three–dimensional consistency; fields assigned to edges

Equation (65) defines a Bäcklund transformation for the potential KdV equation

$$x_t = x_{\xi\xi\xi} - 6x_\xi^2. \tag{68}$$

Similarly, Eq. (67) defines a Bäcklund transformation for the potential MKdV equation

$$X_t = X_{\xi\xi\xi} - 2X_\xi^3, \tag{69}$$

or, alternatively, for the sinh–Gordon equation which belongs to the same hierarchy.

8. Conclusions and Perspectives

Three–dimensional consistency as integrability criterium; Yang–Baxter maps. One of the traditional but rather *ad hoc* definitions of the integrability of two–dimensional systems is based on the notion of the zero–curvature representation. For a system on a quad–graph consisting of Eqs. (1) with fields associated to the *vertices* of the elementary quadrilateral on Fig. 1, the zero curvature representation is usually encoded in the formula like

$$L(y, u, \beta; \lambda)L(u, x, \alpha; \lambda) = L(y, v, \alpha; \lambda)L(v, x, \beta; \lambda),$$

where λ is the spectral parameter, so that the matrices L take values in some loop group. In [8] and independently in [26] it was demonstrated how to derive the zero curvature representation from the three–dimensional consistency. It should be mentioned, however, that to assign fields to the vertices is not the only possibility. Another large class of two–dimensional systems on quad–graphs build those with the fields assigned to the *edges*, see Fig. 9. In this situation it is natural to assume that each elementary quadrilateral carries a map $R : \mathcal{X}^2 \mapsto \mathcal{X}^2$, where $\mathcal{X}$ is the space where the fields a, b take values, so that $(a_2, b_1) = R(a, b; \alpha, \beta)$. The question on the three–dimensional consistency of such maps is also legitimate and, moreover, began to be studied recently. The corresponding property can be encoded in the formula

$$R_{23} \circ R_{13} \circ R_{12} = R_{12} \circ R_{13} \circ R_{23}, \tag{70}$$

where each $R_{ij} : \mathcal{X}^3 \mapsto \mathcal{X}^3$ acts as the map R on the factors i, j of the cartesian product $\mathcal{X}^3$ and acts identically on the third factor. This equation should be understood as follows. The fields a, b are supposed to be attached to the edges parallel to the 1st and the 2^{nd} coordinate axes, respectively. Additionally, consider the fields c attached to the edges parallel to the 3^{rd} coordinate axis. Then the left–hand side of (70) corresponds to the chain of maps along the three rear faces of the cube on Fig. 10:

$$(a, b) \mapsto (a_2, b_1), \quad (a_2, c) \mapsto (a_{23}, c_1), \quad (b_1, c_1) \mapsto (b_{13}, c_{12}),$$

while its right–hand side corresponds to the chain of the maps along the three front faces of the cube:

$$(b, c) \mapsto (b_3, c_2), \quad (a, c_2) \mapsto (a_3, c_{12}), \quad (a_3, b_3) \mapsto (a_{23}, b_{13}).$$

So, Eq. (70) assures that two ways of obtaining (a_{23}, b_{13}, c_{12}) from the initial data (a, b, c) lead to the same results. The maps with this property were introduced by Drinfeld [12] under the name of "set–theoretical solutions of the Yang–Baxter equation", an alternative name is "Yang–Baxter maps" used by Veselov in the recent study [35], see also references therein. Under some circumstances, systems with the fields on vertices can be regarded as systems with the fields on edges or vice versa (this is the case, e.g., for the systems (Q1), $(\text{Q3})_{\delta=0}$, (H1), $(\text{H3})_{\delta=0}$ of our list, for which the variables X enter only in combinations like $a = X - U$ for edges (x, u)), but in general the two classes of systems should be considered as different. The notion of the zero curvature representation makes perfect sense for Yang–Baxter maps: such a map can be called integrable, if it is equivalent to

$$L(b, \beta; \lambda)L(a, \alpha; \lambda) = L(a_2, \alpha; \lambda)L(b_1, \beta; \lambda).$$

The problem of integrability of Yang–Baxter maps in the sense of existence of a zero–curvature representation is under current investigation [33]. Also the problem of classification of Yang–Baxter maps, like the one achieved in the present paper, is of great importance and interest.

A different direction for the development of the ideas of the present paper constitute *quantum systems*, or, more generally, *systems with non–commutative variables*. To remain in the frame of the present paper, these are systems (1), where the fields (x, u, v, y) take values in an arbitrary associative (not necessary commutative) algebra with a unit. It turns out that the notion of the three–dimensional consistency can be formulated also for such non–commutative systems. Also the derivation of the zero curvature representation can be extended to the non–commutative framework [9].

It should be mentioned that in the area of the three–dimensional consistency of classical systems there also remains a number of interesting open problems. For instance, one of the assumptions under which the classification was carried out in the present paper, was less natural, namely the tetrahedron condition. As we pointed out in Sect. 3, there exist three–dimensionally consistent equations without the tetrahedron property, however all examples we are aware of are trivial (linear or linearizable):

$$Q(x, u, v, y) = x + y - u - v = 0 \quad \text{or} \quad Q(x, u, v, y) = xy - uv = 0,$$

or those obtained from these two by the action of a Möbius transformation on all variables. These examples do not contain parameters, and thus the three–dimensional consistency does not give a zero curvature representation with a spectral parameter for them (their integrability is anyway obvious). It would be interesting to find out whether there exist nontrivial examples violating the tetrahedron property.

There is also a vast field of *multi–field* integrable equations on quad–graphs. Existing examples indicate that their study is very promising.

Four–dimensional consistency of three–dimensional systems. Very promising is also the application of the consistency approach to the three–dimensional integrability. The role of an *ad hoc* definition of integrability, played in two dimensions by the zero curvature representation, now goes to the so called *local Yang–Baxter equation*, introduced by Maillet and Nijhoff [22]. The role of the transition matrices from the zero curvature representation is played in this novel structure by certain tensors attached to the elementary two–dimensional plaquettes of the three–dimensional lattice. There exist a number of results on finding this sort of structure for some three–dimensional integrable systems [19, 16, 17]. It would be desirable to relate this *ad hoc* notion of integrability to some constructive one. In the spirit of the present paper, this constructive notion should be the *four–dimensional consistency.*

In the three–dimensional context there are *a priori* many kinds of systems, according to where the fields are defined: on the vertices, on the edges, or on the elementary squares of the cubic lattice. Consider first the situation when the fields are sitting on the elementary squares. Attach the fields a, b, c to the two–dimensional faces parallel to the coordinate planes 12, 23, 13, respectively, so that a, b, c are sitting on the bottom, the left and the front faces of a cube Fig. 3, and a_3, b_1, c_2 on the top, the right and the back faces. The system under consideration is a map $S : \mathcal{X}^3 \mapsto \mathcal{X}^3$ attached to the cube, so that $S(a, b, c) = (a_3, b_1, c_2)$. The condition of the four–dimensional consistency of such a map can be encoded in the formula

$$S_{134} \circ S_{234} \circ S_{124} \circ S_{123} = S_{123} \circ S_{124} \circ S_{234} \circ S_{134}. \tag{71}$$

This equation should be understood as follows. Additionally to the fields a, b, c, consider the fields d, e, f, attached to the two–dimensional faces parallel to the coordinate planes 24, 14, 34 of the four–dimensional hypercubic lattice. Each map of the type S_{ijk} in (71) is a map on $\mathcal{X}^6(a, b, c, d, e, f)$ acting as S on the factors of the cartesian product $\mathcal{X}^6$ corresponding to the variables sitting on the faces parallel to the planes ij, jk, ik, and acting trivially on the other three factors. Thus the left–hand side of (71) corresponds to the chain of maps

$$(a, b, c) \mapsto (a_3, b_1, c_2), \quad (a_3, d, e) \mapsto (a_{34}, d_1, e_2),$$
$$(b_1, d_1, f) \mapsto (b_{14}, d_{13}, f_2), \quad (c_2, e_2, f_2) \mapsto (c_{24}, e_{23}, f_{21}),$$

while the right–hand side of (71) corresponds to the chain of maps

$$(c, e, f) \mapsto (c_4, e_3, f_1), \quad (b, d, f_1) \mapsto (b_4, d_3, f_{12}),$$
$$(a, d_3, e_3) \mapsto (a_4, d_{13}, e_{23}), \quad (a_4, b_4, c_4) \mapsto (a_{34}, b_{14}, c_{24}).$$

Equation (71) expresses then the fact that two different ways of obtaining the data $(a_{34}, b_{14}, c_{24}, d_{13}, e_{23}, f_{12})$ from the initial data (a, b, c, d, e, f) lead to identical results. This equation is known in the literature as the *functional tetrahedron equation* [19, 17]. (Note that the standard notation used in the literature on the tetrahedron equation is different: the indices $1 \le \alpha, \beta, \gamma \le 6$ of $S_{\alpha\beta\gamma}$ numerate the two–dimensional coordinate planes.) The paper [17] contains also a list of solutions of this equation with $\mathcal{X} = \mathbb{C}$, possessing local Yang–Baxter representations with a certain ansatz for the participating tensors. One of the most remarkable examples is the *star–triangle map*:

$$a_3 = -\frac{a}{ab - bc - ca}, \quad b_1 = -\frac{b}{ab - bc - ca}, \quad c_2 = -\frac{c}{ab - bc - ca}. \tag{72}$$

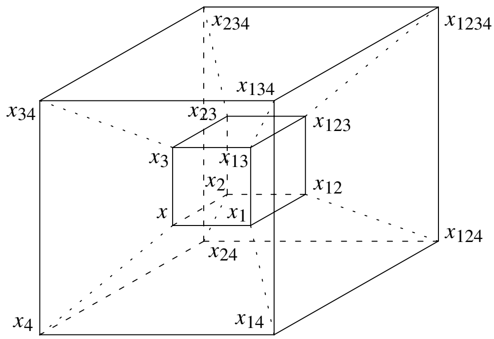

Fig. 11. Hypercube

(Usually this equation is written in a more symmetric form which is obtained by changing $c \mapsto -c$, with all plus signs in all denominators; however in that form the map does not satisfy Eq. (71).) See also [16] for an alternative local Yang–Baxter representation for this system. In [31, 32] an important solution of the functional tetrahedron equation with $\mathcal{X} = \mathbb{C}^2$ was introduced and studied in detail. There seems to be no method available for *deriving* the local Yang–Baxter representation for a given map satisfying the functional tetrahedron equation.

Further, consider three–dimensional systems with the fields sitting on the vertices. In this case each elementary cube carries just one equation

$$Q(x, x_1, x_2, x_3, x_{12}, x_{23}, x_{13}, x_{123}) = 0, \tag{73}$$

relating the fields in all its vertices. Such an equation should be solvable for any of its arguments in terms of the other seven ones. The four–dimensional consistency of such equations is defined as follows:

– Starting with the initial data x, x_i $(1 \le i \le 4)$, x_{ij} $(1 \le i < j \le 4)$, Eq. (73) allows us to uniquely determine all fields x_{ijk} $(1 \le i < j < k \le 4)$. Then we have *four* different possibilities to find x_{1234}, corresponding to four three–dimensional cubic faces adjacent to the vertex x_{1234} of the four–dimensional hypercube, see Fig. 8. All four values actually coincide.

So, one can consistently impose Eqs. (73) on all three–dimensional cubes of the lattice $\mathbb{Z}^4$. It is tempting to accept the four–dimensional consistency of equations of type (73) as the constructive definition of their integrability. It will be very important to solve the correspondent classification problem. We expect that this definition will allow one to derive the local Yang–Baxter representation, as a replacement for the zero curvature representation characteristic for the two–dimensional integrability.

We give here some examples. Consider the equation

$$\frac{(x_1 - x_3)(x_2 - x_{123})}{(x_3 - x_2)(x_{123} - x_1)} = \frac{(x - x_{13})(x_{12} - x_{23})}{(x_{13} - x_{12})(x_{23} - x)}. \tag{74}$$

This equation appeared for the first time in [29, 18], along with a geometric interpretation. It is not difficult to see that Eq. (74) admits a symmetry group D_8 of the cube. This equation can be uniquely solved for a field at an arbitrary vertex of a three–dimensional cube, provided the fields at the other seven vertices are known. The fundamental fact not mentioned in [18] is:

- Eq. (74) is four–dimensionally consistent in the above sense.

This is closely related (in fact, almost synonymous) to the functional tetrahedron equation for the star–triangle map (72), since the plaquette variables (a, b, c) of the latter can be "factorized" into combinations of vertex variables x of Eq. (74). More precisely, given a solution of (74) and setting

$$a = \frac{x_{12} - x}{x_1 - x_2}, \quad b = \frac{x_{23} - x}{x_2 - x_3}, \quad c = \frac{x_{13} - x}{x_1 - x_3}$$

we arrive at a solution of (72).

A different "factorization" of the plaquette variables into the vertex ones leads to another remarkable three–dimensional system known as the *discrete BKP equation* [25, 18]. For any solution $x : \mathbb{Z}^4 \mapsto \mathbb{C}$ of (74), define a function $\tau : \mathbb{Z}^4 \mapsto \mathbb{C}$ by the equations

$$\frac{\tau_i \tau_j}{\tau \tau_{ij}} = \frac{x_{ij} - x}{x_i - x_j}, \quad i < j. \tag{75}$$

Equation (74) assures that this can be done in an essentially unique way (up to initial data on coordinate axes whose influence is a trivial scaling of the solution). The function τ satisfies on any three–dimensional cube the discrete BKP equation:

$$\tau \tau_{ijk} - \tau_i \tau_{jk} + \tau_j \tau_{ik} - \tau_k \tau_{ij} = 0, \quad i < j < k. \tag{76}$$

The following holds:

- Equation (76) is four–dimensionally consistent. Moreover, for the value τ_{1234} one finds a remarkable equation:

$$\tau \tau_{1234} - \tau_{12} \tau_{34} + \tau_{13} \tau_{24} - \tau_{23} \tau_{34} = 0, \tag{77}$$

  which essentially reproduces the discrete BKP equation. So, τ_{1234} does not actually depend on the values τ_i, $1 \le i \le 4$. This can be considered as an analog of the tetrahedron property of Sect. 2.

Notice that usually the discrete BKP equation (76) is written in a slightly different and more symmetric form, with all plus signs on the left–hand side. On every three–dimensional subspace these two forms are easily transformed into one another. However, this cannot be done on the whole of $\mathbb{Z}^4$. Equation (76) with all plus signs on the left–hand side *does not* possess the property of the four–dimensional consistency.

Further, we mention systems of the geometrical origin (discrete analogs of conjugate and orthogonal coordinate systems) [11, 6], which also have the property of four–dimensional consistency. We plan to address various aspects of four–dimensional consistency in our future publications.

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