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Discrete z^γ and Painlevé Equations

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1 Introduction

Circle patterns as discrete analogs of conformal mappings is a fast-developing field of research on the border of analysis and geometry. Recent progress in their investigation was initiated by Thurston's idea (see [18]) about approximating the Riemann mapping by circle packings. The corresponding convergence was proven by Rodin and Sullivan in [15]. For hexagonal packings, it was established by He and Schramm in [9] that the convergence is C^∞. Classical circle packings comprised by disjoint open disks were later generalized to circle patterns, where the disks may overlap (see, for example, [8]). In [16], Schramm introduced and investigated circle patterns with the combinatorics of the square grid and orthogonal neighboring circles. In particular, a maximum principle for these patterns was established, which allowed global results to be proven.

On the other hand, not very much is known about analogs of standard holomorphic functions. Doyle constructed a discrete analogue of the exponential map with the hexagonal combinatorics in [5], and the discrete versions of exponential and erf-function, with underlying combinatorics of the square grid, were found in [16]. The discrete logarithm and z^2 have been conjectured by Schramm and Kenyon (see [17]).

In a conformal setting, Schramm's circle patterns are governed by a difference equation that turns out to be the stationary Hirota equation (see [16], [3]). This equation is an example of an integrable difference equation. It first appeared in a different branch of mathematics—the theory of integrable systems (see [19] for a survey). Moreover, it is easy to show that the lattice comprised by the centers of the circles of a Schramm's pattern and by their intersection points is a special discrete conformal mapping (see

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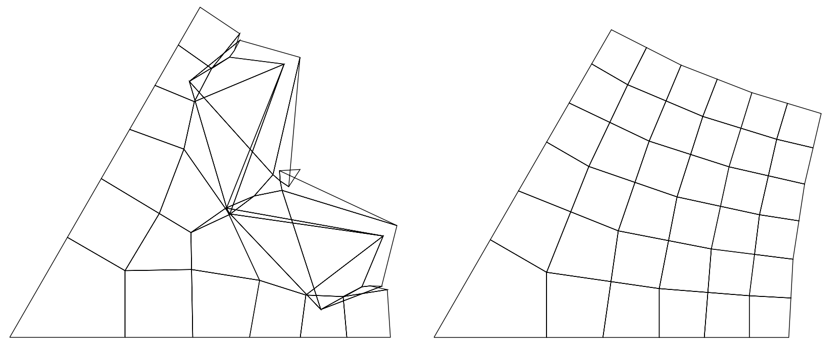

Figure 1 Two discrete conformal maps with close initial data $n = 0,\ m = 0$. The second lattice describes a discrete version of the holomorphic mapping $z^{2/3}$.

Definition 1 below). The latter were introduced in [2] in the setting of discrete integrable geometry, originally without any relation to circle patterns.

The present paper is devoted to the discrete analogue of the function $f(z) = z^\gamma$, first suggested in [1]. We show that the corresponding Schramm's circle patterns can be naturally described by methods developed in the theory of integrable systems. Let us recall the definition of a discrete conformal map from [2].

Definition 1. $f : \mathbf{Z}^2 \to \mathbf{R}^2 = \mathbf{C}$ is called a *discrete conformal map* if all its elementary quadrilaterals are conformal squares; i.e., their cross-ratios are equal to -1:

$$q(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1}) := \frac{(f_{n,m} - f_{n+1,m})(f_{n+1,m+1} - f_{n,m+1})}{(f_{n+1,m} - f_{n+1,m+1})(f_{n,m+1} - f_{n,m})} = -1. \tag{1}$$

This definition is motivated by the following properties: (1) It is Möbius invariant, and (2) a smooth map $f : D \subset \mathbf{C} \to \mathbf{C}$ is conformal (holomorphic or antiholomorphic) if and only if $\forall\,(x, y) \in D$,

$$\lim_{\epsilon\to 0} q\big(f(x,y), f(x+\epsilon, y) f(x+\epsilon, y+\epsilon) f(x, y+\epsilon)\big) = -1.$$

For some examples of discrete conformal maps and for their applications in differential geometry of surfaces, see [2] and [10].

A naive method to construct a discrete analogue of the function $f(z) = z^\gamma$ is to start with $f_{n,0} = n^\gamma,\ n \geq 0$, $f_{0,m} = (im)^\gamma,\ m \geq 0$, then compute $f_{n,m}$ for any $n, m > 0$ using equation (1). But a map that has been determined to be so has a behavior that is far from that of the usual holomorphic maps. Different elementary quadrilaterals overlap (see the left lattice in Figure 1).

Definition 2. A discrete conformal map $f_{n,m}$ is called an *immersion* if the interiors of adjacent elementary quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ are disjoint.

To construct an immersed discrete analogue of z^γ, which is the right lattice presented in Figure 1, a more complicated approach is needed. Equation (1) can be supplemented with the nonautonomous constraint

$$\begin{aligned}\gamma f_{n,m} = 2n\frac{(f_{n+1,m}-f_{n,m})(f_{n,m}-f_{n-1,m})}{(f_{n+1,m}-f_{n-1,m})} \\ + 2m\frac{(f_{n,m+1}-f_{n,m})(f_{n,m}-f_{n,m-1})}{(f_{n,m+1}-f_{n,m-1})},\end{aligned} \tag{2}$$

which plays a crucial role in this paper. This constraint, as well as its compatibility with (1), is derived from some monodromy problem (see Section 2). Let us assume $0 < \gamma < 2$ and denote $\mathbf{Z}^2_+ = \{(n,m) \in \mathbf{Z}^2 : n, m \geq 0\}$. Motivated by the asymptotics of the constraint (2) as $n, m \to \infty$, and by the properties

$$z^\gamma(\mathbf{R}_+) = \mathbf{R}_+, \qquad z^\gamma(i\mathbf{R}_+) = e^{\gamma\pi i/2}\mathbf{R}_+$$

of the holomorphic mapping z^γ, we use the following definition (see [1], [3]) of the *discrete* z^γ.

Definition 3. The discrete conformal map $Z^\gamma : \mathbf{Z}^2_+ \to \mathbf{C},\ 0 < \gamma < 2$ is the solution of (1) and (2) with the initial conditions

$$Z^\gamma(0,0) = 0, \qquad Z^\gamma(1,0) = 1, \qquad Z^\gamma(0,1) = e^{\gamma\pi i/2}. \tag{3}$$

Obviously, $Z^\gamma(n,0) \in \mathbf{R}_+$ and $Z^\gamma(0,m) \in e^{\gamma\pi i/2}(\mathbf{R}_+)$ for any $n, m \in \mathbf{N}$.

Figure 2 suggests that Z^γ is an immersion. The corresponding theorem is the main result of this paper.

Theorem 1. The discrete map Z^γ for $0 < \gamma < 2$ is an immersion. □

The proof is based on analysis of geometric and algebraic properties of the corresponding lattices. In Section 3, we show that Z^γ corresponds to a circle pattern of Schramm's type. (The circle pattern corresponding to $Z^{2/3}$ is presented in Figure 1.) Next, analyzing the equations for the radii of the circles, we show that in order to prove that Z^γ is an immersion, it is enough to establish a special property of a separatrix solution of the following ordinary difference equation of Painlevé type:

$$(n+1)\left(x_n^2-1\right)\left(\frac{x_{n+1}-ix_n}{i+x_nx_{n+1}}\right) - n\left(x_n^2+1\right)\left(\frac{x_{n-1}+ix_n}{i+x_{n-1}x_n}\right)\gamma x_n.$$

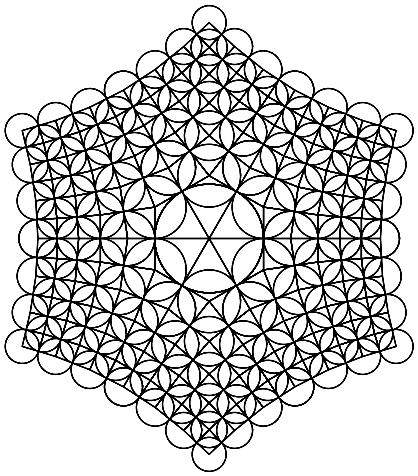

Figure 2 Schramm's circle pattern corresponding to $Z^{2/3}$.

Namely, in Section 4, it is shown that Z^γ is an immersion if and only if the unitary solution $x_n = e^{i\alpha_n}$ of this equation with $x_0 = e^{i\gamma\pi/4}$ lies in the sector $0 < \alpha_n < \pi/2$. Similar problems have been studied in the setting of the isomonodromic deformation method (see [11], [5]). In particular, connection formulas were derived. These formulas describe the asymptotics of solutions x_n for $n \to \infty$ as a function of x_0 (see, in particular, [7]). These methods seem to be insufficient for our purposes since we need to control x_n for finite n's as well. The geometric origin of this equation permits us to prove the property of the solution x_n mentioned above by purely geometric methods. Based on results established for Z^γ, we show in Section 5 how to obtain discrete immersed analogs of z^2 and $\log z$ as limiting cases of Z^γ with $\gamma \to 2$ and $\gamma \to 0$, respectively. Finally, discrete analogs of Z^γ for $\gamma > 2$ are discussed in Section 6.

2 Discrete Z^γ via a monodromy problem

Equation (1) is the compatibility condition of the Lax pair

$$\Psi_{n+1,m} = U_{n,m}\Psi_{n,m}, \qquad \Psi_{n,m+1} = V_{n,m}\Psi_{n,m} \tag{4}$$

found by Nijhoff and Capel in [13]:

$$U_{n,m} = \begin{pmatrix} 1 & -u_{n,m} \\ \lambda/u_{n,m} & 1 \end{pmatrix} \quad V_{n,m} = \begin{pmatrix} 1 & -v_{n,m} \\ -\lambda/v_{n,m} & 1 \end{pmatrix}, \tag{5}$$

where

$$u_{n,m} = f_{n+1,m} - f_{n,m}, \qquad v_{n,m} = f_{n,m+1} - f_{n,m}.$$

Whereas equation (1) is invariant with respect to fractional linear transformations $f_{n,m} \to (pf_{n,m} + q)/(rf_{n,m} + s)$, the constraint (2) is not. By applying a fractional linear transformation and shifts of n and m, (2) is generalized to the following form:

$$\begin{aligned}\beta f_{n,m}^2 + \gamma f_{n,m} + \delta = 2(n-\phi)&\frac{(f_{n+1,m} - f_{n,m})(f_{n,m} - f_{n-1,m})}{(f_{n+1,m} - f_{n-1,m})}\\ + 2(m-\psi)&\frac{(f_{n,m+1} - f_{n,m})(f_{n,m} - f_{n,m-1})}{(f_{n,m+1} - f_{n,m-1})},\end{aligned} \tag{6}$$

where $\beta, \gamma, \delta, \phi, \psi$ are arbitrary constants.

Theorem 2. $f : \mathbf{Z}^2 \to \mathbf{C}$ is a solution to the system $(1, 6)$ if and only if there exists a solution $\Psi_{n,m}$ to $(4, 5)$ satisfying the following differential equation in λ:

$$\frac{d}{d\lambda}\Psi_{n,m} = A_{n,m}\Psi_{n,m}, \qquad A_{n,m} = -\frac{B_{n,m}}{1+\lambda} + \frac{C_{n,m}}{1-\lambda} + \frac{D_{n,m}}{\lambda}, \tag{7}$$

with λ-independent matrices $B_{n,m}, C_{n,m}, D_{n,m}$. The matrices $B_{n,m}, C_{n,m}, D_{n,m}$ in (7) are of the following structure:

$$\begin{aligned}B_{n,m} &= -\frac{n-\phi}{u_{n,m} + u_{n-1,m}}\begin{pmatrix} u_{n,m} & u_{n,m}u_{n-1,m} \\ 1 & u_{n-1,m}\end{pmatrix} - \frac{\phi}{2}I,\\ C_{n,m} &= -\frac{m-\psi}{v_{n,m} + v_{n,m-1}}\begin{pmatrix} v_{n,m} & v_{n,m}v_{n,m-1} \\ 1 & v_{n,m-1}\end{pmatrix} - \frac{\psi}{2}I,\\ D_{n,m} &= \begin{pmatrix} -(\gamma/4) - (\beta/2)f_{n,m} & -(\beta/2)f_{n,m}^2 - (\gamma/2)f_{n,m} - (\delta/2) \\ -\beta/2 & (\gamma/4) + (\beta/2)f_{n,m}\end{pmatrix}.\end{aligned}$$

The constraint (6) is compatible with (1). □

The proof of this theorem is straightforward but involves some computations. It is presented in Appendix A.

Note that the identity

$$\det \Psi_{n,m}(\lambda) = (1+\lambda)^n(1-\lambda)^m \det \Psi_{0,0}(\lambda)$$

for determinants implies

$$\operatorname{tr} A_{n,m}(\lambda) = \frac{n}{1+\lambda} - \frac{m}{1-\lambda} + a(\lambda), \tag{8}$$

where $a(\lambda)$ is independent of n and m. Thus, up to the term $D_{n,m}/\lambda$, equation (7) is the simplest one possible.

Further, we deal with the special case in (6) where $\beta = \delta = \phi = \psi = 0$, leading to the discrete Z^γ. The constraint (2) and the corresponding monodromy problem were obtained in [12] for the case $\gamma = 1$, and generalized to the case of arbitrary γ in [3].

3 Circle patterns and Z^γ

In this section, we show that Z^γ of Definition 3 is a special case of circle patterns with the combinatorics of the square grid as defined by Schramm in [16].

Lemma 1. A discrete $f_{n,m}$ satisfying (1) and (2) with initial data $f_{0,0} = 0$, $f_{1,0} = 1$, $f_{0,1} = e^{i\alpha}$ has the equidistant property

$$f_{2n,0} - f_{2n-1,0} = f_{2n+1,0} - f_{2n,0}, \qquad f_{0,2m} - f_{0,2m-1} = f_{0,2m+1} - f_{0,2m}$$

for any $n \geq 1$, $m \geq 1$. □

Proof. For $m = 0$ or $n = 0$, the constraint (2) is an ordinary difference equation of the second order. The lemma is proved by induction. ■

Given initial $f_{0,0}$, $f_{0,1}$, and $f_{1,0}$, the constraint (2) allows us to compute $f_{n,0}$ and $f_{0,m}$ for all $n, m \geq 1$. Now using equation (1), one can successively compute $f_{n,m}$ for any $n, m \in \mathbf{N}$. Observe that if $|f_{n+1,m} - f_{n,m}| = |f_{n,m+1} - f_{n,m}|$, then the quadrilateral $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ is also of the kite form—it is inscribed in a circle and is symmetric with respect to the diameter of the circle $[f_{n,m}, f_{n+1,m+1}]$. If the angle at the vertex $f_{n,m}$ is $\pi/2$, then the quadrilateral $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ is also of the kite form. In this case, the quadrilateral is symmetric with respect to its diagonal $[f_{n,m+1}, f_{n+1,m}]$.

Proposition 1. Let $f_{n,m}$ satisfy (1) and (2) in $\mathbf{Z}^2_+$ with initial data $f_{0,0} = 0$, $f_{1,0} = 1$, $f_{0,1} = e^{i\alpha}$. Then all the elementary quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ are of the kite form. All edges at the vertex $f_{n,m}$ with $n + m = 0 \pmod 2$ are of the same length,

$$|f_{n+1,m} - f_{n,m}| = |f_{n,m+1} - f_{n,m}| = |f_{n-1,m} - f_{n,m}| = |f_{n,m-1} - f_{n,m}|.$$

All angles between the neighboring edges at the vertex $f_{n,m}$ with $n + m = 1 \pmod 2$ are equal to $\pi/2$. □

Proof. The proof follows from Lemma 1 and from the above observation by induction. ■

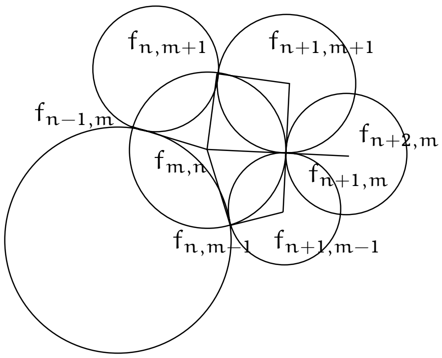

Figure 3 Discrete conformal maps of Schramm type; sublattices and kite-quadrilaterals, $n + m = 0 \pmod 2$.

Proposition 1 implies that for any n, m such that $n + m = 0 \pmod 2$, the points $f_{n+1,m}, f_{n,m+1}, f_{n-1,m}, f_{n,m-1}$ lie on a circle with the center $f_{n,m}$.

Corollary 1. The circumscribed circles of the quadrilaterals $(f_{n-1,m}, f_{n,m-1}, f_{n+1,m}, f_{n,m+1})$ with $n + m = 0 \pmod 2$ form a circle pattern of Schramm type (see [16]); i.e., the circles of neighboring quadrilaterals intersect orthogonally and the circles of half-neighboring quadrilaterals with common vertex are tangent (see Figure 3). □

Proof. Consider the sublattice $\{n, m : n + m = 0 \pmod 2\}$ and denote by $\mathbf{V}$ its quadrant

$$\mathbf{V} = \{z = N + iM : N, M \in \mathbf{Z},\ M \geq |N|\},$$

where

$$N = \frac{(n-m)}{2}, \qquad M = \frac{(n+m)}{2}.$$

We use complex labels $z = N + iM$ for this sublattice. Denote by $C(z)$ the circle of the radius

$$R(z) = |f_{n,m} - f_{n+1,m}| = |f_{n,m} - f_{n,m+1}| = |f_{n,m} - f_{n-1,m}| = |f_{n,m} - f_{n,m-1}|, \tag{9}$$

with the center at $f_{N+M,M-N} = f_{n,m}$. From Proposition 1, it follows that any two circles $C(z)$, $C(z')$ with $|z - z'| = 1$ intersect orthogonally, and any two circles $C(z)$, $C(z')$ with $|z - z'| = \sqrt{2}$ are tangent. Thus, the corollary is proved. ■

Let $\{C(z)\}$, $z \in \mathbf{V}$ be a circle pattern of Schramm type on the complex plane. Define $f_{n,m} : \mathbf{Z}^2_+ \to \mathbf{C}$ in the following manner.

(a) If $n+m=0 \pmod 2$, then $f_{n,m}$ is the center of

$$C\left(\frac{n-m}{2}+i\frac{n+m}{2}\right).$$

(b) If $n+m=1 \pmod 2$, then

$$\begin{aligned} f_{n,m} := C\left(\frac{n-m-1}{2}+i\frac{n+m-1}{2}\right) \cap C\left(\frac{n-m+1}{2}+i\frac{n+m+1}{2}\right) \\ = C\left(\frac{n-m+1}{2}+i\frac{n+m-1}{2}\right) \cap C\left(\frac{n-m-1}{2}+i\frac{n+m+1}{2}\right). \end{aligned}$$

Since all elementary quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ are of the kite form, equation (1) is satisfied automatically. In what follows, the function $f_{n,m}$, defined as above by (a) and (b), is called *a discrete conformal map corresponding to the circle pattern* $\{C(z)\}$.

Theorem 3. Let $f_{n,m}$, satisfying (1) and (2) with initial data $f_{0,0}=0$, $f_{0,1}=1$, $f_{0,1}=e^{i\alpha}$, be an immersion. Then $R(z)$ defined by (9) satisfies the following equations:

$$\begin{aligned} R(z)R(z+1)(-2M-\gamma) + R(z+1)R(z+1+i)\big(2(N+1)-\gamma\big) \\ + R(z+1+i)R(z+i)\big(2(M+1)-\gamma\big) + R(z+i)R(z)(-2N-\gamma) = 0 \end{aligned} \tag{10}$$

for[1] $z \in \mathbf{V}_l := \mathbf{V} \cup \{-N+i(N-1) \mid N \in \mathbf{N}\}$, and

$$\begin{aligned} (N+M)\big(R(z)^2 - R(z+1)R(z-i)\big)\big(R(z+i)+R(z+1)\big) \\ + (M-N)\big(R(z)^2 - R(z+i)R(z+1)\big)\big(R(z+1)+R(z-i)\big) = 0 \end{aligned} \tag{11}$$

for $z \in \mathbf{V}_{\text{int}} := \mathbf{V} \backslash \{\pm N + iN \mid N \in \mathbf{N}\}$.

Conversely, let $R(z) : \mathbf{V} \to \mathbf{R}_+$ satisfy (10) for $z \in \mathbf{V}_l$ and (11) for $z \in \mathbf{V}_{\text{int}}$. Then $R(z)$ defines an immersed circle packing with the combinatorics of the square grid. The corresponding discrete conformal map $f_{n,m}$ is an immersion and satisfies (2). □

Proof. Suppose that the discrete net determined by $f_{n,m}$ is immersed; i.e., the open discs of tangent circles do not intersect. Consider $n+m=1 \pmod 2$ and denote $f_{n+1,m} = f_{n,m} + r_1 e^{i\beta}$, $f_{n,m+1} = f_{n,m} + ir_2 e^{i\beta}$, $f_{n-1,m} = f_{n,m} - r_3 e^{i\beta}$, $f_{n,m-1} = f_{n,m} - ir_4 e^{i\beta}$, where $r_i > 0$ are the radii of the corresponding circles. The constraint (2) reads as follows:

$$\gamma f_{n,m} = e^{i\beta}\left(2n\frac{r_1 r_3}{r_1+r_3} + 2im\frac{r_2 r_4}{r_2+r_4}\right). \tag{12}$$

[1]Note that although $R(-N+i(N-1))$ are not defined, equation (10) also makes sense for $z=-N+i(N-1)$. At these points, it reads as (14).

The kite form of elementary quadrilaterals implies

$$f_{n+1,m+1} = f_{n+1,m} - e^{i\beta} r_1 \frac{(r_1 - ir_2)^2}{r_1^2 + r_2^2},$$
$$f_{n+1,m-1} = f_{n+1,m} - e^{i\beta} r_1 \frac{(r_1 + ir_4)^2}{r_1^2 + r_4^2}.$$

Computing $f_{n+2,m}$ from the constraint (12) at the point $(n+1, m)$ and inserting it into the identity $|f_{n+2,m} - f_{n+1,m}| = r_1$, after some transformations, one arrives at

$$\begin{aligned} r_1 r_2(n+m+1-\gamma) + r_2 r_3(-n+m+1-\gamma) \\ + r_3 r_4(-n-m+1-\gamma) + r_4 r_1(n-m+1-\gamma) = 0. \end{aligned} \tag{13}$$

This equation coincides with (10).

Now let $f_{n+2,m+1} = f_{n+1,m+1} + R_1 e^{i\beta'}$, $f_{n+1,m+2} = f_{n+1,m+1} + iR_2 e^{i\beta'}$, $f_{n,m+1} = f_{n+1,m+1} - R_3 e^{i\beta'}$, $f_{n+1,m} = f_{n+1,m+1} - iR_4 e^{i\beta'}$. Since all elementary quadrilaterals are of the kite form, we have

$$R_4 = r_1, \qquad R_3 = r_2, \qquad e^{i\beta'} = -ie^{i\beta} \frac{(r_2 + ir_1)^2}{r_1^2 + r_2^2}.$$

Substituting these expressions and (12) into the constraint (2) for $(n+1, m+1)$ and using (13), we arrive at

$$R_1 = \frac{(n+1)r_1^2(r_2 + r_4) + mr_2(r_1^2 - r_2 r_4)}{(n+1)r_2(r_2 + r_4) - m(r_1^2 - r_2 r_4)},$$
$$R_2 = \frac{(m+1)r_2^2(r_1 + r_3) + nr_1(r_2^2 - r_1 r_3)}{(m+1)r_1(r_1 + r_3) - n(r_2^2 - r_1 r_3)}.$$

These equations, together with $R_4 = r_1,\ R_3 = r_2$, describe the evolution $(n, m) \to (n+1, m+1)$ of the crosslike figure formed by $f_{n,m}, f_{n\pm1,m}, f_{n,m\pm1}$ with $n+m = 1 \pmod 2$. The equation for R_2 coincides with (11). We have considered internal points $z \in \mathbf{V}_{\text{int}}$; now we consider those that are not. Equation (10) at $z = N + iN$ and $z = -N + i(N-1),\ N \in \mathbf{N}$ reads as

$$R\big(\pm(N+1) + i(N+1)\big) = \frac{2N + \gamma}{2(N+1) - \gamma} R(\pm N + iN). \tag{14}$$

The converse claim of the theorem is based on the following lemma.

Lemma 2. Let $R(z) : \mathbf{V} \to \mathbf{R}_+$ satisfy (10) for $z \in \mathbf{V}_l$ and (11) for $z = iM,\ M \in \mathbf{N}$. Then $R(z)$ satisfies the following:

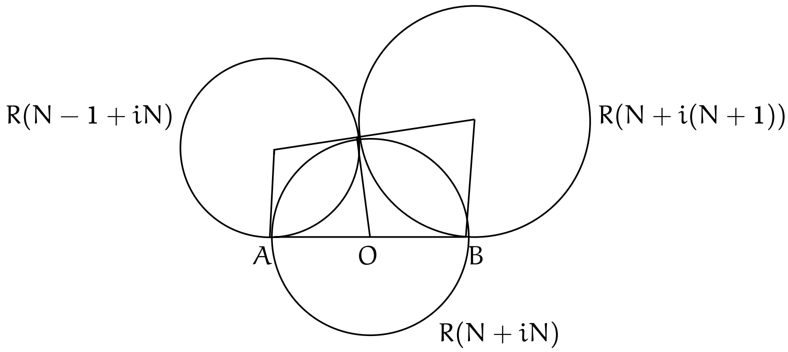

Figure 4 Straight line $f_{n,0}$.

(a) equation (11) for $z \in \mathbf{V}\backslash\{N+iN \mid N \in \mathbf{N}\}$;

(b) equation

$$\begin{aligned}(N{+}M)\big(R(z)^2 - R(z+i)R(z-1)\big)\big(R(z-1) + R(z-i)\big)\\ + (M-N)\big(R(z)^2 - R(z-1)R(z-i)\big)\big(R(z+i) + R(z-1)\big) = 0\end{aligned} \tag{15}$$

for $z \in \mathbf{V}\backslash\{-N+iN \mid N \in \mathbf{N}\}$;

(c) equation

$$R(z)^2 = \frac{\left(\frac{1}{R(z+1)} + \frac{1}{R(z+i)} + \frac{1}{R(z-1)} + \frac{1}{R(z-i)}\right)R(z{+}1)R(z{+}i)R(z{-}1)R(z{-}i)}{R(z{+}1) + R(z{+}i){+}R(z{-}1){+}R(z{-}i)} \tag{16}$$

for $z \in \mathbf{V}_{\text{int}}$. □

The proof of this lemma is technical and is presented in Appendix B.

Let $R(z)$ satisfy (10, 11); then item (c) of Lemma 2 implies that at $z \in \mathbf{V}_{\text{int}}$, equation (16) is fulfilled. In [16], it was proven that, given $R(z)$ satisfying (16), the circle pattern $\{C(z)\}$ with radii of the circles $R(z)$ is immersed. Thus, the discrete conformal map $f_{n,m}$ corresponding to $\{C(z)\}$ is an immersion. Item (b) of Lemma 2 implies that $R(z)$ satisfies (15) at $z = N + iN,\ N \in \mathbf{N}$, which reads

$$R(N-1+iN)R\big(N+i(N+1)\big) = R^2(N+iN). \tag{17}$$

This equation implies that the center O of $C(N+iN)$ and two intersection points A, B of $C(N+iN)$ with $C(N-1+iN)$ and $C(N+i(N+1))$ lie on a straight line (see Figure 4). Thus all the points $f_{n,0}$ lie on a straight line. Using equation (10) at $z = N+iN$, one gets by induction that $f_{n,m}$ satisfies (2) at $(n,0)$ for any $n \geq 0$. Similarly, item (a) of Lemma 2, equation (11) at $z = -N + iN,\ N \in \mathbf{N}$, and equation (10) at $z = -N + i(N-1),\ N \in \mathbf{N}$ imply that $f_{n,m}$ satisfies (2) at $(0,m)$. Now Theorem 2 implies that $f_{n,m}$ satisfies (2) in $\mathbf{Z}^2_+$, and Theorem 3 is proved. ∎

Remark. Equation (16) is a discrete analogue of the equation $\Delta \log(R) = 0$ in the smooth case. Similarly, equations (11) and (15) can be considered discrete analogs of the equation $xR_y - yR_x = 0$, and equation (10) is a discrete analogue of the equation $xR_x + yR_y = (\gamma - 1)R$.

From the initial condition (3), we have

$$R(0) = 1, \qquad R(i) = \tan\frac{\gamma\pi}{4}. \tag{18}$$

Theorem 3 allows us to reformulate the immersion property of the circle lattice completely in terms of the system (10, 11). Namely, to prove Theorem 1, one should show that the solution of the system (10, 11) with initial data (18) is positive for all $z \in \mathbf{V}$. Equation (14) implies

$$R(\pm N + iN) = \frac{\gamma(2+\gamma)\cdots(2(N-1)+\gamma)}{(2-\gamma)(4-\gamma)\cdots(2N-\gamma)}. \tag{19}$$

Proposition 2. Let the solution $R(z)$ of (11) and (10) in $\mathbf{V}$ with initial data

$$R(0) = 1, \qquad R(i) = \tan\frac{\gamma\pi}{4}$$

be positive on the imaginary axis; i.e., $R(iM) > 0$ for any $M \in \mathbf{Z}_+$. Then $R(z)$ is positive everywhere in $\mathbf{V}$. □

Proof. Since the system of equations for $R(z)$ defined in Theorem 3 has the symmetry $N \to -N$, it is sufficient to prove the proposition for $N \geq 0$. Equation (10) can be rewritten as

$$R(z+1+i) = \frac{R(z)R(z+1)(2M+\gamma) + R(z)R(z+i)(2N+\gamma)}{R(z+1)(2N+2-\gamma) + R(z+i)(2M+2-\gamma)}.$$

For $\gamma \leq 2$, $N \geq 0$, $M > 0$, and positive $R(z), R(z+1), R(z+i)$, we get $R(z+1+i) > 0$. Using $R(N+iN) > 0$ for all $N \in \mathbf{N}$, one obtains the conclusion by induction. ∎

4 Z^γ and a discrete Painlevé equation

Due to Proposition 2, the discrete Z^γ is an immersion if and only if $R(iM) > 0$ for all $M \in \mathbf{N}$. To prove the positivity of the radii on the imaginary axis, it is more convenient to use equation (2) for $n = m$.

Proposition 3. The map $f : \mathbf{Z}^2_+ \to \mathbf{C}$ satisfying (1) and (2) with initial data $f_{0,0} = 0$, $f_{1,0} = 1$, $f_{0,1} = e^{i\alpha}$ is an immersion if and only if the solution x_n of the equation

$$(n+1)(x_n^2 - 1)\left(\frac{x_{n+1} - ix_n}{i + x_n x_{n+1}}\right) - n(x_n^2 + 1)\left(\frac{x_{n-1} + ix_n}{i + x_{n-1}x_n}\right) = \gamma x_n, \tag{20}$$

with $x_0 = e^{i\alpha/2}$, is of the form $x_n = e^{i\alpha_n}$, where $\alpha_n \in (0, \pi/2)$. □

Proof. Let $f_{n,m}$ be an immersion. Define $R_n := R(in) > 0$, and define $\alpha_n \in (0, \pi/2)$ through $f_{n,n+1} - f_{n,n} = e^{2i\alpha_n}(f_{n+1,n} - f_{n,n})$. By symmetry, all the points $f_{n,n}$ lie on the diagonal $\arg f_{n,n} = \alpha/2$.

Taking into account that all elementary quadrilaterals are of the kite form, one obtains

$$f_{n+2,n+1} = e^{i\alpha/2}(g_{n+1} + R_{n+1}e^{-i\alpha_{n+1}}), \qquad f_{n+1,n+2} = e^{i\alpha/2}(g_{n+1} + R_{n+1}e^{i\alpha_{n+1}}),$$
$$f_{n+1,n} = e^{i\alpha/2}(g_{n+1} - iR_{n+1}e^{-i\alpha_n}), \qquad f_{n,n+1} = e^{i\alpha/2}(g_{n+1} + iR_{n+1}e^{i\alpha_n}),$$

and

$$R_{n+1} = R_n \tan \alpha_n, \tag{21}$$

where $g_{n+1} = |f_{n+1,n+1}|$ (see Figure 5). Now the constraint (2) for $(n+1, n+1)$ is equivalent to

$$\gamma g_{n+1} = 2(n+1)R_{n+1}\left(\frac{e^{i\alpha_n} + ie^{i\alpha_{n+1}}}{i + e^{i(\alpha_n + \alpha_{n+1})}}\right).$$

Similarly,

$$\gamma g_n = 2nR_n\left(\frac{e^{i\alpha_{n-1}} + ie^{i\alpha_n}}{i + e^{i(\alpha_{n-1} + \alpha_n)}}\right).$$

Putting these expressions into the equality

$$g_{n+1} = g_n + e^{-i\alpha_n}(R_n + iR_{n+1})$$

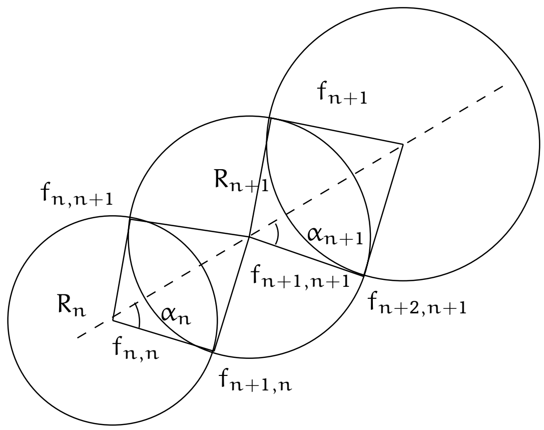

Figure 5 Diagonal circles.

and using (21), one obtains (20) with $x_n = e^{i\alpha_n}$. This proves the necessity part.

Now let us suppose that there is a solution $x_n = e^{i\alpha_n}$ of (20) with $\alpha_n \in (0, \pi/2)$. This solution determines a sequence of orthogonal circles along the diagonal $e^{i\alpha/2}\mathbf{R}_+$, and thus the points $f_{n,n}, f_{n\pm1,n}, f_{n,n\pm1}$ for $n \geq 1$. Now equation (1) determines $f_{n,m}$ in $\mathbf{Z}^2_+$. Since $\alpha_n \in (0, \pi/2)$, the inner parts of the quadrilaterals $(f_{n,n}, f_{n+1,n}, f_{n+1,n+1}, f_{n,n+1})$ on the diagonal, and of the quadrilaterals $(f_{n,n-1}, f_{n+1,n-1}, f_{n+1,n}, f_{n,n})$ are disjoint. That means that we have positive solution $R(z)$ of $(10, 11)$ for $z = iM,\ z = 1 + iM,\ N \in \mathbf{N}$. (See the proof of Theorem 3.) Given $R(iM)$, equation (10) determines $R(z)$ for all $z \in \mathbf{V}$. Due to Lemma 2, $R(z)$ satisfies $(10, 11)$. From Proposition 2, it follows that $R(z)$ is positive. Theorem 3 implies that the discrete conformal map $g_{n,m}$ corresponding to the circle pattern $\{C(z)\}$ determined by $R(z)$ is an immersion and satisfies (2). Since $g_{n,n} = f_{n,n}$ and $g_{n\pm1,n} = f_{n\pm1,n}$, equation (1) implies $f_{n,m} = g_{n,m}$. This proves the theorem. ∎

Remark. Note that although (20) is a difference equation of the second order, a solution x_n of (20) for $n \geq 0$ is determined by its value $x_0 = e^{i\alpha/2}$. From the equation for $n = 0$, one gets

$$x_1 = \frac{x_0(x_0^2 + \gamma - 1)}{i\big((\gamma - 1)x_0^2 + 1\big)}. \tag{22}$$

Remark. Equation (20) is a special case of an equation that has already appeared in the literature, although in a completely different context. Namely, it is related to the discrete Painlevé equation

$$\frac{2\zeta_{n+1}}{1 - X_{n+1}X_n} + \frac{2\zeta_n}{1 - X_nX_{n-1}} = \mu + \nu + \zeta_{n+1} + \zeta_n$$

$$+\frac{(\mu-\nu)(r^2-1)X_n+r(1-X_n^2)\left[\frac{1}{2}(\zeta_n+\zeta_{n+1})+(-1)^n(\zeta_n-\zeta_{n+1}-2m)\right]}{(r+X_n)(1+rX_n)},$$

which was considered in [14], and is called the *generalized* d-PII *equation*. The corresponding transformation[2] is

$$X=\frac{(1+i)(x-i)}{\sqrt{2}(x+1)}$$

with $\zeta_n=n$, $r=-\sqrt{2}$, $\mu=0$, $(\zeta_n-\zeta_{n+1}-2m)=0$, $\gamma=(2\nu-\zeta_n+\zeta_{n+1})$.

Equation (20) can be written in the following recurrent form:

$$x_{n+1}=\varphi(n,x_{n-1},x_n):=\\ -x_{n-1}\frac{nx_n^{-2}+i(\gamma-1)x_{n-1}^{-1}x_n^{-1}+(\gamma-1)+i(2n+1)x_{n-1}^{-1}x_n+(n+1)x_n^2}{nx_n^2-i(\gamma-1)x_{n-1}x_n+(\gamma-1)-i(2n+1)x_{n-1}x_n^{-1}+(n+1)x_n^{-2}}. \tag{23}$$

Obviously, this equation possesses unitary solutions.

Theorem 4. There exists a unitary solution x_n of equation (20) with $x_n\in A_I\backslash\{1,i\}\in S^1$, $\forall n\geq 0$, where

$$A_I:=\left\{e^{i\beta}\mid\beta\in\left[0,\frac{\pi}{2}\right]\right\}.\qquad\square$$

Proof. Let us study the properties of the function $\varphi(n,x,y)$ restricted to the torus $T^2=S^1\times S^1=\{(x,y):x,y\in\mathbf{C},\ |x|=|y|=1\}$. ■

1 The function $\varphi(n,x,y)$ is continuous on $A_I\times A_I$ $\forall n\geq 0$.

(Continuity on the boundary of $A_I\times A_I$ is understood to be one-sided.) The points of discontinuity must satisfy

$$n+1+(\gamma-1)y^2-i(2n+1)xy-i(\gamma-1)xy^3+ny^4=0.$$

The last identity never holds for unitary x,y with $n\in\mathbf{N}$ and $0<\gamma<2$. For $n=0$, the right-hand side of (22) is also continuous on A_I.

[2]We are thankful to A. Ramani and B. Grammaticos for this identification of the equations.

2 For $(x, y) \in A_I \times A_I$, we have $\varphi(n, x, y) \in A_I \cup A_{II} \cup A_{IV}$, where $A_{II} := \{e^{i\beta} \mid \beta \in (\pi/2, \pi]\}$ and $A_{IV} := \{e^{i\beta} \mid \beta \in [-\pi/2, 0)\}$.

To show this, it is convenient to use the substitution

$$u_n = \tan\frac{\alpha_n}{2} = \frac{x_n - 1}{i(x_n + 1)}.$$

In the u-coordinates, (23) takes the form

$$u_{n+1} = F(n, u_{n-1}, u_n) := \frac{(u_n + 1)\big(u_{n-1}P_1(n, u_n) + P_2(n, u_n)\big)}{(u_n - 1)\big(u_{n-1}P_3(n, u_n) + P_4(n, u_n)\big)},$$

where

$$\begin{aligned}
P_1(n, v) &= (2n+\gamma)v^3 - (2n+4+\gamma)v^2 + (2n+4+\gamma)v - (2n+\gamma),\\
P_2(n, v) &= -(2n+\gamma)v^3 + (6n+4-\gamma)v^2 + (6n+4-\gamma)v - (2n+\gamma),\\
P_3(n, v) &= (2n+\gamma)v^3 + (6n+4-\gamma)v^2 - (6n+4-\gamma)v - (2n+\gamma),\\
P_4(n, v) &= -(2n+\gamma)v^3 - (2n+4+\gamma)v^2 - (2n+4+\gamma)v - (2n+\gamma).
\end{aligned}$$

Identity (22) reads as

$$u_1 = \frac{(u_0 + 1)(\gamma u_0^2 - 4u_0 + \gamma)}{(u_0 - 1)(\gamma u_0^2 + 4u_0 + \gamma)}. \tag{24}$$

We have to prove that for $(u, v) \in [0, 1] \times [0, 1]$, the values $F(n, u, v)$ lie in the interval $[-1, +\infty]$. The function $F(n, u, v)$ is smooth on $(0, 1) \times (0, 1)$ and has no critical points, in $(0, 1) \times (0, 1)$. Indeed, for critical points, we have $\partial F(n, u, v)/\partial u = 0$, which yields $P_1(n, v)P_4(n, v) - P_2(n, v)P_3(n, v) = 0$ and, after some calculations, $v = 0, 1, -1$. On the other hand, one can easily check that the values of $F(n, u, v)$ on the boundary of $[0, 1] \times [0, 1]$ lie in the interval $[-1, +\infty]$.

For $n = 0$, using (24) and exactly the same considerations as for $F(n, 0, v)$, one shows that $-1 \le u_1 \le +\infty$ for $u_0 \in [0, 1]$.

Now let us introduce

$$\begin{aligned}
S_{II}(k) &:= \big\{x_0 \in A_I \mid x_k \in A_{II},\ x_l \in A_I\ \forall\, l\ 0 < l < k\big\},\\
S_{IV}(k) &:= \big\{x_0 \in A_I \mid x_k \in A_{IV},\ x_l \in A_I\ \forall\, l\ 0 < l < k\big\},
\end{aligned}$$

where x_n is the solution of (20). From property 1, it follows that $S_{II}(k)$ and $S_{IV}(k)$ are open sets in the induced topology of A_I. Denote

$$S_{II} = \cup S_{II}(k), \qquad S_{IV} = \cup S_{IV}(k),$$

which are open, too. These sets are nonempty since $S_{II}(1)$ and $S_{IV}(1)$ are nonempty. Finally, introduce

$$S_I := \{x_0 \in A_I : x_n \in A_I \ \forall n \in \mathbf{N}\}.$$

It is obvious that S_I, S_{II}, and S_{IV} are mutually disjoint. Property 2 implies

$$S_I \cup S_{II} \cup S_{IV} = A_I.$$

This is impossible for $S_I = \emptyset$. Indeed, the connected set A_I cannot be covered by two open disjoint subsets S_{II} and S_{IV}. So there exists x_0 such that the solution $x_n \in A_I \ \forall \ n$. From

$$\varphi(n, x, 1) \equiv -i, \qquad \varphi(n, x, i) \equiv -1, \tag{25}$$

it follows that (for this solution) $x_n \neq 1, \ x_n \neq i$. This proves the theorem.

To complete the proof of Theorem 1, it is necessary to show $e^{i\gamma\pi/4} \in S_I$. This problem can be treated in terms of the method of isomonodromic deformations (see, for example, [7] for a treatment of a similar problem). One could probably compute the asymptotics of solutions x_n for $n \to \infty$ as functions of x_0 and show that the solution with $x_0 \neq e^{i\gamma\pi/4}$ cannot lie in S_I. The geometric origin of equation (20) allows us to prove the result using just elementary geometric arguments.

Proposition 4. The set S_I consists of only one element, namely, $S_I\{e^{i\gamma\pi/4}\}$. □

Proof. We have shown that S_I is not empty. Take a solution $x_n \in S_I$ and consider the corresponding circle pattern (see Theorem 4 and Theorem 3). Equations (10) and (15) for $N = M$ make it possible to find $R(N + iN)$ and $R(N + i(N + 1))$ in a closed form. We now show that substituting the asymptotics of $R(z)$ at these points into equation (11) for $M = N + 1$, for immersed $f_{n,m}$, one necessarily gets $R(i) = \tan(\gamma\pi/4)$.

Indeed, formula (19) yields the following representation in terms of the Γ-function:

$$R(N + iN) = c(\gamma)\frac{\Gamma\left(N + \frac{\gamma}{2}\right)}{\Gamma\left(N + 1 - \frac{\gamma}{2}\right)},$$

where

$$c(\gamma) = \frac{\gamma\Gamma\left(1 - \frac{\gamma}{2}\right)}{2\Gamma\left(1 + \frac{\gamma}{2}\right)}. \tag{26}$$

From the Stirling formula (see [6]),

$$\Gamma(s) = \sqrt{\frac{2\pi}{s}}\left(\frac{s}{e}\right)^s\left(1+\frac{1}{12s}+O\left(\frac{1}{s^2}\right)\right), \tag{27}$$

one obtains

$$R(N+iN) = c(\gamma)N^{\gamma-1}\left(1+O\left(\frac{1}{N}\right)\right), \qquad N\to\infty. \tag{28}$$

Now let $R(i) = a\tan(\gamma\pi/4)$, where a is a positive constant. Equation (15) for $M = N,\ N \geq 0$ reads

$$R(N-1+iN)R\big(N+i(N+1)\big) = R^2(N+iN).$$

This is equivalent to the fact that the centers of all the circles $C(N+iN)$ lie on a straight line. This equation yields

$$\begin{aligned} &R\big(N+i(N+1)\big)\\ &\quad= \left(a\tan\frac{\gamma\pi}{4}\right)^{(-1)^N}\left(\frac{\big(2(N-1)+\gamma\big)\big(2(N-3)+\gamma\big)\big(2(N-5)+\gamma\big)\cdots}{(2N-\gamma)\big(2(N-2)-\gamma\big)\big(2(N-4)-\gamma\big)\cdots}\right)^2. \end{aligned}$$

Using the product representation for $\tan x$,

$$\tan x = \frac{\sin x}{\cos x} = \frac{x\left(1-\frac{x^2}{\pi^2}\right)\cdots\left(1-\frac{x^2}{(k\pi)^2}\right)\cdots}{\left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{(3\pi)^2}\right)\cdots\left(1-\frac{4x^2}{\big((2k-1)\pi\big)^2}\right)\cdots},$$

one arrives at

$$R\big(N+i(N+1)\big) = a^{(-1)^N}c(\gamma)N^{\gamma-1}\left(1+O\left(\frac{1}{N}\right)\right). \tag{29}$$

Solving equation (11) with respect to $R^2(z)$, we get

$$\begin{aligned} R^2(z) &= G\big(N,M,R(z+i),R(z+1),R(z-i)\big)\\ &:= \frac{R(z+i)R(z+1)R(z-i)+R^2(z+1)\left(\frac{M+N}{2M}R(z-i)+\frac{M-N}{2M}R(z+i)\right)}{R(z+1)+\frac{M+N}{2M}R(z+i)+\frac{M-N}{2M}R(z-i)}. \end{aligned} \tag{30}$$

For $z \in \mathbf{V}$, $R(z+i) \geq 0$, $R(z+1) \geq 0$, $R(z-i) \geq 0$, the function G is monotonic:

$$\frac{\partial G}{\partial R(z+i)} \geq 0, \qquad \frac{\partial G}{\partial R(z+1)} \geq 0, \qquad \frac{\partial G}{\partial R(z-i)} \geq 0.$$

Thus, any positive solution $R(z)$, $z \in \mathbf{V}$ of (4) must satisfy

$$R^2(z) \geq G\big(N, M, 0, R(z+1), R(z-i)\big).$$

Substituting (28) and (29), the asymptotics of R, into this inequality and taking the limit $K \to \infty$, for $N = 2K$, we get $a^2 \geq 1$. Similarly, for $N = 2K+1$, one obtains $1/a^2 \geq 1$, and finally $a = 1$. This completes the proof of the Proposition and the proof of Theorem 1. ■

Remark. Taking further terms from the Stirling formula (27), one gets the asymptotics for Z^γ,

$$Z^\gamma_{n,k} = \frac{2c(\gamma)}{\gamma}\left(\frac{n+ik}{2}\right)^\gamma \left(1 + O\left(\frac{1}{n^2}\right)\right), \quad n \to \infty,\ k = 0, 1, \tag{31}$$

having a proper smooth limit. Here the constant $c(\gamma)$ is given by (26).

Due to representation (7), the discrete conformal map Z^γ can be studied by the isomonodromic deformation method. In particular, applying a technique of [7], one can probably prove the following conjecture.

Conjecture. The discrete conformal map Z^γ has the following asymptotic behavior:

$$Z^\gamma_{n,m} = \frac{2c(\gamma)}{\gamma}\left(\frac{n+im}{2}\right)^\gamma \left(1 + o\left(\frac{1}{\sqrt{n^2+m^2}}\right)\right), \qquad n^2 + m^2 \to \infty. \qquad \square$$

For $0 < \gamma < 2$, this would imply the asymptotic embeddedness of Z^γ at $n, m \to \infty$ and, combined with Theorem 1, the embeddedness[3] of $Z^\gamma : \mathbf{Z}^2_+ \to \mathbf{C}$ conjectured in [1] and [3].

5 The discrete maps Z^2 and Log. Duality

Definition 3 was given for $0 < \gamma < 2$. For $\gamma < 0$ or $\gamma > 2$, the radius $R(1+i) = \gamma/(2-\gamma)$ of the corresponding circle patterns becomes negative and some elementary quadrilaterals around $f_{0,0}$ intersect. But for $\gamma = 2$, one can renormalize the initial values of f so that the corresponding map remains an immersion. Let us consider Z^γ, with $0 < \gamma < 2$, and

[3] A discrete conformal map $f_{n,m}$ is called an embedding if inner parts of different elementary quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ do not intersect.

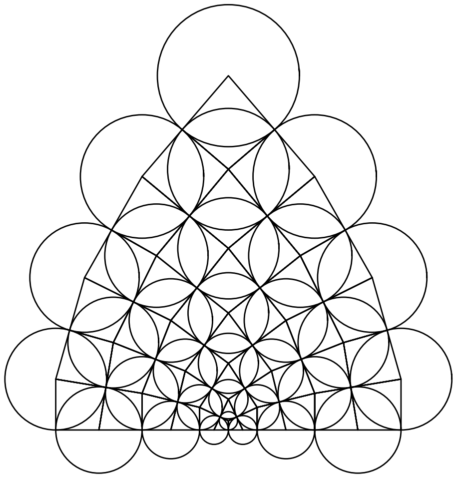

Figure 6 Discrete Z^2.

make the following renormalization for the corresponding radii: $R \to ((2-\gamma)/\gamma)R$. Then as $\gamma \to 2-0$ from below, we have

$$R(0) = \frac{2-\gamma}{\gamma} \longrightarrow +0, \qquad R(1+i) = 1, \qquad R(i) = \frac{2-\gamma}{\gamma}\tan\frac{\gamma\pi}{4} \longrightarrow \frac{2}{\pi}.$$

Definition 4. $Z^2 : \mathbf{Z}^2_+ \to \mathbf{R}^2 = \mathbf{C}$ is the solution of $(1,2)$ with $\gamma = 2$ and the initial conditions

$$Z^2(0,0) = Z^2(1,0) = Z^2(0,1) = 0, \qquad Z^2(2,0) = 1,$$
$$Z^2(0,2) = -1, \qquad Z^2(1,1) = i\frac{2}{\pi}.$$

In this definition, equations (1) and (2) are understood to be regularized through multiplication by their denominators. Note that for the radii on the border, one has $R(N+iN) = N$. Equation (16) has the symmetry $R \to 1/R$.

Proposition 5. Let $R(z)$ be a solution of the system $(10,11)$ for some γ. Then $\tilde{R}(z) = 1/R(z)$ is a solution of $(10,11)$ with $\tilde{\gamma} = 2-\gamma$. □

This proposition reflects the fact that for any discrete conformal map f, there is a *dual discrete conformal map* f^* defined by (see [3])

$$f^*_{n+1,m} - f^*_{n,m} = -\frac{1}{f_{n+1,m} - f_{n,m}}, \qquad f^*_{n,m+1} - f^*_{n,m} = \frac{1}{f_{n,m+1} - f_{n,m}}. \tag{32}$$

Obviously, this transformation preserves the kite form of elementary quadrilaterals and therefore is well defined for Schramm's circle patterns. The smooth limit of the duality (32) is

$$(f^*)' = -\frac{1}{f'}.$$

The dual of $f(z) = z^2$ is, up to a constant, $f^*(z) = \log z$. Motivated by this observation, we define the discrete logarithm as the discrete map dual to Z^2, i.e., the map corresponding to the circle pattern with radii

$$R_{\text{Log}}(z) = \frac{1}{R_{Z^2}(z)},$$

where R_{Z^2} are the radii of the circles for Z^2. Here one has $R_{\text{Log}}(0) = \infty$; i.e., the corresponding circle is a straight line. The corresponding constraint (2) can be also derived as a limit. Indeed, consider the map

$$g = \frac{2-\gamma}{\gamma} Z^\gamma - \frac{2-\gamma}{\gamma}.$$

This map satisfies (1) and the constraint

$$\gamma\left(g_{n,m} + \frac{2-\gamma}{\gamma}\right) = 2n\frac{(g_{n+1,m} - g_{n,m})(g_{n,m} - g_{n-1,m})}{(g_{n+1,m} - g_{n-1,m})} + 2m\frac{(g_{n,m+1} - g_{n,m})(g_{n,m} - g_{n,m-1})}{(g_{n,m+1} - g_{n,m-1})}.$$

Keeping in mind the limit procedure used to determine Z^2, it is natural to define the discrete analogue of $\log z$ as the limit of g as $\gamma \to +0$. The corresponding constraint becomes

$$1 = n\frac{(g_{n+1,m} - g_{n,m})(g_{n,m} - g_{n-1,m})}{(g_{n+1,m} - g_{n-1,m})} + m\frac{(g_{n,m+1} - g_{n,m})(g_{n,m} - g_{n,m-1})}{(g_{n,m+1} - g_{n,m-1})}. \tag{33}$$

Definition 5. Log is the map $\text{Log} : \mathbf{Z}^2_+ \to \mathbf{R}^2 = \overline{\mathbf{C}}$ satisfying (1) and (33) with the initial conditions

$$\text{Log}(0,0) = \infty, \qquad \text{Log}(1,0) = 0, \qquad \text{Log}(0,1) = i\pi,$$
$$\text{Log}(2,0) = 1, \qquad \text{Log}(0,2) = 1 + i\pi, \qquad \text{Log}(1,1) = i\frac{\pi}{2}.$$

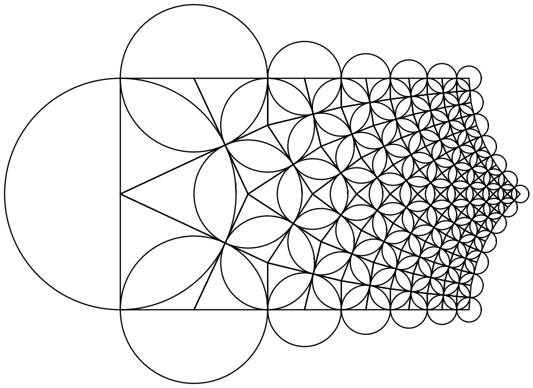

Figure 7 Discrete Log.

The circle patterns corresponding to the discrete conformal mappings Z^2 and Log were conjectured by O. Schramm and R. Kenyon (see [17]), but it was not proved that they are immersed.

Proposition 6. Discrete conformal maps Z^2 and Log are immersions. □

Proof. Consider the discrete conformal map $((2-\gamma)/\gamma)Z^\gamma$ with $0 < \gamma < 2$. The corresponding solution x_n of (20) is a continuous function of γ. So there is a limit as $\gamma \to 2-0$, of this solution with $x_n \in A_I$, $x_0 = i$, and $x_1 = (-1 + i\pi/2)/(1 + i\pi/2) \in A_I$. The solution x_n of (20) with the property $x_n \in A_I$ satisfies $x_n \neq 1$, $x_n \neq i$ for $n > 0$ (see (25)). Now, reasoning as in the proof of Proposition 3, we get that Z^2 is an immersion. The only difference is that $R(0) = 0$. The circle $C(0)$ lies on the border of $\mathbf{V}$, so Schramm's result (see [16]) claiming that the corresponding circle pattern is immersed is true. Log corresponds to the dual circle pattern, with $R_{\mathrm{Log}}(z) = 1/R_{Z^2}(z)$, which implies that Log is also an immersion. ■

6 Discrete maps Z^γ with $\gamma \notin [0, 2]$

Starting with Z^γ, $\gamma \in [0, 2]$ defined in the previous sections, one can easily define Z^γ for arbitrary γ by applying some simple transformations of discrete conformal maps and Schramm's circle patterns. Denote by S_γ the Schramm's circle pattern associated to Z^γ, $\gamma \in (0, 2]$. Applying the inversion of the complex plane $z \mapsto \tau(z) = 1/z$ to S_γ, one obtains a circle pattern τS_γ, which is also of Schramm's type. It is natural to define the discrete conformal map $Z^{-\gamma}$, $\gamma \in (0, 2]$ through the centers and intersection points of circles of τS_γ. On the other hand, constructing the dual Schramm circle pattern (see Proposition 5) for $Z^{-\gamma}$, we arrive at a natural definition of $Z^{2+\gamma}$. Intertwining the inversion and the dualization described above, one constructs circle patterns corresponding to Z^γ for any γ. To define immersed Z^γ, one should discard some points near $(n, m) = (0, 0)$ from the definition domain.

To give a precise description of the corresponding discrete conformal maps in terms of the constraint (2) and initial data for arbitrarily large γ, a more detailed consideration is required. To any Schramm circle pattern S, there corresponds a 1-complex parameter family of discrete conformal maps described in [3]. Take an arbitrary point $P_\infty \in \mathbf{C} \cup \infty$. Reflect it through all the circles of S. The resulting extended lattice is a discrete conformal map and is called a *central extension* of S. As a special case, choosing $P_\infty = \infty$, one obtains the centers of the circles, and thus the discrete conformal map considered in Section 3.

Composing the discrete map $Z^\gamma : \mathbf{Z^2}_+ \to \mathbf{C}$ with the inversion $\tau(z) = 1/z$ of the complex plane, one obtains the discrete conformal map $G(n, m) = \tau(Z^\gamma(n, m))$ satisfying the constraint (2) with the parameter $\gamma_G = -\gamma$. This map is the central extension of τS_γ corresponding to $P_\infty = 0$. Let us define $Z^{-\gamma}$ as the central extension of τS_γ corresponding to $P_\infty = \infty$, i.e., the extension described in Section 3. The map $Z^{-\gamma}$ defined in this way also satisfies the constraint (2) due to the following lemma.

Lemma 3. Let S be a Schramm's circle pattern, and let $f^\infty : \mathbf{Z}^2_+ \to \mathbf{C}$ and $f^0 : \mathbf{Z}^2_+ \to \mathbf{C}$ be its two central extensions corresponding to $P_\infty = \infty$ and $P_\infty = 0$, respectively. Then f^∞ satisfies (2) if and only if f^0 satisfies (2). □

Proof. If f^∞ (or f^0) satisfies (2), then $f^\infty_{n,0}$ (respectively, $f^0_{n,0}$) lie on a straight line, and so do $f^\infty_{0,m}$ (respectively, $f^0_{0,m}$). A straightforward computation shows that $f^\infty_{n,0}$ and $f^0_{n,0}$ satisfy (2) simultaneously, and the same statement holds for $f^\infty_{0,m}$ and $f^0_{0,m}$. Since (1) is compatible with (2), f^0 (respectively, f^∞) satisfy (2) for any $n, m \geq 0$. ■

Let us now describe Z^K for $K \in \mathbf{N}$ as special solutions of (1, 2).

Definition 6. $Z^K : \mathbf{Z}^2_+ \to \mathbf{R}^2 = \mathbf{C}$, where $K \in \mathbf{N}$, is the solution of (1, 2) with $\gamma = K$ and the initial conditions

$$Z^K(n, m) = 0 \quad \text{for } n + m \leq K - 1,\ (n, m) \in \mathbf{Z}^2_+, \tag{34}$$

$$Z^K(K, 0) = 1, \tag{35}$$

$$Z^K(K-1, 1) = i\frac{2^{K-1}\Gamma^2\left(\frac{K}{2}\right)}{\pi\Gamma(K)}. \tag{36}$$

The initial condition (34) corresponds to the identity

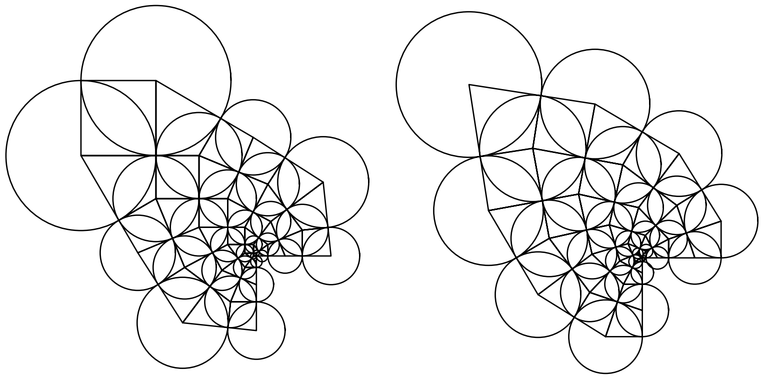

Figure 8 Discrete Z^3.

$$\frac{d^k z^K}{dz^k}(z=0)=0, \quad k<K$$

in the smooth case. For odd $K=2N+1$, condition (36) reads

$$Z^{2N+1}(2N,1)=i\frac{(2N-1)!!}{(2N)!!},$$

and follows from constraint (2). For even $K=2N$, any value of $Z^K(K-1,1)$ is compatible with (2). In this case, formula (36) can be derived from the asymptotics

$$\lim_{N\to\infty}\frac{R(N+iN)}{R\big(N+i(N+1)\big)}=1$$

and reads

$$Z^{2N}(2N-1,1)=i\frac{2}{\pi}\frac{(2N-2)!!}{(2N-1)!!}.$$

We conjecture that so-defined Z^K are immersed.

Note that for odd integer $K=2N+1$, discrete Z^{2N+1} in Definition 6 is slightly different from the one previously discussed in this section. Indeed, by intertwining the dualization and the inversion (as described above), one can define two different versions of Z^{2N+1}. One is obtained from the circle pattern corresponding to discrete $Z(n,m)=n+im$ with centers in $n+im, n+m=0 \pmod 2$. The second one comes from Definition 6 and is obtained by the same procedure from $Z(n,m)=n+im$, but in this case, the centers of the circles of the pattern are chosen in $n+im, n+m=1 \pmod 2$. These two versions

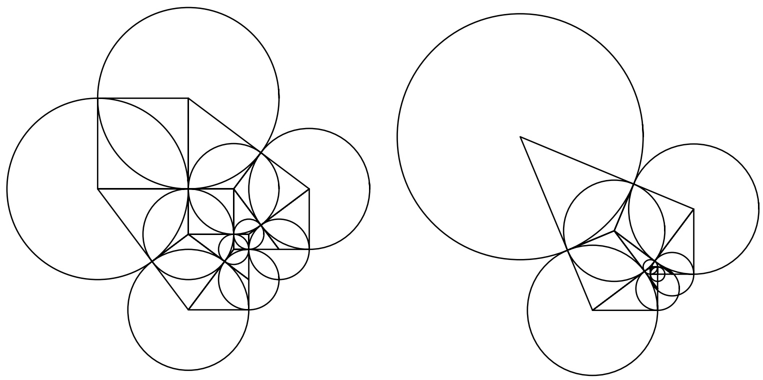

Figure 9 Detailed view of two versions of discrete Z^3.

of Z^3 are presented in Figure 8. The left figure shows Z^3 obtained through Definition 6. Note that this map is immersed, in contrast to the right lattice of Figure 8, which has overlapping quadrilaterals at the origin (see Figure 9).

Appendix A: Proof of Theorem 2

Compatibility

Direct, but rather long, computation (authors used Mathematica[4] computer algebra to perform it) shows that if the constraint (6) holds for 3 vertices of an elementary quadrilateral, it holds for the fourth vertex. A map $f : \mathbf{Z}^2 \to \mathbf{C}$ satisfying equation (1) and the constraint (6) is uniquely determined by its values at four vertices, for example, $f_{n_0,m_0}, f_{n_0,m_0\pm1}, f_{n_0+1,m_0}$. Indeed, starting with this data and consequently applying (6) and (1), one determines $f_{n,m_0}, f_{n,m_0\pm1}$ for all n. Now, applying (6), we get the values $f_{n,m_0\pm2}$, $\forall n$. Note that, due to the observation above, equation (1) is automatically satisfied for all obtained elementary quadrilaterals. Proceeding further as above, one determines $f_{n,m_0\pm3}$, $f_{n,m_0\pm4}, \dots$, and thus $f_{n,m}$, for all n, m.

Necessity

Now let $f_{n,m}$ be a solution to the system (1, 6). Define $\Psi_{0,0}(\lambda)$ as a nontrivial solution of linear equation (7) with $A(\lambda)$ given by Theorem 2. Equations (4) determine $\Psi_{n,m}(\lambda)$ for any n, m. By direct computation, one can check that the compatibility conditions of (7) and (4),

[4]Mathematica Version 3, Wolfram, Champaign, Ill., 1996.

$$
\begin{aligned}
&U_{n,m+1}V_{n,m} = V_{n+1,m}U_{n,m},\\
&\frac{d}{d\lambda}U_{n,m} = A_{n+1,m}U_{n,m} - U_{n,m}A_{n,m},\\
&\frac{d}{d\lambda}V_{n,m} = A_{n,m+1}V_{n,m} - V_{n,m}A_{n,m},
\end{aligned}
\tag{37}
$$

are equivalent to (1, 6).

Sufficiency

Conversely, let $\Psi_{n,m}(\lambda)$ satisfy (7) and (4) with some λ-independent matrices $B_{n,m}$, $C_{n,m}$, $D_{n,m}$. From (8), it follows that $\operatorname{tr} B_{n,m} = -n$, $\operatorname{tr} C_{n,m} = -m$. Equations (37) are equivalent to equations for their principal parts at $\lambda = 0$, $\lambda = -1$, $\lambda = 1$, $\lambda = \infty$:

$$D_{n+1,m}\begin{pmatrix}1 & -u_{n,m}\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & -u_{n,m}\\ 0 & 1\end{pmatrix}D_{n,m}, \tag{38}$$

$$D_{n,m+1}\begin{pmatrix}1 & -v_{n,m}\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & -v_{n,m}\\ 0 & 1\end{pmatrix}D_{n,m}, \tag{39}$$

$$B_{n+1,m}\begin{pmatrix}1 & -u_{n,m}\\ -1/u_{n,m} & 1\end{pmatrix} = \begin{pmatrix}1 & -u_{n,m}\\ -1/u_{n,m} & 1\end{pmatrix}B_{n,m}, \tag{40}$$

$$B_{n,m+1}\begin{pmatrix}1 & -v_{n,m}\\ 1/v_{n,m} & 1\end{pmatrix} = \begin{pmatrix}1 & -v_{n,m}\\ 1/v_{n,m} & 1\end{pmatrix}B_{n,m}, \tag{41}$$

$$C_{n+1,m}\begin{pmatrix}1 & -u_{n,m}\\ 1/u_{n,m} & 1\end{pmatrix} = \begin{pmatrix}1 & -u_{n,m}\\ 1/u_{n,m} & 1\end{pmatrix}C_{n,m}, \tag{42}$$

$$C_{n,m+1}\begin{pmatrix}1 & -v_{n,m}\\ -1/v_{n,m} & 1\end{pmatrix} = \begin{pmatrix}1 & -v_{n,m}\\ -1/v_{n,m} & 1\end{pmatrix}C_{n,m}, \tag{43}$$

$$\left(D_{n+1,m} - B_{n+1,m} - C_{n+1,m}\right)\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\left(D_{n,m} - B_{n,m} - C_{n,m}\right) = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \tag{44}$$

$$\left(D_{n,m+1} - B_{n,m+1} - C_{n,m+1}\right)\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\left(D_{n,m} - B_{n,m} - C_{n,m}\right) = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \tag{45}$$

From (40, 41) and $\operatorname{tr} B_{n,m} = -n$, it follows that

$$B_{n,m} = -\frac{n-\phi}{u_{n,m}+u_{n-1,m}}\begin{pmatrix}u_{n,m} & u_{n,m}u_{n-1,m}\\ 1 & u_{n-1,m}\end{pmatrix} - \frac{\phi}{2}I.$$

Similarly, (42, 43) and $\operatorname{tr} C_{n,m} = -m$ imply

$$C_{n,m} = -\frac{m-\psi}{v_{n,m}+v_{n,m-1}}\begin{pmatrix} v_{n,m} & v_{n,m}v_{n,m-1} \\ 1 & v_{n,m-1} \end{pmatrix} - \frac{\psi}{2}I.$$

Here, ϕ and ψ are constants independent of n, m. The function $a(\lambda)$ in (8), independent of n and m, can be normalized to vanish identically; i.e., $\operatorname{tr} D_{n,m} = 0$. Substitution of

$$D = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

into equations (38) and (39) yields

$$c_{n+1,m} = c_{n,m}, \qquad c_{n,m+1} = c_{n,m}, \tag{46}$$

$$a_{n+1,m} = a_{n,m} - u_{n,m}c_{n,m}, \qquad a_{n,m+1} = a_{n,m} - v_{n,m}c_{n,m}, \tag{47}$$

$$b_{n+1,m} = b_{n,m} + u_{n,m}(a_{n,m} + a_{n+1,m}), \qquad b_{n,m+1} = b_{n,m} + v_{n,m}(a_{n,m} + a_{n,m+1}). \tag{48}$$

Thus, c is a constant independent of n, m. Equations (47) can be easily integrated:

$$a_{n,m} = -cf_{n,m} + \theta,$$

where θ is independent of n, m (recall that $u_{n,m} = f_{n+1,m} - f_{n,m}$, $v_{n,m} = f_{n,m+1} - f_{n,m}$). Substituting this expression into (48) and integrating, we get

$$b_{n,m} = -cf_{n,m}^2 + 2\theta f_{n,m} + \mu,$$

for some constant μ. Now (44) and (45) imply

$$b_{n,m} = -\frac{n-\phi}{u_{n,m}+u_{n-1,m}}u_{n,m}u_{n-1,m} - \frac{m-\psi}{v_{n,m}+v_{n,m-1}}v_{n,m}v_{n,m-1},$$

which is equivalent to the constraint (6) after identifying $c = \beta/2$, $\theta = -(\gamma/4)$, $\mu = -(\delta/2)$.

Appendix B

The proof of Lemma 2 uses the following technical lemma.

Lemma 4. For positive R, the following hold.

(1) Equations (11) and (10) at z and equation (10) at $z - i$ imply (15) at $z + 1$.

(2) Equation (15) at $N + iN$ and equation (10) at $N + iN$ and at $N - 1 + iN$ imply

$$
\begin{aligned}
&(N+M)\big(R(z)^2 - R(z+1)R(z-i)\big)\big(R(z-i)+R(z-1)\big) \\
&\qquad + (N-M)\big(R(z)^2 - R(z-i)R(z-1)\big)\big(R(z+1)+R(z-i)\big) = 0,
\end{aligned}
\tag{49}
$$

at $z = N + iM$, for $M = N + 1$.

(3) Equations (15) and (49) at $z = N + iM, N \neq \pm M$, imply (16) at z.

(4) Equations (15) and (16) at $z = N + iM, N \neq \pm M$, imply

$$
\begin{aligned}
&(N+M)\big(R(z)^2 - R(z+i)R(z-1)\big)\big(R(z+1)+R(z+i)\big) \\
&\qquad + (N-M)\big(R(z)^2 - R(z+1)R(z+i)\big)\big(R(z+i)+R(z-1)\big) = 0,
\end{aligned}
\tag{50}
$$

and (11) at z.

(5) Equations (50) and (10) at z and equation (10) at $z - 1$ imply (49) at $z + i$. □

Proof. The proof is a direct computation. Let us check, for example, (3). Equations (15) and (49) read

$$
\begin{aligned}
&\xi\big(R(z)^2 - R(z+i)R(z-1)\big) + \eta\big(R(z+i)+R(z-1)\big) = 0, \\
&\xi\big(R(z)^2 - R(z+1)R(z-i)\big) - \eta\big(R(z+1)+R(z-i)\big) = 0,
\end{aligned}
$$

where $\xi = (N+M)(R(z-1)+R(z-i)), \eta = (M-N)(R(z)^2 - R(z-1)R(z-i))$. Since $\xi \neq 0$, we get

$$
\begin{aligned}
&\big(R(z)^2 - R(z+i)R(z-1)\big)\big(R(z+1)+R(z-i)\big) \\
&\qquad + \big(R(z)^2 - R(z+1)R(z-i)\big)\big(R(z+i)+R(z-1)\big) = 0,
\end{aligned}
$$

which is equivalent to (16). ■

Proof of Lemma 2. By symmetry reasons, it is enough to prove the lemma for $N \geq 0$. Let us prove it by induction on N.

For $N = 0$, identity (10) yields $R(-1+iM) = R(1+iM)$. Equation (11) at $z = iM$ implies (15). Now equations (11) and (15) at $z = iM$ imply (16). ■

Induction step $N \to N+1$

Claim (1) of Lemma 4 implies (15) at $z = N+1+iM$. Claims (2), (3), and (4) yield equations (16) and (11) at $z = N+1+i(N+2)$. Now, using (5), (3), and (4) of Lemma 4, one gets, by induction on L, equations (16) and (11) at $z = N+1+i(N+L+1)$ for any $L \in \mathbf{N}$.

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