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# Discrete Indefinite Affine Spheres

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## 1 Introduction

Some sixty years before its rediscovery in a solitonic context, the partial differential equation

$$(\ln h)_{xy} = h - h^{-2} \tag{1.1}$$

was set down by Tzitzeica in a geometric context [23]. Tzitzeica's classical papers are believed to have initiated a new area in mathematics, namely affine differential geometry. Since then, the Tzitzeica equation (1.1) has been the subject of extensive study in both mathematics and physics [6, 7, 9, 10, 11, 13, 21]. However, only recently a discrete integrable analogue of the Tzitzeica equation has been proposed in [19]. As a result, the natural question arises as to whether the discrete Tzitzeica equation may be interpreted in a simple manner in terms of discrete surfaces. Such a link between discrete integrable systems and particular classes of discrete surfaces has been well established. Examples in Euclidean geometry comprise discrete pseudospherical surfaces (K-surfaces ) and discrete surfaces of constant mean curvature (H-surfaces ) (see the contribution of Bobenko and Pinkall [3] and [2, 14, 16]). Discrete isothermic surfaces representing Möbius geometries are also known [4]. Thus, in some instances, discrete integrable systems suggest natural geometric properties. In fact, the theory of integrable systems may provide us with methods of investigating such discrete geometries.

The present paper extends the above-mentioned approach to affine differential geometry[1]. We present a natural geometric discretization of affine spheres with indefinite Blaschke metric and show that the corresponding discrete Gauss–Codazzi equations reduce to the integrable discrete Tzitzeica system/equation set down in [19]. A connection with the Bäcklund transformation for classical affine spheres is recorded. Thus, the Tzitzeica transformation applied to classical affine spheres generates generalized discrete affine spheres. The Tzitzeica transformation is used to construct a Bäcklund transformation for (generalized) discrete affine spheres in a purely algebraic manner.

[1] At this point we would like to mention a recently found nice affine-geometrical interpretation [15] of the KdV equation.

## 2 Classical affine spheres

In this section, we present the well-known description of affine spheres with indefinite Blaschke metric (for more details see e.g. [1, 22, 24]. Here, affine differential geometry is treated in its classical setup, that is as the geometry which investigates properties of surfaces in $\mathbb{R}^3$ invariant under (equi)affine transformations :

$$\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{a}, \quad A \in SL(3,\mathbb{R}), \quad \boldsymbol{a} \in \mathbb{R}^3. \tag{2.1}$$

Let $F : M \to \mathbb{R}^3$ be an immersion. It is easily verified that equiaffine transformations are conformal with respect to the second fundamental form. This implies that asymptotic line parametrizations and the class of immersions with negative Gaussian curvature ($\mathcal{K} < 0$) are affine invariant. Thus, let us consider an oriented immersion

$$\begin{aligned} \boldsymbol{r} : M &\to \mathbb{R}^3 \\ (x,y) &\mapsto \boldsymbol{r}(x,y) \end{aligned} \tag{2.2}$$

given in terms of asymptotic coordinates $(x, y)$, i.e.

$$\operatorname{rank}(d\boldsymbol{r}) = 2, \quad \boldsymbol{r}_{xx}, \boldsymbol{r}_{yy} \in d\boldsymbol{r}(TM). \tag{2.3}$$

By virtue of the orientation-preserving transformation $(x, y) \to (y, -x)$, one may assume without loss of generality that

$$|\boldsymbol{r}_x, \boldsymbol{r}_y, \boldsymbol{r}_{xy}| > 0 \tag{2.4}$$

on $M$, where $|\cdot,\cdot,\cdot|$ denotes the standard determinant in $\mathbb{R}^3$.

**Definition.** *The indefinite metric*

$$\sqrt{|\boldsymbol{r}_x, \boldsymbol{r}_y, \boldsymbol{r}_{xy}|}\, dx\, dy \tag{2.5}$$

*is equiaffine invariant. It is called the Blaschke metric of the immersion.*

The Blaschke metric is conformally equivalent to the second fundamental form of the immersion. It is non-degenerate in the case $\mathcal{K} < 0$ considered here.

**Definition.** *A transversal vector field $\boldsymbol{\xi}$ on an oriented surface $\boldsymbol{r}(M)$ is called affine normal if it satisfies*

- $d\boldsymbol{\xi} \in d\boldsymbol{r}(TM)$
- $|d\boldsymbol{r}(\cdot), d\boldsymbol{r}(\cdot), \boldsymbol{\xi}|$ *is the volume form for the Blaschke metric, i.e. in our case,*

$$|\boldsymbol{r}_x, \boldsymbol{r}_y, \boldsymbol{\xi}| = \sqrt{|\boldsymbol{r}_x, \boldsymbol{r}_y, \boldsymbol{r}_{xy}|}. \tag{2.6}$$

**Definition.** *A non-degenerate surface in* $\mathbb{R}^3$ *is called an affine sphere if all affine normal directions meet at a point. If this point is not infinite it may be chosen as the origin of* $\mathbb{R}^3$ *so that*

$$\boldsymbol{\xi} = \mathcal{H}\boldsymbol{r}, \quad \mathcal{H} : M \to \mathbb{R}. \tag{2.7}$$

$\mathcal{H}$ *is called the affine mean curvature.*

One can prove that in the case of a non-degenerate Blaschke metric the affine mean curvature of the affine sphere must be constant. In the following, it is asssumed that $\mathcal{H} \neq 0$. Consequently, $\mathcal{H}$ may be normalized to $\mathcal{H} = \pm 1$ by using a scaling transformation of the ambient space $\mathbb{R}^3$. Furthermore, modulo a change of the orientation of the surface corresponding to $y \to -y$, one may always set

$$\mathcal{H} = 1. \tag{2.8}$$

Hence, on introducing the function

$$h = |\boldsymbol{r}_x, \boldsymbol{r}_y, \boldsymbol{\xi}| \tag{2.9}$$

and the cubic differentials $a\,dx^3$, $b\,dy^3$, where

$$a = |\boldsymbol{r}_x, \boldsymbol{r}_{xx}, \boldsymbol{\xi}|, \quad b = -|\boldsymbol{r}_y, \boldsymbol{r}_{yy}, \boldsymbol{\xi}|, \tag{2.10}$$

one obtains

$$|\boldsymbol{r}_x, \boldsymbol{r}_y, \boldsymbol{r}_{xy}| = h^2 \tag{2.11}$$

and the following linear system for the immersion $\boldsymbol{r}$ (*Gauss equations* ):

$$\begin{aligned} \boldsymbol{r}_{xx} &= \frac{h_x}{h}\boldsymbol{r}_x + \frac{a}{h}\boldsymbol{r}_y \\ \boldsymbol{r}_{xy} &= h\boldsymbol{r} \\ \boldsymbol{r}_{yy} &= \frac{h_y}{h}\boldsymbol{r}_y + \frac{b}{h}\boldsymbol{r}_x. \end{aligned} \tag{2.12}$$

By analogy with the Euclidean case, one may show [22] that $\{h\,dxdy, a\,dx^3, b\,dy^3\}$ constitutes a complete equiaffine invariant system for indefinite surfaces in $\mathbb{R}^3$ which determines a surface up to equiaffine transformations .

Now, the compatibility conditions for (2.12) yield

$$(\ln h)_{xy} = h - abh^{-2}, \quad a_y = 0, \quad b_x = 0. \tag{2.13}$$

The above system is invariant with respect to the transformation

$$a \to \Lambda a, \quad b \to \Lambda^{-1} b \tag{2.14}$$

with arbitrary $\Lambda \in \mathbb{C}$. This fact gives rise to the following theorem.

**Theorem.** *Every indefinite affine sphere posseses a one-parameter family of deformations preserving the Blaschke metric and the differential* $ab\,dx^3dy^3$. *This deformation is described by the transformation (2.14). The system*

$$\begin{pmatrix} \boldsymbol{r}_x \\ \boldsymbol{r}_y \\ \boldsymbol{r} \end{pmatrix}_x = \begin{pmatrix} h_x h^{-1} & \Lambda a h^{-1} & 0 \\ 0 & 0 & h \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{r}_x \\ \boldsymbol{r}_y \\ \boldsymbol{r} \end{pmatrix}$$
$$\begin{pmatrix} \boldsymbol{r}_x \\ \boldsymbol{r}_y \\ \boldsymbol{r} \end{pmatrix}_y = \begin{pmatrix} 0 & 0 & h \\ \Lambda^{-1} b h^{-1} & h_y h^{-1} & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{r}_x \\ \boldsymbol{r}_y \\ \boldsymbol{r} \end{pmatrix} \tag{2.15}$$

*determines the corresponding family of immersions.*

In the terminology of modern soliton theory, the linear system (2.15) is nothing but the *Lax representation* for the system (2.13). In fact, for $a \neq 0$ and $b \neq 0$, one may reparametrize the asymptotic coordinates according to

$$x \to \tilde{x}(x), \quad y \to \tilde{y}(y) \tag{2.16}$$

in such a way that $a = 1$, $b = \epsilon = \pm 1$ and the orientation is preserved. Thus, we obtain

$$(\ln h)_{xy} = h - \epsilon h^{-2}. \tag{2.17}$$

Finally, the parameter $\epsilon$ may be absorbed if one allows negative-valued solutions $h$. Indeed, if $h(x,y)$ is a solution to the integrable *Tzitzeica equation* [23]

$$(\ln h)_{xy} = h - h^{-2} \tag{2.18}$$

and $\boldsymbol{r}(x,y)$ is the corresponding immersion, then $\tilde{h}(x,y) = \epsilon h(x, \epsilon y)$ is a solution to (2.17) with associated immersion $\tilde{\boldsymbol{r}}(x,y) = \boldsymbol{r}(x, \epsilon y)$.

## 3 Discrete affine spheres

Here, we define discrete analogues of affine spheres (*discrete Tzitzeica surfaces*) in a purely geometric manner. These constitute particular 'discrete surfaces' which are maps

$$\boldsymbol{r} : \mathbb{Z}^2 \to \mathbb{R}^3, \quad (n_1, n_2) \mapsto \boldsymbol{r}(n_1, n_2). \tag{3.1}$$

In the following, we suppress the arguments of functions of $n_1$ and $n_2$ and denote increments of the discrete variables by subscripts, for example,

$$\boldsymbol{r} = \boldsymbol{r}(n_1, n_2), \quad \begin{array}{l} \boldsymbol{r}_1 = \boldsymbol{r}(n_1+1, n_2), \\ \boldsymbol{r}_2 = \boldsymbol{r}(n_1, n_2+1), \end{array} \quad \begin{array}{l} \boldsymbol{r}_{11} = \boldsymbol{r}(n_1+2, n_2) \\ \boldsymbol{r}_{12} = \boldsymbol{r}(n_1+1, n_2+1)\,. \\ \boldsymbol{r}_{22} = \boldsymbol{r}(n_1, n_2+2) \end{array} \tag{3.2}$$

Moreover, decrements are indicated by overbars, that is

$$\boldsymbol{r}_{\bar{1}} = \boldsymbol{r}(n_1 - 1, n_2), \quad \boldsymbol{r}_{\bar{2}} = \boldsymbol{r}(n_1, n_2 - 1) \tag{3.3}$$

and the following notation for difference operators is adopted:

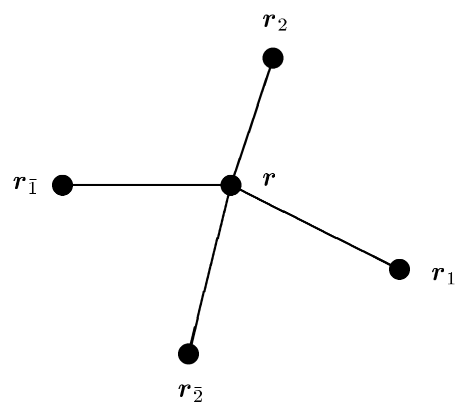
$$\Delta_i \boldsymbol{r} = \boldsymbol{r}_i - \boldsymbol{r}, \quad \Delta_{12}\boldsymbol{r} = \boldsymbol{r}_{12} - \boldsymbol{r}_1 - \boldsymbol{r}_2 + \boldsymbol{r}. \tag{3.4}$$

**Definition (Discrete affine spheres).** *A two-dimensional lattice (net) in three-dimensional Euclidean space*

$$\boldsymbol{r} : \mathbb{Z}^2 \to \mathbb{R}^3 \tag{3.5}$$

*is called a discrete affine sphere if it has the following properties:*

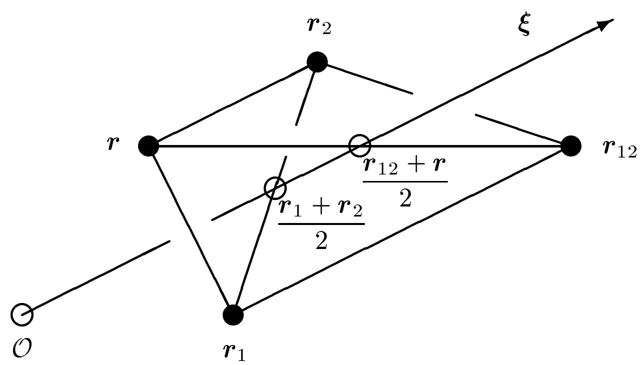
**(a)** *Any point $\boldsymbol{r}(n_1, n_2)$ and its neighbours $\boldsymbol{r}_{\bar{1}}, \boldsymbol{r}_1, \boldsymbol{r}_{\bar{2}}, \boldsymbol{r}_2$ lie on a plane.*




**(b)** *All affine normals $\boldsymbol{\xi}$ whose directions are defined by*

$$\boldsymbol{\xi} \sim \Delta_{12}\boldsymbol{r}$$

*meet at a point $\mathcal{O}$.*




*Remark.* The properties (a) and (b) define two well-known types of net. In fact, the definition of discrete affine spheres may be formulated as

**(a)** $\boldsymbol{r}$ is a discrete *asymptotic* net

**(b)** $\boldsymbol{r}$ is a discrete *affine Lorentz harmonic* net.

Discrete asymptotic nets (discrete A-surfaces ) were first introduced by Sauer in the 1930s in connection with models for asymptotic lines on smooth surfaces (see [17] and references therein). Their significance is seen as follows. Since the points $\boldsymbol{r}, \boldsymbol{r}_{\bar{1}}, \boldsymbol{r}_1, \boldsymbol{r}_{\bar{2}}, \boldsymbol{r}_2$ lie on a plane, one may associate with the point $\boldsymbol{r}$ a unit normal $\boldsymbol{N}$ being orthogonal to this plane and whence

$$\Delta_1 \boldsymbol{N} \cdot \Delta_1 \boldsymbol{r} = 0, \quad \Delta_2 \boldsymbol{N} \cdot \Delta_2 \boldsymbol{r} = 0, \tag{3.6}$$

where $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ denote the unit normals corresponding to $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ respectively. It is evident that these relations constitute a natural discretization of the classical definition of asymptotic coordinates. On the other hand, the property (b) in the definition of discrete affine spheres implies that

$$\boldsymbol{r}_{12} + \boldsymbol{r} \,||\, \boldsymbol{r}_1 + \boldsymbol{r}_2 \tag{3.7}$$

if one chooses $\mathcal{O}$ to be the origin of $\mathbb{R}^3$. Consequently, there exists a function $\rho$ such that

$$\Delta_{12} \boldsymbol{r} = \rho(\boldsymbol{r}_{12} + \boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}) \tag{3.8}$$

which is the usual definition of discrete Lorentz harmonic nets. See for example the contribution of Bobenko and Pinkall [3], where this property of the Gauss map of discrete K-surfaces is discussed.

In analytical terms, condition (a) translates into

$$|\boldsymbol{r}_1 - \boldsymbol{r}, \boldsymbol{r}_2 - \boldsymbol{r}, \boldsymbol{r} - \boldsymbol{r}_{\bar{1}}| = 0, \quad |\boldsymbol{r}_1 - \boldsymbol{r}, \boldsymbol{r}_2 - \boldsymbol{r}, \boldsymbol{r} - \boldsymbol{r}_{\bar{2}}| = 0 \tag{3.9}$$

so that the position vector of the discrete surfaces considered here obeys the discrete 'Gauss equations'

$$\begin{aligned} \boldsymbol{r}_{11} - \boldsymbol{r}_1 &= \alpha(\boldsymbol{r}_1 - \boldsymbol{r}) + \beta(\boldsymbol{r}_{12} - \boldsymbol{r}_1) \\ \boldsymbol{r}_{12} + \boldsymbol{r} &= H(\boldsymbol{r}_1 + \boldsymbol{r}_2) \\ \boldsymbol{r}_{22} - \boldsymbol{r}_2 &= \gamma(\boldsymbol{r}_2 - \boldsymbol{r}) + \delta(\boldsymbol{r}_{12} - r_2) \end{aligned} \tag{3.10}$$

where $\alpha, \beta, \gamma, \delta$, and $H$ are as-yet unspecified functions of $n_i$. However, the compatibility conditions of the above yield

$$\begin{gathered} \alpha = \frac{H_1 - 1}{H_1(H-1)}, \quad \gamma = \frac{H_2 - 1}{H_2(H-1)} \\ \beta_2(H_2 - 1)H = \beta(H-1)H_1, \quad \delta_1(H_1 - 1)H = \delta(H-1)H_2 \\ H_{12} = \frac{H(H-1)}{H^2(H_1 + H_2 - H_1 H_2) - H + \beta\delta H_1 H_2 (H-1)^2}. \end{gathered} \tag{3.11}$$

Hence, on setting

$$A = \beta(H-1), \quad B = \delta(H-1) \tag{3.12}$$

we obtain:

**Theorem (The discrete Tzitzeica system).** *Discrete affine spheres are governed by the discrete Gauss equations*

$$\begin{aligned}
\boldsymbol{r}_{11} - \boldsymbol{r}_1 &= \frac{H_1 - 1}{H_1(H-1)}(\boldsymbol{r}_1 - \boldsymbol{r}) + \frac{A}{H-1}(\boldsymbol{r}_{12} - \boldsymbol{r}_1) \\
\boldsymbol{r}_{12} + \boldsymbol{r} &= H(\boldsymbol{r}_1 + \boldsymbol{r}_2) \\
\boldsymbol{r}_{22} - \boldsymbol{r}_2 &= \frac{H_2 - 1}{H_2(H-1)}(\boldsymbol{r}_2 - \boldsymbol{r}) + \frac{B}{H-1}(\boldsymbol{r}_{12} - \boldsymbol{r}_2).
\end{aligned} \tag{3.13}$$

*They are compatible modulo*

$$\begin{gathered}
A_2 = \frac{H_1}{H}A, \quad B_1 = \frac{H_2}{H}B \\
H_{12} = \frac{H(H-1)}{H^2(H_1 + H_2 - H_1H_2) - H + ABH_1H_2}
\end{gathered} \tag{3.14}$$

*which is termed the discrete Tzitzeica system.*

The justification of the term 'discrete Tzitzeica system' is seen as follows. If one regards a discrete function $f : \mathbb{Z}^2 \to \mathbb{R}$ as an approximation of a smooth function $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$, that is

$$f(n_1, n_2) = \tilde{f}(x, y)\Big|_{(x,y)=(\epsilon_1 n_1, \epsilon_2 n_2)} \tag{3.15}$$

for small $\epsilon_i$, then, on dropping the tilde, the Taylor expansions

$$f_1 = f + \epsilon_1 f_x + O(\epsilon_1^2), \quad f_2 = f + \epsilon_2 f_y + O(\epsilon_2^2) \tag{3.16}$$

apply. Now, the form of the discrete Gauss equations (3.13) suggests the natural expansion

$$H = 1 + \tfrac{1}{2}\epsilon_1\epsilon_2 h, \quad A = \tfrac{1}{2}\epsilon_1^3 a, \quad B = \tfrac{1}{2}\epsilon_2^3 b \tag{3.17}$$

so that the discrete Tzitzeica system reduces to

$$(\ln h)_{xy} = h - abh^{-2}, \quad a = a(x), \quad b = b(y) \tag{3.18}$$

in the limit $\epsilon_i \to 0$. Similarly, the discrete Gauss equations (3.13) become the continuous ones (2.12) in this limit.

*Remark.* In view of the above continuum limit, the relations

$$\begin{gathered}
|\boldsymbol{r}, \Delta_1\boldsymbol{r}, \Delta_2\boldsymbol{r}| = c\frac{H-1}{H}, \quad |\Delta_1\boldsymbol{r}, \Delta_2\boldsymbol{r}, \Delta_{12}\boldsymbol{r}| = 2c\frac{(H-1)^2}{H} \\
|\boldsymbol{r}, \Delta_1\boldsymbol{r}, \Delta_{11}\boldsymbol{r}| = cA, \quad |\boldsymbol{r}, \Delta_2\boldsymbol{r}, \Delta_{22}\boldsymbol{r}| = -cB
\end{gathered} \tag{3.19}$$

may be regarded as the discrete analogues of the formulae (2.9)–(2.11) derived in Section 2. Here, $c$ is a constant of integration.

The linear equations $(3.14)_{1,2}$ for $A$ and $B$ imply the existence of a potential $\tau$ defined according to

$$\begin{aligned} \tau_{11} &= \tilde{c}\frac{\tau_1^2}{\tau A} \quad &\Rightarrow \quad A &= \tilde{c}\frac{\tau_1^2}{\tau\tau_{11}} \\ \tau_{12} &= \frac{\tau_1\tau_2}{\tau H} \quad &\Rightarrow \quad H &= \frac{\tau_1\tau_2}{\tau\tau_{12}} \\ \tau_{22} &= \hat{c}\frac{\tau_2^2}{\tau B} \quad &\Rightarrow \quad B &= \hat{c}\frac{\tau_2^2}{\tau\tau_{22}} \end{aligned} \tag{3.20}$$

where $\tilde{c}$ and $\hat{c}$ are arbitrary constants. This, inserted into the nonlinear equation $(3.14)_3$ for $H$, results in the *discrete Tzitzeica equation*

$$\begin{vmatrix} \tau & \tau_1 & \tau_{11} \\ \tau_2 & \tau_{12} & \tau_{112} \\ \tau_{22} & \tau_{122} & \tau_{1122} \end{vmatrix} + \tilde{c}\hat{c}\tau_{12}^3 = 0. \tag{3.21}$$

It is noted that the Tzitzeica equation

$$(\ln h)_{xy} = h - h^{-2} \tag{3.22}$$

may be brought into the form

$$\begin{vmatrix} \tau & \tau_x & \tau_{xx} \\ \tau_y & \tau_{xy} & \tau_{xxy} \\ \tau_{yy} & \tau_{xyy} & \tau_{xxyy} \end{vmatrix} + \tfrac{1}{4}\tau^3 = 0 \tag{3.23}$$

in terms of a $\tau$-function defined via

$$h = -2(\ln\tau)_{xy}. \tag{3.24}$$

This underlines the analogy between the classical continuous case and the discrete formalism presented here.

In conclusion, it is observed that the discrete Tzitzeica system is invariant under $A \to \Lambda A$, $B \to \Lambda^{-1}B$ where $\Lambda$ is an arbitrary constant. Thus, we have the following theorem.

**Theorem.** *Every solution of the discrete Tzitzeica system (3.14) corresponds to a one-parameter family of discrete affine spheres governed by*

$$\begin{aligned} \boldsymbol{r}_{11} - \boldsymbol{r}_1 &= \frac{H_1 - 1}{H_1(H-1)}(\boldsymbol{r}_1 - \boldsymbol{r}) + \Lambda\frac{A}{H-1}(\boldsymbol{r}_{12} - \boldsymbol{r}_1) \\ \boldsymbol{r}_{12} + \boldsymbol{r} &= H(\boldsymbol{r}_1 + \boldsymbol{r}_2) \\ \boldsymbol{r}_{22} - \boldsymbol{r}_2 &= \frac{H_2 - 1}{H_2(H-1)}(\boldsymbol{r}_2 - \boldsymbol{r}) + \frac{1}{\Lambda}\frac{B}{H-1}(\boldsymbol{r}_{12} - \boldsymbol{r}_2). \end{aligned} \tag{3.25}$$

## 4 Discrete surfaces generated by the Tzitzeica transformation

In the previous section, discrete analogues of the properties of affine spheres led to a definition of discrete affine spheres. One of the basic ingredients in this construction was the discrete asymptotic net. On the other hand, it is known that discrete asymptotic nets may be generated by means of Bäcklund transformations applied to surfaces if the 'tangency condition' is satisfied, that is the line segment which connects corresponding points on neighbouring surfaces $\Sigma$ and $\Sigma_1$ is tangential to both surfaces and the Bianchi diagram associated with the Bäcklund transformation commutes. For instance, it has been shown that the classical Bäcklund transformation applied to pseudospherical surfaces generates the discrete K-surfaces [25]. Thus, in this case, the direct discretization method (similar to that applied in Section 3) and the Bäcklund transformation yield the *same* discrete surfaces. This section is devoted to the natural question as to whether such a remarkable result also holds in the case of affine spheres.

As pointed out in Section 2, the Gauss equations for affine spheres read

$$\begin{aligned} \boldsymbol{r}_{xx} &= \frac{h_x}{h}\boldsymbol{r}_x + \frac{\Lambda}{h}\boldsymbol{r}_y \\ \boldsymbol{r}_{xy} &= h\boldsymbol{r} \\ \boldsymbol{r}_{yy} &= \frac{h_y}{h}\boldsymbol{r}_y + \frac{\Lambda^{-1}}{h}\boldsymbol{r}_x \end{aligned} \tag{4.1}$$

where $h$ is a solution of the Tzitzeica equation. The following variant of the classical Moutard transformation may be used to construct an infinite number of affine spheres [23].

**Theorem (The Tzitzeica transformation ).** *The Gauss equations (4.1) and the Tzitzeica equation are invariant under*

$$\begin{aligned} \boldsymbol{r} \to \boldsymbol{r}_1 &= \boldsymbol{r} - \frac{2}{(\Lambda - \Lambda_1)h}\left(\Lambda\frac{\phi^1_x}{\phi^1}\boldsymbol{r}_y - \Lambda_1\frac{\phi^1_y}{\phi^1}\boldsymbol{r}_x\right) \\ h \to h_1 &= h - 2(\ln\phi^1)_{xy} \end{aligned} \tag{4.2}$$

*where $\phi^1$ is a scalar solution of the Gauss equations with parameter $\Lambda_1$.*

Since $\boldsymbol{r}_x$ and $\boldsymbol{r}_y$ are tangential to the surface $\Sigma = \{\boldsymbol{r} = \boldsymbol{r}(x,y)\}$, it is clear that the line segment $\Delta\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}$ is tangential to $\Sigma$. In addition, it is readily verified that $\Delta\boldsymbol{r}$ is also tangential to the new surface $\Sigma_1 = \{\boldsymbol{r}_1 = \boldsymbol{r}_1(x,y)\}$. Hence, the Tzitzeica transformation obeys the *tangency condition*

$$\Delta\boldsymbol{r} \,||\, \Sigma, \quad \Delta\boldsymbol{r} \,||\, \Sigma_1. \tag{4.3}$$

Now, in order to iterate the Tzitzeica transformation one needs a scalar solution of the transformed Gauss equations. This solution is obtained by inserting another scalar solution $\phi^2$ of the Gauss equations (4.1) with parameter $\Lambda_2 \neq \Lambda_1$ into the transformation formula $(4.2)_1$. Thus, by construction, the quantity

$$\phi_1^2 = \phi^2 - \frac{2}{(\Lambda_2 - \Lambda_1)h}\left(\Lambda_2 \frac{\phi_x^1}{\phi^1}\phi_y^2 - \Lambda_1 \frac{\phi_y^1}{\phi^1}\phi_x^2\right) \tag{4.4}$$

satisfies the Gauss equations associated with $\Sigma_1$. On the other hand, $\phi^2$ may be used to construct a surface $\Sigma_2$ from $\Sigma$ given by

$$\boldsymbol{r}_2 = \boldsymbol{r} - \frac{2}{(\Lambda - \Lambda_2)h}\left(\Lambda \frac{\phi_x^2}{\phi^2}\boldsymbol{r}_y - \Lambda_2 \frac{\phi_y^2}{\phi^2}\boldsymbol{r}_x\right) \tag{4.5}$$

with associated scalar solution

$$\phi_2^1 = \phi^1 - \frac{2}{(\Lambda_1 - \Lambda_2)h}\left(\Lambda_1 \frac{\phi_x^2}{\phi^2}\phi_y^1 - \Lambda_2 \frac{\phi_y^2}{\phi^2}\phi_x^1\right). \tag{4.6}$$

Hence, application of the Tzitzeica transformation to $\Sigma_1$ and $\Sigma_2$ produces two surfaces $\Sigma_{12}$ and $\Sigma_{21}$ whose position vectors read

$$\begin{aligned} \boldsymbol{r}_{12} &= \boldsymbol{r}_1 - \frac{2}{(\Lambda - \Lambda_2)h_1}\left(\Lambda \frac{\phi_{1x}^2}{\phi_1^2}\boldsymbol{r}_{1y} - \Lambda_2 \frac{\phi_{1y}^2}{\phi_1^2}\boldsymbol{r}_{1x}\right) \\ \boldsymbol{r}_{21} &= \boldsymbol{r}_2 - \frac{2}{(\Lambda - \Lambda_1)h_2}\left(\Lambda \frac{\phi_{2x}^1}{\phi_2^1}\boldsymbol{r}_{2y} - \Lambda_1 \frac{\phi_{2y}^1}{\phi_2^1}\boldsymbol{r}_{2x}\right). \end{aligned} \tag{4.7}$$

It turns out that these two surfaces coincide, that is

$$\boldsymbol{r}_{12} = \boldsymbol{r}_{21} \tag{4.8}$$

which implies *closure* of the corresponding *Bianchi diagram* [12].

The fact that the Bianchi diagram associated with the Tzitzeica transformation commutes is paramount to the analysis in the remainder of this paper. Indeed, for fixed parameters $x$ and $y$, iteration of the Tzitzeica transformation generates a two-dimensional lattice in $\mathbb{R}^3$ which we label by

$$\boldsymbol{r}(n_1, n_2) = \boldsymbol{r}(x, y; n_1, n_2) \tag{4.9}$$

so that the Tzitzeica transforms $\boldsymbol{r}_1, \boldsymbol{r}_2$ and $\boldsymbol{r}_{12} = \boldsymbol{r}_{21}$ correspond to increments of $n_1$ and $n_2$ in the sense of Section 3. This together with the tangency condition (4.3) gives rise to the following theorem.

**Theorem.** *The Tzitzeica transformation generates a two-parameter (x,y) family of discrete asymptotic nets. The unit normals of these nets coincide with those of the corresponding affine spheres.*

In order to obtain further properties of these discrete asymptotic nets, it is observed that on use of the expressions for $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ one may remove the tangent vectors in the expression for $\boldsymbol{r}_{12}$ in favour of $\boldsymbol{r}_1, \boldsymbol{r}_2$, and $\boldsymbol{r}$, leading to

$$\begin{aligned} \boldsymbol{r}_{11} - \boldsymbol{r}_1 &= \alpha(\boldsymbol{r}_1 - \boldsymbol{r}) + \beta(\boldsymbol{r}_{12} - \boldsymbol{r}_1) \\ \boldsymbol{r}_{12} + c_0 \boldsymbol{r} &= \hat{H}(c_1 \boldsymbol{r}_1 + c_2 \boldsymbol{r}_2) \\ \boldsymbol{r}_{22} - \boldsymbol{r}_2 &= \gamma(\boldsymbol{r}_2 - \boldsymbol{r}) + \delta(\boldsymbol{r}_{12} - \boldsymbol{r}_2), \end{aligned} \tag{4.10}$$

where

$$\hat{H} = \frac{\Lambda_2 + \Lambda_1}{\Lambda_2 - \Lambda_1}\left[1 - \frac{2}{(\Lambda_2 - \Lambda_1) h \phi^1 \phi^2}(\Lambda_2 \phi^1_x \phi^2_y - \Lambda_1 \phi^1_y \phi^2_x)\right]^{-1} \tag{4.11}$$

and the coefficients $c_i$ are given by

$$c_0 = c_1 c_2, \quad c_1 = -\frac{\Lambda + \Lambda_2}{\Lambda - \Lambda_2}, \quad c_2 = \frac{\Lambda + \Lambda_1}{\Lambda - \Lambda_1}. \tag{4.12}$$

The linear triad (4.10) may be regarded as discrete Gauss equations for the asymptotic nets generated by the Tzitzeica transformation. In fact, the compatibility conditions of this triad results in the parametrization

$$\begin{gathered} \alpha = c_2 \frac{\sigma_1}{\hat{H}_1 \sigma}, \quad \gamma = c_1 \frac{\sigma_2}{\hat{H}_2 \sigma} \\ \hat{A} = \sigma\beta - 1 + c_2^2, \quad \hat{B} = \sigma\delta - 1 + c_1^2 \\ \sigma = 1 + c_0 - (c_1 + c_2)\hat{H} \end{gathered} \tag{4.13}$$

and the nonlinear system

$$\hat{A}_2 = \frac{\hat{H}_1}{\hat{H}}\hat{A}, \quad \hat{B}_1 = \frac{\hat{H}_2}{\hat{H}}\hat{B}, \quad \hat{H}_{12} = F(\hat{H}, \hat{H}_1, \hat{H}_2, \hat{A}, \hat{B}) \tag{4.14}$$

where $F$ is a known function of the indicated arguments. Now, introduction of the scaling

$$\hat{A} = (c_2^2 - 1)A, \quad \hat{B} = (c_1^2 - 1)B, \quad \hat{H} = \frac{1 + c_0}{c_1 + c_2}H \tag{4.15}$$

leads to:

**Theorem (Generalized discrete affine spheres).** *The Tzitzeica transformation generates generalized discrete affine spheres which are defined by the discrete Gauss equations*

$$\begin{aligned} \boldsymbol{r}_{11} - \boldsymbol{r}_1 = {}& c_{11}\frac{H_1 - 1}{H_1(H - 1)}(\boldsymbol{r}_1 - \boldsymbol{r}) + c_{12}\frac{A - 1}{H - 1}(\boldsymbol{r}_{12} - \boldsymbol{r}_1) \\ \boldsymbol{r}_{12} + c_0 \boldsymbol{r} = {}& c_{00} H(c_1 \boldsymbol{r}_1 + c_2 \boldsymbol{r}_2) \\ \boldsymbol{r}_{22} - \boldsymbol{r}_2 = {}& c_{22}\frac{H_2 - 1}{H_2(H - 1)}(\boldsymbol{r}_2 - \boldsymbol{r}) + c_{21}\frac{B - 1}{H - 1}(\boldsymbol{r}_{12} - \boldsymbol{r}_2) \end{aligned} \tag{4.16}$$

*where*

$$c_{11} = \frac{\Lambda + \Lambda_1}{\Lambda - \Lambda_1}\frac{\Lambda_2 - \Lambda_1}{\Lambda_1 + \Lambda_2}, \quad c_{12} = 2\frac{\Lambda - \Lambda_2}{\Lambda - \Lambda_1}\frac{\Lambda_1}{\Lambda_1 + \Lambda_2}, \quad c_{22} = \frac{\Lambda + \Lambda_2}{\Lambda - \Lambda_2}\frac{\Lambda_1 - \Lambda_2}{\Lambda_1 + \Lambda_2}, \quad c_{21} = 2\frac{\Lambda - \Lambda_1}{\Lambda - \Lambda_2}\frac{\Lambda_2}{\Lambda_1 + \Lambda_2}, \quad c_{00} = \frac{\Lambda_2 + \Lambda_1}{\Lambda_2 - \Lambda_1}. \tag{4.17}$$

*Every solution of the generalized Tzitzeica system*

$$A_2 = \frac{H_1}{H}A, \quad B_1 = \frac{H_2}{H}B$$

$$\begin{aligned}&(\Lambda_1 - \Lambda_2)^2(H_{12} - 1)(H - 1)H \\ &+ H_{12}[4\Lambda_1\Lambda_2(H_1A - H)(H_2B - H) - (\Lambda_1 + \Lambda_2)^2(H_1 - 1)(H_2 - 1)H^2] = 0\end{aligned} \tag{4.18}$$

*corresponds to a one-parameter family of generalized discrete affine spheres represented by the parameter* $\Lambda$. *Discrete affine spheres are obtained in the well-defined limit* $\Lambda_1 \to \infty$, $\Lambda_2 \to 0$ *with the substitutions* $A \to -A\Lambda_1/2$ *and* $B \to -B/(2\Lambda_2)$.

*Remark.* The asymptotic nets generated by the Tzitzeica transformation depend on the parameters $\Lambda_1$ and $\Lambda_2$. In principle, these parameters may be functions of $n_1$ and $n_2$ respectively. In this case, the linear triad (4.10)–(4.12) is still valid even though the compatibility conditions are more complicated. It turns out, however, that in a natural continuum limit, the dependence of $\Lambda_i$ on $n_i$ corresponds to the usual ambiguity in the definition of asymptotic parameters on *generalized affine spheres*. A generalized affine sphere is defined by the requirement that the logarithm of the affine distance between the surface and a fixed point in $\mathbb{R}^3$ be a harmonic function. This will be discussed in a forthcoming paper [20].

In conclusion, we point out that, once again, the parametrization

$$A = \frac{\tau_1^2}{\tau\tau_{11}}, \quad H = \frac{\tau_1\tau_2}{\tau\tau_{12}}, \quad B = \frac{\tau_2^2}{\tau\tau_{22}} \tag{4.19}$$

yields a single discrete equation, viz

$$\begin{vmatrix} \tau & \tau_1 & \tau_{11} \\ \tau_2 & \tau_{12} & \tau_{112} \\ \tau_{22} & \tau_{122} & \tau_{1122} \end{vmatrix} = 4\frac{\Lambda_1\Lambda_2}{(\Lambda_1 - \Lambda_2)^2}\begin{vmatrix} \tau_1 & \tau_{11} & \tau_{12} \\ \tau_2 & \tau_{12} & \tau_{22} \\ \tau_{12} & \tau_{112} & \tau_{122} \end{vmatrix}. \tag{4.20}$$

This *generalized discrete Tzitzeica equation* has been recently identified as a symmetry reduction of a four-dimensional discrete equation which, in a natural continuum limit, may be regarded as another form of Plebański's heavenly equation governing self-dual Einstein spaces [19, 18].

## 5 The Bäcklund transformation

It is evident that the Tzitzeica transformation provides an invariance of the Gauss equations for both classical affine spheres and generalized discrete affine spheres. This is due to the fact that the Bianchi diagram associated with the Tzitzeica transformation commutes. Thus, if we apply another Tzitzeica transformation to our Bäcklund lattice of affine spheres $\boldsymbol{r}(n_1, n_2)$, we obtain a second lattice $[\boldsymbol{r}(n_1, n_2)]'$ of affine spheres. The closed Bianchi diagram now guarantees that these affine spheres are again Bäcklund transforms of each other. This may be symbolized by $[\boldsymbol{r}(n_1, n_2)]' = \boldsymbol{r}'(n_1, n_2)$. The explicit construction of $\boldsymbol{r}'$ is given below.

Let $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ be the Tzitzeica transforms of $\boldsymbol{r}$ as given by $(4.2)_1$ and (4.5) and $\psi$ be another solution of the Gauss equations (4.1) with parameter $\mu$. Then, the Tzitzeica transformation generated by $\psi$ reads

$$\boldsymbol{r} \to \boldsymbol{r}' = \boldsymbol{r} - \frac{2}{(\Lambda - \mu)h}\left(\Lambda \frac{\psi_x}{\psi}\boldsymbol{r}_y - \mu \frac{\psi_y}{\psi}\boldsymbol{r}_x\right) \tag{5.1}$$

and the action of the first two Tzitzeica transformations on $\psi$ is given by

$$\begin{aligned}
\psi_1 &= \psi - \frac{2}{(\mu - \Lambda_1)h}\left(\mu \frac{\phi^1_x}{\phi^1}\boldsymbol{r}_y - \Lambda_1 \frac{\phi^1_x}{\phi^1}\boldsymbol{r}_x\right)\\
\psi_2 &= \psi - \frac{2}{(\mu - \Lambda_2)h}\left(\mu \frac{\phi^2_x}{\phi^2}\boldsymbol{r}_y - \Lambda_2 \frac{\phi^2_x}{\phi^2}\boldsymbol{r}_x\right).
\end{aligned} \tag{5.2}$$

These relations and the expressions for $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ may be used to eliminate derivatives of $\boldsymbol{r}$ and $\psi$ in (5.1). One obtains

$$\begin{aligned}
\boldsymbol{r}' = \boldsymbol{r} &- \frac{H}{(\Lambda - \mu)(H - 1)}\\
&\times\left[(\Lambda - \Lambda_1)\frac{\Lambda_2 - \mu}{\Lambda_2 - \Lambda_1}\frac{\Delta_2\psi}{\psi}\Delta_1\boldsymbol{r} - (\Lambda - \Lambda_2)\frac{\Lambda_1 - \mu}{\Lambda_2 - \Lambda_1}\frac{\Delta_1\psi}{\psi}\Delta_2\boldsymbol{r}\right]
\end{aligned} \tag{5.3}$$

on use of the formulae (4.11) and $(4.15)_3$ for $H$. Moreover, it is directly verified that this transformation formula is not only valid for generalized discrete affine spheres generated by the Tzitzeica transformation but also in the generic case.

**Theorem.** *The Gauss equations for generalized discrete affine spheres and the generalized discrete Tzitzeica system are invariant under*

$$\begin{aligned}
\boldsymbol{r} \to \boldsymbol{r}' = \frac{\boldsymbol{S}}{\psi}, &\quad A \to A' = \frac{\psi_1^2}{\psi\psi_{11}}A\\
H \to H' = \frac{\psi_1\psi_2}{\psi\psi_{12}}H, &\quad B \to B' = \frac{\psi_2^2}{\psi\psi_{22}}B
\end{aligned} \tag{5.4}$$

*where*

$$\begin{gathered}\boldsymbol{S}=\psi\boldsymbol{r}-\frac{H}{(\Lambda-\mu)(H-1)}(\kappa_1\Delta_2\psi\Delta_1\boldsymbol{r}-\kappa_2\Delta_1\psi\Delta_2\boldsymbol{r})\\ \kappa_1=(\Lambda-\Lambda_1)\frac{\Lambda_2-\mu}{\Lambda_2-\Lambda_1},\quad \kappa_2=(\Lambda-\Lambda_2)\frac{\Lambda_1-\mu}{\Lambda_2-\Lambda_1}\end{gathered}\tag{5.5}$$

*and $\psi$ is a scalar solution of the Gauss equations (4.16) with parameter $\mu$. The new $\tau$-function is given by*

$$\tau'=\psi\tau\tag{5.6}$$

*without loss of generality.*

As pointed out earlier, the Gauss equations (3.25) for discrete affine spheres are obtained from those for generalized discrete affine spheres in the limit $\Lambda_1\to\infty$, $\Lambda_2\to 0$. Thus, on application of this limiting procedure one obtains a discrete analogue of the Tzitzeica transformation.

**Theorem (The discrete Tzitzeica transformation ).** *The Gauss equations associated with discrete affine spheres and the discrete Tzitzeica system are invariant under*

$$\begin{gathered}\boldsymbol{r}\to\boldsymbol{r}'=\frac{\boldsymbol{S}}{\psi},\quad A\to A'=\frac{\psi_1^2}{\psi\psi_{11}}A\\ H\to H'=\frac{\psi_1\psi_2}{\psi\psi_{12}}H,\quad B\to B'=\frac{\psi_2^2}{\psi\psi_{22}}B\end{gathered}\tag{5.7}$$

*where*

$$\boldsymbol{S}=\frac{\Lambda-\mu}{\Lambda+\mu}\left[\psi\boldsymbol{r}-\frac{H}{(\Lambda-\mu)(H-1)}(\Lambda\Delta_1\psi\Delta_2\boldsymbol{r}-\mu\Delta_2\psi\Delta_1\boldsymbol{r})\right]\tag{5.8}$$

*and $\psi$ is a scalar solution of the Gauss equations (3.25) with parameter $\mu$. The skew-symmetric bilinear quantity $\boldsymbol{S}$ is normalized in such a way that it obeys the identities*

$$\Delta_1\boldsymbol{S}=\psi\boldsymbol{r}_1-\psi_1\boldsymbol{r},\quad \Delta_2\boldsymbol{S}=\psi_2\boldsymbol{r}-\psi\boldsymbol{r}_2.\tag{5.9}$$

*These relations constitute the counterparts of those associated with the classical Tzitzeica transformation.*

## 6 Loop group description

Affine spheres allow a description in terms of loop groups. Here, it is shown that a natural discretization of this description leads to the same definition of discrete affine spheres. Whether the loop group discretization method may be applied to convex affine spheres is under current investigation. Technically, the latter case should be more difficult than the case of the indefinite Blaschke metric [2].

[2]Discrete definite as well as indefinite affine spheres are defined in our recent paper [5] in terms of duality relations involving a dual or co-normal lattice.

The frame equations for the family of immersions (2.15) are gauge equivalent to the matrix system

$$R_x = UR, \quad R_y = VR \tag{6.1}$$

where

$$R = \begin{pmatrix} \frac{1}{\lambda} & 0 & 0 \\ 0 & \frac{\lambda}{h} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{r}_x \\ \boldsymbol{r}_y \\ \boldsymbol{r} \end{pmatrix}, \quad U = \begin{pmatrix} \frac{h_x}{h} & \lambda a(x) & 0 \\ 0 & -\frac{h_x}{h} & \lambda \\ \lambda & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & \frac{h}{\lambda} \\ \frac{b(y)}{h^2\lambda} & 0 & 0 \\ 0 & \frac{h}{\lambda} & 0 \end{pmatrix} \tag{6.2}$$

and $\Lambda = \lambda^3$. The structure of the above linear system allows a natural algebraic interpretation[3]. Thus, introduce the loop group

$$\begin{aligned} G[\lambda] = \{\phi : \mathbb{R}_* \to SL(3, \mathbb{R}) : Q\phi(q\lambda)Q^{-1} = \phi(\lambda), \\ T[\phi(-\lambda)^{-1}]^T T = \phi(\lambda)\} \end{aligned} \tag{6.3}$$

where

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} q & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q = e^{2\pi i/3}. \tag{6.4}$$

The first reduction in (6.3) should be understood in terms of the Laurent series $\phi(\lambda) = \sum_{k\in\mathbb{Z}} \phi_k \lambda^k \in G[\lambda]$ with coefficients of the form

$$\phi_{3n} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}, \quad \phi_{3n+1} = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix}, \quad \phi_{3n+2} = \begin{pmatrix} 0 & 0 & * \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix}. \tag{6.5}$$

The Lie algebra of this group is

$$g[\lambda] = \{\xi : \mathbb{R}_* \to sl(3, \mathbb{R}) : Q\xi(q\lambda)Q^{-1} = \xi(\lambda), T\xi(-\lambda)^T T = -\xi(\lambda)\} \tag{6.6}$$

with the same Laurent series formulation.

The following natural subgroups of $G[\lambda]$ are essential for the construction of discrete affine spheres:

$$\begin{aligned} G^0 &= G^+[\lambda] \cap G^-[\lambda] = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \alpha \in \mathbb{R} \right\} \\ G^+[\lambda] &= \left\{ \phi \in G[\lambda] : \phi(\lambda) = \sum_{k\geq 0} \phi_k \lambda^k \right\} \\ G^-[\lambda] &= \left\{ \phi \in G[\lambda] : \phi(\lambda) = \sum_{k\leq 0} \phi_k \lambda^k \right\}. \end{aligned} \tag{6.7}$$

[3]The reductions (6.3) of the zero curvature representation for the Tzitzeica equation were first introduced in [13].

The corresponding sub-algebras of $g[\lambda]$ are

$$g^0 = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & -A & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A \in \mathbb{R} \right\}$$

$$g^+[\lambda] = \left\{ \xi \in g[\lambda] : \xi(\lambda) = \sum_{k\geq 0} \xi_k \lambda^k \right\} \tag{6.8}$$

$$g^-[\lambda] = \left\{ \xi \in g[\lambda] : \xi(\lambda) = \sum_{k\leq 0} \xi_k \lambda^k \right\}.$$

It is seen that $U, V \in g[\lambda]$, which implies that $R(x,y) \in G[\lambda]$ for all $(x,y) \in \mathbb{R}^2$ if the same holds for some point $(x_0, y_0) \in \mathbb{R}^2$. Conversely, the matrices $U, V$ in (6.2) turn out to be the *simplest* matrices in the corresponding sub-algebras $U \in g^+[\lambda]$, $V \in g^-[\lambda]$, $U, V \notin g^0$.

**Theorem (Loop group description of classical affine spheres).** *Let $\phi : U \to G[\lambda]$ be a smooth map on a domain $U \subset \mathbb{R}^2$ satisfying*

$$\phi_x \phi^{-1} = \lambda A + B, \quad \phi_y \phi^{-1} = \frac{1}{\lambda} C + D, \tag{6.9}$$

*where $A, B, C, D : U \to sl(3, \mathbb{R})$ with $A_{2,3} \neq 0, C_{1,3} \neq 0$ and $(x,y)$ are standard coordinates in $\mathbb{R}^2$. Then, the $G^0$-gauge equivalent matrix*

$$R = \sigma\phi, \quad \sigma = \begin{pmatrix} A_{2,3} & 0 & 0 \\ 0 & A_{2,3}^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.10}$$

*satisfies (6.1) with $a_y = b_x = 0$ and $h = C_{1,3} A_{2,3}$.*

This statement follows directly from $\lambda A + B$, $\frac{1}{\lambda} C + D \in g[\lambda]$ and the compatibility conditions for the system (6.9).

A natural integrable discretization of the system (6.1) is obtained as follows. With each point $(n_1, n_2)$ on a $\mathbb{Z}^2$-lattice one associates $\Phi(n_1, n_2) \in G[\lambda]$. The matrices $\Phi(n_1, n_2)$ corresponding to two neighbouring vertices are related by

$$\Phi_1 = \mathcal{U}\Phi, \quad \Phi_2 = \mathcal{V}\Phi, \tag{6.11}$$

where the usual notation has been used, that is $\Phi = \Phi(n_1, n_2), \Phi_1 = \Phi(n_1 + 1, n_2)$ and $\Phi_2 = \Phi(n_1, n_2 + 1)$. The matrices $\mathcal{U} = \mathcal{U}(n_1, n_2), \mathcal{V} = \mathcal{V}(n_1, n_2) \in G[\lambda]$ are associated with the edges connecting neighbouring lattice points and satisfy the compatibility condition

$$\mathcal{V}_1 \mathcal{U} = \mathcal{U}_2 \mathcal{V}. \tag{6.12}$$

Now, the Gauss equations (3.25) of the discrete affine spheres may be transformed into the matrix system

$$R_1 = \mathcal{U}R, \quad R_2 = \mathcal{V}R \tag{6.13}$$

by means of the gauge transformation

$$R = \sqrt{2}\begin{pmatrix} \frac{1}{\lambda} & 0 & 0 \\ 0 & \lambda\frac{H}{2(H-1)} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \Delta_1 \boldsymbol{r} \\ \Delta_2 \boldsymbol{r} \\ \boldsymbol{r} \end{pmatrix}, \tag{6.14}$$

where $\Lambda = \lambda^3$ and

$$\begin{gathered}\mathcal{U} = \begin{pmatrix} \frac{H_1-1}{H_1(H-1)} + A\lambda^3 & 2\lambda A & 2\lambda^2 A \\ \lambda^2\frac{(H-1)H_1}{2(H_1-1)} & \frac{H_1(H-1)}{(H_1-1)} & \lambda\frac{H_1(H-1)}{(H_1-1)} \\ \lambda & 0 & 1 \end{pmatrix}\\ \mathcal{V} = \begin{pmatrix} H & \frac{2}{\lambda^2}\frac{(H-1)^2}{H} & \frac{2}{\lambda}(H-1) \\ \frac{1}{2\lambda}B\frac{HH_2}{(H-1)(H_2-1)} & \frac{1}{H} + \frac{1}{\lambda^3}B\frac{H_2(H-1)}{H(H_2-1)} & \frac{1}{\lambda^2}B\frac{H_2}{H_2-1} \\ 0 & \frac{2}{\lambda}\frac{H-1}{H} & 1 \end{pmatrix}.\end{gathered} \tag{6.15}$$

Conversely, the matrices (6.15) turn out to be the *simplest* matrices obeying $\mathcal{U} \in G^+[\lambda]$, $\mathcal{V} \in G^-[\lambda]$, $\mathcal{U}, \mathcal{V} \notin G^0$.

**Lemma.** *The general form of cubic elements of* $G[\lambda]$ *is*

$$\begin{gathered}\{\mathcal{A} = \sum_{k=0}^{3}\mathcal{A}_k\lambda^k \in G[\lambda]\} = G^I_{(3)} \cup G^{II}_{(3)}\\ \{\mathcal{B} = \sum_{k=0}^{3}\mathcal{B}_k\lambda^{-k} \in G[\lambda]\} = G^I_{(-3)} \cup G^{II}_{(-3)}\\ \begin{aligned} G^J_{(3)} &= \{\mathcal{A}^J : a,b,c \in \mathbb{R}\}, \\ G^J_{(-3)} &= \{\mathcal{B}^J : d,f,g \in \mathbb{R}\}, \end{aligned} \qquad J = I, II,\end{gathered} \tag{6.16}$$

*where*

$$\begin{aligned} \mathcal{A}^I &= \begin{pmatrix} \frac{1}{a} + \frac{1}{2}\lambda^3 b^2 c & \lambda c & \lambda^2 bc \\ \frac{1}{2}\lambda^2 ab^2 & a & \lambda ab \\ \lambda b & 0 & 1 \end{pmatrix} \\ \mathcal{B}^I &= \begin{pmatrix} \frac{1}{f} + \frac{1}{2\lambda^3}dg^2 & \frac{1}{2\lambda^2}fg^2 & \frac{1}{\lambda}g \\ \frac{1}{\lambda}d & f & 0 \\ \frac{1}{\lambda^2}dg & \frac{1}{\lambda}fg & 1 \end{pmatrix} \end{aligned} \tag{6.17}$$

*and their inverses*

$$\begin{aligned}\mathcal{A}^{II} &= \begin{pmatrix} a & -\lambda c & 0 \\ \frac{1}{2}\lambda^2 ab^2 & \frac{1}{a} - \frac{1}{2}\lambda^3 b^2 c & -\lambda b \\ -\lambda ab & \lambda^2 bc & 1 \end{pmatrix} \\ \mathcal{B}^{II} &= \begin{pmatrix} f & \frac{1}{2\lambda^2} fg^2 & -\frac{1}{\lambda} fg \\ -\frac{1}{\lambda} d & \frac{1}{f} - \frac{1}{2\lambda^3} dg^2 & \frac{1}{\lambda^2} dg \\ 0 & -\frac{1}{\lambda} g & 1 \end{pmatrix}.\end{aligned} \tag{6.18}$$

*Remark.* The linear and quadratic cases, which correspond to $b = g = 0$ and $c = d = 0$ respectively in the formulae above, are trivial. In these cases, the dependence on $\lambda$ may be gauged away and one obtains equations in $G^0$.

Consider a map $\Phi : \mathbb{Z}^2 \to G[\lambda]$ with cubic $\Phi_1\Phi^{-1}$ and $\Phi_2\Phi^{-1}$. We call it *superdiscrete* if the image of the map $\Phi_1\Phi^{-1} : \mathbb{Z}^2 \to G[\lambda]$ does not lie completely in $G^I_{(3)}$ or completely in $G^{II}_{(3)}$, i.e. matrices of both forms $\mathcal{A}^I, \mathcal{A}^{II}$ appear in $\Phi(n_1 + 1, n_2)\Phi(n_1, n_2)^{-1}$ for some $n_1, n_2$, or the image of the map $\Phi_2\Phi^{-1} : \mathbb{Z}^2 \to G[\lambda]$ does not lie completely in $G^I_{(-3)}$ or completely in $G^{II}_{(-3)}$.

Since $\mathcal{A}^I = (\mathcal{A}^{II})^{-1}$, $\mathcal{B}^I = (\mathcal{B}^{II})^{-1}$ it is clear that in the non-superdiscrete case it is sufficient to consider, for example,

$$\Phi_1\Phi^{-1} : \mathbb{Z}^2 \to G^I_{(3)}, \quad \Phi_2\Phi^{-1} : \mathbb{Z}^2 \to G^{II}_{(-3)}.$$

All three other cases may be obtained by the change of variables $n_1 \to -n_1$ or $n_2 \to -n_2$ or $(n_1, n_2) \to -(n_1, n_2)$.

**Theorem (Loop group description of discrete affine spheres).** *Let $\Phi : \mathbb{Z}^2 \to G[\lambda]$ be a non-superdiscrete map with*

$$\Phi_1\Phi^{-1} = \sum_{k=0}^{3} \mathcal{A}_k \lambda^k, \quad \Phi_2\Phi^{-1} = \sum_{k=0}^{3} \mathcal{B}_k \lambda^{-k} \tag{6.19}$$

*and $\mathcal{A}_3 \not\equiv 0$, $\mathcal{B}_3 \not\equiv 0$. Then, up to a change of the orientation of the axes represented by $n_1 \to -n_1$ or $n_2 \to -n_2$ or both,[4] the system (6.19) is gauge equivalent to (6.13)–(6.15).*

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[4] $1 \to \bar{1}$ or $2 \to \bar{2}$ or both in our short notation

In order to prove the theorem, choose the directions of $n_1$ and $n_2$ in such a way that $\Phi_1\Phi^{-1} = \mathcal{A}^I$ and $\Phi_2\Phi^{-1} = \mathcal{B}^{II}$. The transformation

$$R = \sigma\Phi, \quad \sigma = \begin{pmatrix} b & 0 & 0 \\ 0 & b^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{6.20}$$

where $b$ is the coefficient in $\mathcal{A}^I$, gives rise to the admissible normalization $b = 1$. The analysis of the compatibility conditions then provides us with the identifications

$$f = H, \quad g = -2\frac{H-1}{H}, \quad a = \frac{(H-1)H_1}{(H_1-1)}. \tag{6.21}$$

Finally, introduction of

$$A = \frac{c}{2}, \quad B = -2d\frac{(H-1)(H_2-1)}{HH_2} \tag{6.22}$$

produces (6.13)–(6.15).

*Remark.* One of the possible descriptions of Bäcklund (Darboux) transformations is the so called 'dressing up' procedure [8]. In this approach, the transformation of the surface (or the frame of the surface) $\Psi$ is described by

$$\Psi \to \Psi' = Q\Psi, \tag{6.23}$$

where $\Psi$ and $Q$ lie in some loop group $G$. Commutativity of two dressing transformations represented by $Q_2Q_1\Psi = \tilde{Q}_1\tilde{Q}_2\Psi$ implies the compatibility condition

$$Q_2Q_1 = \tilde{Q}_1\tilde{Q}_2 \tag{6.24}$$

in $G$ which may be interpreted as an equation of a discretized surface.

Let us denote by $\mathbb{R}_*G[\lambda]$ the loop group defined in the same way as $G[\lambda]$ but with the target space $\mathbb{R}_*SL(3,\mathbb{R})$ instead of $SL(3,\mathbb{R})$. In this context, the Tzitzeica transformation (4.2) was described in [8] by a cubic polynomial $Q = \sum_{k=0}^{3} Q_k\lambda^k \in \mathbb{R}_*G[\lambda]$, where the leading term $Q_3$ is non-degenerate, i.e. $\det Q_3 \neq 0$. Consequently, the compatibility condition (6.24) implies the definition (4.16)–(4.18) of generalized discrete affine spheres.

## 7 Particular discrete affine spheres

Here, we briefly sketch how the discrete Tzitzeica transformation (5.7)–(5.8) may be used to construct explicitly a particular class of discrete affine spheres. To this end, it is observed that

$$H = \text{const}, \quad A = \text{const}, \quad B = \text{const} \tag{7.1}$$

constitutes a solution of the discrete Tzitzeica system (3.14) if and only if the constraint

$$(H+1)(H-1)^3 = ABH^2 \tag{7.2}$$

is satisfied. In this case, the discrete Gauss equations (3.13) reduce to a system of linear equations with constant coefficients, and hence it is sufficient to seek particular scalar solutions of the form

$$\phi = a^{n_1} b^{n_2} \tag{7.3}$$

where the constants $a$ and $b$ obey the algebraic system

$$\begin{aligned}(a+b)H &= ab+1\\ a(a+1)A &= (a-H)(a-1)(a-H^{-1})\\ b(b+1)B &= (b-H)(b-1)(b-H^{-1}).\end{aligned} \tag{7.4}$$

It is readily shown that the constraint (7.2) is a consequence of (7.4). Thus, for fixed $A, B$, and $H$, the cubic polynomial $(7.4)_2$ possesses at least one real root $\alpha$ with the corresponding real root $\beta$ of $(7.4)_3$ given by $(7.4)_1$. We may use these roots to parametrize $A, B$, and $H$ according to

$$H = \frac{\alpha\beta+1}{\alpha+\beta}, \quad A = \frac{\beta(\alpha+1)(\alpha-1)^3}{\alpha(\alpha+\beta)(\alpha\beta+1)}, \quad B = \frac{\alpha(\beta+1)(\beta-1)^3}{\beta(\alpha+\beta)(\alpha\beta+1)} \tag{7.5}$$

so that the constraint (7.2) is identically satisfied. The remaining roots are then obtained by solving quadratic equations. In this way, we obtain three linearly independent scalar solutions of the discrete Gauss equations (3.13) from which we can construct the position vector $\boldsymbol{r}$ of the seed discrete affine sphere.

In order to apply the discrete Tzitzeica transformation, one needs a scalar solution of the deformed discrete Gauss equations (3.25) with parameter $\mu$. It turns out (as in the continuous case) that the choice $\mu = \mathrm{i}\sigma$ with real $\sigma$ yields interesting discrete surfaces. Thus, the ansatz

$$\psi = p^{n_1} q^{n_2} \tag{7.6}$$

produces the algebraic system

$$\begin{aligned}(p+q)H &= pq+1\\ \mathrm{i}\sigma p(p+1)A &= (p-H)(p-1)(p-H^{-1})\\ -\mathrm{i}\sigma q(q+1)B &= (q-H)(q-1)(q-H^{-1}).\end{aligned} \tag{7.7}$$

It is observed that if $(p,q)$ is a solution of the above system then $(1/\bar{p}, 1/\bar{q})$ is another solution, where the overbar denotes complex conjugation. Hence, there are three pairs of roots

$$\begin{aligned}&p_{(1)}, \quad p_{(2)} = 1/\bar{p}_{(1)}, \quad p_{(3)} = 1/\bar{p}_{(3)}\\ &q_{(1)}, \quad q_{(2)} = 1/\bar{q}_{(1)}, \quad q_{(3)} = 1/\bar{q}_{(3)}\end{aligned} \tag{7.8}$$

related by $(7.7)_1$. This implies that the linear combination

$$\psi = \nu p_{(1)}^{n_1} q_{(1)}^{n_2} + (\bar{\nu}\bar{p}_{(1)}^{n_1}\bar{q}_{(1)}^{n_2})^{-1}, \quad \nu = \text{const} \tag{7.9}$$

is admissible as a solution of (3.25) or, equivalently,

$$\begin{gathered} \psi = \cosh(\gamma)\exp(\mathrm{i}\delta) \\ \gamma = c_1 n_1 + c_2 n_2 + c_3, \quad \delta = c_4 n_1 + c_5 n_2 + c_6, \end{gathered} \tag{7.10}$$

where the real constants $c_i$ may be read off (7.9). A new *real* solution of the discrete Tzitzeica system is therefore given by

$$\begin{aligned} H' &= \frac{\cosh(\gamma + c_1)\cosh(\gamma + c_2)}{\cosh(\gamma)\cosh(\gamma + c_1 + c_2)} H \\ A' &= \frac{\cosh(\gamma + c_1)^2}{\cosh(\gamma)\cosh(\gamma + 2c_1)} A \\ B' &= \frac{\cosh(\gamma + c_2)^2}{\cosh(\gamma)\cosh(\gamma + 2c_2)} B \end{aligned} \tag{7.11}$$

and the corresponding discrete affine sphere may be obtained by means of the transformation formula $(5.7)_1$ in a purely algebraic manner. By virtue of the representation

$$H' - H = -\frac{\sinh(c_1)\sinh(c_2)}{\cosh(\gamma)\cosh(\gamma + c_1 + c_2)} H \tag{7.12}$$

it is seen that $H'$ constitutes a simple discretization of the usual $\text{sech}^2$-shaped one-soliton solution.

The procedure sketched above is now illustrated by way of the simplest example

$$\alpha = \beta > 1, \quad \sigma = 1. \tag{7.13}$$

In this case, the roots of the system (7.4) turn out to be

$$\begin{aligned} a: &\quad \alpha, \quad a_\pm = \frac{\alpha(\alpha+1) \pm \mathrm{i}(\alpha - 1)\sqrt{\alpha(\alpha^2 + \alpha + 1)}}{\alpha(\alpha^2 + 1)} \\ b: &\quad \alpha, \quad b_\pm = a_\mp. \end{aligned} \tag{7.14}$$

Thus, decomposition of either $a_+$ or $a_-$ into

$$a_\pm = \tau e^{\mathrm{i}\Xi} \tag{7.15}$$

leads to three linearly independent real solutions of the discrete Gauss equations associated with the constant solution (7.5) given by

$$\begin{aligned} x &= \tau^{n_1+n_2}\cos[\Xi(n_1 - n_2)] \\ y &= \tau^{n_1+n_2}\sin[\Xi(n_1 - n_2)] \\ z &= \alpha^{n_1+n_2}, \end{aligned} \tag{7.16}$$

where

$$\tau = \alpha^{-1/2}, \quad \cos(\Xi) = \alpha^{1/2}\frac{\alpha+1}{\alpha^2+1}. \tag{7.17}$$

Consequently, the position vector of the corresponding discrete affine sphere reads

$$\boldsymbol{r} = D\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{7.18}$$

where $D$ is an arbitrary but non-singular constant matrix.

The roots of the algebraic equations $(7.7)_{\sigma=1}$ reduce to

$$\begin{gathered} p_{\pm} = \frac{\alpha+1+\mathrm{i}(\alpha-1)}{4\alpha}\left[\alpha+1\pm(\alpha-1)\sqrt{\frac{\alpha^2+4\alpha+1}{\alpha^2+1}}\right], \\ p_{(3)} = \frac{1-\mathrm{i}\alpha}{\alpha-\mathrm{i}}, \quad q_{\pm} = \bar{p}_{\pm}, \quad q_{(3)} = \bar{p}_{(3)}. \end{gathered} \tag{7.19}$$

Hence, on using the parametrization

$$p_{\pm} = \exp(\kappa + \mathrm{i}\chi), \tag{7.20}$$

where

$$\begin{gathered} \cosh(\kappa) = \frac{\alpha+1}{2^{3/2}\alpha}\sqrt{\alpha^2+1} \\ \cos(\chi) = \frac{\alpha+1}{\sqrt{2}\sqrt{\alpha^2+1}}, \quad \sin(\chi) = \frac{\alpha-1}{\sqrt{2}\sqrt{\alpha^2+1}}, \end{gathered} \tag{7.21}$$

we may choose

$$\psi = \cosh[\kappa(n_1+n_2)]\exp[\mathrm{i}\chi(n_1-n_2)] \tag{7.22}$$

as a particular scalar solution of the deformed discrete Gauss equations with parameter $\mu = \mathrm{i}$. Finally, insertion of the seed position vector (7.18) and $\psi$ into the discrete Tzitzeica transformation $(5.7)_{\Lambda=1}$ produces the following three particular solutions of the discrete Gauss equations:

$$\begin{aligned} x' &= \frac{\tau^{n_1+n_2}}{\alpha-1}\left[-1+\frac{\sqrt{2}\,\alpha}{\sqrt{\alpha^2+1}}\frac{\cosh[\kappa(n_1+n_2+1)]}{\cosh[\kappa(n_1+n_2)]}\right]\cos[\Xi(n_1-n_2)] \\ y' &= \frac{\tau^{n_1+n_2}}{\alpha-1}\left[-1+\frac{\sqrt{2}\,\alpha}{\sqrt{\alpha^2+1}}\frac{\cosh[\kappa(n_1+n_2+1)]}{\cosh[\kappa(n_1+n_2)]}\right]\sin[\Xi(n_1-n_2)] \\ z' &= \frac{\alpha^{n_1+n_2}}{\alpha-1}\left[\alpha(\alpha+1)-\sqrt{2}\sqrt{\alpha^2+1}\,\frac{\cosh[\kappa(n_1+n_2+1)]}{\cosh[\kappa(n_1+n_2)]}\right]. \end{aligned} \tag{7.23}$$

Thus, the position vector of the new discrete affine sphere takes the form

$$\boldsymbol{r}' = D' \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}, \quad \det D' \neq 0. \tag{7.24}$$

The discrete affine spheres represented by $\boldsymbol{r}$ and $\boldsymbol{r}'$ may be regarded as 'discrete surfaces of revolution' if $D = D' = \mathbb{1}$ and

$$\Xi = \frac{\pi}{N}, \quad N \in \mathbb{Z}. \tag{7.25}$$

More precisely, if this condition is satisfied then the discrete surfaces possess a $\mathbb{Z}_N$ rotational symmetry. Furthermore, if we apply the translational symmetry

$$n_1 \to n_1 + \tfrac{1}{2}\Sigma, \quad n_2 \to n_2 + \tfrac{1}{2}\Sigma, \quad \Sigma = \text{const} \tag{7.26}$$

to the formulae (7.23), we may choose $\Sigma$ in such a way that there exists an $n \in \mathbb{Z}$ for which

$$\tanh[\kappa(n+\Sigma)] = \epsilon\frac{\sqrt{\alpha^2+1}}{\sqrt{\alpha^2+4\alpha+1}}, \quad \epsilon = \pm 1. \tag{7.27}$$

It is readily shown that this choice leads to

$$\left.\begin{pmatrix} x' \\ y' \end{pmatrix}\right|_{n_1+n_2=n} = 0 \quad \text{or} \quad \Delta_i z'|_{n_1+n_2=n} = 0, \quad i = 1,2, \tag{7.28}$$

depending on the sign in (7.27), that is if $\epsilon = -\mathrm{sgn}(\kappa)$ or $\epsilon = \mathrm{sgn}(\kappa)$ respectively. In geometric terms, the former case corresponds to a local degeneration of the discrete surface to a vertex while the latter implies the existence of a closed planar coordinate polygon.

Seed discrete affine spheres ($\boldsymbol{r}$) together with their Tzitzeica transforms ($\boldsymbol{r}'$) are displayed in Fig. 1 for various values of $N$. The first surface on the right-hand side ($N = 4$) contains a square coordinate polygon parallel to the $(x, y)$-plane. The second surface on the right-hand side ($N = 6$) possesses a vertex. The elementary quadrilaterals which belong to both the upper part and the lower part of the surface are degenerate. In the remaining case ($N = 20$), the choice of $\Sigma$ becomes insignificant since $\kappa$ is sufficiently small. It is noted that for large $N$ the discrete affine spheres resemble their continuous counterparts as depicted in [21]. Indeed, it turns out that the latter are obtained in the limit $N \to \infty$.

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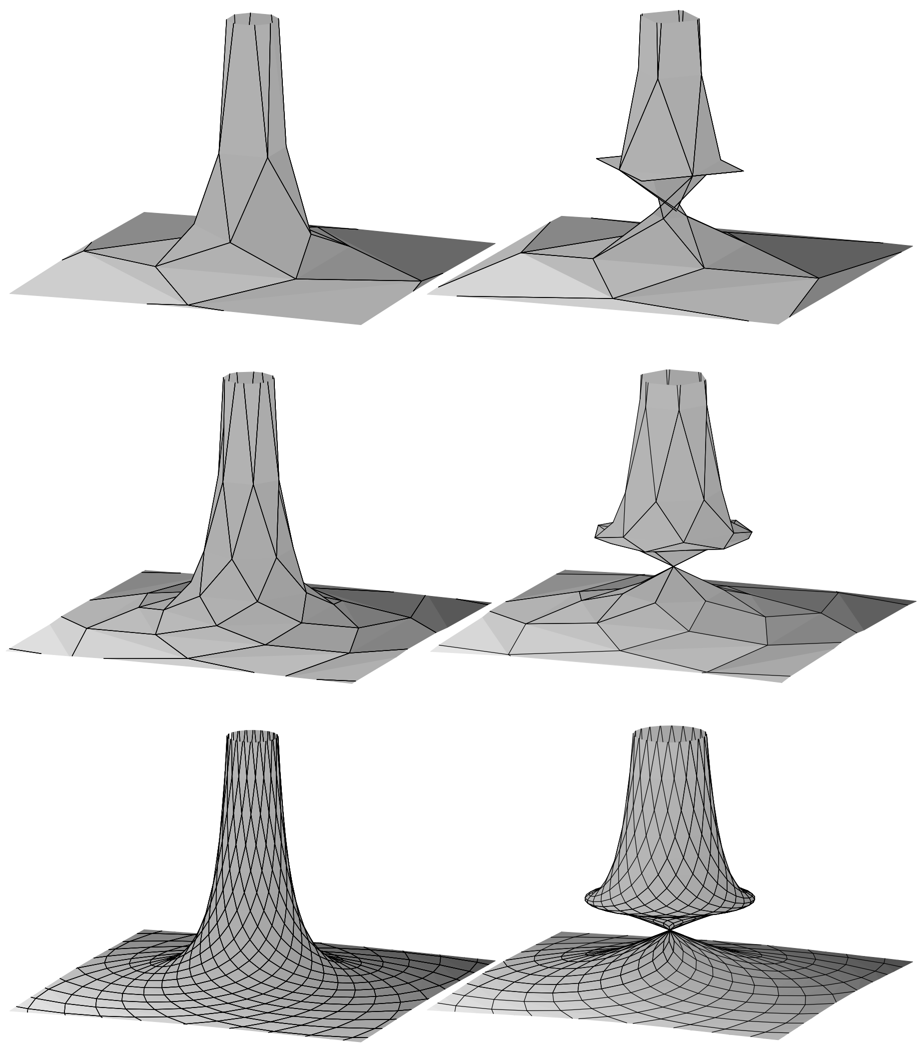

FIG. 1. Discrete affine spheres for $N = 4, 6, 20$

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