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Discretization of Surfaces and Integrable Systems

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1 Introduction

Long before the theory of solitons, geometers used integrable equations to describe various special curves and surfaces. Nowadays this field of research takes advantage of using both geometrical intuition and algebraic and analytic methods of soliton theory in order to study *integrable geometries*, i.e. geometries described by integrable systems.

The question of proper discretization of the geometries mentioned above has recently become a subject of intensive study. Indeed, one can suggest various discrete problems which have the same continuous limit and nevertheless have quite different properties. Is there a particular discretization among them one should choose? Interest in this problem is partially motivated by the importance of discretizations for numerical solution of differential equations or variational problems describing surfaces [1].

Taking into account the connection between the above-mentioned geometries and integrable systems it is natural to suggest two approaches to define proper discrete analogues of integrable geometries:
(i) to postulate natural discrete analogues of characteristic geometrical properties,
(ii) to construct a discrete integrable system corresponding to a given continous one.

These two approaches turn out to be complementary and (as experience shows) in those cases where both approaches are possible yield the same *discrete integrable geometry*, i.e. a geometry corresponding to a discrete integrable system. It is probably more natural to talk about different presentations rather then about different methods. These two presentations, which we call geometric and algebraic, are compared in Table 1.

The fundamental research in discrete systems is still in its infancy and many of the concepts have to be developed by trial and error. This is very much so in the special area of discrete integrable geometry, where we cannot present a developed theory starting with a general definition[2]. Rather we have to work by

[1] At this point we mention a recent construction [24] of compact constant mean curvature surfaces based on a new discretization algorithm [42].

[2] It is remarkable that many definitions in the present chapter have several versions.

Table 1 Geometric versus algebraic descriptions

	Geometric description	Algebraic description
Advantages	• *Consistent* (Definitions and properties are formulated in internal geometric terms no reference to integrable systems is needed) • *Descriptive* and *visual* (Geometric properties are intuitive)	• *General picture* (Provides a sort of Klein's Erlangen program in discrete integrable geometry) • *Computational methods* (Provides direct methods to prove statements and check conjectures)
Disadvantages	Does not explain why different geometries have similar properties	Not as descriptive, intuitive and visual as the geometric description

examples and combinations of geometric and algebraic methods.

For the present chapter we have chosen a historical presentation. In fact, a crucial point is a proper definition of the discrete geometry in question and we show which method first provided us with the corresponding definitions.

The sine-Gordon equation

$$\phi_{xt} - \sin\phi = 0$$

was probably the first known integrable equation. It was derived in differential geometry to describe surfaces with constant negative Gaussian curvature ($K = const < 0$). Another real realization of it — the elliptic sinh-Gordon equation

$$u_{z\bar{z}} + \sinh u = 0$$

is the Gauss equation for surfaces with constant mean curvature ($H = const$) and with constant positive Gaussian curvature . In the late 1980s significant progress in the theory of these surfaces was achieved (see [49, 44, 4, 38]) mainly due to the methods developed in soliton theory. It was natural to try again to make use of the cooperation of differential geometry and theory of integrable equations to define discrete analogues of these surfaces.

As a result of this research, discrete surfaces with constant Gaussian or mean curvature (we call them the discrete K- and H-surfaces) were defined in the early 1990s [3]. The method of derivation, which was based on an integrable discretiza-

[3]The corresponding results were first presented in 1991 in the talks of the authors at the conferences in Granada and Oberwolfach.

tion of the Lax representations, and the corresponding results are presented in Sections 3 and 4. Geometric properties of discrete K-surfaces are rather simple and these surfaces had already been known for about 40 years before we started to study them. First they were defined by Sauer [45] and investigated in detail by Wunderlich [50] by purely geometric methods. Of course, the relation to discrete integrable systems was at that time unknown. For our paper [10] we have chosen the geometric presentation (which is complementary to the presentation in Section 3): from the definition of Sauer and Wunderlich to the Lax representation and finite-gap integration. For a more algebraic loop group description see [43].

Discrete H-surfaces turned out to be important for further progress in the study of discrete integrable geometries. The geometric properties of discrete H-surfaces are not as transparent as those of discrete K-surfaces. As a result, the geometric Definition 11 of discrete H-surfaces, which consists of various ingredients of conformal and Euclidean geometries, wouldn't have been guessed without using the theory of integrable systems.

Keeping in mind that surfaces with constant mean curvature are isothermic and that it is the isothermic parametrization which is discretized in Section 4, it is natural to look for generalizations. Indeed, what are discrete isothermic surfaces and, more generally, discrete curvature line parametrized surfaces? The answer to the last question presented in Section 2.3 is more or less obvious: these are circular lattices [4]. The definition of discrete isothermic surfaces presented in Section 5.2 and suggested[5] in [11] is more complicated and requires the notion of cross-ratio (see Section 2.3). Based on these definitions a purely geometric theory of the Darboux transformations of discrete isothermic surfaces and of discrete H-surfaces has been developed in the contribution by Hertrich-Jeromin, Hoffmann and Pinkall [30].

Circular lattices as a discretization of curvature line parametrized surfaces allow a natural generalization to the three-dimensional case, which is a notion of discrete triply orthogonal coordinate systems introduced in [6]. This generalization based on a natural discretization of the Dupin theorem is discussed in Section 5.1. One can proceed further: namely, a generalization of circular lattices is the quadrilateral lattices introduced in [18]. These lattices provide a natural discretization of conjugate coordinate systems (see the contribution of Doliwa and Santini [19]). Algebraic and geometric descriptions in this case are especially simple.

A special case of discrete H-surfaces ($H = 0$) is discrete isothermic minimal surfaces (see [11] and Section 5.2). Via a discrete version of the Weierstrass representation they are intimately related to discrete conformal mappings .

These mappings considered in Section 5.3 are special circular nets in a plane.

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[4] Remarkably, circular lattices as a discretization of curvature line parametrization have been used in computer-aided surface design [37, 41].

[5] At this point we used the modern treatment of the isothermic surfaces of [15, 13].

In the contribution of Hoffmann [27] it is shown how discrete H-surfaces can be parametrized in terms of discrete conformal mappings . Algebraic properties of discrete conformal mappings (in particular compatible constraints) are studied in the contribution of Nijhoff [40]. Some of the constraints are important geometrically. In particular, a discrete analogue of the power function z^α can be described in this way. This discrete mapping is discussed in [6], Section 6 and the contribution of Hoffmann [27]. The discrete version of z^α belongs to a remarkable subclass of discrete conformal mappings. We finish our paper with the discussion of this subclass, recently introduced by Schramm [47] and suggest a generalization of his circular lattices to three-dimensional Euclidean space, which we call Schramm-isothermic nets.

2 Parametrizations of surfaces and their discretization

2.1 Parametrized surfaces and nets

Surfaces in Euclidean 3-space studied by analytical methods are usually described as maps

$$F : \mathcal{R} \to \mathbb{R}^3,$$

where $\mathcal{R}$ is a two-dimensional manifold. Let $(u, v) : U \to \mathbb{R}^2$ be a local coordinate on a domain $U \subset \mathcal{R}$. In these coordinates the fundamental forms are

$$\begin{aligned} I &= < dF, dF >= Edu^2 + 2Fdudv + Gdv^2 \\ II &= - < dF, dN >= Ldu^2 + 2Mdudv + Ndv^2, \end{aligned} \tag{2.1}$$

where $N : \mathcal{R} \to S^2$ is the Gauss map. The principal curvatures k_1, k_2 are the eigenvalues of the Weingarten operator

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

of an immersion ($EG - F^2 \neq 0$). For the mean and the Gaussian curvature this implies

$$\begin{aligned} H &= \frac{k_1 + k_2}{2} = \frac{EN + LG - 2MF}{2(EG - F^2)} \\ K &= k_1 k_2 = \frac{LN - M^2}{EG - F^2} \end{aligned}$$

In the simplest case $\mathcal{R}$ coincides with $\mathbb{R}^2$ or with some domain in $\mathbb{R}^2$. The theory we are dealing with is essentially a local one. We will not distinguish these two cases by notation and will write

$$F : \mathbb{R}^2 \to \mathbb{R}^3 \tag{2.2}$$

keeping in mind that F might be defined only on a domain in $\mathbb{R}^2$.

To discretize surfaces described by integrable equations and for investigation by analytical methods it is convenient to identify three-dimensional Euclidean space with the space Im$\mathbb{H}$ of imaginary quaternions and to describe immersion in terms of 2×2 matrices [5].

Let us denote the algebra of quaternions by $\mathbb{H}$, the multiplicative quaternion group by $\mathbb{H}_* = \mathbb{H}\setminus\{0\}$ and the standard basis of $\mathbb{H}$ by $\{\mathbf{1},\mathbf{i},\mathbf{j},\mathbf{k}\}$

$$\mathbf{ij}=\mathbf{k},\ \mathbf{jk}=\mathbf{i},\ \mathbf{ki}=\mathbf{j}.$$

Using the standard matrix representation of $\mathbb{H}$ the Pauli matrices σ_α are related to this basis as follows:

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}=i\ \mathbf{i},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}=i\ \mathbf{j},$$
$$\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}=i\ \mathbf{k},\quad \mathbf{1}=\begin{pmatrix}1&0\\0&1\end{pmatrix}. \tag{2.3}$$

The real and imaginary parts of the quaternion

$$q=q_0\mathbf{1}+q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k}$$

are defined by

$$\text{Re}\ q=q_0,\qquad \text{Imq}\ Q=q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k}$$

with the corresponding norms

$$|q|=\sqrt{q_0^2+|\text{Im}\ q|^2},\qquad |\text{Im}\ q|=\sqrt{q_1^2+q_2^2+q_3^2}.$$

The identification

$$X=-i\sum_{\alpha=1}^{3}X_\alpha\sigma_\alpha\in\text{Im}\ \mathbb{H}\ \longleftrightarrow\ X=(X_1,X_2,X_3)\in\mathbb{R}^3. \tag{2.4}$$

of $\mathbb{R}^3$ and Im$\mathbb{H}$ provides us with the following matrix representation

$$X=\begin{pmatrix}-iX_3&-iX_1-X_2\\-iX_1+X_2&iX_3\end{pmatrix} \tag{2.5}$$

of vectors in $\mathbb{R}^3$. For the scalar and vector products of vectors in terms of quaternions one has

$$X\times Y=\frac{1}{2}[X,Y],$$
$$<X,Y>=-\text{Re}\ (XY)=-\frac{1}{2}\text{tr}\ XY, \tag{2.6}$$
$$XY=-<X,Y>\mathbf{1}+X\times Y.$$

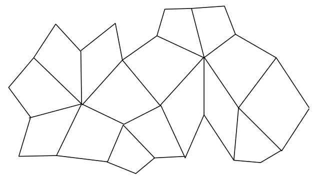


FIG. 1. Quad-graph

In the following we do not distinguish between vectors in $\mathbb{R}^3$, imaginary quaternions or their matrix representation (2.5). In particular this convention will be used for the immersion F and the Gauss map N.

Considering surfaces in $\mathbb{R}^3$ there is no need to deal with the most general parametrization (2.1). It is well known [17] that surfaces with negative Gaussian curvature allow asymptotic line parametrizations , i.e. parametrizations with $L = N = 0$ in (2.1). On the other hand, any surface without umbilic points (i.e. $k_1 \neq k_2$ for all points of the surface) allows curvature line parametrizations $F = M = 0$.

Discrete analogues of these two kinds of parametrizations are presented in the next two sections. Having in mind applications to the theory of integrable systems, by a discrete surface we mean a "quadrilateral surface", i.e. a surface "made out of quadrilaterals". More precisely, a discrete surface in the present paper is a map

$$F : G \to \mathbb{R}^3, \tag{2.7}$$

where G is a graph of special topology, which we call a *quad-graph* . Let us describe the topology of quad-graph, an example of which is presented in Fig. 1. Introduce the following notations:

$v \in V$–a vertex of G,
$V = \{$ vertices of $G\}$,
$e = [v, v'] \in E$ — the edge connecting the vertices $v, v' \in V$,
$E = \{$ edges of $G\}$,
$q = (v, v', v'', v''') \in Q$ the elementary quadrilateral of G with the vertices $v, v', v'', v''' \in V$
$Q = \{$ quadrilaterals of $G\}$.

The vertices of an elementary quadrilateral are connected by exactly four edges $[v, v'], [v', v''], [v'', v'''], [v''', v] \in E$ (in particular $[v, v''], [v', v'''] \notin E$). Each edge of a quad-graph belongs to either exactly one or exactly two elementary quadrilaterals. In the first case we say that the edge lies on the boundary ∂G of G.

Remark. Since the edges of the discrete surfaces studied in this chapter are analogs of the asymptotic or of the curvature lines on smooth surfaces, one can additionally assume that the number of edges meeting at each vertex is even.

The case $G = \mathbb{Z}^2$ or $G \subset \mathbb{Z}^2$ closest to (2.2) has been elaborated most extensively. To make the notation shorter we will write

$$F : \mathbb{Z}^2 \to \mathbb{R}^3 \tag{2.8}$$

also in the case when F is defined on a subset of $\mathbb{Z}^2$. We use the following notation for the elements of discrete surfaces (n, m are integer labels):

$F_{n,m}$ for the vertices,
$[F_{n+1,m}, F_{n,m}], [F_{n,m+1}, F_{n,m}]$ for the edges,
$(F_{n,m}, F_{n+1,m}, F_{n+1,m+1}, F_{n,m+1})$ for the elementary quadrilaterals.

Each vertex $F_{n,m}$ has 4 neighbours $F_{n-1,m}, F_{n+1,m}, F_{n,m-1}, F_{n,m+1}$.

2.2 Discrete A-surfaces (asymptotic line nets)

Let us consider a surface $\mathcal{F}$ with negative Gaussian curvature. For each regular point of $\mathcal{F}$ there are two directions where the normal curvature vanishes. These are called asymptotic directions. We use asymptotic line parametrizations of $\mathcal{F}$

$$\begin{aligned} F : \mathbb{R}^2 &\to \mathbb{R}^3, \\ (u, v) &\mapsto F(u, v). \end{aligned} \tag{2.9}$$

For such a parametrization one has $L = N = 0$ in (2.1), i.e. the vectors F_u, F_v, F_{uu}, F_{vv} are orthogonal to the normal vector N

$$F_u, F_v, F_{uu}, F_{vv} \perp N. \tag{2.10}$$

The fundamental forms are as follows:

$$\begin{aligned} I &= < dF, dF >= A^2 du^2 + 2AB \cos\phi \, du \, dv + B^2 dv^2, \\ II &= - < dF, dN >= 2M du \, dv, \end{aligned} \tag{2.11}$$

where ϕ is the angle between asymptotic lines and

$$A = |F_u|, \qquad B = |F_v|. \tag{2.12}$$

Our goal is to find a proper discrete version of this parametrization. Let us mention two important geometric properties of asymptotic line parametrizations which are easy to check and which we want to retain in the discrete case.

Property 1 *Asymptotic coordinates (2.9, 2.11) can be characterized in terms of F only:*

$$F_{uu}, \; F_{vv} \in \mathit{span}\ \{F_u, F_v\}. \tag{2.13}$$

The next property shows that the asymptotic line parametrization is natural in affine geometry.

Property 2 *(Affine invariance). Let $F : \mathcal{R} \to \mathbb{R}^3$ be a surface parametrized by asymptotic lines and $\mathcal{A}$ an affine transformation of Euclidean 3-space. Then $\tilde{F} \equiv \mathcal{A} \circ F : \mathcal{R} \to \mathbb{R}^3$ is also a parametrization by asymptotic lines.*

Motivated by these two properties we define discrete asymptotic line parametrizations (we call them *discrete A-surfaces* because of *asymptotic line* and *affine*) as follows:

Definition 1 *(Narrow definition of discrete A-surfaces) A discrete A-surface is a map*

$$F : \mathbb{Z}^2 \to \mathbb{R}^3$$

such that for each point $F_{n,m}$ there is a (tangent) plane $\mathcal{P}_{n,m}$ which contains $F_{n,m}$ and all its neighbouring points

$$F_{n,m}, F_{n+1,m}, F_{n-1,m}, F_{n,m+1}, F_{n,m-1} \in \mathcal{P}_{n,m}. \tag{2.14}$$

This definition agrees with the two properties mentioned above of asymptotic line parametrizations. The property (2.14) is obviously affinely invariant. The discrete versions $F_{n+1,m} - 2F_{n,m} + F_{n-1,m}$, $F_{n,m+1} - 2F_{n,m} + F_{n,m-1}$ of the second derivatives F_{uu}, F_{vv} lie in the tangent plane $\mathcal{P}_{n,m}$ of the vertex $F_{n,m}$, which agrees with (2.13).

Definition 1 can be easily generalized for the case (2.7) of a quad-graph G. In this way one can for example define discrete analogues of surfaces with non-positive curvature and with a more complicated asymptotic line net. Let us denote by $NN(v) \in V$ the set of the nearest neighbours of the vertex v, i.e. the set of vertices of G which have common edges with v.

Definition 2 *(Wide definition of discrete A-surfaces) A discrete A-surface is a map of a quad-graph*

$$F : G \to \mathbb{R}^3$$

such that for each point $F(v)$ there is a (tangent) plane $\mathcal{P}_v$, which contains $F(v) \in \mathcal{P}_v$ and all its neighbouring points

$$F(v') \in \mathcal{P}_v, \qquad \forall v' \in NN(v). \tag{2.15}$$

Special classes of discrete A-surfaces are discrete K- surfaces considered in Section 3 and discrete indefinite affine spheres considered in the contribution of Bobenko and Schief [12].

2.3 Discrete C-surfaces (curvature line nets)

Let $\mathcal{F}$ be a surface without umbilics and

$$\begin{aligned} F : \mathbb{R}^2 &\to \mathbb{R}^3, \\ (u,v) &\mapsto F(u,v). \end{aligned} \tag{2.16}$$

be a curvature line parametrization of $\mathcal{F}$. For a curvature line parametrization both fundamental forms are diagonal: $F = M = 0$,

$$\begin{aligned} I &= < dF, dF >= E\, du^2 + G\, dv^2, \\ II &= - < dF, dN >= L\, du^2 + N\, dv^2. \end{aligned} \tag{2.17}$$

Property 3 *Curvature line coordinates (2.16, 2.17) can be characterized in terms of the immersion function F only*

$$< F_u, F_v >= 0, \quad F_{uv} \in span\ \{F_u, F_v\}. \tag{2.18}$$

The next property shows that the notion of a curvature line parametrization belongs to conformal (Möbius) geometry.

Property 4 *(Möbius invariance). Let $F : \mathcal{R} \to \mathbb{R}^3$ be a surface parametrized by curvature lines and $\mathcal{M}$ a Möbius transformation of Euclidean 3-space. Then $\tilde{F} \equiv \mathcal{M} \circ F : \mathcal{R} \to \mathbb{R}^3$ is also a parametrization by curvature lines.*

Proof. Since the Möbius group is generated by inversions in spheres it is enough to prove Property 4 for the case of the inversion $\mathcal{M}$ in the unit sphere

$$\tilde{F} = \frac{F}{< F, F >}. \tag{2.19}$$

The direct calculation shows that $< \tilde{F}_u, \tilde{F}_v >= 0$ and

$$\tilde{F}_{uv} = \left(\alpha - 2\frac{< F, F_v >}{< F, F >}\right) \tilde{F}_u + \left(\beta - 2\frac{< F, F_u >}{< F, F >}\right) \tilde{F}_v,$$

where α and β are defined by

$$F_{uv} = \alpha F_u + \beta F_v.$$

To define discrete surfaces parametrized by curvature lines (we call these surfaces *discrete C-surfaces* because of *curvature line* and *conformal*) one needs the notion of the cross-ratio of a quadrilateral (X_1, X_2, X_3, X_4) in three-dimensional Euclidean space. The notion of the cross-ratio

$$q = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

of four complex numbers $z_1, z_2, z_3, z_4 \in \mathbb{C}$ can be easily extended to points in $\mathbb{R}^3$ by indentifying a sphere S, passing through X_1, X_2, X_3, X_4, with the Riemann sphere $\mathbb{CP}^1$. We usually will just speak of the "cross-ratio $q \in \mathbb{C}$". One has to keep in mind that q is only well defined up to complex conjugation, since S is not oriented.

There is a quaternionic description [11] of the cross-ratio based on the isomorphism (2.4).

Definition 3 *Let $X_1, X_2, X_3, X_4 \in \mathrm{Im}\ \mathbb{H}$ be four points in $\mathbb{R}^3$ and Q be the quaternion*

$$Q = (X_1 - X_2)(X_2 - X_3)^{-1}(X_3 - X_4)(X_4 - X_1)^{-1}. \tag{2.20}$$

The unordered pair of complex numbers

$$\{q, \bar{q}\} = \mathrm{Re}\ Q \pm i\ |\mathrm{Im}\ Q|$$

is called the cross-ratio of the quadrilateral (X_1, X_2, X_3, X_4).

Lemma 1 *The cross-ratio is invariant with respect to the Möbius transformations of* $\mathbb{R}^3$.

Lemma 2 *The cross-ratio of a quadrilateral is real iff it is inscribed in a circle*[6]. *The quadrilateral is embedded (i.e. its opposite edges do not intersect) iff* $q < 0$.

Definition 4 *Let* $F : \mathcal{R} \to \mathbb{R}^3$ *be a parametrized surface,* $(u, v) : U \subset \mathcal{R} \to \mathbb{R}^2$ *a local coordinate. A two-parameter* $\varepsilon = (\epsilon_1, \epsilon_2)$ *family of quadrilaterals* $F^\varepsilon = (F_1, F_2, F_3, F_4)$ *with vertices*

$$\begin{aligned} F_1 &= F(u - \epsilon_1, v - \epsilon_2), \\ F_2 &= F(u + \epsilon_1, v - \epsilon_2), \\ F_3 &= F(u + \epsilon_1, v + \epsilon_2), \\ F_4 &= F(u - \epsilon_1, v + \epsilon_2) \end{aligned}$$

is called an infinitesimal quadrilateral at (u, v).

We consider the limit $\epsilon \to 0$,

$$\epsilon_1 = \epsilon \Delta_1, \qquad \epsilon_2 = \epsilon \Delta_2 \tag{2.21}$$

with some fixed $\Delta_1, \Delta_2 \in \mathbb{R}$. Up to terms of order $O(\epsilon^3)$ the vertices of the infinitesimal quadrilateral coincide with

$$\begin{aligned} F_1 &= F + (-\epsilon_1 F_u - \epsilon_2 F_v) + \frac{1}{2}(\epsilon_1^2 F_{uu} + \epsilon_2^2 F_{vv} + 2\epsilon_1 \epsilon_2 F_{uv}) + O(\epsilon^3), \\ F_2 &= F + (\epsilon_1 F_u - \epsilon_2 F_v) + \frac{1}{2}(\epsilon_1^2 F_{uu} + \epsilon_2^2 F_{vv} - 2\epsilon_1 \epsilon_2 F_{uv}) + O(\epsilon^3), \\ F_3 &= F + (\epsilon_1 F_u + \epsilon_2 F_v) + \frac{1}{2}(\epsilon_1^2 F_{uu} + \epsilon_2^2 F_{vv} + 2\epsilon_1 \epsilon_2 F_{uv}) + O(\epsilon^3), \\ F_4 &= F + (-\epsilon_1 F_u + \epsilon_2 F_v) + \frac{1}{2}(\epsilon_1^2 F_{uu} + \epsilon_2^2 F_{vv} - 2\epsilon_1 \epsilon_2 F_{uv}) + O(\epsilon^3), \end{aligned}$$

where $F, F_u, ..., F_{vv}$ are the values of the immersion function and its derivatives at (u, v). The following remark is trivial:

Lemma 3

(i) $q(F^\varepsilon) = q + O(\epsilon), q \in \mathbb{R} \Longleftrightarrow Q(F^\varepsilon) = qI + O(\epsilon), \epsilon \to 0$
(ii) $q(F^\varepsilon) = q + O(\epsilon^2), q \in \mathbb{R} \Longleftrightarrow Q(F^\varepsilon) = qI + O(\epsilon^2), \epsilon \to 0$

Theorem 1 *Orthogonal and curvature line parametrized immersions* F *are characterized in terms of infinitesimal quadrilaterals as follows:*

(i) $Q(F^\varepsilon) = qI + O(\epsilon), q < 0, \epsilon \to 0 \Longleftrightarrow F$ *is orthogonally parametrized,*
(ii) $Q(F^\varepsilon) = qI + O(\epsilon^2), q < 0, \epsilon \to 0 \Longleftrightarrow F$ *is curvature line parametrized.*

[6] a straight line is a special case

Proof. To calculate the cross-ratio of the infinitesimal quadrilateral we note that F^ε up to scaling is a translation of the quadrilateral with vertices at

$$X_1 = 0, \quad X_2 = \Delta_1 F_u - \epsilon \Delta_1 \Delta_2 F_{uv} + O(\epsilon^2),$$
$$X_3 = \Delta_1 F_u + \Delta_2 F_v + O(\epsilon^2), \quad X_4 = \Delta_2 F_v - \epsilon \Delta_1 \Delta_2 F_{uv} + O(\epsilon^2).$$

Inverting it by the transformation (2.19) we map one of the points to infinity

$$\tilde{X}_1 = \infty, \quad \tilde{X}_2 = \frac{F_u - \epsilon \Delta_2 F_{uv}}{\Delta_1 \|F_u - \epsilon \Delta_2 F_{uv}\|^2} + O(\epsilon^2),$$
$$\tilde{X}_3 = \frac{\Delta_1 F_u + \Delta_2 F_v}{\|\Delta_1 F_u + \Delta_2 F_v\|^2} + O(\epsilon^2), \quad \tilde{X}_4 = \frac{F_v - \epsilon \Delta_1 F_{uv}}{\Delta_2 \|F_v - \epsilon \Delta_1 F_{uv}\|^2} + O(\epsilon^2).$$

The condition

$$Q(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4) = qI + O(\epsilon^l), \qquad l = 1, 2, \ q \in \mathbb{R}$$

is equivalent to

$$\tilde{X}_3 = \alpha \tilde{X}_2 + \beta \tilde{X}_4 + O(\epsilon^l), \qquad \alpha = -\frac{q}{1-q}, \ \beta = \frac{1}{1-q}.$$

This identity can be rewritten as

$$2\frac{\Delta_1 F_u + \Delta_2 F_v}{\|\Delta_1 F_u + \Delta_2 F_v\|^2} = \frac{\alpha}{\Delta_1} \frac{F_u - \epsilon \Delta_2 F_{uv}}{\|F_u\|^2} \left(1 + 2\epsilon \Delta_2 \frac{< F_u, F_{uv} >}{\|F_u\|^2}\right) \tag{2.22}$$
$$+ \frac{\beta}{\Delta_2} \frac{F_v - \epsilon \Delta_1 F_{uv}}{\|F_v\|^2} \left(1 + 2\epsilon \Delta_1 \frac{< F_v, F_{uv} >}{\|F_v\|^2}\right) + O(\epsilon^l).$$

One can easily check that the zero oder (ϵ^0) term of this identity is equivalent to $< F_u, F_v >= 0, \ q < 0$, which proves the first statement of the theorem. The term of order ϵ in the case $l = 2$ is equivalent to the condition that the vector F_{uv} lies in the tangential plane. Due to Property 3 of curvature line parametrizations this completes the proof.

This theorem motivates the following definition of discrete curvature line nets (discrete C-surfaces):

Definition 5 *(Narrow definition of discrete C-surfaces) A discrete C-surface is a map*

$$F : \mathbb{Z}^2 \to \mathbb{R}^3$$

such that all elementary quadrilaterals $(F_{n,m}, F_{n+1,m}, F_{n+1,m+1}, F_{n,m+1})$ *have negative cross-ratios*

$$Q(F_{n,m}, F_{n+1,m}, F_{n+1,m+1}, F_{n,m+1}) = q_{n,m}, \qquad q_{n,m} < 0. \tag{2.23}$$

The definition can be reformulated more geometrically using Lemma 2. It can be also generalized[7] for the case of a quad-graph (2.7), which is for example useful to define discrete analogues of surfaces with umbilic points .

Definition 6 *(Wide definition of discrete C-surfaces) A discrete C-surface is a map of a quad-graph*

$$F : G \to \mathbb{R}^3$$

such that all elementary quadrilaterals are inscribed in circles.

Special classes of discrete C-surfaces are discrete H-, I- and M-surfaces considered in Sections 4, 5.2, and 5.3.

3 Discrete K-surfaces (constant negative Gaussian curvature surfaces)

Discrete K-surfaces are natural discrete analogues of surfaces with constant negative Gaussian curvature. In this section we define discrete K-surfaces, study their properties and construct some examples. More geometric presentation of the theory can be found in [10]. For the loop group description see [43].

3.1 Smooth surfaces with constant negative Gaussian curvature

Here we present some fragments of the theory of smooth surfaces with constant negative Gaussian curvature , most of which are classical. More detailed presentations can be found in [2, 5].

Let

$$F : \mathcal{R} \to \mathbb{R}^3.$$

be an asymptotic line parametrization (2.11) of a surface $\mathcal{F}$ with negative Gaussian curvature. For the constant Gaussian curvature case $K = \det II / \det I = -1$ we get the second fundamental form

$$II = 2AB \sin\phi \, du \, dv$$

and the following Gauss–Codazzi equations :

$$\phi_{uv} - AB \sin\phi = 0, \tag{3.1}$$

$$A_v = B_u = 0. \tag{3.2}$$

Such a parametrization with non-vanishing A and B is called a weak Chebyshev net .

The Gauss–Codazzi equations are invariant with respect to the transformation

$$A \to \lambda A, \qquad B \to \lambda^{-1} B, \qquad \lambda \in \mathbb{R}. \tag{3.3}$$

Every surface with constant negative Gaussian curvature posesses a one-parameter family of deformations preserving the second fundamental form, the Gaussian curvature and the angle ϕ between the asymptotic lines. This deformation

[7] In this generalized definition the embeddedness of the quadrilaterals is not required.

is described by the transformation (3.3). This one-parameter family of surfaces is called the *associated family.*

Equations (3.1, 3.2) can be represented as the compatibility condition

$$U_v - V_u + [U, V] = 0$$

for the following system:

$$\Psi_u = U\Psi, \qquad \Psi_v = V\Psi, \tag{3.4}$$

$$\begin{aligned} U &= \frac{i}{2}\begin{pmatrix} \phi_u/2 & -A\lambda e^{-i\phi/2} \\ -A\lambda e^{i\phi/2} & -\phi_u/2 \end{pmatrix}, \\ V &= \frac{i}{2}\begin{pmatrix} -\phi_v/2 & B\lambda^{-1}e^{i\phi/2} \\ B\lambda^{-1}e^{-i\phi/2} & \phi_v/2 \end{pmatrix}. \end{aligned} \tag{3.5}$$

It can easily be checked that (for more detail see [5]) the following formulas describe the moving frame of a surface with $K = -1,\ |F_u| = \lambda A,\ |F_v| = \lambda^{-1}B$ if one uses the isomorphism (2.4):

$$F_u = -i\lambda A \Psi^{-1}\begin{pmatrix} 0 & e^{-i\phi/2} \\ e^{i\phi/2} & 0 \end{pmatrix}\Psi, \tag{3.6}$$

$$F_v = -i\frac{B}{\lambda}\Psi^{-1}\begin{pmatrix} 0 & e^{i\phi/2} \\ e^{-i\phi/2} & 0 \end{pmatrix}\Psi, \tag{3.7}$$

$$N = -i\Psi^{-1}\sigma_3\Psi. \tag{3.8}$$

Matrices (3.5) belong to the loop algebra

$$g_K[\lambda] = \{\xi : \mathbb{R}_* \to su(2) : \xi(-\lambda) = \sigma_3\xi(\lambda)\sigma_3\}$$

and Ψ in (3.6–3.8) lies in the corresponding loop group

$$G_K[\lambda] = \{\phi : \mathbb{R}_* \to SU(2) : \phi(-\lambda) = \sigma_3\phi(\lambda)\sigma_3\}. \tag{3.9}$$

The Sym formula [48] allows us to integrate (3.6, 3.7).

Theorem 2 *Let $\phi(u, v), A(u), B(v)$ be a solution of (3.1). Then the corresponding immersion with $K = -1, |F_u| = \lambda A, |F_v| = \lambda^{-1}B$ is given by*

$$F = 2\Psi^{-1}\frac{\partial\Psi}{\partial t}, \qquad \lambda = e^t, \tag{3.10}$$

where $\Psi(u, v, \lambda = e^t) \in SU(2)$ is a solution of (3.4, 3.5). The Gauss map is given by

$$N = -i\Psi^{-1}\sigma_3\Psi. \tag{3.11}$$

We consider not only immersions but more generally weakly regular surfaces, i.e. the surfaces with $A \neq 0, B \neq 0$ for all u, v. In this case the change of coordinates $u \to \tilde{u}(u), v \to \tilde{v}(v)$ reparametrizes the surface so that the asymptotic lines are parametrized by arc-lengths (generally different for u and v directions)

$$A = |F_u| = \text{const}, \quad B = |F_v| = \text{const}. \tag{3.12}$$

This parametrization is called an anisotropic Chebyshev net (a Chebyshev net if $A = B$). In this parametrization the Gauss equation and the system (3.4, 3.5) become the sine-Gordon equation with the standard Lax representation [21].

Finally we also mention a well-known fact, which can be easily checked.

Proposition 1 *The Gauss map $N : \mathbb{R}^2 \to S^2$ of the surface with $K = -1$ is Lorentz-harmonic, i.e.*

$$N_{uv} = \rho N, \quad \rho : \mathbb{R}^2 \to \mathbb{R}. \tag{3.13}$$

It forms in S^2 the same kind of Chebyshev net as the immersion function does in $\mathbb{R}^3$:

$$|N_u| = \lambda A, \quad |N_v| = \lambda^{-1} B. \tag{3.14}$$

3.2 Discrete K-surfaces from a discrete Lax representation

Discrete surfaces with constant Gaussian curvature (we will also call these surfaces discrete weak Chebyshev nets or *discrete K-surfaces*) are defined by natural discrete analogues of the properties (2.10, 3.2) of the corresponding smooth surfaces.

Definition 7 *(Geometric definition of discrete K-surfaces) A discrete K-surface is a discrete A-surface (see Definition 2)*

$$F : G \to \mathbb{R}^3,$$

such that the lengths of the opposite edges of the elementary quadrilaterals are equal.
In particular in the case $G = \mathbb{Z}^2$ one has:

(i) *For each point $F_{n,m}$ there is a plane $\mathcal{P}_{n,m}$ such that*

$$F_{n,m}, F_{n+1,m}, F_{n-1,m}, F_{n,m+1}, F_{n,m-1} \in \mathcal{P}_{n,m},$$

(ii) *the lengths of the opposite edges of the elementary quadrilaterals are equal*

$$\begin{aligned} ||F_{n+1,m} - F_{n,m}|| &= ||F_{n+1,m+1} - F_{n,m+1}|| = A_n \neq 0, \\ ||F_{n,m+1} - F_{n,m}]| &= ||F_{n+1,m+1} - F_{n+1,m}|| = B_m \neq 0, \end{aligned}$$

where we have incorporated into the notation that A_n does not depend on m and B_m not on n.

In our paper [10], we start with this geometrically motivated definition suggested by Sauer (see [46] and references therein, in particular [45]) and studied by Wunderlich [50] and show how integrable discrete systems appear. In the present chapter we follow another path, which comes from the theory of integrable systems: we discretize the Lax representation (3.4, 3.5) in a natural way, preserving its loop group structure, and show how Definition 7 appears in this approach [8].

A natural integrable discretization of the system (3.4, 3.5) looks as follows. To each point (n, m) of the $\mathbb{Z}^2$-lattice one associates a matrix $\Psi_{n,m}$. These matrices at two neighbouring vertices are related by

$$\Psi_{n+1,m} = \mathcal{U}_{n,m}\Psi_{n,m}, \qquad \Psi_{n,m+1} = \mathcal{V}_{n,m}\Psi_{n,m}, \tag{3.15}$$

where the matrices $\mathcal{U}_{n,m}$ and $\mathcal{V}_{n,m}$ are associated with the edges connecting the points $(n+1, m), (n, m)$ and $(n, m+1), (n, m)$ respectively. Having in mind the continuous limit (ϵ is a characteristic size of edges)

$$\mathcal{U} = I + \epsilon U + \cdots, \qquad \mathcal{V} = I + \epsilon V + \cdots$$

with U, V of the form (3.5), and preserving the group structure and the dependence of λ of the continuous case, it is natural to set

$$\mathcal{U}_{n,m} = \begin{pmatrix} a_{n,m} & \lambda b_{n,m} \\ -\lambda \bar{b}_{n,m} & \bar{a}_{n,m} \end{pmatrix}, \qquad \mathcal{V}_{n,m} = \begin{pmatrix} c_{n,m} & \lambda^{-1} d_{n,m} \\ -\lambda^{-1}\bar{d}_{n,m} & \bar{c}_{n,m} \end{pmatrix},$$

where the fields a, b, c, d live at the corresponding edges. The compatibility condition

$$\mathcal{V}_{n+1,m}\mathcal{U}_{n,m} = \mathcal{U}_{n,m+1}\mathcal{V}_{n,m} \tag{3.16}$$

in terms of these fields reads as follows:

$$\begin{gathered} a_{n,m+1}c_{n,m} - b_{n,m+1}\bar{d}_{n,m} = c_{n+1,m}a_{n,m} - d_{n+1,m}\bar{b}_{n,m}, \\ b_{n,m+1}\bar{c}_{n,m} = c_{n+1,m}b_{n,m}, \qquad a_{n,m+1}d_{n,m} = d_{n+1,m}\bar{a}_{n,m}. \end{gathered}$$

By a λ-independent gauge transformation

$$\mathcal{U}_{n,m} \to G_{n+1,m}\mathcal{U}_{n,m}G_{n,m}^{-1}, \qquad \mathcal{V}_{n,m} \to G_{n,m+1}\mathcal{V}_{n,m}G_{n,m}^{-1}$$

with the matrices

$$G_{n,m} = \begin{pmatrix} g_{n,m} & 0 \\ 0 & \bar{g}_{n,m} \end{pmatrix} \tag{3.17}$$

at vertices one can normalize

$$b_{n,m} = i, \qquad c_{n,m} = 1 \qquad \text{for all } n, m.$$

[8] We would like to mention that this method and not the one presented in [10] was how we came to the notion of the discrete K-surfaces.

Given $g_{0,0}$ this condition specifies all $g_{n,m}$ in a unique way. The equations in this gauge become as follows:

$$a_{n,m+1} - i\bar{d}_{n,m} = a_{n,m} + id_{n+1,m},$$
$$a_{n,m+1}d_{n,m} = d_{n+1,m}\bar{a}_{n,m}.$$

To simplify this system further, let us remark that the zeros of the left- and right-hand sides of the equality (which itself follows from (3.16))

$$v_{n+1,m}(\lambda)u_{n,m}(\lambda) = u_{n,m+1}(\lambda)v_{n,m}(\lambda),$$
$$u_{n,m}(\lambda) = \det \mathcal{U}_{n,m}, \; v_{n,m}(\lambda) = \det \mathcal{V}_{n,m},$$

considered as functions of λ, should coincide. Both terms in the products above have a pair of symmetric zeros $\lambda_0, -\lambda_0$. We suppose that the zeros of $u_{n,m+1}(\lambda)$ and $u_{n,m}(\lambda)$ coincide. This is equivalent to saying that the zeros $\lambda = \pm 1/p_n$ of $u_{n,m}(\lambda)$ are m-independent and the zeros $\lambda = \pm q_m$ of $v_{n,m}(\lambda)$ are n-independent

$$u_{n,m}(\lambda = \pm 1/p_n) = 0, \qquad v_{n,m}(\lambda = \pm q_m) = 0.$$

Applying these arguments one gets

$$a_{n,m} = \frac{1}{p_n}e^{i\alpha_{n,m}}, \qquad d_{n,m} = iq_m e^{i\delta_{n,m}}$$

and the equations[9]

$$\alpha_{n,m+1} + \alpha_{n,m} \equiv \delta_{n+1,m} - \delta_{n,m},$$
$$e^{i\alpha_{n,m+1}} - e^{i\alpha_{n,m}} = p_n q_m (e^{-i\delta_{n,m}} - e^{i\delta_{n+1,m}}).$$

The first equation can be easily resolved as

$$\alpha_{n,m} \equiv h_{n+1,m} - h_{n,m}, \quad \delta_{n,m} \equiv h_{n,m+1} + h_{n,m}, \tag{3.18}$$

where $h_{n,m}$ can now be associated with the corresponding vertices. Finally, $h_{n,m}$ satisfies the equation

$$\exp(ih_{n+1,m+1} + ih_{n,m}) - \exp(ih_{n+1,m} + ih_{n,m+1}) =$$
$$p_n q_m (1 - \exp(ih_{n+1,m+1} + ih_{n+1,m} + ih_{n+1,m} + ih_{n,m})). \tag{3.19}$$

This equation first appeared in a paper by Hirota [25] without any relation to geometry. In the exponential form $H_{n,m} = \exp(ih_{n,m})$ it looks as follows:

$$H_{n+1,m+1}H_{n,m} = \frac{k_{n,m} + H_{n+1,m}H_{n,m+1}}{1 + k_{n,m}H_{n+1,m}H_{n,m+1}}, \qquad k_{n,m} = p_n q_m. \tag{3.20}$$

[9] by $\alpha \equiv \alpha'$ we mean $\alpha = \alpha' \pmod{2\pi}$

Having in mind the loop group interpretation of Section 3.1, our goal is to define in the discrete case a map $\Phi : \mathbb{Z}^2 \to G_K[\lambda]$, where $G_K[\lambda]$ is the loop group (3.9). Let us multiply $\mathcal{U}_{n,m}$ and $\mathcal{V}_{n,m}$ by scalar factors to make their determinants equal to 1:

$$\overset{0}{\mathcal{U}}_{n,m} = \frac{1}{\sqrt{p_n^{-2}+\lambda^2}}\mathcal{U}_{n,m}, \quad \overset{0}{\mathcal{V}}_{n,m} = \frac{1}{\sqrt{\lambda^{-2}q^2{}_m+1}}\mathcal{V}_{n,m}, \tag{3.21}$$

$$\mathcal{U}_{n,m}(\lambda) = \begin{pmatrix} \frac{1}{p_n}e^{ih_{n+1,m}-ih_{n,m}} & i\lambda \\ i\lambda & \frac{1}{p_n}e^{-ih_{n+1,m}+ih_{n,m}} \end{pmatrix}, \tag{3.22}$$

$$\mathcal{V}_{n,m}(\lambda) = \begin{pmatrix} 1 & \frac{iq_m}{\lambda}e^{ih_{n,m+1}+ih_{n,m}} \\ \frac{iq_m}{\lambda}e^{-ih_{n,m+1}-ih_{n,m}} & 1 \end{pmatrix}. \tag{3.23}$$

Evidently $\overset{0}{\mathcal{U}}_{n,m}, \overset{0}{\mathcal{V}}_{n,m}$ satisfy the compatibility condition (3.16).

Theorem 3 *(Algebraic definition of discrete K-surfaces) Let $h_{n,m}$ be a solution of (3.19) and $\Phi_{n,m} : \mathbb{Z}^2 \to G_K[\lambda]$ be a solution of the system*

$$\Phi_{n+1,m} = \overset{0}{\mathcal{U}}_{n,m} \Phi_{n,m}, \;\; \Phi_{n,m+1} = \overset{0}{\mathcal{V}}_{n,m} \Phi_{n,m}. \tag{3.24}$$

Then the discrete surface described by the formula

$$F_{n,m} = 2\Phi^{-1}_{n,m}\frac{\partial \Phi_{n,m}}{\partial t}, \qquad \lambda = e^t \tag{3.25}$$

is a discrete K-surface in a sense of Definition 7. The Gauss map $N_{n,m}$ of this surface ($N_{n,m}$ is defined as a unit vector orthogonal to the plane $\mathcal{P}_{n,m}$) is given by

$$N_{n,m} = -i\Phi^{-1}_{n,m}\sigma_3\Phi_{n,m}. \tag{3.26}$$

Proof. For the edges defined by (3.25) one has

$$F_{n+1,m} - F_{n,m} = 2\Phi^{-1}_{n,m}(\overset{0}{\mathcal{U}}{}^{-1}_{n,m}\frac{\partial \overset{0}{\mathcal{U}}_{n,m}}{\partial t})\Phi_{n,m},$$

$$F_{n,m+1} - F_{n,m} = 2\Phi^{-1}_{n,m}(\overset{0}{\mathcal{V}}{}^{-1}_{n,m}\frac{\partial \overset{0}{\mathcal{V}}_{n,m}}{\partial t})\Phi_{n,m},$$

$$F_{n-1,m} - F_{n,m} = -2\Phi^{-1}_{n,m}(\frac{\partial \overset{0}{\mathcal{U}}_{n-1,m}}{\partial t}\overset{0}{\mathcal{U}}{}^{-1}_{n-1,m})\Phi_{n,m},$$

$$F_{n,m-1} - F_{n,m} = -2\Phi^{-1}_{n,m}(\frac{\partial \overset{0}{\mathcal{V}}_{n,m-1}}{\partial t}\overset{0}{\mathcal{V}}{}^{-1}_{n,m-1})\Phi_{n,m}.$$

All these vectors as well as $N_{n,m}$ (forming a frame associated with the vertex $F_{n,m}$) have common factors $\Phi_{n,m}{}^{-1}$ on the left and $\Phi_{n,m}$ on the right, which describe a rotation of this frame as a whole. Considering the local geometry of the frame we can neglect this rotation. Direct calculation yields

$$
\begin{aligned}
2\,\overset{0}{\mathcal{U}}{}^{-1}_{n,m}\frac{\partial \overset{0}{\mathcal{U}}_{n,m}}{\partial t} &= i\sin\Delta^u_n(\lambda)\begin{pmatrix} 0 & e^{ih_{n,m}-ih_{n+1,m}} \\ e^{-ih_{n,m}+ih_{n+1,m}} & 0 \end{pmatrix},\\
2\,\overset{0}{\mathcal{V}}{}^{-1}_{n,m}\frac{\partial \overset{0}{\mathcal{V}}_{n,m}}{\partial t} &= -i\sin\Delta^v_m(\lambda)\begin{pmatrix} 0 & e^{ih_{n,m}+ih_{n,m+1}} \\ e^{-ih_{n,m}-ih_{n,m+1}} & 0 \end{pmatrix},\\
-2\frac{\partial \overset{0}{\mathcal{U}}_{n-1,m}}{\partial t}\overset{0}{\mathcal{U}}{}^{-1}_{n-1,m} &= -i\sin\Delta^u_n(\lambda)\begin{pmatrix} 0 & e^{ih_{n,m}-ih_{n-1,m}} \\ e^{-ih_{n,m}+ih_{n-1,m}} & 0 \end{pmatrix},\\
-2\frac{\partial \overset{0}{\mathcal{V}}_{n,m-1}}{\partial t}\overset{0}{\mathcal{V}}{}^{-1}_{n,m-1} &= i\sin\Delta^v_m(\lambda)\begin{pmatrix} 0 & e^{ih_{n,m}+ih_{n,m-1}} \\ e^{-ih_{n,m}-ih_{n,m-1}} & 0 \end{pmatrix},
\end{aligned}
\tag{3.27}
$$

where we defined

$$
\sin\Delta^u_n(\lambda) := \frac{2\lambda p_n^{-1}}{\lambda^2+p_n^{-2}}, \qquad \sin\Delta^v_m(\lambda) := \frac{2\lambda^{-1}q_m}{1+\lambda^{-2}q_m^2}. \tag{3.28}
$$

The vectors given by (3.27) are orthogonal to $-i\sigma_3$, which proves the orthogonality of the corresponding edges to $N_{n,m}$. The property (ii) of Definition 7 of the discrete K-surfaces evidently also holds.

3.3 Gauss map of discrete K-surfaces

Definition 8 *A map $N:\mathbb{Z}^2\to S^2$ is called a Chebyshev net if $< N_{n+1,m}, N_{n,m} >$ is independent of m and $< N_{n,m+1}, N_{n,m} >$ is independent of n.*

Corollary 1 *The Gauss map (3.26) forms a Chebyshev net. Under the action of the associated family (λ-family) the edges and the normals of the discrete K-surface described in Theorem 3 transform as follows*

$$
\begin{aligned}
< N_{n+1,m}, N_{n,m} > &= \cos\Delta^u_n(\lambda), \qquad & \|F_{n+1,m}-F_{n,m}\| &= \sin\Delta^u_n(\lambda),\\
< N_{n,m+1}, N_{n,m} > &= \cos\Delta^v_m(\lambda), \qquad & \|F_{n,m+1}-F_{n,m}\| &= \sin\Delta^v_m(\lambda),
\end{aligned}
$$

where the angles $\Delta(\lambda)$ are determined by (3.28).

This corollary allows us to interpret $\Delta^u_n(\lambda)$ and $\Delta^v_m(\lambda)$ as the angles between the planes $\mathcal{P}_{n+1,m}, \mathcal{P}_{n,m}$ and between the planes $\mathcal{P}_{n,m+1}, \mathcal{P}_{n,m}$ respectively.

Corollary 2 *The vectors of the normals and the edges of a discrete K-surface as described in Theorem 3 are related as follows:*

$$
\begin{aligned}
F_{n+1,m}-F_{n,m} &= N_{n+1,m}\times N_{n,m},\\
F_{n,m+1}-F_{n,m} &= -N_{n,m+1}\times N_{n,m}.
\end{aligned}
\tag{3.29}
$$

Proof. Use the isomorphism (2.4) and the moving frame in the proof of Theorem 3.

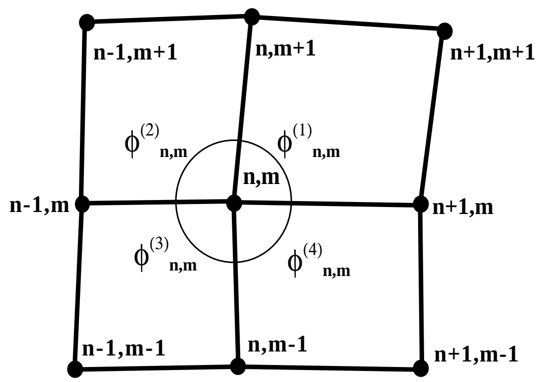


FIG. 2. Angles between edges of a discrete K-surface

Definition 9 *A map $N : \mathbb{Z}^2 \to S^2$ is called Lorentz-harmonic if for any n, m*

$$N_{n+1,m+1} - N_{n+1,m} - N_{n,m+1} + N_{n,m} = \rho_{n,m}(N_{n+1,m+1} + N_{n+1,m} + N_{n,m+1} + N_{n,m}) \tag{3.30}$$

with some $\rho : \mathbb{Z}^2 \to \mathbb{R}$.

A direct computation with the frames of Theorem 3 proves the following

Corollary 3 *The Gauss map (3.26) is Lorentz-harmonic*

3.4 The discrete sine-Gordon equation

The formulas (3.27) allow us to determine the angles between all edges (see Fig. 2 for the notations of the angles)

$$\begin{aligned} \phi^{(1)}_{n,m} &\equiv -h_{n,m+1} - h_{n+1,m} + \pi, \\ \phi^{(2)}_{n,m} &\equiv h_{n-1,m} + h_{n,m+1}, \\ \phi^{(3)}_{n,m} &\equiv -h_{n,m-1} - h_{n-1,m} + \pi, \\ \phi^{(4)}_{n,m} &\equiv h_{n+1,m} + h_{n,m-1}. \end{aligned} \tag{3.31}$$

Let us consider again a small piece of a discrete K-surface and derive a difference equation for the angles between edges, which can be regarded as a difference analogue of the sine-Gordon equation (3.1). Now if we orient the lattice diagonally (Fig. 3), the following theorem holds.

Theorem 4 *The neighbouring angles between the edges of a discrete K-surface satisfy the equation*

$$\phi_u + \phi_d - \phi_l - \phi_r \equiv 2\arg(1 - k_l e^{-i\phi_l}) + 2\arg(1 - k_r e^{-i\phi_r}), \tag{3.32}$$

where

$$k_l = p_l q_l, \;\; k_r = p_r q_r$$

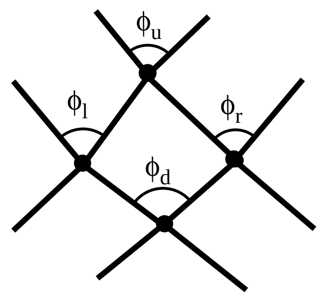


FIG. 3. Diagonally oriented lattice

are the products (see Section 3.2) associated with the quadrilaterals corresponding to ϕ_l and ϕ_r respectively.

Proof. Using the symmetry of the quadrilateral

$$\phi^{(1)}_{n,m} = \phi^{(3)}_{n+1,m+1}, \ \phi^{(2)}_{n+1,m} = \phi^{(4)}_{n,m+1}, \tag{3.33}$$

the Hirota equation (3.19)

$$\exp(i\phi^{(1)}_{n,m} + i\phi^{(2)}_{n+1,m}) + 1 = k_{n,m}(\exp(i\phi^{(1)}_{n,m}) + \exp(i\phi^{(2)}_{n+1,m})), \ k_{n,m} = p_n q_m,$$

and the fact that the sum of angles around a vertex is equal to 2π

$$\phi^{(1)}_{n,m} + \phi^{(2)}_{n,m} + \phi^{(3)}_{n,m} + \phi^{(4)}_{n,m} \equiv 0,$$

one derives for $\phi^{(1)}_{n,m}$

$$\phi^{(1)}_{n,m} + \phi^{(1)}_{n-1,m-1} - \phi^{(1)}_{n-1,m} - \phi^{(1)}_{n,m-1} =$$
$$2\arg(1 - k_{n-1,m}\exp(-i\phi^{(1)}_{n-1,m})) + 2\arg(1 - k_{n,m-1}\exp(-i\phi^{(1)}_{n,m-1})).$$

Turning the lattice by 45° we get this equation in the form (3.32). For symmetry reasons all the angles $\phi^{(2)}$, $\phi^{(3)}$, $\phi^{(4)}$ satisfy the same equation (3.32).

For obvious reasons equation (3.32) is called the discrete sine-Gordon equation. In the exponential form $Q = \exp(i\phi)$ this equation reads as follows:

$$Q_u Q_d = \frac{Q_l - k_l}{1 - k_l Q_l}\frac{Q_r - k_r}{1 - k_r Q_r}. \tag{3.34}$$

Let us consider now a Lorentz-harmonic Chebyshev net in S^2 generated by the Gauss map $N_{n,m}$. The angles ψ between the arcs of the great circles in S^2 generated by the corresponding normals satisfy the following difference equation

$$\psi_u + \psi_d - \psi_l - \psi_r \equiv 2\arg(1 + k_l e^{-i\psi_l}) + 2\arg(1 + k_r e^{-i\psi_r}). \tag{3.35}$$

Applications of the discrete sine-Gordon equation (3.34) extend beyond differential geometry. This equation can be considered over a finite field. A cellular automaton with a Lax representation has been constructed in this way in [7]. Equation (3.34), in contrast with (3.20), possesses a natural local Hamiltonian structure. A quantum version of (3.34) has been derived in [9] based on the results of [22]. For further results on quantization of this and similar models see the contributions of Faddeev and Volkov [23], Kashaev and Reshetikhin [32], and Kellendonk, Kutz, and Seiler [34]. A Lagrangian formalism for the discrete sine-Gordon equation is presented in the contribution of Kutz [36].

3.5 Construction of discrete K-surfaces

A simple geometrical method described below allows us to construct all discrete K-surfaces with periodic Gauss map. In particular, this class includes all discrete K-cylinders. One constructs these surfaces solving the Cauchy problem with an initial stairway (see Fig. 5) loop

$$N_{n,m}, N_{n,m+1}, N_{n+1,m+1}, \ldots, N_{n+N,m+M} = N_{n,m}$$

on S^2.

All $N_{n,m},\ n,m \in \mathbb{Z}$ can be reconstructed using the property (3.30) of N to be Lorentz-harmonic. Equation (3.30) uniquely determines $N_{n,m+1}$ by $N_{n,m}$, $N_{n+1,m}$, $N_{n+1,m+1}$

$$N_{n,m+1} = -N_{n+1,m} + \frac{< N_{n+1,m}, N_{n,m} + N_{n+1,m+1} >}{(1+ < N_{n,m}, N_{n+1,m+1} >)}(N_{n,m} + N_{n+1,m+1}).$$

Obviously, the N-loop remains closed under this evolution. Finally, the formulas (3.29) describe the corresponding discrete K-surface.

Note that to obtain a cylinder one should kill the translational period of the immersion $F_{n,m} = F_{n+N,m+M}$. Besides cylinders one can construct by this method also discrete Amsler surfaces [26]. Geometrically, the discrete Amsler surfaces can be characterized by the condition that they contain two straight asymptotic lines. Analytically, this implies that the discrete sine-Gordon equation reduces to a discrete version of the Painlevé III equation .

This simple geometrical method does not allow us to control the global behaviour of the surface in the direction of the evolution of the N-loop (for example, to control the periodicity). Thus this method is inappropriate if one wants to construct, for example, compact discrete K-surfaces . Compact discrete K-surfaces were constructed in [10] using analytic methods of the finite-gap integration theory of equations (3.20, 3.32). An example of a discrete torus is presented in Plate 1.

The Bäcklund transformation for discrete K-surfaces is of the same geometric and analytic nature as for the smooth surfaces with constant negative Gaussian curvature and is discussed in [50, 10, 29].

4 Discrete H-surfaces (constant mean curvature surfaces)

4.1 Smooth constant mean curvature surfaces

Here we present some fragments of the theory of smooth surfaces with constant mean curvature (CMC-surfaces), most of which are classical. For details see [5].

Let $\mathcal{F}$ be a smooth surface in $\mathbb{R}^3$ and

$$F : \mathcal{R} \to \mathbb{R}^3$$

a conformal parametrization of $\mathcal{F}$. The fundamental forms are as follows:

$$\begin{aligned} I &= < dF, dF >= e^u \, dz \, d\bar{z}, \\ II &= - < dF, dN >= \\ &= (He^u + Q + \bar{Q}) \, dz^2 + 2i(Q - \bar{Q}) \, dz \, d\bar{z} + (He^u - Q - \bar{Q}) \, d\bar{z}^2, \end{aligned}$$

where z is a conformal coordinate, Q and H denote the Hopf differential and the mean curvature

$$Q =< F_{zz}, N >, \qquad < F_{z\bar{z}}, N >= \frac{1}{2} He^u. \tag{4.1}$$

The Gauss–Codazzi equations have the following form:

$$\begin{aligned} u_{z\bar{z}} + \frac{1}{2} H^2 e^u - 2Q\bar{Q}e^{-u} &= 0, \\ Q_{\bar{z}} - \frac{1}{2} H_z e^u &= 0. \end{aligned}$$

In the CMC case $H = const$ the Hopf differential is holomorphic $Q_{\bar{z}} = 0$. In the absence of umbilic points $Q \neq 0$, by a holomorphic change of coordinates $z \to \tilde{z}(z)$ the Hopf differential can be normalized to constant $Q = const \neq 0$.

Similar to (3.3) the Gauss–Codazzi equations of CMC-surfaces are invariant with respect to the transformation

$$Q \to \Lambda Q, \qquad \bar{Q} \to \Lambda^{-1} \bar{Q}, \qquad |\Lambda| = 1.$$

Every CMC-surface possesses a one parameter family of isometries preserving the mean curvature. This Λ-family is called the *associated family.*

A conformal frame of a CMC-surface with the Hopf differential λQ is given by

$$F_z = -ie^{u/2} \hat{\Psi}^{-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \hat{\Psi}, \qquad F_{\bar{z}} = -ie^{u/2} \hat{\Psi}^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \hat{\Psi}, \tag{4.2}$$

$$N = \hat{\Psi}^{-1} \mathbf{k} \hat{\Psi},$$

where $\hat{\Psi}(z, \bar{z}, \Lambda)$ satisfies

$$\hat{\Psi}_z = \mathsf{U} \hat{\Psi}, \qquad \hat{\Psi}_{\bar{z}} = \mathsf{V} \hat{\Psi},$$

$$\mathsf{U} = \begin{pmatrix} \frac{u_z}{4} & -\Lambda Q e^{-u/2} \\ \frac{1}{2} He^{u/2} & -\frac{u_z}{4} \end{pmatrix}, \quad \mathsf{V} = \begin{pmatrix} -\frac{u_{\bar{z}}}{4} & -\frac{1}{2} He^{u/2} \\ \frac{1}{\Lambda} \bar{Q} e^{-u/2} & \frac{u_{\bar{z}}}{4} \end{pmatrix}. \tag{4.3}$$

The Sym formula [4, 5] allows us to integrate (4.2, 4.3).

Theorem 5 *Let $\hat{\Psi}(z,\bar{z},\Lambda=e^{2it})$ be a solution of (4.2). Then $\check{F}$ and N, defined by the formulas*

$$\check{F} = -\frac{1}{H}(\hat{\Psi}^{-1}\frac{\partial}{\partial t}\hat{\Psi} - i\hat{\Psi}^{-1}\sigma_3\hat{\Psi}), \qquad N = -i\hat{\Psi}^{-1}\sigma_3\hat{\Psi}$$

describe a CMC-surface with the metric e^u, the mean curvature H and the Hopf differential $e^{2it}Q$ and its Gauss map.

We also mention a well known fact, which can be easily checked.

Proposition 2 *The Gauss map $N:\mathcal{R}\to S^2$ of the CMC-surface is harmonic, i.e.*

$$N_{z\bar{z}} = qN, \qquad q:\mathcal{R}\to\mathbb{R}.$$

Having in mind a proper discretization as our goal, we will use (as in Section 1) special parametrizations of the CMC-surfaces. If the Hopf differential is normalized to be real $Q\in\mathbb{R}$, then the preimages of the curvature lines are the lines $x=const$ and $y=const$ in the parameter domain and one obtains a conformal curvature line parametrization . Such a parametrization and the surface which admits it are called *isothermic*. This class of surfaces is more general then CMC and is also described and discretized within the frames of the theory of integrable systems. We come to this description later in Section 5.2.

Thus, umbilic-free CMC-surfaces are isothermic. Without loss of generality we normalize

$$H=1, \qquad Q=\frac{1}{2}. \tag{4.4}$$

It is a classical result that surfaces parallel to a CMC-surface and lying in the normal direction at distances $1/(2H)$ and $1/H$ are of constant Gaussian and of constant mean curvature respectively. To describe them let us introduce gauge equivalent frames

$$\Psi = \begin{pmatrix} \frac{1}{\sqrt{\lambda}} & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}\hat{\Psi}, \qquad \check{\Psi} = \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \lambda \end{pmatrix}\hat{\Psi},$$

where

$$\lambda = e^{it}, \qquad \Lambda = \lambda^2.$$

Theorem 6 *The formulas*

$$\hat{F} = -\hat{\Psi}^{-1}\frac{\partial}{\partial t}\hat{\Psi}, \qquad F = -\Psi^{-1}\frac{\partial}{\partial t}\Psi, \qquad \check{F} = -\check{\Psi}^{-1}\frac{\partial}{\partial t}\check{\Psi} \tag{4.5}$$

describe three parallel surfaces $\hat{\mathcal{F}},\mathcal{F},\check{\mathcal{F}}$

$$\check{F} = F - \frac{N}{2}, \qquad \hat{F} = F + \frac{N}{2}, \qquad \hat{F} = \check{F} + N,$$

where

$$N = \Psi^{-1}\mathbf{k}\Psi$$

is the Gauss map of $\check{\mathcal{F}}$.[10] *The surfaces $\hat{\mathcal{F}}, \check{\mathcal{F}}$ are of constant mean curvature $H = 1$. The surface $\mathcal{F}$ is of constant Gaussian curvature $K = 4$. Variation of t preserves both principal curvatures of $\hat{\mathcal{F}}, \mathcal{F}, \check{\mathcal{F}}$; for $\hat{\mathcal{F}}, \check{\mathcal{F}}$ it is an isometry, whereas for $\mathcal{F}$ the second fundamental form is preserved. For $t = 0$ the parametrization of $\hat{\mathcal{F}}, \check{\mathcal{F}}$ is isothermic.*

This theorem can be proven by a direct computation [5] as well as the following

Proposition 3 *The surfaces $\hat{\mathcal{F}}$ and $\check{\mathcal{F}}$ are dual isothermic surfaces.*[11]

In the normalized (4.4) isothermic coordinates the frame equations for Ψ become

$$\Psi_z = U\Psi, \qquad \Psi_{\bar{z}} = V\Psi, \tag{4.6}$$

$$\begin{aligned} U &= \frac{1}{2}\begin{pmatrix} -\frac{i}{2}u_y & -\lambda e^{-u/2} - \frac{1}{\lambda}e^{u/2} \\ \lambda e^{u/2} + \frac{1}{\lambda}e^{-u/2} & \frac{i}{2}u_y \end{pmatrix}, \\ V &= \frac{1}{2}\begin{pmatrix} \frac{i}{2}u_x & -i\lambda e^{-u/2} + \frac{i}{\lambda}e^{u/2} \\ i\lambda e^{u/2} - \frac{i}{\lambda}e^{-u/2} & -\frac{i}{2}u_x \end{pmatrix}. \end{aligned} \tag{4.7}$$

The matrices (4.7) belong to the loop algebra

$$g_H[\lambda] = \{\xi : S^1 \to su(2) : \xi(-\lambda) = \sigma_3\xi(\lambda)\sigma_3\}$$

and Ψ in (4.6) lies in the corresponding loop group

$$G_H[\lambda] = \{\phi : S^1 \to SU(2) : \phi(-\lambda) = \sigma_3\phi(\lambda)\sigma_3\}. \tag{4.8}$$

Here S^1 is the set $|\lambda| = 1$.

4.2 Discrete H-surfaces from a discrete Lax representation

Let us discretize the CMC-surfaces in exactly the same way as we discretized the surfaces with constant negative Gausssian curvature in Section 3. Our goal is to define a map

$$\Phi : \mathbb{Z}^2 \to G_H[\lambda],$$

[10] The Gauss maps of $\hat{\mathcal{F}}, \mathcal{F}, \check{\mathcal{F}}$ coincide. To make the mean curvature of $\mathcal{F}$ positive one should change direction of the normal $\check{N} = -N$.

[11] For a definition of dual isothermic surfaces see Section 5.2.

where $G_H[\lambda]$ is the loop group (4.8), such that the matrices

$$\mathcal{U}_{n,m} = \Phi_{n+1,m}\Phi_{n,m}^{-1}, \qquad \mathcal{V}_{n,m} = \Phi_{n,m+1}\Phi_{n,m}^{-1} \tag{4.9}$$

depend on λ in "the same way" as the elements (4.7) of the corresponding loop algebra.

A natural choice is

$$\begin{aligned} \mathcal{U} &= \frac{1}{\alpha}\begin{pmatrix} a & -\lambda c - \frac{1}{\lambda} f \\ \lambda \bar{f} + \frac{1}{\lambda}\bar{c} & \bar{a} \end{pmatrix}, \\ \mathcal{V} &= \frac{1}{\beta}\begin{pmatrix} b & -\lambda d - \frac{1}{\lambda} g \\ \lambda \bar{g} + \frac{1}{\lambda}\bar{d} & \bar{b} \end{pmatrix}, \end{aligned} \tag{4.10}$$

where a, b, c, d, f, g are complex-valued fields defined on the corresponding edges. α and β are chosen to normalize $\det\mathcal{U} = \det\mathcal{V} = 1$. If $cd \neq 0, dg \neq 0$ by a diagonal gauge transformation (3.17) one can normalize

$$f = \frac{1}{c}, \qquad g = \frac{1}{d}. \tag{4.11}$$

For α, β this implies

$$\begin{aligned} \alpha^2 &= \lambda^2 \frac{c}{\bar{c}} + \lambda^{-2}\frac{\bar{c}}{c} + |a|^2 + |c|^2 + |c|^{-2}, \\ \beta^2 &= \lambda^2 \frac{d}{\bar{d}} + \lambda^{-2}\frac{\bar{d}}{d} + |b|^2 + |d|^2 + |d|^{-2}. \end{aligned}$$

In the sequel we basically make our calculations considering an elementary quadrilateral. Sometimes we supress the arguments n, m of the functions of n and m and denote increments and decrements of the discrete variables by subscripts and overbars respectively, for example

$$\begin{gathered} \Phi_{\bar{1}2} = \Phi_{2\bar{1}} = \Phi_{n-1,m+1}\Phi_2 = \Phi_{n,m+1}\Phi_{12} = \Phi_{21} = \Phi_{n+1,m+1} \\ \Phi_{\bar{1}} = \Phi_{n-1,m} \quad \Phi = \Phi_{n,m} \quad \Phi_1 = \Phi_{n+1,m} \end{gathered}$$

To distinguish the fields defined on edges and vertices we use different notations for the matrices $\mathcal{U}, \mathcal{V}$ and their coefficients[12] (see Fig. 4)

$$\mathcal{U} = \mathcal{U}_{n,m},\ \mathcal{V} = \mathcal{V}_{n,m},\ \mathcal{U}' = \mathcal{U}_{n,m+1},\ \mathcal{V} = \mathcal{V}_{n+1,m}.$$

The singularities of the left- and right-hand sides of the compatibility condition

$$\mathcal{V}'\mathcal{U} = \mathcal{U}'\mathcal{V}, \tag{4.12}$$

[12] $\mathcal{U}' = \mathcal{U}_2, \mathcal{V}' = \mathcal{V}_1$ identifies these notations.

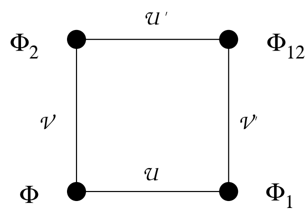


FIG. 4. Notations

considered as functions of λ, should coincide. Having in mind the smooth limit it is natural to assume that the zeros of $\alpha(\lambda)$ coincide with the zeros of $\alpha'(\lambda)$. This is equivalent to say that the zeros of $\alpha_{n,m}(\lambda)$ are m-independent and the zeros of $\beta_{n,m}(\lambda)$ are n-independent. In particular this implies that

$$\begin{aligned} &\arg c_{n,m} \qquad \text{independent of } m, \\ &\arg d_{n,m} \qquad \text{independent of } n. \end{aligned} \tag{4.13}$$

Using $\Phi : \mathbb{Z}^2 \to G_H[\lambda]$ defined by (4.9, 4.10, 4.11, 4.13) one can define three "parallel" discrete surfaces $\hat{\mathcal{F}}, \mathcal{F}, \check{\mathcal{F}}$ by the Sym formulas (4.5). The nets $\hat{\mathcal{F}}, \check{\mathcal{F}}$ constructed in this way can be treated as discrete conformal CMC nets. An analysis of the geometric properties of these nets might be helpful in looking for a definition of general discrete conformal nets, which is still missing.

The algebraic description allows us to specify our discrete CMC nets further. Taking into account the frame equations (4.16) in the isothermic parametrization it is natural to perform the following additional reduction on the coefficients of $\mathcal{U}, \mathcal{V}$: c is real-valued, d is purely imaginary. Later on we will show that this constraint is compatible with (4.12).

Introducing on the edges the real-valued fields u, v

$$c = u, \qquad d = iv,$$

we have

$$\begin{aligned} \mathcal{U} &= \frac{1}{\alpha}\begin{pmatrix} a & -\lambda u - \dfrac{1}{\lambda}u^{-1} \\ \lambda u^{-1} + \dfrac{1}{\lambda}u & \bar{a} \end{pmatrix}, \\ \mathcal{V} &= \frac{1}{\beta}\begin{pmatrix} b & -i\lambda v + \dfrac{i}{\lambda}v^{-1} \\ i\lambda v^{-1} - \dfrac{i}{\lambda}v & \bar{b} \end{pmatrix}, \end{aligned} \tag{4.14}$$

$$\alpha^2 = \lambda^2 + \lambda^{-2} + |a|^2 + u^2 + u^{-2}, \qquad \beta^2 = -\lambda^2 - \lambda^{-2} + |b|^2 + v^2 + v^{-2}.$$

Definition 10 *(Algebraic definition of discrete H-surfaces). Let* $\Phi : \mathbb{Z}^2 \to G_H[\lambda]$ *be a map with* $\mathcal{U}_{n,m} = \Phi_{n+1,m}\Phi^{-1}_{n,m}, \mathcal{V}_{n,m} = \Phi_{n,m+1}\Phi^{-1}_{n,m}$ *of the form (4.9). We call the nets given by*

$$\hat{F}_{n,m} = -\Phi^{-1}_{n,m}\frac{\partial}{\partial t}\Phi_{n,m} + \frac{1}{2}N_{n,m}, \tag{4.15}$$

$$\check{F}_{n,m} = -\Phi^{-1}_{n,m}\frac{\partial}{\partial t}\Phi_{n,m} - \frac{1}{2}N_{n,m}, \tag{4.16}$$

$$N_{n,m} = \Phi^{-1}_{n,m}\mathbf{k}\Phi_{n,m}, \qquad \lambda = e^{it}, t = 0, \tag{4.17}$$

discrete H-surfaces (discrete isothermic CMC surfaces) and the central net

$$F_{n,m} = -\Phi^{-1}_{n,m}\frac{\partial}{\partial t}\Phi_{n,m} \tag{4.18}$$

a discrete surface with constant positive Gaussian curvature. The Gauss map of these surfaces defined at vertices is given by the formula (4.17):

$$\check{N}_{n,m} = N_{n,m} = -\hat{N}_{n,m}.$$

Remark. In the next section we will show that in addition one can assume $u > 0, v > 0$ in this algebraic definition of nets. This assumption is natural in view of the continuum limit (4.7).

In the following sections we prove the existence of the surfaces defined above and study their geometric properties. The analysis of these geometric properties will provide us with natural geometric definitions of discrete isothermic and discrete H-surfaces.

4.3 Compatibility conditions

The compatibility conditions (4.12, 4.14) read as follows:

$$uu' = vv', \tag{4.19}$$

$$b'a - a'b = i(u'v + v'u - \frac{1}{u'v} - \frac{1}{v'u}), \tag{4.20}$$

$$\bar{b}u' - b'u = i(\bar{a}v' - a'v), \tag{4.21}$$

$$b'u' - \bar{b}u = i(\bar{a}v - a'v'). \tag{4.22}$$

The first of these equations can be resolved by introducing a function w at vertices:

$$u = ww_1,\ u' = w_2w_{12},\ v = ww_2,\ v' = w_1w_{12}. \tag{4.23}$$

Let us express a', b' using (4.21, 4.22)

$$\begin{aligned} a' &= \frac{\bar{a}w_1w_2(w_{12}^2 + w^2) + i\bar{b}(w_2^2w_{12}^2 - w_1^2w^2)}{ww_{12}(w_1^2 + w_2^2)}, \\ b' &= \frac{\bar{b}w_1w_2(w_{12}^2 + w^2) + i\bar{a}(w^2w_2^2 - w_1^2w_{12}^2)}{ww_{12}(w_1^2 + w_2^2)} \end{aligned} \tag{4.24}$$

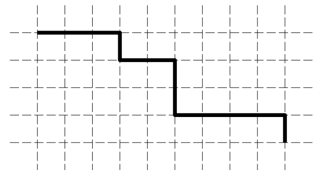


FIG. 5. Stairway in $\mathbb{Z}^2$

and substitute these expressions into (4.20). The equation obtained can be resolved with respect to w_{12}:

$$w_{12}^2 = \frac{w^2(|a|^2w_2^2 + |b|^2w_1^2 + 2\mathrm{Im}(a\bar{b})w_1w_2) + \left(\dfrac{w_1}{w_2} + \dfrac{w_2}{w_1}\right)^2}{w^2(w_1^2+w_2^2)^2 + |a|^2w_1^2 + |b|^2w_2^2 - 2\mathrm{Im}(a\bar{b})w_1w_2}. \tag{4.25}$$

The system (4.24, 4.25) is invariant with respect to the transformation

$$w \to sw,\ w_{12} \to sw_{12},\ w_1 \to s^{-1}w_1,\ w_2 \to s^{-1}w_2,$$

which preserves u, v. Up to this transformation the systems (4.19–4.22) and (4.24, 4.25) are equivalent.

Similarly to Section 3.5, discrete H-surfaces can be constructed by solving the corresponding Cauchy problem for the system (4.24–4.25).

Theorem 7 *Given a periodic*[13] *stairway in $\mathbb{Z}^2$ (see Fig. 5) with positive w at its vertices and complex a and b defined at its horizontal and vertical edges respectively, there exists a unique solution of (4.24, 4.25) in the same class of functions on all lattice $w : \mathbb{Z}^2 \to \mathbb{R}_+,\ a, b : \mathbb{Z}^2 \to \mathbb{C}$.*

Proof. Consider an elementary quadrilateral with given $w, w_1, w_2 > 0, a, b \in \mathbb{C}$. The numerator and denominator in (4.25) are both positive and this equation uniquely determines $w_{12} > 0$. After that, a' and b' are determined by (4.24). One can describe the evolution in the opposite direction similarly. This evolution determines a unique global solution on the whole lattice.

Integrating the frame equations (4.9) we construct by (4.15, 4.16) discrete H-surfaces.

4.4 Geometric properties of discrete H-surfaces

Proposition 4 *The maps $\hat{\mathcal{F}}, \check{\mathcal{F}}, \mathcal{F} : \mathbb{Z}^2 \to \mathbb{R}^3$ and $N : \mathbb{Z}^2 \to S^2$ defined by (4.15–4.18) are discrete C-surfaces in the sence of the narrow Definition 5. The cross-ratios of the elementary quadrilaterals of these surfaces are equal to*

$$Q(\hat{F}, \hat{F}_1, \hat{F}_2, \hat{F}_{12}) = -\frac{\beta^2}{\alpha^2}, \tag{4.26}$$

[13] The period can be infinite.

$$
\begin{aligned}
Q(\check{F},\check{F}_1,\check{F}_2,\check{F}_{12}) &= -\frac{\beta^2}{\alpha^2}, \\
Q(F,F_1,F_2,F_{12}) &= -\frac{\beta^2}{\alpha^2}\frac{(u-u^{-1})(u'-u'^{-1})}{(v+v^{-1})(v'+v'^{-1})}, \\
Q(N,N_1,N_2,N_{12}) &= -\frac{(u+u^{-1})(u'+u'^{-1})}{(v-v^{-1})(v'-v'^{-1})},
\end{aligned}
\tag{4.27}
$$

where

$$\alpha^2 = 2+|a|^2+u^2+u^{-2}, \qquad \beta^2 = -2+]b]^2+v^2+v^{-2}$$

are independent of m *and independent of* n *respectively:* $\alpha=\alpha', \beta=\beta'$.

Proof. The proposition is proven by direct computation. We present it for $\hat{F}$. As in the smooth case let us introduce

$$\hat{\Phi} = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \dfrac{1}{\sqrt{\lambda}} \end{pmatrix} \Phi.$$

It satisfies $\hat{\Phi}_1 = \hat{\mathcal{U}}\Phi, \hat{\Phi}_2 = \hat{\mathcal{V}}\Phi$ with

$$\hat{\mathcal{U}} = \frac{1}{\alpha}\begin{pmatrix} a & -\Lambda u - u^{-1} \\ u^{-1}+\dfrac{1}{\Lambda}u & \bar{a} \end{pmatrix}, \qquad \hat{\mathcal{V}} = \frac{1}{\beta}\begin{pmatrix} b & -i\Lambda v + iv^{-1} \\ iv^{-1}-\dfrac{i}{\Lambda}v & \bar{b} \end{pmatrix}.$$

For the edges of $\hat{F} = -\hat{\Phi}^{-1}\hat{\Phi}_t|_{t=0}$ we have

$$
\begin{aligned}
\hat{F}_1 - \hat{F} &= \hat{\Phi}^{-1}\hat{\mathcal{U}}^{-1}\hat{\mathcal{U}}_t\hat{\Phi} \\
\hat{F}_2 - \hat{F} &= \hat{\Phi}^{-1}\hat{\mathcal{V}}^{-1}\hat{\mathcal{V}}_t\hat{\Phi} \\
\hat{F}_{12} - \hat{F}_1 &= \hat{\Phi}^{-1}\hat{\mathcal{U}}^{-1}\hat{\mathcal{V}}'^{-1}\hat{\mathcal{V}}'_t\hat{\mathcal{U}}\hat{\Phi} \\
\hat{F}_{12} - \hat{F}_2 &= \hat{\Phi}^{-1}\hat{\mathcal{V}}^{-1}\hat{\mathcal{U}}'^{-1}\hat{\mathcal{U}}'_t\hat{\mathcal{V}}\hat{\Phi}.
\end{aligned}
$$

Considering the local geometry we can neglect the common rotation $\hat{\Phi}^{-1}\cdots\hat{\Phi}$ and set $\hat{\Phi}=I$. Substituting these expressions into formula (2.20) for the cross-ratio and using the compatibility conditions (4.19) we obtain

$$
\begin{aligned}
Q &= (\hat{F}-\hat{F}_1)(\hat{F}_1-\hat{F}_{12})^{-1}(\hat{F}_{12}-\hat{F}_2)(\hat{F}_2-\hat{F})^{-1} \\
&= (\hat{\mathcal{U}}^{-1})_t(\hat{\mathcal{V}}')_t^{-1}\hat{\mathcal{U}}'_t((\hat{\mathcal{V}}_t^{-1})^{-1}.
\end{aligned}
\tag{4.28}
$$

For $t=0$ the derivatives are

$$
\begin{aligned}
\hat{\mathcal{U}}_t &= -\frac{2iu}{\alpha}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -(\hat{\mathcal{U}}^{-1})_t, \\
\hat{\mathcal{V}}_t &= \frac{2v}{\beta}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -(\hat{\mathcal{V}}^{-1})_t.
\end{aligned}
$$

Finally, substituting these expressions into (4.28) and using (4.12) we prove the first formula of the proposition.

Remark. The curvature lines of the three parallel smooth surfaces $\hat{\mathcal{F}}, \mathcal{F}, \check{\mathcal{F}}$ (see Section 4.1) correspond. The property of the discrete surfaces $\hat{F}, F, \check{F}$ to be discrete C-surfaces may be regarded as a discrete analogue of this property.

The next proposition follows directly from (4.15, 4.16).

Proposition 5 *The discrete H-surfaces $\hat{F}$ and $\check{F}$ are at a constant distance*

$$\|\hat{F}_{n,m} - \check{F}_{n,m}\| = 1.$$

In Section 5.2 we use the factorization property (4.26, 4.27) of the cross-ratio to define discrete isothermic surfaces.

Proposition 6 *The "parallel" discrete H-surfaces $\hat{F}$ and $\check{F}$ are dual discrete isothermic surfaces*

$$\hat{F} = \check{F}^*.$$

Proof. The corresponding edges of these surfaces are equal to

$$\begin{aligned}
\hat{F}_1 - \hat{F} &= -\frac{2iu}{\alpha^2}\Phi^{-1}\begin{pmatrix} u+u^{-1} & \bar{a} \\ a & -u-u^{-1}\end{pmatrix}\Phi, \\
\check{F}_1 - \check{F} &= \frac{2iu^{-1}}{\alpha^2}\Phi^{-1}\begin{pmatrix} u+u^{-1} & \bar{a} \\ a & -u-u^{-1}\end{pmatrix}\Phi, \\
\hat{F}_2 - \hat{F} &= -\frac{2iv}{\beta^2}\Phi^{-1}\begin{pmatrix} v-v^{-1} & i\bar{c} \\ -ic & v^{-1}-v\end{pmatrix}\Phi, \\
\check{F}_2 - \check{F} &= -\frac{2iv^{-1}}{\beta^2}\Phi^{-1}\begin{pmatrix} v-v^{-1} & i\bar{c} \\ -ic & v^{-1}-v\end{pmatrix}\Phi
\end{aligned}$$

with the lengths

$$\begin{aligned}
\|\hat{F}_1 - \hat{F}\| &= \frac{2u}{\alpha}, \qquad \|\check{F}_1 - \check{F}\| = \frac{2u^{-1}}{\alpha}, \\
\|\hat{F}_2 - \hat{F}\| &= \frac{2v}{\beta}, \qquad \|\check{F}_2 - \check{F}\| = \frac{2v^{-1}}{\beta}.
\end{aligned}$$

The edges are related by the formulas (5.7) for dual discrete isothermic surfaces.

Also by direct computation one can prove

Proposition 7 *The distances between the neighbouring vertices of the discrete H-surfaces $\hat{F}$ and $\check{F}$ are*

$$\begin{aligned}
\|\hat{F}_1 - \check{F}\| = \|\check{F}_1 - \hat{F}\| &= \frac{\sqrt{\alpha^2-4}}{\alpha}, \\
\|\hat{F}_2 - \check{F}\| = \|\check{F}_2 - \hat{F}\| &= \frac{\sqrt{\beta^2+4}}{\beta}.
\end{aligned} \tag{4.29}$$

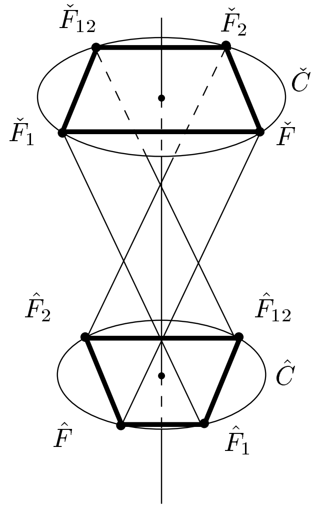


FIG. 6. Elementary hexahedron of a pair of discrete H-surfaces

Proposition 8

(i) *An elementary hexahedron* $(\hat{F}, \hat{F}_1, \hat{F}_{12}, \hat{F}_2, \check{F}, \check{F}_1, \check{F}_{12}, \check{F}_2)$ *lies on a sphere.*

(ii) *The quadrilaterals* $(\hat{F}, \hat{F}_1, \check{F}_1, \check{F})$ *and* $(\hat{F}, \hat{F}_2, \check{F}_2, \check{F})$ *are isosceles trapezoids of two different types (non-embedded and embedded) (see Fig. 6).*

(iii) *Define the axis of the hexahedron as the straight line connecting the centres of the circles* $\hat{C}$ *and* $\check{C}$ *of the circular quadrilaterals* $(\hat{F}, \hat{F}_1, \hat{F}_{12}, \hat{F}_2)$ *and* $(\check{F}, \check{F}_1, \check{F}_{12}, \check{F}_2)$ *respectively. The axis of the hexahedron is orthogonal to the planes of both quadrilaterals.*

Proof. The first statement follows from the third one, and (ii) follows from Propositions 5, 7. To prove (iii) one should build the planes passing orthogonally through the middle points of the edges of the quadrilaterals $(\hat{F}, \hat{F}_1, \hat{F}_{12}, \hat{F}_2)$ and $(\check{F}, \check{F}_1, \check{F}_{12}, \check{F}_2)$. The axis of the hexahedron is the intersection line of these planes.

Propositions 5, 6 suggest the following natural definition.

Definition 11 *(Geometric definition of discrete H-surfaces)*

A discrete isothermic surface $F : \mathbb{Z}^2 \to \mathbb{R}^3$ *is called a discrete H-surface if there is a dual discrete isothermic surface* $F^* : \mathbb{Z}^2 \to \mathbb{R}^3$ *at a constant distance*

$$\|F_{n,m} - F^*_{n,m}\|^2 = \frac{1}{H^2}.$$

Starting with this definition Hertrich-Jeromin, Hoffmann, and Pinkall [30] derived geometrical properties as well as the Darboux transformation of discrete H-surfaces. Discrete analogues of the Delaunay surfaces are described by Hoffmann in [28] using a discrete version of the classical rolling ellipse construction.

4.5 A definition of the mean curvature for discrete isothermic surfaces

It is desirable to give a geometric definition of discrete H-surfaces completely in internal terms without referring to dual isothermic surfaces. In the present section we discuss the notion of the mean curvature function in the discrete case.

Lemma 4 *Let $(x,y) \mapsto F(x,y)$ be a conformal immersion with the Gauss map $(x,y) \mapsto N(x,y)$. The point $S(x,y)$ is the centre of the mean curvature sphere (central sphere) at the point $F(x,y)$*

$$S(x,y) = F(x,y) + \frac{1}{H} N(x,y)$$

if for $\epsilon \to 0$

$$\begin{aligned} \frac{\delta_2}{\delta_1}\|S(x,y) - F(x+\epsilon\delta_1, y)\|^2 + \frac{\delta_1}{\delta_2}\|S(x,y) - F(x, y+\epsilon\delta_2)\|^2 = \\ \left(\frac{\delta_1}{\delta_2} + \frac{\delta_2}{\delta_1}\right)\|S(x,y) - F(x,y)\|^2 + o(\epsilon^2), \quad (4.30) \\ \|S(x,y) - F(x, y+\epsilon\delta_2)\| = \|S(x,y) - F(x, y-\epsilon\delta_2)\| + o(\epsilon^2), \\ \|S(x,y) - F(x+\epsilon\delta_1, y)\| = \|S(x,y) - F(x-\epsilon\delta_1, y)\| + o(\epsilon^2), \end{aligned}$$

Proof. Use the Taylor series expansion

$$\|S(x,y) - F(x+\epsilon\delta_1, y)\|^2 = \left(\frac{1}{H} - \frac{\epsilon^2\delta_1^2}{2} < F_{xx}, N >\right)^2 + (\|F_x\|\epsilon\delta_1)^2 + o(\epsilon^2)$$

and the definition of the mean curvature

$$H = \frac{< F_{xx} + F_{yy}, N >}{2\|F_x\|^2}, \; \|F_x\| = \|F_y\|.$$

In the discrete case the relations (4.30) can be used as defining the centre of the mean curvature sphere.

Lemma 5 *Let $F, F_{\bar{1}}, F_1, F_{\bar{2}}, F_2$ be five neighbouring points of a discrete surface $F : \mathbb{Z}^2 \to \mathbb{R}^3$ in general position*[14]*. For any δ_1, δ_2 there exists exactly one point S, such that*

$$\frac{\delta_2}{\delta_1}\|S - F_1\|^2 + \frac{\delta_1}{\delta_2}\|S - F_2\|^2 = \left(\frac{\delta_1}{\delta_2} + \frac{\delta_2}{\delta_1}\right)\|S - F\|^2, \quad (4.31)$$

$$\|S - F_1\| = \|S - F_{\bar{1}}\|, \qquad \|S - F_2\| = \|S - F_{\bar{2}}\|. \quad (4.32)$$

Proof. Let l be the straight line for which the identities (4.32) are satisfied

$$l = \{P \in \mathbb{R}^3 : \quad \|P - F_1\| = \|P - F_{\bar{1}}\|, \|P - F_2\| = \|P - F_{\bar{2}}\|\}.$$

[14] We assume that the planes equidistant from F_1 and $F_{\bar{1}}$ and from F_2 and $F_{\bar{2}}$ intersect.

Make l a coordinate axis with an origin O. Let $F^\perp$ be the orthogonal projection of F to l. With each point F we associate two coordinates (r,z) with the corresponding labels. Here $r = \|F - F^\perp\| > 0$ is the distance of F from the axis and $z \in \mathbb{R}$ is the axis coordinate of $F: \ |z| = \|F^\perp\|$. Note that $z_1 = z_{\bar{1}}, z_2 = z_{\bar{2}}$. In these coordinates equations (4.31, 4.32) read as follows

$$\frac{\delta_2}{\delta_1}((x-z_1)^2 + r_1^2) + \frac{\delta_1}{\delta_2}((x-z_2)^2 + r_2^2) = \left(\frac{\delta_1}{\delta_2} + \frac{\delta_2}{\delta_1}\right)(x-z)^2,$$

where $(0,x)$ are the coordinates of S. This is a linear equation with respect to x

$$2x\left(\frac{\delta_2}{\delta_1}(z-z_1) + \frac{\delta_1}{\delta_2}(z-z_2)\right) = \frac{\delta_2}{\delta_1}(z^2 - r_1^2 - z_1^2) + \frac{\delta_1}{\delta_2}(z^2 - r_2^2 - z_2^2).$$

In particular, x is infinite if

$$\delta_2^2(z-z_1) + \delta_1^2(z-z_2) = 0. \tag{4.33}$$

Definition 12 *Let $F: \mathbb{Z}^2 \to \mathbb{R}^3$ be a discrete isothermic surface (see Section 5.2) with constant cross-ratio of elementary quadrilaterals*

$$-\frac{\delta_1^2}{\delta_2^2} = Q(F_{n,m}, F_{n+1,m}, F_{n+1,m+1}, F_{n,m+1}).$$

The point $S_{n,m}$ of Lemma 5 is the centre of the mean curvature sphere at the point $F_{n,m}$ and

$$H_{n,m} = \frac{1}{\|F_{n,m} - S_{n,m}\|}$$

is called the mean curvature at this point.

Remark. In the geometrically most natural case $\delta_1 = \delta_2$, the centre of the mean curvature sphere is defined by

$$\|S-F_1\|^2 + \|S-F_2\|^2 = 2\|S-F\|^2,$$
$$\|S-F_1\| = \|S-F_{\bar{1}}\|, \qquad \|S-F_2\| = \|S-F_{\bar{2}}\|.$$

The mean curvature vanishes ($H = 0$) if $z = (z_1 + z_2)/2$, i.e. if there exists a plane $\mathcal{P}_{n,m}$ (tangent plane) passing through the point $F_{n,m}$ such that the points $F_{n-1,m}, F_{n+1,m}, F_{n,m-1}, F_{n,m+1}$ lie at the same distance from $\mathcal{P}_{n,m}$ and two pairs $F_{n-1,m}, F_{n+1,m}$ and $F_{n,m-1}, F_{n,m+1}$ lie to the different sides of $\mathcal{P}_{n,m}$.

For minimal surfaces ($H = 0$) in the general case $\delta_1 \neq \delta_2$ formula (4.33) implies Definition 20.

Theorem 8 *The discrete surfaces $\hat{F}, \check{F}$ defined in Section 4.4 have constant mean curvature $H = 1$ in the sense of Definition 12.*

Proof. According to (4.26) $\delta_1 = \beta, \delta_2 = \alpha$. Substitution of (4.29) into (4.31) yields the result.

5 Other integrable nets

5.1 Discrete O-systems (orthogonal coordinate systems)

The definition of discrete C-surfaces in Section 2.3 allows a natural generalization to the three-dimensional case, which gives a notion of discrete triply orthogonal coordinate systems. This generalization, based on a "discretization" of the Dupin theorem (see below), was first suggested in [6].

We recall that an orthogonal coordinate system in three-dimensional Euclidean space is an immersion

$$\begin{aligned} F : U \subset \mathbb{R}^3 &\to \mathbb{R}^3, \\ (u, v, w) &\mapsto F(u, v, w). \end{aligned} \tag{5.1}$$

such that for all points of U the three vectors

$$\frac{\partial F}{\partial u}, \frac{\partial F}{\partial v}, \frac{\partial F}{\partial w}$$

form an orthogonal basis.

The notion of orthogonal coordinate systems belongs to conformal geometry because of the following obvious

Property 5 *(Möbius invariance). Let $F : U \subset \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal coordinate system and $\mathcal{M}$ a Möbius transformation of Euclidean 3-space. Then $\tilde{F} \equiv \mathcal{M} \circ F : U \subset \mathbb{R}^3 \to \mathbb{R}^3$ is also an orthogonal coordinate system.*

The general theory of these coordinate systems, developed at the end of the 19th and the beginning of the 20th century, as well as many concrete examples, can be found in the fundamental book by Darboux [16] and in the book by Bianchi [2]. A fundamental result in this theory is a theorem according to which the coordinate surfaces of a triply orthogonal coordinate system cross along their curvature lines.

Theorem 9 *(Dupin). The immersion (5.1) forms a triply orthogonal coordinate system iff for any point $(u_0, v_0, w_0) \in U$ the three coordinate surfaces*

$$F(u_0, v, w), F(u, v_0, w), F(u, v, w_0)$$

are curvature line parametrized.

This description can be discretized in a natural way using the notion of a discrete C-surface.

Definition 13 *(Wide definition of discrete O-systems as circular lattices) A discrete orthogonal system is a map*

$$\begin{aligned} F : \mathbb{Z}^3 &\to \mathbb{R}^3, \\ (k, l, m) &\mapsto F(k, l, m) \end{aligned} \tag{5.2}$$

all elementary quadrilaterals of which are inscribed in circles.

One can give a more restrictive definition.

Definition 14 *(Narrow definition of discrete O-systems) A discrete orthogonal system is a map (5.2) such that all elementary quadrilaterals have negative cross-ratios*

$$\begin{aligned} Q(F_{k,l,m}, F_{k+1,l,m}, F_{k+1,l+1,m}, F_{k,l+1,m}) &< 0, \\ Q(F_{k,l,m}, F_{k+1,l,m}, F_{k+1,l,m+1}, F_{k,l,m+1}) &< 0, \\ Q(F_{k,l,m}, F_{k,l+1,m}, F_{k,l+1,m+1}, F_{k,l,m+1}) &< 0. \end{aligned} \tag{5.3}$$

Lemma 2 shows that (5.3) is equivalent to the requirement that the elementary quadrilaterals are inscribed in circles and embedded (i.e. the opposite edges of the quadrilaterals do not intersect).

These definitions are Möbius invariant. Obviously, Theorem 9 also holds in the discrete case.

Corollary 4 *For any point $(k_0, l_0, m_0) \in \mathbb{Z}^3$ of a discrete O-system (5.2) the three coordinate surfaces*

$$F(k_0, l, m), F(k, l_0, m), F(k, l, m_0)$$

are discrete C-surfaces (in the sense of the wide and narrow definitions respectively).

We call the hexahedrons with vertices $\{F_{k,l,m}, F_{k+1,l,m}, F_{k+1,l+1,m}, F_{k,l+1,m}, F_{k,l,m+1}, F_{k+1,l,m+1}, F_{k+1,l+1,m+1}, F_{k,l+1,m+1}\}$ *elementary hexahedrons.* Let us say also that a hexahedron with planar faces is embedded if all its faces are embedded.

Corollary 5 *Each elementary hexahedron of a discrete orthogonal system lies on a sphere (and is embedded in the case of the narrow definition).*

Proof. The circles $C(F_{k,l,m}, F_{k+1,l,m}, F_{k+1,l+1,m}, F_{k,l+1,m})$ and $C(F_{k,l,m}, F_{k+1,l,m}, F_{k+1,l,m}, F_{k,l,m+1})$ passing through the indicated points determine a unique sphere[15] S containing them. The circle $C(F_{k,l,m}, F_{k,l+1,m}, F_{k,l+1,m+1}, F_{k,l,m+1})$ also lies on S since $F_{k,l,m}, F_{k,l+1,m}, F_{k,l,m+1} \in S$. This implies $F_{k,l+1,m+1} \in S$. The same proof holds for all remaining points of the elementary hexahedron.

We call a discrete orthogonal system non-degenerate if the spheres of neighbouring elementary hexahedrons are distinct.

Theorem 10 *$F : \mathbb{Z}^3 \supset U \to \mathbb{R}^3$ is a non-degenerate discrete O-system iff all its elementary hexahedrons lie on spheres. It is a discrete O-system in the sense of Definition 14 iff the elementary hexahedrons are embedded.*

Proof. The spheres of two neighbouring elementary hexahedrons intersect along a circle. The embeddedness is also obvious.

[15]A plane is a special case.

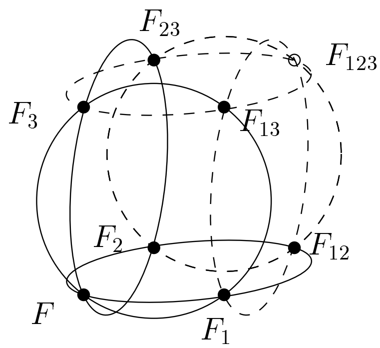


FIG. 7. Miguel theorem

Remark. This theorem can be used as a definition of discrete O-systems.

Recently discrete orthogonal nets and their generalization—discrete conjugate nets—have become a focus of interest in the theory of integrable systems (see the contribution of Doliwa and Santini in this volume and [14, 33, 20, 18]). In particular, Cieśliński, Doliwa, and Santini [14] have proven the following theorem.

Theorem 11 *(Cauchy problem for discrete O-systems). A discrete O-system in the sense of Definition 13 is uniquely determined by its three coordinate surfaces*

$$F(0,\bullet,\bullet),\ F(\bullet,0,\bullet),\ F(\bullet,\bullet,0):\mathbb{Z}^2\to\mathbb{R}^3,$$

which are discrete C-surfaces of Definition 6.

The proof is based on:

Theorem 12 *(Miguel) [1]. Let*

$$F, F_1, F_2, F_3, F_{12}, F_{13}, F_{23}$$

be seven points in Euclidean 3-space such that the vertices of each of the quadrilaterals

$$(F, F_1, F_{12}, F_2),\ (F, F_1, F_{13}, F_3),\ (F, F_2, F_{23}, F_3)$$

are concircular and the corresponding circles do not coincide. Then the circles given by the point triples

$$\{F_1, F_{12}, F_{13}\},\ \{F_2, F_{12}, F_{23}\},\ \{F_3, F_{13}, F_{23}\}$$

intersect at one point $=: F_{123}$. *In particular, all eight points lie on a sphere (see Fig. 7).*

It is easy to see that the embeddedness property is not necessarily preserved by the evolution described by the Miguel theorem . Therefore the Cauchy problem of Cieśliński, Doliwa, and Santini may not have a solution if we stick to the narrow Definitions 5 and 14.

Algebraically, the same circular lattices have been obtained by Konopelchenko and Schief [33] as a special case of discrete conjugate nets (planar quadrilaterals) by assuming a certain algebraic constraint inherited from the smooth case. Bobenko and Hertrich-Jeromin described in [8] discrete O-systems as well as discrete Ribaucour sphere congruences in terms of the Clifford algebra using the spinor representation of the conformal group.

Note that the dimension of the Euclidean space is not important for the considerations above. In exactly the same way one can consider circular lattices of arbitrary dimension. Another generalization is given by O-systems in spaces of constant curvature, which can be obtained by stereographic projection from the circular lattices described above.

5.2 Discrete I-surfaces (isothermic surfaces)

Discrete isothermic surfaces are natural discrete analogues of isothermic surfaces. The discrete I-surfaces as well as the discrete M-surfaces of the next section have been defined in our paper [11], where one should look for details and for missing proofs.

An isothermic surface is a surface the curvature lines of which comprise infinitesimal squares.

Definition 15 *A conformal curvature line parametrization $(x,y) \mapsto F(x,y)$ is called isothermic. For the fundamental forms this implies*

$$< dF, dF >= e^u(dx^2+dy^2), \qquad -< dF, dN >= e^u(k_1 dx^2 + k_2 dy^2).$$

A surface which admits isothermic coordinates is called isothermic.

Let us mention two important geometric properties of isothermic surfaces which persist in the discrete case. Isothermic surfaces belong to conformal geometry due to:

Property 6 *(Möbius invariance). Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be an isothermic immersion and $\mathcal{M}$ a Möbius transformation of Euclidean 3-space. Then $\tilde{F} \equiv \mathcal{M} \circ F : \mathbb{R}^2 \to \mathbb{R}^3$ is also isothermic.*

Special Euclidean properties of isothermic surfaces are established in:

Property 7 *(Dual surface). Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be an isothermic immersion. Then the immersion $F^* : \mathbb{R}^2 \to \mathbb{R}^3$ defined by the formulas*

$$F^*_x = -e^{-u} F_x, \quad F^*_y = e^{-u} F_y \tag{5.4}$$

is isothermic. The Gauss maps of F and F^* are antipodal

$$N = -N^*.$$

The map $F \to F^*$ is an involution $F^{**} = F$ and the fundamental forms of F^* are as follows

$$\begin{aligned} < dF^*, dF^* > &= e^{-u}(dx^2+dy^2), \\ -< dF^*, dN^* > &= k_1 dx^2 - k_2 dy^2. \end{aligned}$$

Definition 16 *The immersion $F^*: \mathbb{R}^2 \to \mathbb{R}^3$ defined above is called dual to F.*

Surfaces of revolution, quadrics, constant mean curvature surfaces without umbilics, and Bonnet surfaces are isothermic. It is natural to define discrete isothermic surfaces in such a way that this set includes the discrete H-surfaces of Section 4. We do this by postulating property (4.26) of discrete H-surfaces as a definition of discrete I-surfaces. Again, as in many previous cases, we have narrow and wide definitions of discrete isothermic surfaces.

Definition 17 *(Wide definition of discrete I-surfaces) A discrete isothermic surface is a discrete C-surface $F: \mathbb{Z}^2 \to \mathbb{R}^3$ such that the cross-ratios*

$$q_{n,m} = Q(F_{n,m}, F_{n+1,m}, F_{n+1,m+1}, F_{n,m+1})$$

of elementary quadrilaterals are negative $q_{n,m} < 0$ and satisfy the factorization condition

$$q_{n,m} q_{n+1,m+1} = q_{n+1,m} q_{n,m+1}. \tag{5.5}$$

Equivalently, the cross-ratio $q_{n,m}$ is a product of two factors

$$q_{n,m} = -\frac{\beta_m^2}{\alpha_n^2}, \tag{5.6}$$

where α_n does not depend on m and β_m not on n.

A special case is a more geometric definition.

Definition 18 *(Narrow definition of discrete I-surfaces) A discrete I-surface is a discrete C-surface for which all elementary quadrilaterals are conformal squares, i.e. they have cross-ratio -1.*

The cross-ratio is Möbius invariant.

Theorem 13 *(Möbius invariance). Let $F: \mathbb{Z}^2 \to \mathbb{R}^3$ be a discrete isothermic surface and $\mathcal{M}$ a Möbius transformation of Euclidean 3-space. Then $\tilde{F} \equiv \mathcal{M} \circ F: \mathbb{Z}^2 \to \mathbb{R}^3$ is also isothermic.*

Property 7 of smooth isothermic surfaces also persists in the discrete case.

Theorem 14 *(Dual discrete I-surface). Let $F: \mathbb{Z}^2 \to \mathbb{R}^3$ be a discrete isothermic surface with cross-ratios (5.6). Then the discrete surface $F^*: \mathbb{Z}^2 \to \mathbb{R}^3$ defined (up to translation) by the formulas*

$$\begin{aligned} F^*_{n+1,m} - F^*_{n,m} &= -\frac{1}{\alpha_n^2} \frac{F_{n+1,m} - F_{n,m}}{\|F_{n+1,m} - F_{n,m}\|^2}, \\ F^*_{n,m+1} - F^*_{n,m+1} &= \frac{1}{\beta_m^2} \frac{F_{n,m+1} - F_{n,m}}{\|F_{n,m+1} - F_{n,m}\|^2}. \end{aligned} \tag{5.7}$$

is isothermic. The cross-ratios of the corresponding quadrilaterals of F and F^* coincide

$$Q(F^*_{n,m}, F^*_{n+1,m}, F^*_{n+1,m+1}, F^*_{n,m+1}) = -\frac{\beta_m^2}{\alpha_n^2} \tag{5.8}$$

The discretization based on the loop group description of isothermic surfaces is very similar to that of K- and H-surfaces. In the framework of conformal geometry isothermic surfaces are described in terms of certain sphere congruences [3, 13]. The Lax representation of isothermic surfaces is given in four-dimensional matrices, because it is based on the spinor representation of the conformal group. It turns out that, as in the cases considered in Sections 3 and 4, this algebraic discretization provides us with the same definition (5.5) of discrete isothermic surfaces. The details of the loop group discretization as well as the corresponding Sym formula can be found in [11].

Remark. One can generalize the definition (5.5) to the case of discrete C-surfaces given by Definition 5, i.e. assuming

$$q_{n,m} = \frac{\beta_m}{\alpha_n} \in \mathbb{R}$$

only. These nets possess natural multidimensional generalizations which are special discrete orthogonal systems in the sense of Definition 13.

Definition 19 *A discrete I-system is a map $F : \mathbb{Z}^3 \to \mathbb{R}^3$ for which*

$$\begin{aligned} Q(F_{k,l,m}, F_{k+1,l,m}, F_{k+1,l+1,m}, F_{k,l+1,m}) &= \frac{\beta_l}{\alpha_k}, \\ Q(F_{k,l,m}, F_{k+1,l,m}, F_{k+1,l,m+1}, F_{k,l,m+1}) &= \frac{\gamma_m}{\alpha_k}, \\ Q(F_{k,l,m}, F_{k,l+1,m}, F_{k,l+1,m+1}, F_{k,l,m+1}) &= \frac{\gamma_m}{\beta_l} \end{aligned}$$

holds with some $\alpha, \beta, \gamma : \mathbb{Z} \to \mathbb{R}$.

All the coordinate surfaces of a discrete I-system are discrete I-surfaces. A discrete I-system is uniquely determined by its Cauchy data

$$F(\bullet, 0, 0),\ F(0, \bullet, 0),\ F(0, 0, \bullet) : \mathbb{Z} \to \mathbb{R}^3, \qquad \alpha, \beta, \gamma : \mathbb{Z} \to \mathbb{R}.$$

A direct geometric proof of this is presented in the contribution by Hertrich-Jeromin, Hoffmann, and Pinkall [30].

5.3 Discrete M-surfaces (minimal isothermic surfaces)

A special class of discrete H-surfaces is provided by discrete isothermic minimal surfaces (discrete M-surfaces). Setting the mean curvature H of a discrete H-surface equal to zero (see the Remark at the end of Section 4.5) we obtain the following:

Definition 20 *Let $F : \mathbb{Z}^2 \to \mathbb{R}^3$ be a discrete I-surface with constant cross-ratio of elementary quadrilaterals*

$$Q(F_{n,m}, F_{n+1,m}, F_{n+1,m+1}, F_{n,m+1}) = -\frac{\beta^2}{\alpha^2}.$$

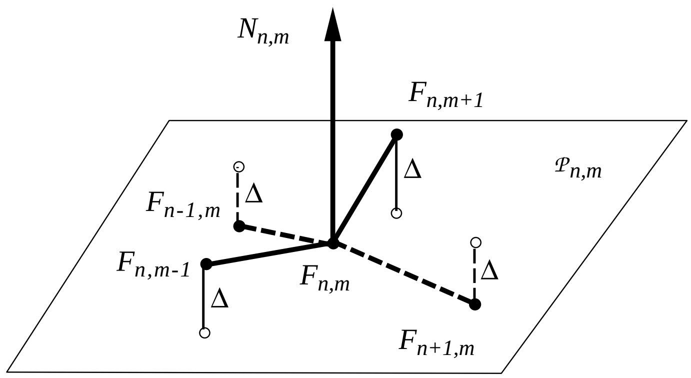


FIG. 8. Definition of discrete minimal isothermic surfaces

The surface F is called a discrete M-surface (minimal isothermic) if all the mean curvature spheres of Definition 12 are planes. Equivalently, at each vertex $F_{n,m}$ there is a "normal" vector $N_{n,m}$ such that[16]

$$< F_{n+1,m} - F_{n,m}, N_{n,m} >=< F_{n-1,m} - F_{n.m}, N_{n,m} >= \frac{\Delta_{n,m}}{\alpha^2}$$
$$< F_{n,m+1} - F_{n,m}, N_{n,m} >=< F_{n,m-1} - F_{n,m}, N_{n,m} >= -\frac{\Delta_{n,m}}{\beta^2}.$$

Figure 8 visualizes this definition in the special case of the narrow definition $\alpha = \beta = 1$ of discrete I-surfaces.

Investigating the dual isothermic surface one can show[17] that F^* lies on a sphere and that for all points of a discrete M-surface $\Delta_{n,m} = \Delta$ is constant on $U \subset \mathbb{Z}^2$. The normalization

$$\Delta = \frac{1}{2} \tag{5.9}$$

implies that the sphere of F^* has unit radius.

Theorem 15 *[11]. The following statements are equivalent:*

(i) $F : \mathbb{Z}^2 \to \mathbb{R}^3$ is a discrete isothermic minimal surface normalized by (5.9).
(ii) The dual surface $F^* : \mathbb{Z}^2 \to \mathbb{R}^3$ lies on a sphere and without loss of generality one can assume that it coincides with the Gauss map N

$$F^* = N : \mathbb{Z}^2 \to S^2.$$

This theorem allows us to reduce the dimension of the problem and to parametrize discrete minimal surfaces by "holomorphic" data N. Indeed, applying

[16] As in Section 4.5 one can justify these formulas by considering infinitesimal curvature line quadrilaterals of minimal surfaces.

[17] The proof is the same as the one presented in [11] for the case $\alpha = \beta = 1$.

stereographic projection $S^2 \to \mathbb{C}$ to N, we obtain a discrete isothermic net[18] $g : \mathbb{Z}^2 \to \mathbb{C}$ on the complex plane. The isothermic Gauss map $N : \mathbb{Z}^2 \to S^2$ is the stereographic projection of g

$$(N_1 + iN_2, N_3) = \left(\frac{2g}{1+|g|^2}, \frac{|g|^2 - 1}{|g|^2 + 1} \right). \tag{5.10}$$

It has the same cross-ratio as F:

$$\frac{(g_{n+1,m} - g_{n,m})(g_{n,m+1} - g_{n+1,m+1})}{(g_{n+1,m+1} - g_{n+1,m})(g_{n,m} - g_{n,m+1})} = -\frac{\beta^2}{\alpha^2}.$$

$g_{n,m}$ are complex numbers here.

Combining formulas (5.10) and (5.7) one gets an analogue of the Weierstrass representation in the discrete case.

Theorem 16 *Let $g : \mathbb{Z}^2 \to \mathbb{C}$ be discrete conformal. Then the formulas*

$$F_{n+1,m} - F_{n,m} =$$
$$-\frac{1}{2\alpha^2}\mathrm{Re}\left(\frac{1}{g_{n+1,m} - g_{n,m}} (1 - g_{n+1,m}g_{n,m}, i(1 + g_{n+1,m}g_{n,m}), g_{n+1,m} + g_{n,m}) \right)$$
$$F_{n,m+1} - F_{n,m} =$$
$$\frac{1}{2\beta^2}\mathrm{Re}\left(\frac{1}{g_{n,m+1} - g_{n,m}} (1 - g_{n,m+1}g_{n,m}, i(1 + g_{n,m+1}g_{n,m}), g_{n,m+1} + g_{n,m}) \right)$$

describe a discrete minimal isothermic surface. All discrete minimal isothermic surfaces are described in this way.

Discrete H-surfaces can also be parametrized in terms of discrete conformal mappings . This has been shown by Hoffmann [27] using a discrete version of the Dorfmeister–Pedit–Wu factorization method.

Naturally we now come to investigation of the simplest integrable nets studied in the present chapter, which are discrete conformal maps, described in the next section.

6 Discrete conformal maps

6.1 Discrete isothermic nets in $\mathbb{C}$

Definition 21 *[11] (Wide definition of discrete conformal maps) A discrete isothermic map in $\mathbb{C}$*

$$f : \mathbb{Z}^2 \to \mathbb{R}^2 = \mathbb{C}$$

is called discrete conformal.

[18]There are several reasons to call this net a discrete conformal mapping (see Section 6). One of these reasons is the discrete Weierstrass representation of Theorem 16.

In this section we stick to the narrow Definition 18

$$q_{n,m} = \frac{(f_{n,m} - f_{n+1,m})(f_{n+1,m+1} - f_{n,m+1})}{(f_{n+1,m} - f_{n+1,m+1})(f_{n,m+1} - f_{n,m})} = -1 \tag{6.1}$$

of discrete isothermic nets.

Definition 21 is motivated by the following properties:

- $f: D \subset \mathbb{C} \to \mathbb{C}$ is a (smooth) conformal (holomorphic or antiholomorphic) map if and only if $\forall (x,y) \in D$

$$\lim_{\epsilon \to 0} q\left(f(x,y), f(x+\epsilon,y), f(x+\epsilon,y+\epsilon), f(x,y+\epsilon)\right) = -1.$$

- Definition 21 is Möbius invariant and the dual discrete conformal map $f^*: \mathbb{Z}^2 \to \mathbb{C}$ is defined (see Section 5.2) by

$$f^*_{n+1,m} - f^*_{n,m} = -\frac{1}{\overline{f_{n+1,m}} - \overline{f_{n,m}}}, \quad f^*_{n,m+1} - f^*_{n,m} = \frac{1}{\overline{f_{n,m+1}} - \overline{f_{n,m}}}. \tag{6.2}$$

  The smooth limit of this duality is

$$(\overline{f^*})' = -\frac{1}{f'}, \tag{6.3}$$

  where f is holomorphic and f^* is antiholomorphic.
- Equation (6.1) is integrable. The Lax pair

$$\Psi_{n+1,m} = U_{n,m}\Psi_{n,m} \qquad \Psi_{n,m+1} = V_{n,m}\Psi_{n,m} \tag{6.4}$$

  found by Nijhoff and Capel (see the contribution of Nijhoff [40]) is of the form

$$U_{n,m} = \begin{pmatrix} 1 & -u_{n,m} \\ \dfrac{\lambda}{u_{n,m}} & 1 \end{pmatrix}, \quad V_{n,m} = \begin{pmatrix} 1 & -v_{n,m} \\ -\dfrac{\lambda}{v_{n,m}} & 1 \end{pmatrix}, \tag{6.5}$$

  where

$$u_{n,m} = f_{n+1,m} - f_{n,m}, \qquad v_{n,m} = f_{n,m+1} - f_{n,m}.$$

Definition 21 of a discrete conformal map as a discrete isothermic map is too general for some purposes. For example, there clearly are isothermic nets $f: \mathbb{Z}^2 \to \mathbb{C}$ which are far (see Fig. 9) from the behaviour of usual holomorphic maps. (We are grateful to Tim Hoffmann who produced these and all other Mathematica pictures in this section.)

Another class of maps $f: \mathbb{Z}^2 \to \mathbb{C}$ introduced by Schramm [47] has properties that bring them much closer to the well-known world of complex analysis. Schramm's maps can be viewed (in a sense to be made precise; see Section 6.4) as a subclass of all discrete isothermic nets in $\mathbb{C}$.

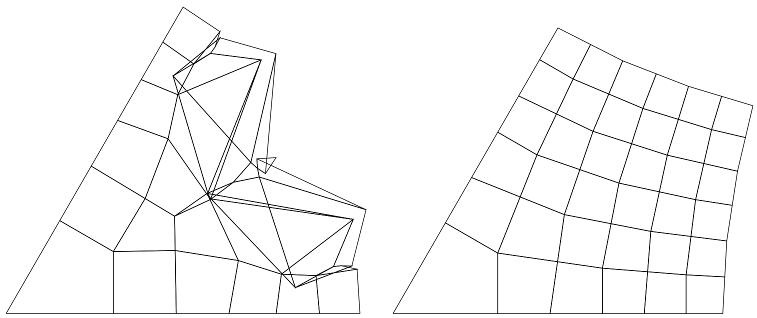


FIG. 9. Two discrete conformal maps with close initial data $n = 0, m = 0$. The second lattice describes a discrete version of the holomorphic mapping $z^{2/3}$.

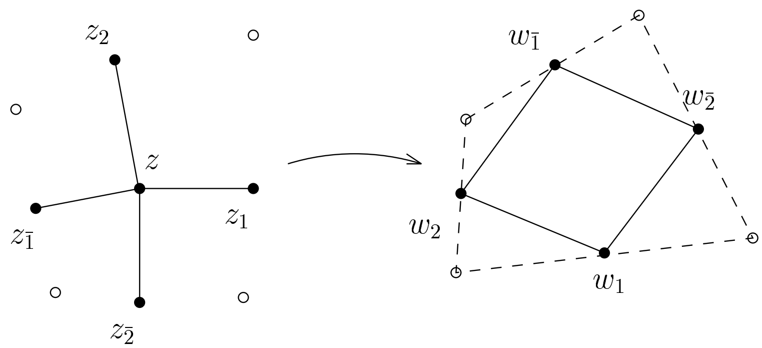


FIG. 10. Parallelogram property

6.2 Discrete P-surfaces (nets with the parallelogram property)

First we show that a discrete isothermic net $F : \mathbb{Z}^2 \to \mathbb{R}^3$ can (almost) be reconstructed from half of the points. Colour the points as black and white in a checkerboard pattern and forget the white points. The black points form a lattice in their own right. To investigate the property of this lattice, look at a black point and its four neighbouring faces (Fig. 10). Mapping the central point to infinity by an inversion, we see a quadrilateral of four white points and the black points are the centres (recall that the cross-ratios $q_{n,m}$ are -1) of the edges of this quadrilateral.

Obviously, these four black points form a parallelogram. We call this property of a net the *parallelogram property* and a net possessing it a *P-net* or *discrete P-surface.*

Lemma 6

- *Both sublattices (black and white) of a discrete I-surface form P-nets.*
- *A (black) lattice possesing the parallelogram property and an additional*

(white) point can be uniquely extended to a discrete I-surface.

Proof. The white quadrilateral in Fig. 10 is uniquely determined by one of its vertices and the black parallelogram. Moreover, due to the parallelogram property the quadrilateral exists for any choice of this (white) vertex. By continuation of this process one reconstructs the whole isothermic net.

Remark. One can interpret the extension construction described above as a transformation of P-nets. Indeed, given such a (black) lattice, choose an arbitrary (white) point, extend them to a discrete I-surface, and delete the original (black) sublattice. We end up with a new (white) P-net.

6.3 Equations for cross-ratios of P-nets in $\mathbb{C}$

As we have seen above, a discrete conformal map is almost determined by its (black) sublattice, which is a lattice in $\mathbb{C}$ with the parallelogram property. Denote the last lattice by

$$z : \mathbb{Z}^2 \to \mathbb{C}.$$

To describe it algebraically let us introduce the cross-ratios

$$s_{n,m} = \frac{(z_{n+1,m} - z_{n,m})(z_{n-1,m} - z_{n,m+1})}{(z_{n,m+1} - z_{n+1,m})(z_{n,m} - z_{n-1,m})},$$
$$t_{n,m} = \frac{(z_{n,m+1} - z_{n,m})(z_{n+1,m} - z_{n+1,m+1})}{(z_{n+1,m+1} - z_{n,m+1})(z_{n,m} - z_{n+1,m})}.$$

Note that whereas the t are cross-ratios of the lattice faces, the s are cross-ratios of the corresponding parallelograms in Fig. 10:

$$s = \frac{(z_1 - z)(z_{\bar{1}} - z_2)}{(z_2 - z_1)(z - z_{\bar{1}})} = \frac{w_{\bar{1}} - w_2}{w_2 - w_1}.$$

Here we use the notations of Section 4.4. It is natural to think about s and t as defined on vertices and faces of the lattice respectively.

It is convenient to use the graphical notations of Fig. 11 for these cross-ratios.

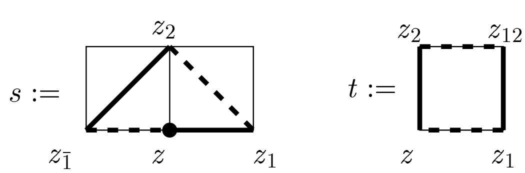
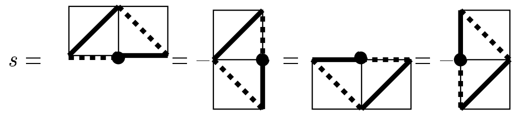


FIG. 11. Graphical representation for cross-ratios

The terms in the numerator and in the denominator of the cross-ratios are denoted by solid and dashed lines respectively. The orientation of edges is cyclic.

The parallelogram property can be reformulated in terms of the cross-ratio s.

Lemma 7 *The following symmetries of the cross-ratio*



are equivalent to the parallelogram property of the lattice.

This implies that under a $\pi/2$ rotation both cross-ratios s, t transform the same way:

$$s \to -\frac{1}{s}, \qquad t \to \frac{1}{t}. \tag{6.6}$$

In order to derive algebraic equations for s, t it is convenient to introduce one more cross-ratio on the faces, which differs from t by a modular transformation $T = 1/(1-t)$.

$T :=$  $= \dfrac{1}{1-t}$

Let us consider two horizontally neighbouring faces and denote by t_l, t_r, s_u, s_d the cross-ratios associated with their faces and common vertices. The labels here denote the l(eft) and r(ight) faces and u(p) and d(own) common vertices of these faces (Fig. 12). Do the same for vertical neighbours. Finally, considering a quadrilateral with its four neighbouring quadrilaterals, we denote their cross-ratios as shown in the third diagram of Fig. 12.

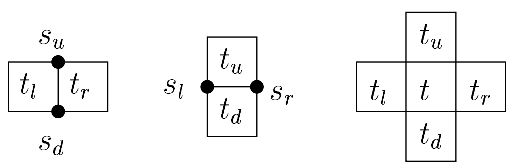


FIG. 12. Labels in the identities (6.7) and (6.8)

Lemma 8 *The neighbouring cross-ratios of a P-net in* $\mathbb{C}$ *satisfy the following identities:*

$$\frac{s_u}{s_d} = \frac{1-t_r}{1-t_l}, \qquad \frac{s_r}{s_l} = \frac{1-t_u^{-1}}{1-t_d^{-1}}, \tag{6.7}$$

$$t^2 = \frac{(1-t_r)(1-t_l)}{(1-t_u^{-1})(1-t_d^{-1})}. \tag{6.8}$$

In these three equations the notations of the corresponding three diagrams of Fig.12 are used.

Proof. Using the graphical notation for the cross-ratios one can immediately see (Fig. 13) that

$$\frac{s_u}{s_d} = \frac{T_l}{T_r}.$$

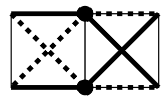


FIG. 13. Proof of the identity $s_u/s_d = T_l/T_r$

Substituting the definition of T we obtain the first of the identities (6.7). The second one follows from the transformation property (6.6). The system (6.7) is linear with respect to s. Eliminating s we obtain the compatibility condition (6.8).

Equation (6.8) is the stationary (one discrete variable is excluded) 3D-Hirota equation (for the 3D-Hirota equation see for example [35]).

The description of Lemma 8 is conformal: Möbius equivalent lattices correspond to the same solution of (6.7, 6.8).

Remark. A Cauchy problem.

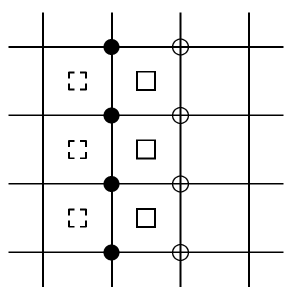


FIG. 14. A well-posed Cauchy problem for system (6.7)

A canonical Cauchy problem for system (6.7) is obtained by prescribing s on a vertical chain of lattice points and t on adjoining faces, that is

$$s_{0,m}, \quad t_{0,m}. \tag{6.9}$$

This is schematically indicated in Fig. 14. Here, bullets and boxes represent the Cauchy data $s_{0,m}$ and $t_{0,m}$ respectively. Obviously, both $s_{1,m}$ (circles) and

$t_{-1,m}$ (dashed boxes) may be calculated from the Cauchy data. It is evident that iterative application of this procedure uniquely determines the solution of (6.7).

Conversely, a solution of (6.7) determines a P-net uniquely up to Möbius transformation. We summarize the results in the following (compare with [47])

Theorem 17

- *The t and s invariants of a P-net in $\mathbb{C}$ satisfy the system (6.7).*
- *Conversely, given a solution (s, t) to the system (6.7) there exists a P-net with these s, t as the lattice invariants.*
- *Two P-nets with the same invariants s, t differ by a Möbius transformation.*
- *Suppose s, t satisfy (6.7). Then t satisfies the stationary 3D-Hirota equation (6.8).*
- *Conversely, suppose that t satisfies (6.8). Then there is a field s defined on vertices such that s and t together satisfy (6.7). Moreover, s is unique, up to multiplication by a complex constant.*

This theorem can be proven by considerations similar to those used above for the Cauchy problem. One starts with three arbitrary points (this ambiguity corresponds to an arbitrary Möbius transformation of the lattice) and builds the whole lattice using the corresponding cross-ratios s, t. The symmetries of Lemma 7 are required; therefore given s at some point and three points of the corresponding "cross" the remaining two points are uniquely determined. Identities (6.7) guarantee the compatibility of the construction.

Remark. Let us note that the last point of the theorem describes a one-parameter family (associated family) of lattices corresponding to the same solution of (6.8). The family parameter plays the role of the spectral parameter.

6.4 Schramm's constraint

Our first observation deals with general discrete P-surfaces in Euclidean 3-space $F : \mathbb{Z}^2 \to \mathbb{R}^3$. With each vertex F one can associate four circles $C^{(1)}, C^{(2)}, C^{(3)}$, $C^{(4)}$, determined by the corresponding triples of points:

$$F, F_1, F_2 \in C^{(1)},\ F, F_2, F_{\bar{1}} \in C^{(2)},\ F, F_{\bar{1}}, F_{\bar{2}} \in C^{(3)},\ F, F_{\bar{2}}, F_1 \in C^{(4)}.$$

The parallelogram property is equivalent to the condition that opposite circles touch (with the same tangent line) each other. In Section 6.5 we study a special class of discrete P-nets in $\mathbb{R}^3$. Before doing that let us continue the investigation of a simpler case of P-nets in $\mathbb{C}$, where we have the algebraic description of the previous section in our disposal.

The following lemma provides us with natural geometric and algebraic subclasses of discrete P-nets in $\mathbb{C}$.

Proposition 9 *The system (6.7) is compatible with the following constraints:*

- *Circular constraint: $t \in \mathbb{R}$.*

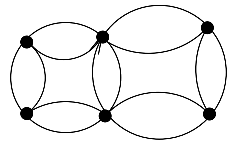


FIG. 15. Schramm's constraint

- *Circular orthogonal constraint:* $t \in \mathbb{R}$ *and* $is \in \mathbb{R}$.
- *Schramm's constraint:* $t < 0, is < 0$.

Geometrically the circular constraint is equivalent to the condition that the elementary quadrilaterals are circular. The orthogonal circular constraint implies that the circles intersect orthogonally. Schramm's constraint implies in addition to the previous two that the elementary quadrilaterals are embedded (the opposite edges do not intersect) and the quadrilateral lattice is immersed.

The last claim is clarified in Fig. 15. Consider closed elementary quadrilaterals with the edges comprised by the corresponding arcs of neighbouring circles. These neighbouring quadrilaterals intersect only in their common vertices iff $is < 0$. It is natural to call a circular lattice satisfying this condition *immersed.*

The (orthogonal) circular constraint is compatible even with the Cauchy problem (6.7, 6.9): given real $t_{0,m}, is_{0,m}$ the solution $t_{n,m}, is_{n,m}$ of the Cauchy problem is real for all $n, m \in \mathbb{Z}$. Note that in the case of the circular constraint,

$$\arg\ s_{n,m} = \text{const}$$

on the whole lattice.

Schramm's constraint is more delicate (and also more interesting) to investigate. Schramm has defined his circle patterns with combinatorics of the square grid [47] coming from questions in approximation theory. He has shown that negative solutions $t < 0$ of equation (6.8) satisfy a maximum principle, which allows us to prove global results. In particular he has proved [47] that the only embedding of the whole $\mathbb{Z}^2$ is the standard circle pattern (where all circles have constant radius). In terms of solutions of the Hirota equation (6.8) this implies that $t \equiv -1$ is the only strictly negative solution on the whole lattice $t : \mathbb{Z}^2 \to \mathbb{R}_- = \{t < 0\}$.

We call discrete P-nets in $\mathbb{C}$ satisfying Schramm's constraint *S-nets* in $\mathbb{C}$. In a wide version of this definition only circular and orthogonal constraints are required.

It is possible to generalize Schramm's circle patterns, replacing $\mathbb{Z}^2$ by a quad-graph (see Section 2.1). Instead of having four vertices on every circle, one allows various numbers N of vertices (and as a consequence N neighbouring and N half-neighbouring circles. It is natural to call [6] such a singular point a *branch point* of order $N/4 - 1$.

6.5 Discrete S-surfaces in $\mathbb{R}^3$ (Schramm isothermic surfaces)

The constraints of the previous section are too restrictive for discrete P-surfaces in $\mathbb{R}^3$. Indeed, it is easy to see that a discrete P-surface in $\mathbb{R}^3$ with circular quadrilaterals (discrete C-surface) must lie on a sphere S^2. Identifying S^2 with $\mathbb{C}$ we end up with a circular P-net in $\mathbb{C}$ of the already considered type.

Definition 22 *A discrete P-surface is called orthogonal if all parallelograms defined as in Fig. 10 are rectangles. Equivalently, the circles $C^{(1)}, \ldots, C^{(4)}$ defined in the previous section intersect orthogonally $C^{(i)} \perp C^{(i+1)}$.*

Let us observe another possible characterization of circular orthogonal P-nets in $\mathbb{C}$. A simple corollary of equations (6.7) is the following:

Lemma 9 *An orthogonal P-net in $\mathbb{C}$ with one circular elementary quadrilateral (i.e. one real $t_{0,0} \in \mathbb{R}$) is orthogonal circular, i.e. $t_{n,m} \in \mathbb{R}$, $\forall n, m$.*

This lemma motivates us to suggest the following generalization in the case of space nets.

Definition 23 *A discrete S-surface is a discrete orthogonal P-surface $F : \mathbb{Z}^2 \to \mathbb{R}^3$ for which half of the quadrilaterals, say those of the form*

$$(F_{n,m}, F_{n+1,m}, F_{n+1,m+1}, F_{n,m+1}) \qquad \text{with}^{19}\ n + m \equiv 0 \tag{6.10}$$

are circular.

Let us denote the circles of (6.10) by $C_{n,m}$ and assume that other quadrilaterals are non-circular, i.e. they uniquely define spheres $S_{n,m}$ such that

$$F_{n,m}, F_{n+1,m}, F_{n+1,m+1}, F_{n,m+1} \in S_{n,m} \qquad \text{with } n + m \equiv 1.$$

In this generic case at each point $F_{n,m}$ we have a pair of touching spheres and a pair of touching circles, which intersect these spheres orthogonally.

A discrete S-surface can be described as a sphere packing with this property. Given a packing of touching spheres the C-circles are well defined, due to the following:

Lemma 10 *Let S_1, S_2, S_3, S_4 be four spheres touching at the points $P_1 = S_1 \cap S_2, P_2 = S_2 \cap S_3, P_3 = S_3 \cap S_4, P_4 = S_4 \cap S_4$. Then the points P_1, P_2, P_3, P_4 lie on a circle.*

Proposition 10 *(Sphere packing definition of discrete S-surfaces). A discrete S-surface is described by a packing of touching spheres $S_{n,m}$ $(n + m \equiv 1)$ such that the circles $C_{n,m}$ $(n + m \equiv 0)$ defined in Lemma 10 cut them orthogonally.*

A natural subclass of these sphere packings are packings of mutually disjoint touching balls with orthogonal circles as in Proposition 10. This is a proper three-dimensional generalization of Schramm's constraint.

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[19] By $a \equiv 0$ we denote a=0 (mod 2).

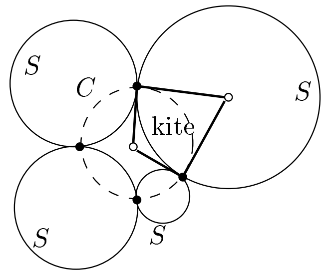

FIG. 16. Discrete S-surface and its central extension

We have defined discrete P-surfaces as sublattices of discrete I-surfaces. Let us describe the reconversion process for discrete S-surfaces. Let us add to a discrete S-surface the centres of the spheres S and of the circles C (Fig. 16). Obviously, all elementary quadrilaterals of the extended lattice are kites. In particular, they are conformal squares ($q = -1$) and hence form a discrete I-surface. All the notions here except the "kite form" are Möbius invariant. This implies that the construction above holds for any choice of the "infinity point". Indeed, choose an arbitrary point $P_\infty \in \mathbb{R}^3 \cup \{\infty\}$. Reflect it in all the spheres S and all the circles [20] C. We call the resulting extended lattice a *central extension*. We have proved the first statement of the following:

Lemma 11 *A central extension of a discrete S-surface is discrete isothermic. All isothermic extensions of a discrete S-surface are central extensions with some point* $P_\infty \in \mathbb{R}^3 \cup \{\infty\}$.

Due to Lemma 6 an isothermic extension of a discrete P-surface is uniquely determined by one additional point. Take such a point and invert it in the corresponding sphere or circle to determine P_∞. The uniqueness implies that the central extension with P_∞ is the discrete I-surface we started with.

The class of discrete I-surfaces which are central extensions with $P_\infty = \infty$ (i.e. with kite faces) is invariant with respect to the dualization transformation of Theorem 14 (note that we are dealing with the narrow definition of discrete I-surfaces $q = -1$ and $\alpha = \beta = 1$ in (5.7). This allows us to define a *dual* discrete S-surface as follows:

1. extend a discrete S-surface to kites (i.e. build the central extension with $P_\infty = \infty$,
2. dualize the kites using (5.7),
3. throw away the centres.

Proposition 11 *The dualization transformation described above applied to a discrete S-surface F yields another discrete S-surface (which we call dual to F).*

[20] The reflection in C is defined as follows. Consider the sphere $\tilde{S}$ passing through C and P_∞. Denote by S_C the orthogonal sphere $S_C \perp \tilde{S}$ containing C. Reflect P_∞ in S_C. The image lies on $\tilde{S}$.

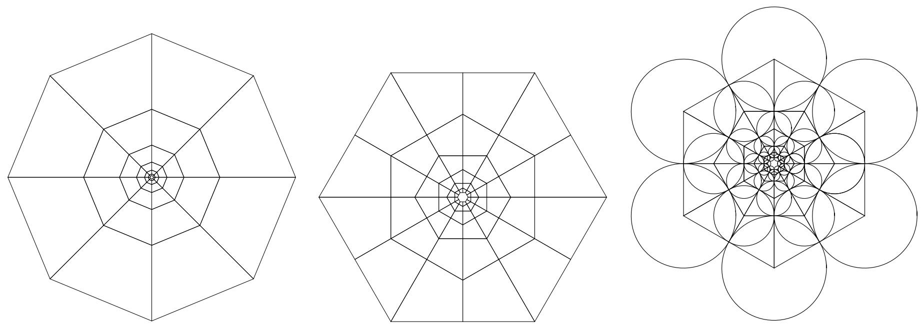

FIG. 17. Discrete conformal maps: EXP and S-EXP without and with circles.

One should prove that step 3 of the dualization procedure provides us with an S-net. Central extensions with $P_\infty = \infty$ are characterized by the property that their faces are of the kite form and that the centres of touching spheres are collinear with the point of contact. Both these properties are preserved by the transformation (5.7).

Remark. The duality transformation preserves the planarity of lattices; therefore it is well defined for S-nets in $\mathbb{C}$.

6.6 Examples of discrete conformal mappings

Keeping in mind the relation to holomorphic mappings, it is natural to look for discrete relatives of the simplest holomorphic functions.

- Z:= discrete z

$$Z(n,m) := n + im.$$

The standard lattice belongs to all classes considered in the present paper.

- EXP:= discrete e^z

$$\mathrm{EXP}_\gamma(n,m) := \exp(2n \ \mathrm{arcsinh}\gamma + 2im \arcsin\gamma), \quad \gamma \in \mathbb{R}$$

is a discrete conformal map. The Schramm exponent S-EXP is a little bit less symmetric (see Fig. 17).

- TANH:= discrete $\tanh z$

is a Möbius transformation of EXP (see [27]).

- Z^{N+1} := discrete z^{N+1}, $N \in \mathbb{Z}$

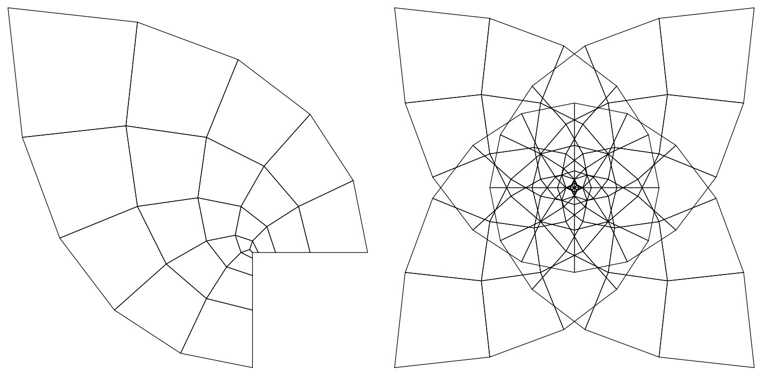

FIG. 18. A sector and the whole lattice of the discrete conformal map Z^3.

is a discrete conformal mapping, obtained from the standard lattice by intertwining the inversion $1/z$ and the dualization.[21] For the first one in this series we obtain

$$\mathrm{Z}(n,m) \overset{1/z}{\to} \frac{1}{\mathrm{Z}}(n,m) = \frac{1}{n+im} \overset{\bar{*}}{\to} \frac{1}{3}\mathrm{Z}^3(n,m) = \frac{1}{3}((n+im)^3 - (n-im)) \overset{1/z}{\to} \ldots$$

The discrete conformal map $\mathrm{Z}^3 = z^3 - \bar{z}$ is close to the smooth z^3. A sector of this map as well as the whole lattice is presented in Fig. 18.

Remark. Since the lattices Z and 1/Z are both discrete conformal and S-nets one can build a similar sequence Z_S^{2N+1} in the Schramm class replacing the transformation (6.2) we used above by the duality transformation of S-nets described in Proposition 11.

- $\mathrm{Z}^\gamma :=$ discrete z^γ, $0 < \gamma < 2$ (for details see [6]).

Equation (6.1) can be supplemented with the following nonautonomous constraint:

$$\gamma f_{n,m} = 2n \frac{(f_{n+1,m} - f_{n,m})(f_{n,m} - f_{n-1,m})}{f_{n+1,m} - f_{n-1,m}} + 2m \frac{(f_{n,m+1} - f_{n,m})(f_{n,m} - f_{n,m-1})}{f_{n,m+1} - f_{n,m-1}}. \tag{6.11}$$

Proposition 12 *$f : \mathbb{Z}^2 \to \mathbb{C}$ is a solution to the system (6.1, 6.11) iff there exists a solution to (6.4, 6.5), which satisfies the following differential equation in λ:*

$$\frac{d}{d\lambda}\Psi_{n,m} = A\Psi_{n,m}, \quad A = \frac{1}{\lambda}A_0 + \frac{1}{\lambda-1}A_1 + \frac{1}{\lambda+1}A_{-1}, \tag{6.12}$$

[21] It is natural to combine the dualization transformation (6.2) with complex conjugation to obtain maps close to holomorphic ones (compare with (6.3)). We denote this combination by $\bar{*}$.

where the matrices A_0, A_1, A_{-1} *are as follows:*

$$A_0 = \begin{pmatrix} -\dfrac{\gamma}{4} & n\dfrac{u_{n,m}u_{n-1,m}}{u_{n,m}+u_{n-1,m}} + m\dfrac{v_{n,m}v_{n,m-1}}{v_{n,m}+v_{n,m-1}} \\ 0 & \dfrac{\gamma}{4} \end{pmatrix},$$

$$A_1 = \frac{m}{v_{n,m}+v_{n,m-1}} \begin{pmatrix} v_{n,m} & v_{n,m}v_{n,m-1} \\ 1 & v_{n,m-1} \end{pmatrix}$$

$$A_{-1} = \frac{n}{u_{n,m}+u_{n-1,m}} \begin{pmatrix} u_{n,m} & u_{n,m}u_{n-1,m} \\ 1 & u_{n-1,m} \end{pmatrix}.$$

The constraint (6.11) is compatible with (6.1).

In the case $\gamma = 1$ the constraint (6.11) and the corresponding monodromy problem (6.12) were obtained in [39] (see also the contribution of Nijhoff [40]).

Remark. The monodromy problem (6.12) coincides with the monodromy problem of the Painlevé VI equation [31], which shows that the system can be solved in terms of the Painlevé transcendents.

Let us assume $\gamma < 2$ and denote $\mathbb{Z}^2_+ = \{(n,m) \in \mathbb{Z}^2 : n, m \geq 0\}$. Motivated by the asymptotics of the constraint (6.11) at $n, m \to \infty$ and the properties

$$z^\gamma(\mathbb{R}_+) \in \mathbb{R}_+, \quad z^\gamma(i\mathbb{R}_+) \in e^{\gamma\pi i/2}\mathbb{R}_+,$$

of the holomorphic z^γ it is natural to give the following definition of the "discrete z^γ" which we denote by Z^γ.

Definition 24 $Z^\gamma : \mathbb{Z}^2 \to \mathbb{C}$ *is the solution of (6.1, 6.11) with initial conditions*

$$Z^\gamma(0,0) = 0, \quad Z^\gamma(1,0) = 1, \quad Z^\gamma(0,1) = e^{\gamma\pi i/2}.$$

It is easy to see that $Z^\gamma(n,0) \in \mathbb{R}_+, Z^\gamma(0,m) \in e^{\gamma\pi i/2}\mathbb{R}_+, \ \forall n, m \in \mathbb{N}$.

It is not difficult to check that the discrete conformal map Z^γ with $\gamma = 4/N$, $N \in \mathbb{N}$, $N > 4$ is a generalized Schramm circle pattern. (Recall that the central points of the circles are also included). In this case the only branch point is at the origin. We call the combinatorics of this pattern combinatorics of the plane with one branch point of order $N/4 - 1$.

In the discrete as well as in the smooth case (up to a constant factor) one has

$$(Z^\gamma)^{\bar{*}} = Z^{2-\gamma}.$$

Conjecture 1

- $Z^\gamma : \mathbb{Z}^2_+ \to \mathbb{C}$ *is an embedding, i.e. different open elementary quadrilaterals of the pattern* $Z^\gamma(\mathbb{Z}^2_+)$ *do not intersect.*
- Z^γ *is the only embedded discrete conformal map* $f : \mathbb{Z}^2_+ \to \mathbb{C}$ *with*

$$f(0,0) = 0, \quad f(n,0) \in \mathbb{R}_+, \quad f(0,m) \in e^{\gamma\pi i/2}\mathbb{R}_+ \quad \forall n, m \in \mathbb{N}.$$

- *Up to a similarity $Z^{4/N}$ is the only embedded generalized Schramm circle pattern with the combinatorics of the plane with one branch point of order $N/4-1$.*

Computer experiments made by Tim Hoffmann confirm the first conjecture of the list. The second lattice in Fig. 9 is a sector of $Z^{2/3}$.

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