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Bonnet surfaces and Painlevé equations

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Introduction

Let $\mathscr{F}$ be a surface in Euclidean 3-space without umbilic points. This paper studies the following

Problem. To classify non-trivial one-parameter families $\mathscr{F}_\tau$, $\tau \in (-\varepsilon, \varepsilon)$ of isometries of $\mathscr{F} = \mathscr{F}_0$ preserving both principal curvatures.

Since the Gaussian curvature is preserved by isometries one can reformulate the problem, replacing "both principle curvatures" by "the mean curvature function". Let us specify what we mean by a non-trivial family. We consider families of surfaces which do not differ by rigid motions. We assume also that the surfaces and isometries are sufficiently smooth. The case of surfaces with constant mean curvature (CMC surfaces), which all possess non-trivial isometries (the associated family) is also excluded from our consideration. We suppose that the mean curvature is a non-constant function on $\mathscr{F}$.

It turns out that the condition of possessing a one-parameter family $\mathscr{F}_\tau$ of isometries, preserving H, implies restrictive conditions on $\mathscr{F}$. Moreover, it was shown by Cartan [C] that all the family $\mathscr{F}_\tau$ can be described (see section 2) as a reparametrization of $\mathscr{F}$ itself. Thus the problem is reduced to the problem of classification of the surfaces $\mathscr{F}$. Since the problem formulated at the beginning of this introduction was first studied by Bonnet, we call these surfaces *Bonnet surfaces*.

The problem is classical and many mathematicians contributed to its solution. O. Bonnet himself showed in [Bo] that besides the CMC surfaces there is a class of surfaces, depending of finitely many parameters, which allow non-trivial H preserving isometries. These results were developed further by L. Raffy, who proved that the Bonnet surfaces are isothermic (i.e., allow conformal curvature-line parametrization) and isometric to surfaces of revolution. In 1899 J. N. Hazzidakis [H] showed that the mean curvature function H satisfies an ordinary differential equation of the third order and was able to integrate

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it once. Graustein [G] proved that all Bonnet surfaces are Weingarten surfaces, i.e. the mean and the Gaussian curvature are related by $dH \wedge dK = 0$. He also found a convenient alternative description for the Bonnet surfaces. Namely, he showed that these surfaces can be characterized as isothermic surfaces where for x and y conformal curvature-line coordinates the function $1/Q$ with

$$Q = \frac{1}{4}\langle F_{xx} - F_{yy}, N\rangle, \quad F \text{ the immersion}, \quad N \text{ the normal vector field on } F,$$

is harmonic, meaning

$$(\partial_{xx} + \partial_{yy})(1/Q) = 0 .$$

In modern notation Q is the Hopf differential, written in isothermic coordinates x, y.

Later the problem was treated by E. Cartan in [C], where the most detailed results concerning the Bonnet surfaces are presented. Cartan gave a modern definition of these surfaces and classified them into 3 cases A, B and C. The mean curvature function $H(t)$ satisfies the Hazzidakis equation

$$\left(\frac{H''}{H'}\right)' - H' = |Q|^2\left(2 - \frac{H^2}{H'}\right), \tag{1}$$

where $|Q|^2$ is a fixed function which is different for each of the three cases A, B and C (see section 2):

$$\begin{aligned} |Q_A|^2 &= \frac{4}{\sin^2(2t)}, \\ |Q_B|^2 &= \frac{4}{\sinh^2(2t)}, \\ |Q_C|^2 &= \frac{1}{t^2}. \end{aligned} \tag{2}$$

Equation (1) is the Gauss equation of the Bonnet surfaces. After the result of Hazzidakis, who reduced this equation to equations of the second order for all 3 cases A, B and C there was no progress in investigation of (1). Cartan finished his paper by the phrase: "An investigation of the singularities of the differential equation (1) seems to be difficult." We mention also a more recent paper by S. Chern [Ch], where it was shown in particular, that the argument of the Hopf differential written in any conformal coordinates is harmonic.

It turns out that Cartan was right in his estimation of equation (1), (2). In this paper we show that the Hazzidakis equation (1) with $|Q|^2$ given by (2) is isomorphic to the Painlevé equations: namely to Painlevé VI equations in cases A and B and to a special case of the Painlevé V equation, which can be reduced to a Painlevé III equation, in case C. The isomorphism is given by explicit formulas (81), (79) in cases A and B and by (49), (53) in case C.

Although the formulas establishing the isomorphism can be checked directly, they would hardly be found without using the theory of integrable systems. The starting point of the present paper is an observation made in [B94] that the frame equation for the Bonnet surfaces written via 2×2 matrices has the same structure as the Lax representation of the Painlevé equations given in [J-M], [I-N]. Here we develop this observation and describe the corresponding isomorphism explicitly.

Modern achievements in the global asymptotic analysis of the Painlevé equations make it possible to evaluate in closed form (in terms of elementary functions and their quadratures) asymptotic connection formulae for the corresponding solution manifolds. This is a characteristic analytical property of the special functions. In other words, the current status of the Painlevé transcendents should be considered to be the same as that of the hypergeometric functions and their degenerations. If a problem can be solved in terms of the Painlevé transcendents, the solution should be treated as an explicit one. In more details, this point of view is presented in the review papers [I92], [I94]. Therefore we solve the Bonnet problem mentioned at the beginning of this introduction *explicitly*.

In Appendix A a special case is discussed, when the Bonnet surfaces can be described in hypergeometric functions. Appendix B deals with the asymptotes of the families of type A. In particular we show that they are helicoidal surfaces and give a geometrical meaning of Hazzidaki's integral of (1) in this case.

After a preprint version [B-E-94] of the present paper has appeared further progress in investigation of the Bonnet surfaces has been achieved. In particular, the Bonnet surfaces in space forms were studied in [CL]. They are dual to natural classes of surfaces with harmonic inverse mean curvature. This aspect was investigated in [B-E-K]. The Schlesinger transformation for the Bonnet surfaces is constructed in [Eit96]. Possible umbilics on the Bonnet surfaces are discussed in [Rou]. Finally, let us mention also a recent paper [K-P-P] where the Bonnet pairs (i.e. pairs of isometric surfaces with coinciding mean curvature functions) are described in terms of isothermic surfaces.

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1. Quaternionic description of surfaces in Euclidean 3-space

To study surfaces in $\mathbb{R}^3$ by analytical methods it is convenient to describe them in terms of 2×2 matrices (for more details see [B94]). In section 2 and 3 this description allows us to identify the equations for the moving frame of the Bonnet surfaces with the Lax representation of the Painlevé equations.

Let $F: \mathscr{R} \to \mathbb{R}^3$ be a conformal parametrization of an orientable surface:

$$(F_z, F_z) = (F_{\bar{z}}, F_{\bar{z}}) = 0, \quad (F_z, F_{\bar{z}}) = \frac{1}{2} e^u .$$

Here $\mathscr{R}$ is a Riemann surface with the induced complex structure,

$$(v, w) = v_1 w_1 + v_2 w_2 + v_3 w_3 ,$$

z is a complex coordinate and F_z and $F_{\bar{z}}$ are the partial derivatives

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The vectors F_z, $F_{\bar{z}}$ and N define a moving frame on the surface. The fundamental forms are

$$(3) \qquad \begin{aligned} (dF, dF) &= e^u dz\, d\bar{z}, \\ -(dF, dN) &= Q dz^2 + H e^u dz\, d\bar{z} + \bar{Q} d\bar{z}^2 , \end{aligned}$$

where $Q = (F_{zz}, N)$ is the Hopf differential and H the mean curvature function on F.

The compatibility conditions (the Gauss-Codazzi equations) of the moving frame equations are

$$(4) \qquad \begin{aligned} u_{z\bar{z}} + \frac{H^2}{2} e^u - 2Q\bar{Q}e^{-u} &= 0 \qquad \text{(Gauss equation)}, \\ \bar{Q}_z = \frac{H_{\bar{z}}}{2} e^u, \quad Q_{\bar{z}} &= \frac{H_z}{2} e^u \quad \text{(Codazzi equations)}. \end{aligned}$$

Let us denote the algebra of quaternions by $\mathbb{H}$ and the standard basis by $\{\mathbb{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$

$$(5) \qquad \mathbf{k} = \mathbf{ij} = -\mathbf{ji}, \quad \mathbf{i} = \mathbf{jk} = -\mathbf{kj}, \quad \mathbf{j} = \mathbf{ki} = -\mathbf{ik} .$$

We will use the standard matrix representation of $\mathbb{H}$:

$$(6) \quad \mathbf{i} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We identify the 3-dimensional Euclidean space with the space of imaginary quaternions

$$\mathbf{Im}\, \mathbb{H} = \mathrm{su}(2) = \mathrm{span}(\mathbf{i}, \mathbf{j}, \mathbf{k})$$

by

$$(7) \qquad X = (x_1, x_2, x_3)^t = x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k} \in \mathrm{su}(2) .$$

The scalar product of vectors in terms of matrices is then

$$(X, Y) = -\frac{1}{2} \mathrm{tr}(XY) .$$

Let us take $\Phi \in \mathrm{SU}(2)$ which transforms the basis $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ into the orthonormal frame $e^{-u/2}F_x$, $e^{-u/2}F_y$, N:

$$(8) \qquad e^{-u/2}F_x = \mathbf{\Phi}^{-1}\mathbf{i}\mathbf{\Phi}\,, \quad e^{-u/2}F_y = \mathbf{\Phi}^{-1}\mathbf{j}\mathbf{\Phi}\,, \quad N = \mathbf{\Phi}^{-1}\mathbf{k}\mathbf{\Phi}\,.$$

Deriving (8) with respect to x and y, respectively, using the moving frame equations for F_x, F_y and N, and the condition that $F_{xy} = F_{yx}$ we find for the complex coordinate $z = x + iy$, $\bar{z} = x - iy$ that

$$(9) \quad \mathbf{\Phi}_z\mathbf{\Phi}^{-1} = \frac{1}{4}\begin{pmatrix} u_z & -4Qe^{-u/2} \\ 2He^{u/2} & -u_z \end{pmatrix}, \quad \mathbf{\Phi}_{\bar{z}}\mathbf{\Phi}^{-1} = \frac{1}{4}\begin{pmatrix} -u_{\bar{z}} & -2He^{u/2} \\ 4\bar{Q}e^{-u/2} & u_{\bar{z}} \end{pmatrix}.$$

For a more detailed discussion of this see [B94].

2. Differential equations of Bonnet surfaces

Definition 1. If a surface $\mathscr{F}$ posesses a 1-parameter family of isometries $\mathscr{F}_\tau$

$$\mathscr{F}_0 = \mathscr{F}, \quad \tau \in (-\varepsilon, \varepsilon), \quad \varepsilon > 0,$$

preserving the mean curvature function, $\mathscr{F}$ is called a Bonnet surface.

Let $F_\tau : \mathscr{R} \to \mathbb{R}^3$ be a conformal parametrization of $\mathscr{F}_\tau$ and

$$z : V \subset \mathscr{R} \to U \subset \mathbb{C}$$

be a local parameter. Locally in terms of $z \in U$ we get a map

$$\begin{array}{rccc} F\colon & (-\varepsilon,\varepsilon)\times U \subseteqq \mathbb{C} & \to & \mathbb{R}^3, \\ & (\tau, z, \bar{z}) & \mapsto & F(\tau, z, \bar{z})\,. \end{array}$$

We suppose also that F is umbilic-free and analytic.

Remark 1. The property that F be umbilic-free is necessary since we are dealing with conformal curvature line coordinates which exist only away from umbilics. Requiring the surfaces to be analytic ensures that the set of those exceptional points is discrete.

Remark 2. Since isometries preserve the Gauss curvature, both principal curvatures are preserved.

General cylinders and surfaces of revolution are examples of Bonnet surfaces. They are generated by a rigid motion acting in a plane curve. The plane curve as well as the curve given by the rigid motion action are curvature lines on these surfaces. The action is an isometry and because it preserves the curvature lines it preserves the mean curvature.

For CMC (constant mean curvature) surfaces it is well-known that there is an isometric 1-parameter family of CMC surfaces with the same mean curvature value – the so-called associated family. Thus these are also examples for Bonnet surfaces.

We restrict our discussion to non-constant mean curvature surfaces not generated by a rigid motion.

Let us denote the Hopf differential of $F(\tau,z,\bar z)$ by $Q_\tau(z,\bar z)$. The Gauss equation (4) implies that $|Q_\tau(z,\bar z)|$ is invariant under the deformation, so

$$Q_\tau(z,\bar z)=e^{i\varphi(\tau,z,\bar z)}\,q_0(z,\bar z)\,,$$

where $q_0(z,\bar z)$ is real-valued. The Codazzi equations in (4) yield

$$(Q_\tau-Q_0)_{\bar z}=\{(e^{i\varphi(\tau,z,\bar z)}-e^{i\varphi(0,z,\bar z)})\,q_0(z,\bar z)\}_{\bar z}=\,0\,,$$

which gives

$$(10)\qquad Q_\tau-Q_0=(e^{i\varphi(\tau,z,\bar z)}-e^{i\varphi(0,z,\bar z)})\,q_0(z,\bar z)=f(\tau,z)\,.$$

Thus we find

$$\frac{\partial}{\partial\tau}Q_\tau(z,\bar z)=iQ_\tau(z,\bar z)\frac{\partial}{\partial\tau}\varphi(\tau,z,\bar z)=g(\tau,z)$$

is holomorphic for all τ. Solving this for Q_τ we find

$$(11)\qquad Q_\tau(z,\bar z)=-ig(\tau,z)\left(\frac{\partial}{\partial\tau}\varphi(\tau,z,\bar z)\right)^{-1},$$

which is a product of the real-valued function (the derivative of the argument function φ with respect to τ) and a holomorphic function $-ig(\tau,z)$ (the $-i$ multiple of the derivative of $f(\tau,z)$ with respect to τ).

It is not hard to see that $(F_{xy},N)=-2\,\mathrm{Im}(Q)$ for a conformal immersion F and its normal vector field N. So the vanishing of the imaginary part of Q is equivalent to the fact that the conformal coordinates x and y are curvature line parameters as well.

The following lemma shows that the form (11) for the Hopf differential means that the surface is isothermic, i.e., admits conformal curvature line coordinates.

Lemma 1. *Let $Q(z,\bar z)=f(z)\,q_0(z,\bar z)$ be the Hopf differential of an immersion with $f(z)$ holomorphic and $q_0(z,\bar z)$ a real-valued function. Then with respect to the conformal coordinate*

$$w=\int\sqrt{f(z)}\,dz$$

the corresponding Hopf differential is $\tilde Q(w,\bar w)=q_0\big(z(w),\bar z(\bar w)\big)$.

The next theorem shows that $1/Q$ is harmonic.

Theorem 1 (Graustein [G]). *Any umbilic-free analytic Bonnet surface is isothermic. With respect to isothermic coordinates*, $1/Q$ *is harmonic*:

$$Q(z,\bar z) = \frac{1}{h(z) + \bar h(\bar z)}, \tag{12}$$

where $h(z)$ is holomorphic.

Proof. Let us consider a conformal curvature line parametrization of $\mathscr{F} = \mathscr{F}_0$. Then $Q_0(z,\bar z)$ is real. On the other hand our previous observations imply

$$Q_\tau \bar Q_\tau = Q_0^2, \quad Q_\tau = Q_0 + f(\tau, z), \tag{13}$$

with $f(\tau,z)$ holomorphic as above. Solving (13) one gets for all τ

$$Q_0 = -\frac{f\bar f}{f+\bar f}, \quad Q_\tau = -\frac{f}{\bar f} Q_0.$$

We identify

$$h(z) = -\frac{1}{f(\tau,z)} + iT(\tau) \tag{14}$$

and find

$$Q_\tau(z,\bar z) = Q_T(z,\bar z) = \left(\frac{1 - iT\bar h}{1 + iTh}\right)\left(\frac{1}{h+\bar h}\right), \quad T \in \mathbb{R} \cup \{\pm\infty\} \tag{15}$$

for the deformation family in terms of a new deformation parameter $T = T(\tau)$.

Finally we show, that $h(z)$ can be made τ-independent by a proper choice of $T(\tau)$. Since $Q_0 = 1/\big(h(z) + \bar h(\bar z)\big)$ is τ-independent, $h(z)$ as well as $1/f(\tau,z)$ must be sums of a τ-independent holomorphic function and a purely imaginary function depending only on τ. Using this fact we can assume that $h(z)$ in (14) is τ-independent.

Remark 3. The dependence of T on τ might be rather complicated. We consider T as a new deformation parameter that can take any value in $\mathbb{R}P^1$. Formula (15) shows that the dependence of the family on T is smooth.

In our argument we do not prefer any special Bonnet surface of a Bonnet family. Therefore the above holds for any Bonnet surface of a Bonnet family but with different functions h and different curvature line coordinates z and $\bar z$. For any such surface the compatibility conditions (4) are as follows:

$$Q(z,\bar z) = \frac{1}{h(z) + \bar h(\bar z)}, \tag{16}$$

$$\left(\frac{H_{z\bar z}}{H_z}\right)_{\bar z} - \frac{H_{\bar z}}{h_z} = 2\frac{|h_z|^2}{(h+\bar h)^2} - \frac{H^2 h_z}{H_{\bar z}(h+\bar h)^2} \quad \text{(Gauss equation)}, \tag{17}$$

$$h_z H_z = \bar h_{\bar z} H_{\bar z} \quad \text{(Codazzi equation)}, \tag{18}$$

$$e^{u(z,\bar z)} = -\frac{2h_z}{(h+\bar h)^2 H_{\bar z}} \quad \text{(Codazzi equation)}. \tag{19}$$

Remark 4. The isothermic coordinate z in Theorem 1 is unique up to $z \to \alpha z + \beta$, where α is a non zero real or purely imaginary number and $\beta \in \mathbb{C}$ arbitrary.

Lemma 2. *The transformation of the surface $\mathscr{F} \to \alpha\mathscr{F}$, $\alpha \in \mathbb{R}\backslash\{0\}$ transforms*

$$h(z) \to 1/\alpha h(z)\,.$$

Proof. A scaling of the surface by α changes the coefficients of the fundamental forms in the following way:

$$e^u \to \alpha^2 e^u, \quad H \to \frac{1}{\alpha} H, \quad Q \to \alpha Q\,. \tag{20}$$

Applying these rules to (16) shows that $h(z)$ must change in the described manner. Then (17), (18) and (19) are satisfied when the metric and the mean curvature behaves in the way stated in (20).

We will not distinguish Bonnet surfaces which differ by a scaling transformation.

Theorem 2. *Let $\mathscr{F} = \mathscr{F}_0$ be a Bonnet surface with isothermic coordinates z, $\bar{z}$. Then*

$$w = w(z) = \int \frac{1}{h_z(z)}\, dz\,, \tag{21}$$

is also a conformal coordinate, and the mean curvature function H, the metric u and the modulus of the Hopf differential $|Q|$ are functions of

$$t = w + \bar{w} \tag{22}$$

only.

Proof. By the chain rule we get $H_w = h_z H_z$ and $H_{\bar{w}} = \bar{h}_{\bar{z}} H_{\bar{z}}$ which implies by (18) that $H(w, \bar{w})$ is a function of t only. Since $Q(z,\bar{z})\,dz^2 = \tilde{Q}(w,\bar{w})\,dw^2$ we get

$$\tilde{Q}(w,\bar{w}) = \big(w'(z)\big)^2 Q(z,\bar{z}) = \frac{h_z^2(z)}{h(z) + \bar{h}(\bar{z})}\,. \tag{23}$$

We rename $\tilde{Q}$ by Q. This should not cause ambiguities since now everything is thought to be expressed in terms of the coordinates w, $\bar{w}$. A simple calculation shows

$$|Q|^2 = -Q_{\bar{w}} = -\bar{Q}_w\,,$$

which implies that the metric function satisfies

$$e^u = \frac{2Q_{\bar{w}}}{H_w} = -\frac{2|Q|^2}{H'}\,, \tag{24}$$

where H' is the derivative of H with respect to t defined in (22). If we reformulate the Gauss equation (see (17)) in terms of the mean curvature function H we get the following equation:

$$\left(\frac{H''}{H'}\right)' - H' = |Q|^2 \left(2 - \frac{H^2}{H'}\right). \tag{25}$$

Here $|Q|$ should be independent of $w - \bar{w}$ because otherwise $(H''/H')' - H'$ and $2 - H^2/H'$ must vanish identically. But this implies $H' > 0$ in contradiction to (24). Finally the metric too, depends only on t.

Remark 5. An interesting interpretation of the corollary above is that all Bonnet surfaces are Weingarten surfaces [G]. This should be clear since the dependence of H and u on only one real variable implies the same property for the Gaussian curvature K, and so ∇H and ∇K are parallel.

With respect to w the coefficient functions of the fundamental forms of the deformation family are

$$Q_T(w, \bar{w}) = \left(\frac{1 - iT\bar{h}(\bar{z})}{1 + iTh(z)}\right)\left(\frac{h_z^2(z)}{h(z) + \bar{h}(\bar{z})}\right), \quad e^{u(w,\bar{w})} = -\frac{2|h_z|^4}{(h(z) + \bar{h}(\bar{z}))^2 H'(t)}, \tag{26}$$

and $H(w, \bar{w}) = H(t)$ is a solution of the ordinary differential equation (25). In particular for $T = 0$ we get for the Hopf differential just Q as in (23).

Let us now fix a particular surface in a deformation family of Bonnet surfaces and call this F_0. The corresponding coefficients of the fundamental forms are as in (26) with $T = 0$.

Theorem 3. *The holomorphic function $h = h(z)$ satisfies the differential equation*

$$h_{zz}(h + \bar{h}) - h_z^2 = \bar{h}_{\bar{z}\bar{z}}(h + \bar{h}) - \bar{h}_{\bar{z}}^2 . \tag{27}$$

Proof. Since $|Q_T| = |Q_0|$ ($Q_0 = Q$ in (23)) depends only on $t = w + \bar{w}$ we have

$$\left(\frac{|h_z(z)|^2}{h(z) + \bar{h}(\bar{z})}\right)_w = \left(\frac{|h_z(z)|^2}{h(z) + \bar{h}(\bar{z})}\right)_{\bar{w}},$$

which implies (27).

Theorem 4. *Up to normalization by transformations described by Remark* 4 *and Lemma* 2 *any solution of* (27) *is of one of the following five forms*:

$$\begin{aligned} h_1(z) &= z, \\ h_2(z) &= -iz^2, \\ h_3(z) &= e^z, \\ h_4(z) &= 2\cosh(z), \\ h_5(z) &= 2\sinh(z). \end{aligned} \tag{28}$$

Proof. First we reformulate equation (27):

$$h_z^2 - \bar{h}_{\bar{z}}^2 = (h + \bar{h})(h_{zz} - \bar{h}_{\bar{z}\bar{z}}) .$$

Since the left hand side is harmonic the same must hold for the right hand side which leads to the condition

$$\frac{\bar{h}_{\bar{z}\bar{z}\bar{z}}}{\bar{h}_{\bar{z}}} = \frac{h_{zzz}}{h_z} = \varrho \in \mathbb{R} \text{ fixed} . \tag{29}$$

Here $h(z)$ cannot be a constant because this would mean that H is a constant too. So there are two different cases to consider, $\varrho = 0$ and $\varrho \neq 0$. The general solution of (29) is

$$\begin{aligned} h(z) &= A_1 z^2 + A_2 z + A_3 \quad \text{for } \varrho = 0 , \\ h(z) &= B_1 \cosh(\sqrt{\varrho} z) + B_2 \sinh(\sqrt{\varrho} z) \quad \text{for } \varrho \neq 0 . \end{aligned} \tag{30}$$

After resetting this in the equation (27) we get some conditions for the coefficients. To normalize the remaining freedoms we use Remark 4 and Lemma 2. In this way we get h_1 and h_2 from the polynomial solutions ($\varrho = 0$) and the other three functions from the hyperbolic solutions ($\varrho \neq 0$), which completes the proof.

By use of (21) and (26) we find for these five cases and $T = 0$:

$$\begin{aligned} Q_0^1(w, \bar{w}) &= \frac{1}{w + \bar{w}} , \quad |Q_0^1(w, \bar{w})|^2 = \frac{1}{t^2} , \\ Q_0^2(w, \bar{w}) &= 2\left(\frac{\cos(2(w + \bar{w}))}{\sin(2(w + \bar{w}))} + i\right) , \quad |Q_0^2(w, \bar{w})|^2 = \frac{4}{\sin^2(2t)} , \\ Q_0^3(w, \bar{w}) &= -\frac{\bar{w}}{w} \frac{1}{w + \bar{w}} , \quad |Q_0^3(w, \bar{w})|^2 = \frac{1}{t^2} , \\ Q_0^4(w, \bar{w}) &= -2 \frac{\sinh(2\bar{w})}{\sinh(2w)} \frac{1}{\sinh(2(w + \bar{w}))} , \quad |Q_0^4(w, \bar{w})|^2 = \frac{4}{\sinh^2(2t)} , \\ Q_0^5(w, \bar{w}) &= -2 \frac{\sin(2\bar{w})}{\sin(2w)} \frac{1}{\sin(2(w + \bar{w}))} , \quad |Q_0^5(w, \bar{w})|^2 = \frac{4}{\sin^2(2t)} . \end{aligned} \tag{31}$$

The following theorem classifies umbilic free Bonnet surfaces.

Theorem 5 (Cartan [C]). *There are three different types of umbilic free Bonnet families classified by whether they contain one, two or four surfaces*:

Type A: $|Q_T^4(w, \bar{w})|^2 = \dfrac{4}{\sin^2(2t)}$ *containing four surfaces in each family namely* $F_{T = \pm\frac{1}{2}}$ *with* Q_0^2 *and* $\bar{Q}_0^2$, $F_{T=0}$ *with* Q_0^5 *and* $F_{T=\infty}$ *with* $Q_0^5(w + \pi/4, \bar{w} + \pi/4)$ *from* (31). *The surfaces* $F_{T = \pm\frac{1}{2}}$ *are helicoids.*

Type B: $|Q_T^B(w,\bar{w})|^2 = \dfrac{4}{\sinh^2(2t)}$ *containing only one quasi-periodic surface in each family.*

Type C: $|Q_T^C(w,\bar{w})|^2 = \dfrac{1}{t^2}$ *containing two surfaces in each family, namely* $F_{T=0}$ *with* Q_0^1 *and* $F_{T=\infty}$ *with* Q_0^3 *from* (31). *The surface* $F_{T=0}$ *is a cylinder or a surface of revolution.*

These surfaces represent all the corresponding one-parameter families of isometries, which are described by the translations on the surfaces

$$(32)\quad w \to w + i\hat{T}(T), \quad \bar{w} \to \bar{w} - i\hat{T}(T), \quad \textit{where } \hat{T}(T) \textit{ a function of } T \textit{ only.}$$

Remark 6. By the notation F_T, $T = \infty, 0, \pm\frac{1}{2}$ we mean the surfaces one gets by integrating the moving frame equation with the coefficient functions $H(t)$, $e^{u(t)}$ and the Hopf differential $Q_T(w,\bar{w})$ in (26) for $T \to \infty, 0, \pm\frac{1}{2}$, respectively, and with the function $h(z)$ from (28).

Before we start with the proof we should illustrate our strategy. The main idea is, to show that the coefficient of the fundamental forms, H, e^u and the Hopf differential are the same up to the shift (32) for all $T \in \mathbb{R}P^1$. Actually this is true for the mean curvature function H and the metric e^u from our assumption that they are the same for all surfaces of the family. Therefore the only thing to show is, that for any $h(z)$ in (28) the corresponding $Q_T(w,\bar{w})$ given by (26) satisfies

$$(33)\qquad Q_T(w,\bar{w}) = Q_{T_0}\big(w + i\hat{T}(T), \bar{w} - i\hat{T}(T)\big), \quad T_0 \in \mathbb{R}P^1 \text{ fixed},$$

for all $T \in \mathbb{R}P^1$ – with the possible exception of some special values. We call these values the exceptional values and the corresponding surfaces the exceptional value surfaces or just asymptotes. Twice the number of these exceptional values is the maximal number of distinct surfaces in each family. More precisely: The shift (32) preserves the mean curvature function H as well as the metric e^u and is therefore an isometry we are looking for. On the other hand it is a reparametrization that according to (33) also preserves the Hopf differential. Thus the Bonnet-theorem yields that the surfaces can differ only by a rigid motion, which means, that they are the same – in the sense of Euclidean differential geometry.

Actually we will show that the families of surfaces generated by

- $h_2(z)$ coincide with those generated by $h_5(z)$ and vice versa. There are two exceptional values, $T = \pm\frac{1}{2}$. The Hopf differentials at these exceptional values are different. Moreover the Hopf differential for $T \in (-1/2, 1/2)$ is different from that for $T > 1/2$ and $T < -1/2$. But the Hopf differentials on both intervals are the same. Therefore each family of this type contains four different surfaces (case A).

- $h_4(z)$ have no exceptional value surfaces. Thus each family of this type contains only one surface (case B).

• $h_1(z)$ coincide with the exceptional value surfaces for $h_3(z)$ and vice versa. The exceptional values are $T = \pm\infty$. The corresponding Hopf differential at these exceptional values are the same. Therefore the family contains two distinct surfaces (case C).

Proof. We start with the case C. Here we have from (26) and (28)

$$Q_T^1(w,\bar w) = Q_0^3\left(w - \frac{i}{T}, \bar w + \frac{i}{T}\right), \quad Q_\infty^1(w,\bar w) = Q_0^3(w,\bar w)\,,$$
$$Q_0^1(w,\bar w) = Q_\infty^3(w,\bar w), \quad Q_\infty^1(w+iT,\bar w - iT) = Q_{\frac{1}{T}}^1(w,\bar w)\,,$$

which shows that the first and the third cases of (31) are the same. Moreover we have

$$|Q_T^1(w,\bar w)|^2 = |Q_T^3(w,\bar w)|^2 = \frac{1}{t^2}, \quad \text{for all } T \in \mathbb{R}P^1\,. \tag{34}$$

We see that there are only two different surfaces (up to translation (32) of the conformal coordinates), one with $Q_0^1(w,\bar w)$ and the other with $Q_0^3(w,\bar w)$ as the Hopf differential.

The surface corresponding to $Q_0^1(w,\bar w)$ is either a cylinder or a surface of revolution, since $Q_0^1(w,\bar w)$ is real, w is a conformal curvature line coordinate on this surface, which means that the isometry preserves the curvature lines. Any cylinder or surface of revolution is a Bonnet surface. We shall not consider them.

Surfaces of type B: Here with $h(z) = 2\cosh(z)$ and $e^z = -\dfrac{e^{2w}+1}{e^{2w}-1}$ we get

$$\begin{aligned} Q_T^4(w,\bar w) &= -\frac{2}{\sinh\big(2(w+\bar w)\big)}\left(\frac{1-2iT\cosh(\bar z)}{1+2iT\cosh(z)}\right)\left(\frac{\sinh(2\bar w)}{\sinh(2w)}\right) \\ &= -\frac{2}{\sinh\big(2(w+\bar w)\big)}\left(\frac{e^{2\bar w} - \left(\dfrac{1-2iT}{1+2iT}\right)e^{-2\bar w}}{\left(\dfrac{1-2iT}{1+2iT}\right)e^{2w} - e^{-2w}}\right) \\ &= Q_0^4(w+i\hat T, \bar w - i\hat T) \end{aligned} \tag{35}$$

where $\hat T = -\dfrac{i}{2}\log\left(\sqrt{\dfrac{1-2iT}{1+2iT}}\right) = \dfrac{1}{2}\arg\left(\sqrt{\dfrac{1-2iT}{1+2iT}}\right)$. Clearly $\hat T(T)$ is unique only up to an addition of $k\pi/2$, $k\in\mathbb{Z}$. In particular

$$Q_\infty^4(w,\bar w) = Q_0^4\left(w + (2k+1)i\frac{\pi}{4}, \bar w - (2k+1)i\frac{\pi}{4}\right), \quad k\in\mathbb{Z}\,,$$

which shows the $\dfrac{\pi}{2}$-periodicity of Q_0^4. Thus any family of this type contains only one surface. This is totally described by one period of $\hat T$ and a fixed motion in $\mathbb{R}^3$. In other words the surfaces of this type are quasi-periodic because the mean curvature function and the metric are periodic in the direction generated by $\mathrm{Re}(w) = \mathrm{const}$ and the Hopf differential has a $\frac{\pi}{2}$-periodicity in the same direction.

Finally from (31) we find that the modulus of the Hopf differential is just what is stated in the Theorem.

Surfaces of type A. First we recognize

$$\begin{aligned} Q_T^2(w,\bar{w}) &= h_2'^2 \frac{1-iT\bar{h}_2}{1+iTh_2}\frac{1}{h_2+\bar{h}_2} = Q_0^5(w+i\hat{T}, w-i\hat{T}) \quad \text{with } \hat{T} = -\frac{1}{2}\log(\sqrt{2T}), \\ Q_T^5(w,\bar{w}) &= h_5'^2 \frac{1-iT\bar{h}_5}{1+iTh_5}\frac{1}{h_5+\bar{h}_5} = Q_{\hat{T}}^2(w,\bar{w}) \quad \text{with } \hat{T} = \frac{1}{2}\left(\frac{1-2T}{1+2T}\right). \end{aligned} \tag{36}$$

This shows that both cases coincide. So we can focus only on Q_T^5 to describe the whole family. By doing so we get:

$$Q_T^5(w,\bar{w}) = \begin{cases} Q_0^5\big(w+\hat{T}(T), \bar{w}-\hat{T}(T)\big) & \text{if } T \in (\frac{1}{2}, -\frac{1}{2}), \\ Q_0^5\big(w+\pi/4+\hat{T}(T), \bar{w}+\pi/4-\hat{T}(T)\big) & \text{if } T \in \mathbb{R}\backslash[\frac{1}{2}, -\frac{1}{2}], \\ Q_0^2(w,\bar{w}) & \text{if } T = \frac{1}{2}, \\ \bar{Q}_0^2(w,\bar{w}) & \text{if } T = -\frac{1}{2}, \end{cases}$$

with

$$\hat{T}(T) = -\frac{i}{2}\log\left(\sqrt{\frac{1-2T}{1+2T}}\right).$$

Furthermore we find directly

$$|Q_T^5|^2 = |Q_0^5|^2 = \frac{4}{\sin^2(2t)}.$$

The two surfaces corresponding to the exceptional values $T = \pm\frac{1}{2}$ are helicoids. This will be shown in Appendix B.

Table 1 presents the fundamental functions and the ordinary differential equation to be solved.

Type	Q	H solves	e^u
A_1	$Q_0^5(w,\bar{w}) = -2\dfrac{\sin(2\bar{w})}{\sin(2w)}\dfrac{1}{\sin(2(w+\bar{w}))}$	$\left(\left(\dfrac{H''}{H'}\right)' - H'\right)\dfrac{\sin^2(2t)}{4} = 2 - \dfrac{H^2}{H'}$	$-\dfrac{8}{\sin^2(2t)H'}$
A_2	$Q_\infty^5(w,\bar{w}) = 2\dfrac{\cos(2\bar{w})}{\cos(2w)}\dfrac{1}{\sin(2(w+\bar{w}))}$		
B	$Q_0^4(w,\bar{w}) = -2\dfrac{\sinh(2\bar{w})}{\sinh(2w)}\dfrac{1}{\sinh(2(w+\bar{w}))}$	$\left(\left(\dfrac{H''}{H'}\right)' - H'\right)\dfrac{\sinh^2(2t)}{4} = 2 - \dfrac{H^2}{H'}$	$-\dfrac{8}{\sinh^2(2t)H'}$
C	$Q_0^3(w,\bar{w}) = -\dfrac{\bar{w}}{w}\dfrac{1}{w+\bar{w}}$	$\left(\left(\dfrac{H''}{H'}\right)' - H'\right)t^2 = 2 - \dfrac{H^2}{H'}$	$-\dfrac{2}{t^2H'}$

Table 1. Table of fundamental functions

Remark 7. Even though we started with a local description of Bonnet surfaces, Theorem 5 shows that our ansatz can be considered to be global.

Remark 8. The amazing result of Theorem 5 is that beyond the associated families of CMC (constant mean curvature) surfaces there are no real smooth 1-parameter families of distinct isometric surfaces which have the same principle curvatures.

3. Bonnet surfaces of type C and Painlevé V(III) equations

Let us return to the description of the moving frame as in (9). In the variables

$$t = w + \bar{w}, \quad \lambda = \frac{w}{w + \bar{w}} \tag{37}$$

this system reads as

$$\begin{aligned} \Phi_\lambda(\lambda, t)\Phi^{-1}(\lambda, t) &= t\begin{pmatrix} a(t) & \varphi(t) \\ \varphi(t) & -a(t) \end{pmatrix} + e^{-u(t)/2}\begin{pmatrix} 0 & \frac{1}{\lambda} \\ -\frac{1}{\lambda - 1} & 0 \end{pmatrix}, \\ \Phi_t(\lambda, t)\Phi^{-1}(\lambda, t) &= \lambda\begin{pmatrix} a(t) & \varphi(t) \\ \varphi(t) & -a(t) \end{pmatrix} + \begin{pmatrix} -\frac{a(t)}{2} & -\varphi(t) \\ 0 & \frac{a(t)}{2} \end{pmatrix} \end{aligned} \tag{38}$$

with the functions $a(t)$, $\varphi(t)$ and $e^{-u(t)/2}$ given by

$$\begin{aligned} a(t) &= \frac{u'(t)}{2} = -\frac{1}{t} - \frac{H''(t)}{2H'(t)}, \\ \varphi(t) &= \frac{H(t) + tH'(t)}{2}\, e^{u(t)/2}, \\ e^{-u(t)/2} &= t\sqrt{-\frac{H'(t)}{2}}. \end{aligned} \tag{39}$$

Remark 9. The equation $A'(t) + [A(t), B(t)] = 0$ with some $B(t)$ implies that the eigenvalues of A are constant.

Using this remark one can easily see, that the compatibility conditions imply also, that the determinant of the matrix

$$A(t) = \begin{pmatrix} a(t) & \varphi(t) \\ \varphi(t) & -a(t) \end{pmatrix}$$

is independent of t. Here we come to a result, first obtained by Hazzidakis in [H].

Lemma 3. *Equation* (25) *in the case* C *has the first integral* μ, *given by*

$$a(t)^2 + \varphi(t)^2 = \left(\frac{\mu}{2}\right)^2 \tag{40}$$

with $a(t)$ *and* $\varphi(t)$ *as in* (39).

We have a system of matrix dimension 2×2 with the following dependence on λ

$$\begin{aligned} \Phi_\lambda \Phi^{-1} &= tA(t) + \frac{1}{\lambda} A_0(t) + \frac{1}{\lambda - 1} A_1(t), \\ \Phi_t \Phi^{-1} &= \lambda A(t) + C(t). \end{aligned} \tag{41}$$

Here specialists in the theory of the Painlevé equations immediately recognize the Lax representation for the Painlevé V equation (see for example [I-N]). In the rest of this section we carry out in detail this identification, showing as a result how the Bonnet surfaces of C-type can be described in terms of the Painlevé transcendents.

Theorem 6 (Jimbo-Miwa [J-M]). *Let us consider the system* (41) *such that* $A(t)$ *has two different eigenvalues. Then the compatibility conditions for this system are equivalent to the Painlevé* V *equation*

$$\begin{aligned} y''(t) = {} & \left(\frac{1}{2y(t)} + \frac{1}{y(t)-1}\right) y'^2(t) - \frac{y'(t)}{t} + \frac{\big(y(t)-1\big)^2}{t^2}\left(\alpha y(t) + \frac{\beta}{y(t)}\right) \\ & + \frac{\gamma y(t)}{t} + \frac{\delta y(t)\big(y(t)+1\big)}{y(t)-1} \end{aligned} \tag{42}$$

which implies that the coefficients of the matrices $A(t)$, $A_0(t)$, $A_1(t)$ *as well as of the matrix* $C(t)$ *can be expressed in terms of this function, and the constants* α, β, γ *and* δ *depend on the eigenvalues of the matrices* $A(t)$, $A_0(t)$ *and* $A_1(t)$ *only.*

Proof. First let us normalize the matrices in the first equation of (41) to be traceless by

$$\Phi \to \Psi = e^{-\tau t \lambda/2} \lambda^{-\tau_0/2} (\lambda - 1)^{-\tau_1/2} \Phi \tag{43}$$

with $\tau = \operatorname{tr}\big(A(t)\big)$, $\tau_0 = \operatorname{tr}\big(A_0(t)\big)$ and $\tau_1 = \operatorname{tr}\big(A_1(t)\big)$. The transformed A-matrix has two non vanishing eigenvalues and we can bring it by another gauge transformation to a diagonal form. So we may assume that the system (41) is of the form

$$\begin{aligned} \Psi_\lambda \Psi^{-1} &= \frac{ti\mu}{2}\mathbf{k} + \frac{1}{\lambda} A_0(t) + \frac{1}{\lambda - 1} A_1(t), \\ \Psi_t \Psi^{-1} &= \frac{\lambda i \mu}{2}\mathbf{k} + C(t) \end{aligned} \tag{44}$$

with $\mu \neq 0$. We set $A_\nu(t) = \big(a_{ij}^\nu(t)\big)$ and $\det\big(A_\nu(t)\big) = -\theta_\nu^2/4$ for $\nu = 0, 1$ and

$$\theta_\infty = -2\bigl(a^0_{11}(t) + a^1_{11}(t)\bigr)\,.$$

It is easy to check that these θs are constants. Now define

$$z(t) = a^0_{11}(t) - \frac{\theta_0}{2}, \quad y(t) = -\frac{a^0_{21}(t)}{a^1_{21}(t)}\left(1 + \frac{\theta_0 + \theta_1 + \theta_\infty}{2z(t)}\right). \tag{45}$$

Finally one can prove that the compatibility conditions of (44) are given by

$$\begin{aligned} tz'(t) &= -\frac{1}{y(t)}\bigl(z(t) + \theta_0\bigr)\left(z(t) + \frac{\theta_0 + \theta_1 + \theta_\infty}{2}\right) \\ &\quad + y(t)z(t)\left(z(t) + \frac{\theta_0 - \theta_1 + \theta_\infty}{2}\right), \end{aligned} \tag{46}$$

$$ty'(t) = \mu t y(t) - 2z(t)\bigl(y(t)-1\bigr)^2 - \bigl(y(t)-1\bigr)\left(\frac{\theta_0 - \theta_1 + \theta_\infty}{2}y(t) - \frac{3\theta_0 + \theta_1 + \theta_\infty}{2}\right).$$

This system can be rewritten as an equation in $y(t)$ only, which gives (42) with

$$\begin{aligned} \alpha &= \frac{1}{2}\left(\frac{\theta_0 - \theta_1 + \theta_\infty}{2}\right)^2, \quad \beta = \frac{1}{2}\left(\frac{\theta_0 - \theta_1 - \theta_\infty}{2}\right)^2, \\ \gamma &= \mu(1 - \theta_0 - \theta_1), \quad \delta = -\frac{1}{2}\mu^2\,. \end{aligned} \tag{47}$$

Now we apply the proof of Theorem 6 to our system (38).

Theorem 7. *Let $H(t)$ be a solution of* (25) *in the* C*-case. Then*

$$y(t) = \frac{2a(t) - \mu}{2a(t) + \mu}$$

with $a(t)$ defined in (39) *and μ a root of* (40) *defines a solution of the Painlevé* V *equation* (42) *with*

$$\alpha = 0, \quad \beta = 0, \quad \gamma = \mu \quad \text{and} \quad \delta = -\frac{1}{2}\mu^2\,. \tag{48}$$

On the other hand let $y(t)$ be an arbitrary solution of (42) *with constants as in* (48), $\mu \neq 0$. *Then*

$$H(t) = -\frac{t\bigl(y'(t)^2 - \mu^2 y(t)^2\bigr)}{2y(t)\bigl(y(t)-1\bigr)^2} \tag{49}$$

is a solution of (25). *If finally* $0 \geqq y(t)$ *and not the solution* $y(t) = Ce^{\mu t}$, *the solution is geometrical and the metric is given by*

$$e^{u(t)} = -\frac{4y(t)\bigl(y(t)-1\bigr)^2}{t^2\bigl(y'(t) - \mu y(t)\bigr)^2}\,. \tag{50}$$

Proof. First let us normalize (38) to the special form (44) with the gauge transformation

$$\Phi \to \Psi = D\Phi = \begin{pmatrix} a + \frac{\mu}{2} & \varphi \\ a - \frac{\mu}{2} & \varphi \end{pmatrix} \Phi . \tag{51}$$

Since $\det(D) = \varphi\mu$ this is a regular transformation if $\varphi \neq 0$. If there were a domain in $\mathbb{C}$ where φ vanishes identically then $H(t) = c/t$ and consequently $a(t)$ also vanishes identically. However this would require $\mu = 0$ in contradiction to our assumption. By this gauge transformation (51), equation (38) becomes

$$\begin{aligned} \Psi_\lambda \Psi^{-1} &= \frac{it\mu}{2}\mathbf{k} + \frac{\varphi e^{-u/2}}{\mu(\lambda-1)} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} + \frac{e^{-u/2}}{\varphi\mu\lambda} \begin{pmatrix} \frac{\mu^2}{4} - a^2 & \left(a + \frac{\mu}{2}\right)^2 \\ -\left(a - \frac{\mu}{2}\right)^2 & a^2 - \frac{\mu^2}{4} \end{pmatrix}, \\ \Psi_t \Psi^{-1} &= \frac{i\lambda\mu}{2}\mathbf{k} + \begin{pmatrix} -\frac{\mu}{4} & -\frac{1}{2}\left(a + \frac{\mu}{2}\right) \\ -\frac{1}{2}\left(a - \frac{\mu}{2}\right) & \frac{\mu}{4} \end{pmatrix} + D'(t)\,D(t)^{-1} . \end{aligned} \tag{52}$$

We see that $\theta_v = 0$, for $v = 0, 1, \infty$ and by (45) we get

$$y(t) = \frac{2a(t) - \mu}{2a(t) + \mu}, \quad z(t) = -\frac{\varphi e^{-u/2}}{\mu} . \tag{53}$$

These functions solve

$$\begin{aligned} tz'(t) &= z^2(t)\left(y(t) - \frac{1}{y(t)}\right), \\ ty'(t) &= \mu t y(t) - 2z(t)\big(y(t) - 1\big)^2 , \\ y''(t) &= \left(\frac{3y(t) - 1}{2y(t)\big(y(t) - 1\big)}\right) y'^2(t) - \frac{y'(t)}{t} + \mu\frac{y(t)}{t} - \mu^2 \frac{y(t)\big(y(t) + 1\big)}{2\big(y(t) - 1\big)} . \end{aligned} \tag{54}$$

This proves the first part of the theorem. We shall remark that we get the following formulas by the Theorem 6

$$\begin{aligned} a(t) &= -\frac{\mu\big(y(t) + 1\big)}{2\big(y(t) - 1\big)}, \\ \varphi(t)\,e^{u/2} &= -\frac{2\mu y(t)}{t\big(y'(t) - \mu y(t)\big)}, \\ \varphi(t)\,e^{-u/2} &= \frac{\mu t\big(y'(t) - \mu y(t)\big)}{2\big(y(t) - 1\big)^2} . \end{aligned} \tag{55}$$

On the other hand $a(t)$, $\varphi(t)e^{-u(t)/2}$ and $\varphi(t)e^{u(t)/2}$ can be expressed (39) in terms of the functions H, H' and H''. It can be interpreted as a linear system for these functions, which is uniquely solvable since $\varphi \neq 0$. Comparing (39) with (55) we get (49), (50).

It is simple to prove that $y(t) = Ce^{\mu t}$ is a solution of (54). But it is the only negative solution for which neither (49) nor (50) makes sense.

Remark 10. For another real reduction of the Painlevé V equation the equation in the form (40) appeared already in papers devoted to the calculation of the correlation functions for the Bose gas [M-T].

Remark 11. For any solution $H(t)$ of (25) in the C-case

$$\tilde{H}(\tilde{t}) = \frac{1}{\alpha} H(t), \quad \text{with} \quad \tilde{t} = \alpha t \tag{56}$$

is also a solution of (25). That implies that we can fix μ to some special value in (40). Geometrically this is only a scaling of $\mathbb{R}^3$.

From now on we fix $\mu = 4$. The case $\mu = 0$ is considered in Appendix A.

It turns out that in the case $\alpha = \beta = 0$ the Painlevé V equation can be reduced to the Painlevé III equation. The following three statements can be proved by direct calculations.

Corollary 1 ([Ki]). *Let $y(t)$ be a solution of* (42) *with $\alpha = \beta = 0$ and $\mu = 4$. The function $p(t)$ defined by*

$$y(t) = \left(\frac{p(t)+1}{p(t)-1}\right)^2 \tag{57}$$

solves the Painlevé III

$$p''(t) = \frac{p'^2(t)}{p(t)} - \frac{p'(t)}{t} - \frac{p^2(t)}{t} + \frac{1}{t} + p^3(t) - \frac{1}{p(t)}. \tag{58}$$

The mean curvature and the metric are

$$\begin{aligned} H(t) &= -\frac{t\left(p'^2(t) - \left(p^2(t)-1\right)^2\right)}{2p^2(t)}, \\ e^{u(t)} &= -\frac{4p^2(t)}{t^2\left(p'(t)+p^2(t)-1\right)^2}. \end{aligned} \tag{59}$$

Remark 12. The reduction (57) holds in general for any Painlevé V equation (42) with $\alpha = \beta = 0$ and arbitrary γ and δ.

Remark 13. The geometrical solutions $p(t)$ are of modulus 1 and we have to exclude the solutions

$$p(t) = \tanh(2t + c) \quad \text{and} \quad p(t) = \pm 1.$$

These two functions are solutions of (58). They come via the transformation (57) from the solution $y(t) = Ce^{4t}$ of the Painlevé V (42) with $\alpha = \beta = 0$ and $\mu = 4$, which we have excluded in Theorem 7.

Corollary 2. *Let $p(t)$ be a solution of* (58) *of modulus* 1. *Then its argument $\phi(t)$*

$$p(t) = e^{i\phi(t)}$$

solves

$$t\big(\phi''(t) - 2\sin\big(2\phi(t)\big)\big) + \phi'(t) + 2\sin\big(\phi(t)\big) = 0\,. \tag{60}$$

The solutions of (60), *which are not solutions of $\phi' + 2\sin(\phi) = 0$ are geometrical. The mean curvature and the metric are given by*

$$H(t) = 2t\left(\frac{\phi'(t)^2}{4} - \sin^2\big(\phi(t)\big)\right), \quad e^{u(t)} = \frac{4}{t^2\big(\phi'(t) + 2\sin\big(\phi(t)\big)\big)^2}\,. \tag{61}$$

Corollary 3. *Let $y(t)$ be a solution of the third equation in* (54) *and $z(t)$ be defined by the second equation in* (54). *Then equation* (52) *is of the form*

$$\begin{gathered}\Psi_\lambda \Psi^{-1} = 2it\mathbf{k} + \frac{1}{\lambda}\begin{pmatrix} z(t) & -\dfrac{z(t)}{y(t)} \\ z(t)y(t) & -z(t)\end{pmatrix} + \frac{1}{\lambda - 1}\begin{pmatrix} -z(t) & z(t) \\ -z(t) & z(t)\end{pmatrix}, \\ \Psi_t \Psi^{-1} = 2i\lambda\mathbf{k} + \frac{1}{t}\begin{pmatrix} z(t)\big(y(t)-1\big) - t\,\dfrac{3y(t)-1}{y(t)-1} & z(t)\left(1 - \dfrac{1}{y(t)}\right) \\ z(t)\big(y(t)-1\big) & z(t)\left(1 - \dfrac{1}{y(t)}\right) + t\,\dfrac{y(t)-3}{y(t)-1}\end{pmatrix}.\end{gathered} \tag{62}$$

The Ψ-function, which satisfies (62) is related to the geometrical frame Φ by the following transformation:

$$\Phi(t,\lambda) = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -\sqrt{-y(t)} & -\dfrac{1}{\sqrt{-y(t)}}\end{pmatrix}\Psi(t,\lambda)\,. \tag{63}$$

4. Bonnet surfaces of types A and B and Painlevé VI equation

In this part we will study Bonnet families of type A (containing A_1 and A_2 as subcases) and B in more detail. First we repeat the ordinary differential equations to be solved by the mean curvature function $H(t)$ in the cases A and B. These are for $t = w + \bar{w}$ (see Table 1)

$$\left(\frac{H''(t)}{H'(t)}\right)' - H'(t) = \frac{4}{\sin^2(2t)}\left(2 - \frac{H^2(t)}{H'(t)}\right), \tag{64}$$

$$\left(\frac{H''(t)}{H'(t)}\right)' - H'(t) = \frac{4}{\sinh^2(2t)}\left(2 - \frac{H^2(t)}{H'(t)}\right). \tag{65}$$

The solutions of these two equations are simply related. Let $H_B(t)$ a solution of (65). Then

$$H_A(t) \equiv -iH_B(it) \tag{66}$$

solves (64). Now we start again with (9). In all three cases A_1, A_2 and B (see Table 1) we get:

$$\begin{aligned}
\Phi_\lambda(\lambda, s)\,\Phi^{-1}(\lambda, s) &= \frac{1}{\lambda}\begin{pmatrix} sf'(s) & \varphi_1(s) \\ \varphi_2(s) & -sf'(s) \end{pmatrix} \\
&\quad + \frac{\varphi_2(s) - \varphi_1(s)}{\lambda - 1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{\varphi_2(s) - \varphi_1(s)}{\lambda - s}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \\
\Phi_s(\lambda, s)\,\Phi^{-1}(\lambda, s) &= \begin{pmatrix} -\dfrac{f'(s)}{2} & \dfrac{\varphi_1(s) - s\varphi_2(s)}{s(s-1)} \\ \dfrac{\varphi_2(s) - \varphi_1(s)}{s-1} & \dfrac{f'(s)}{2} \end{pmatrix} \\
&\quad + \frac{\varphi_2(s) - \varphi_1(s)}{\lambda - s}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\end{aligned} \tag{67}$$

where the coefficients are presented in Table 2.

	A_1	A_2	B
s	$e^{4i(w+\bar{w})}$		$e^{4(w+\bar{w})}$
λ	e^{4iw}	$-e^{4iw}$	e^{4w}
$B(s)$	$H_A\left(-\frac{i}{4}(\log(s))\right)$		$H_B\left(\frac{1}{4}(\log(s))\right)$
$f(s)$	$\frac{1}{2}u_A\left(-\frac{i}{4}(\log(s))\right)$		$\frac{1}{2}u_B\left(\frac{1}{4}(\log(s))\right)$
$\varphi_1(s)$	$-i\frac{B(s)}{8}e^{f(s)} - \frac{s}{s-1}e^{-f(s)}$		$\frac{B(s)}{8}e^{f(s)} - \frac{s}{s-1}e^{-f(s)}$
$\varphi_2(s)$	$-i\frac{B(s)}{8}e^{f(s)} - \frac{1}{s-1}e^{-f(s)}$		$\frac{B(s)}{8}e^{f(s)} - \frac{1}{s-1}e^{-f(s)}$

Table 2. Coefficients in (67)

Let us write down the system (67) in a more general form as

$$\Phi_\lambda \Phi^{-1} = \frac{1}{\lambda - s}A_s + \frac{1}{\lambda - 1}A_1 + \frac{1}{\lambda}A_0 = U, \quad \Phi_s \Phi^{-1} = -\frac{1}{\lambda - s}A_s + C = V. \tag{68}$$

The compatibility conditions for this system for $\lambda \to 0, 1, s$ are

$$
\begin{aligned}
A_s' + \left[A_s, C + \frac{1}{s} A_0 + \frac{1}{s-1} A_1 \right] &= 0\,, \\
A_1' + \left[A_1, C + \frac{1}{s-1} A_s \right] &= 0\,, \\
A_0' + \left[A_0, C + \frac{1}{s} A_s \right] &= 0\,.
\end{aligned}
\tag{69}
$$

The last equation implies that the determinant of A_0 is independent of s (see Remark 9), and as for the Bonnet surfaces of type C we get the following first integral, first found by Hazzidakis [H].

Lemma 4. *Equation* (25) *in the cases* A *and* B *has the first integral* μ, *given by*

$$
s^2 f'(s)^2 + \varphi_1(s)\,\varphi_2(s) = \left(\frac{\mu}{2} \right)^2 \tag{70}
$$

with $f(s)$, $\varphi_1(s)$ *and* $\varphi_2(s)$ *as in Table* 2. *If we formulate this equation in terms of the function* $B(s)$ *we get in the* B-*case*

$$
s^2 \left(\frac{B''(s)}{2B'(s)} + \frac{1}{s-1} \right)^2 - \frac{sB'(s)}{8} - \frac{B^2(s)}{8B'(s)(s-1)^2} - \frac{B(s)}{8}\,\frac{s+1}{s-1} = \frac{\mu^2}{4} \tag{71}
$$

and in the A-*cases*

$$
s^2 \left(\frac{B''(s)}{2B'(s)} + \frac{1}{s-1} \right)^2 + i\,\frac{sB'(s)}{8} + i\,\frac{B^2(s)}{8B'(s)(s-1)^2} + i\,\frac{B(s)}{8}\,\frac{s+1}{s-1} = \frac{\mu^2}{4}\,. \tag{72}
$$

In contrast to the case C the parameter μ seems to be an essential parameter of the surface. In the case B μ can be real, purely imaginary, or zero. In Appendix B it will be shown that equation (72) is up to a factor the equation that describes the relation between the curvature and the torsion of a family of helices on the exceptional helicoidal surfaces (see Theorem 5). Along these curves these surfaces are isometric. It comes out that μ^2 must be strictly negative and thus $\mu \in i\mathbb{R}$.

Remark 14. In fact (71) and (72) can be found directly from (25) by multiplying this equation with an integrating factor. This factor in case A is

$$
\frac{H''(t)}{H'(t)} + 4\cot(2t) \tag{73}
$$

and the same in case B – but with $\coth(2t)$ instead of $\cot(2t)$. This factor vanishes identically for $H(t) = C_1 + C_2 \cot(2t)$. Thus $B(s) = H(t) = C_1 + C_2 \cot(2t)$ is a 2-parameter family of solutions of (72) (again with $\coth(2t)$ instead of $\cot(2t)$ we also have a

2-parameter family of solutions of (71)). However, it comes out, that non of them is a solution of (25).

Now let us return to the more general system (68). Again as in the case C here one can recognize a Lax representation for a Painlevé equation, but now for the Painlevé VI equation.

Theorem 8 (Jimbo-Miva [J-M]). *Let us consider the system* (68) *such that*

$$A_\infty := -A_0 - A_1 - A_s$$

has two different eigenvalues. Then the compatibility conditions of this system are equivalent to the Painlevé VI *equation. That implies that all the coefficients of the matrices in* (68) *can be expressed in terms of the function* $y(s)$, *which satisfies the Painlevé* VI *equation*

$$y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-s}\right)y'^2 - \left(\frac{1}{s} + \frac{1}{s-1} + \frac{1}{y-s}\right)y' \\ + \frac{y(y-1)(y-s)}{s^2(s-1)^2}\left(\alpha + \beta\frac{s}{y^2} + \gamma\frac{s-1}{(y-1)^2} + \delta\frac{s(s-1)}{(y-s)^2}\right) \tag{74}$$

with some constants α, β, γ *and* δ *depending on the traces and determinants of the matrices* A_ν, *for* $\nu = 0, 1, s$.

Proof. Due to Remark 9 we know that the determinants of the matrices A_ν are constant. First we normalize system (68) to make the matrices traceless (as in the previous section). Then we can reach $\det(A_\nu) = 0$ for $\nu = 0, 1, s$ by the gauge-transformation

$$\Phi \to \left(\lambda^{d_0}(\lambda-1)^{d_1}(\lambda-t)^{d_t}\right)\Phi = \tilde{\Phi}$$

with $d_\nu = \sqrt{\det(A_\nu)}$, $\nu = 0, 1, s$. Therefore we can assume that the A_ν-matrices are of the form

$$A_\nu = \begin{pmatrix} z_\nu(s) + \theta_\nu & -u_\nu(s)\, z_\nu(s) \\ \dfrac{z_\nu(s) + \theta_\nu}{u_\nu(s)} & -z_\nu(s) \end{pmatrix} \tag{75}$$

with $\theta_\nu = 2d_\nu$. Now let us consider A_∞ defined as in the theorem. Both gauge-transformations we used to create the special form (75) preserve the inequality of the eigenvalues of this matrix. Let us finally assume that A_∞ is diagonal with the diagonal elements κ_1 and κ_2 and $\theta_\infty = \kappa_1 - \kappa_2$. The matrix C must be diagonal.

Define

$$y(s) = -\frac{s u_0(s)\, z_0(s)}{s u_s(s)\, z_s(t) + u_1(s)\, z_1(s)}, \tag{76}$$

$$z(s) = \frac{z_0(s) + \theta_0}{y(s)} + \frac{z_1(s) + \theta_1}{y(s) - 1} + \frac{z_s(s) + \theta_s}{y(s) - s},$$

$$\tilde{z}(s) = z(s) - \frac{\theta_0}{y(s)} - \frac{\theta_1}{y(s) - 1} - \frac{\theta_\infty}{y(s) - s}.$$

Now the diagonality of A_∞ implies

$$\begin{aligned}
z_0 &= \frac{y}{s\theta_\infty}\{y(y-1)(y-s)\tilde{z}^2 + \big(\theta_1(y-s) + s\theta_s(y-1) - 2\kappa_2(y-1)(y-s)\big)\tilde{z} \\
&\quad + \kappa_2^2(y-s-1) - \kappa_2(\theta_1 + s\theta_s)\},\\
z_1 &= \frac{1-y}{(s-1)\theta_\infty}\{y(y-1)(y-s)\tilde{z}^2 + \big((\theta_1+\theta_\infty)(y-s) + s\theta_s(y-1) \\
&\quad - 2\kappa_2(y-1)(y-s)\big)\tilde{z} + \kappa_2^2(y-s) - \kappa_2(\theta_1 + s\theta_s) - \kappa_1\kappa_2\},\\
z_s &= \frac{y-s}{s(s-1)\theta_\infty}\{y(y-1)(y-s)\tilde{z}^2 + \big(\theta_1(y-s) + s(\theta_s+\theta_\infty)(y-1) \\
&\quad - 2\kappa_2(y-1)(y-s)\big)\tilde{z} + \kappa_2^2(y-1) - \kappa_2(\theta_1 + s\theta_s) - s\kappa_1\kappa_2\}.
\end{aligned} \tag{77}$$

The compatibility conditions in terms of $z(s)$ and $y(s)$ read as follows:

$$\begin{aligned}
z' &= \frac{1}{s(s-1)}\big((-3y^2 + 2(1+s)y - s)z^2 \\
&\quad + \big(\theta_0(2y-s-1) + \theta_1(2y-s) + (2y-1)(\theta_s-1)\big)z - \kappa_1(\kappa_2+1)\big),\\
y' &= \frac{y(y-1)(y-s)}{s(s-1)}\left(2z - \frac{\theta_0}{y} - \frac{\theta_1}{y-1} - \frac{\theta_t - 1}{y-t}\right),
\end{aligned} \tag{78}$$

which can be formulated as (74) with

$$\alpha = \frac{(\theta_\infty - 1)^2}{2}, \quad \beta = -\frac{\theta_0^2}{2}, \quad \gamma = \frac{\theta_1^2}{2}, \quad \delta = \frac{1-\theta_s^2}{2}.$$

Now we apply this to our case.

Theorem 9. *Let $H(t) = B(s)$ be a solution of* (65) *or* (64), *the functions $f(s)$, $\varphi_1(s)$ and $\varphi_2(s)$ be defined as in Table* 2 *and $\mu \neq 0$ be a root of* (72) *respectively* (71). *Then*

$$y(s) = \frac{s\big(2s^2f'(s)^2 + \varphi_1(s)^2 + \varphi_1(s)\varphi_2(s) - \mu s f'(s)\big)}{\big(2s^3f'(s)^2 + \varphi_1(s)^2 + s\varphi_1(s)\varphi_2(s) - \mu s^2 f'(s)\big)} \tag{79}$$

solves the Painlevé VI (74) *with the coefficients*

$$\alpha = \frac{(\mu+1)^2}{2}, \quad \beta = -\frac{\mu^2}{2}, \tag{80}$$

$$\gamma = 0\,, \quad \delta = \frac{1}{2}\,.$$

On the other hand let $y(s)$ *be any solution of* (74) *with the above constants. Then*

$$\begin{aligned} H(w+\bar{w}) &= B(s) \\ &= 2\,\frac{\mu^2\big(s+y^2(s)-y(s)(1+s)\big)^2-\big(y(s)\big(1-y(s)\big)+sy'(s)(s-1)\big)^2}{(s-1)\,y(s)\big(y(s)-1\big)\big(y(s)-s\big)} \end{aligned} \tag{81}$$

solves the differential equation of the B-*case* (65). *The transformation* (66) *yields the solution to the differential equation of the case* A (64). *In case* B *for* $\mu \in \mathbb{R}$ *and for* $1 < y(s) < s$ *we get a geometrical solution. The metric is then given by*

$$e^{u(w+\bar{w})} = e^{2f(s)} = -\frac{4s\big(y(s)-1\big)\big(y(s)-s\big)}{\big(sy'(s)(s-1)-\mu\big(\mu(s)-1\big)\big(\mu(s)-s\big)-y(s)\big(y(s)-1\big)\big)^2}\,. \tag{82}$$

Proof. By the gauge transformation

$$\Phi(\lambda,s) \to \lambda^{\mu/2}\begin{pmatrix} \dfrac{2\varphi_1(s)}{2sf'(s)-\mu} & -1 \\ -1 & -\dfrac{2\varphi_2(s)}{2sf'(s)-\mu} \end{pmatrix}\Phi(\lambda,s) = \Psi(\lambda,s) \tag{83}$$

the system (67) reads as follows:

$$\begin{aligned} \Psi_\lambda\Psi^{-1} = {} & \frac{1}{\mu\lambda}\begin{pmatrix} (\varphi_1-\varphi_2)^2+\mu^2 & \mu(\varphi_2-\varphi_1)+\dfrac{2\varphi_1(\varphi_2-\varphi_1)^2}{2sf'-\mu} \\ \mu(\varphi_1-\varphi_2)+\dfrac{2\varphi_2(\varphi_1-\varphi_2)^2}{2sf'-\mu} & -(\varphi_1-\varphi_2)^2 \end{pmatrix} \\ & + \frac{\varphi_2-\varphi_1}{\mu(\lambda-1)}\begin{pmatrix} \varphi_1(s) & \dfrac{2\varphi_1^2}{2sf'-\mu} \\ -\dfrac{1}{2}(2sf'-\mu) & -\varphi_1 \end{pmatrix} \\ & + \frac{\varphi_2-\varphi_1}{\mu(\lambda-s)}\begin{pmatrix} -\varphi_2 & \dfrac{1}{2}(2sf'-\mu) \\ -\dfrac{2\varphi_2^2}{2sf'-\mu} & \varphi_2 \end{pmatrix}, \\ \Psi_s\Psi^{-1} = {} & \frac{\varphi_2-\varphi_1}{\mu(\lambda-s)}\begin{pmatrix} \varphi_2 & -\dfrac{1}{2}(2sf'-\mu) \\ \dfrac{2\varphi_2^2}{2sf'-\mu} & -\varphi_2 \end{pmatrix} + B_0(s)\,. \end{aligned} \tag{84}$$

Here we do not specify the diagonal matrix B_0 because the definition of $y(s)$ is independent of this. With $\kappa_1 = -\mu$, $\kappa_2 = 0$ we get (76), (79) with the coefficients given by (80), which proves the first part for the theorem.

As in the case C we can interpret the definitions of $f'(s)$, $\varphi_1(s)\,e^{-f(s)}$ and $\varphi_2(s)\,e^{f(s)}$ in Table 2 as a linear system for the functions $B(s)$, $B'(s)$ and $B''(s)$, which can be solved explicitly for these functions. On the other hand we find

$$\begin{aligned} z_0(s) &= \frac{(\varphi_1(s)-\varphi_2(s))^2}{\mu}, & e^{-f} &= +\sqrt{\mu z_0}, \\ z_1(s) &= \varphi_1(s)\frac{(\varphi_2(s)-\varphi_1(s))}{\mu}, \quad \text{and so} & \varphi_1 e^{-f} &= \mu z_1, \\ z_s(s) &= \varphi_2(s)\frac{(\varphi_1(s)-\varphi_2(s))}{\mu}, & \varphi_2 e^{f} &= -\mu z_s e^{2f}. \end{aligned} \tag{85}$$

Because of (77) this gives (81) and (82). For the case $\mu = 0$ in the case B Theorem 8 does not hold, since the eigenvalues of A_∞ are not distinct and so Theorem 8 could not be applied to (67). But by a simple calculation one can show that for this case (79) as well as (81) and (82) are correct.

Remark 15. Formulas (81) in the cases A and B deal in general with the complex-valued functions $y(s)$ and $B(s)$. The Bonnet surfaces are characterized by the condition, that for s defined in Table 2, $B(s)$ and $f(s)$ in (81), (82) should be real-valued. It seems to be rather difficult to describe the variety of the geometrical solutions in terms of $y(s)$.

In order to integrate the Bonnet surfaces of type A and B we have to solve our special Painlevé VI and the frame equation (84) for Ψ under the conditions that the functions B and f as defined in (81) and (82) are both real, and e^{2f} is strictly positive. By the inverse left-multiplication of (83) with the functions given in (85) and (77) we find the geometrical frame, which finally has to be integrated.

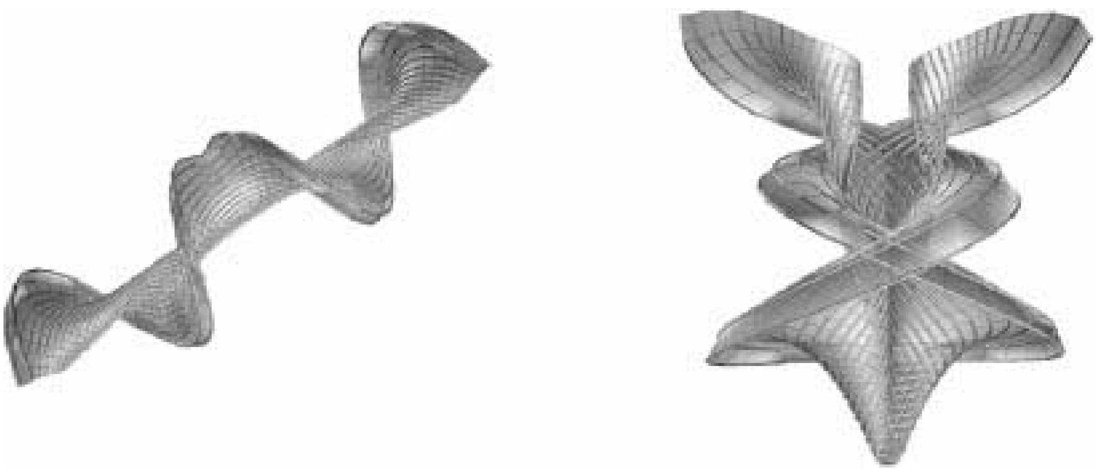

Figure 1. A Bonnet A family

In Fig. 1 and 2 surfaces of both types A and B are presented. They are generated numerically. We use the formulas for the coefficient of the fundamental functions given in Table 1 and a 4-step Runge-Kutta method with step-size correction to integrate the ordinary differential equation for H. In case C it is convenient to use the Painlevé functions (namely $\phi(t)$ as solution of (60) and the relations (61)) to construct the surfaces.

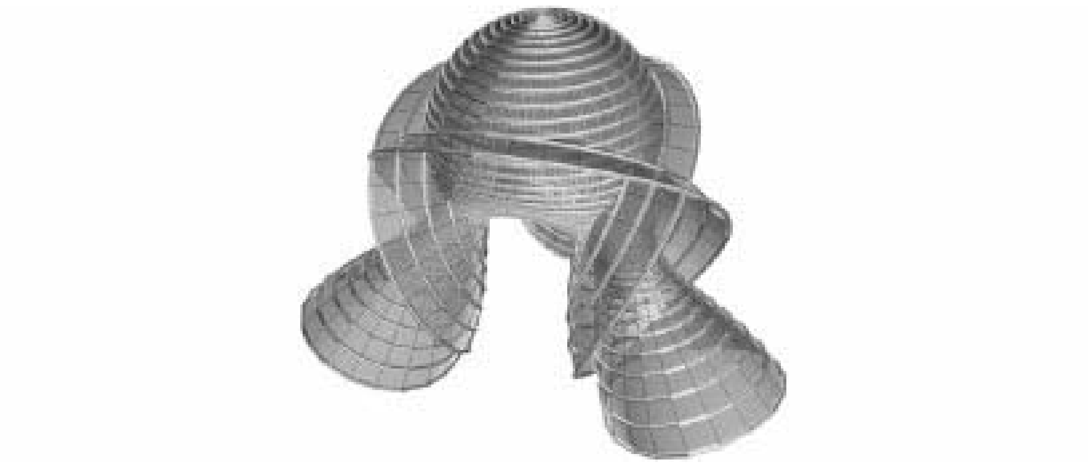

Figure 2. A Bonnet B surface with parameter curves

The isometry preserving the mean curvature corresponds to going along the closed parameter curves $t = w + \bar{w} = \text{const}$. The quasi-periodicity seems to be a periodicity, since the surface closed up in direction of the isometry lines. (Usually the surfaces are only quasi-periodic along these lines. It seems to be a rather complicated problem to find closing-up conditions.)

Concluding remarks. The isomorphism between the Gauss equation of the Bonnet surfaces and the Painlevé V (III) and VI equations we established in this paper allows one to apply the modern theory of the Painlevé equations to describe global properties of the Bonnet surfaces. The main tool of the theory of the Painlevé equations is the Ψ-function, which is a solution of the corresponding linear system (see [J-M]). This function is also well investigated. It is worth mentioning, that the frame of the Bonnet surfaces is described by a quaternion, which differs from Ψ just by gauge transformation (63), (83).

Let us mention two geometrical problems, which is now possible to solve and which we plan to discuss in the future. It is well known that the Painlevé equations possess Schlesinger transformations. For example, if $y(s)$ is a solution of the Painlevé VI equation with some constants α, β, γ, δ, then there is a transformation

$$(86) \qquad (y, y', \alpha, \beta, \gamma, \delta) \to (y_N, y_N', \alpha_N, \beta_N, \gamma_N, \delta_N)\,,$$

which yields a solution $y_N(s)$ to the Painlevé VI equation with some constants $\alpha_N, \beta_N, \gamma_N, \delta_N$. In (86)

$$(y_N, y_N') = \mathrm{R}(y, y')\,,$$

where R is a rational function. The corresponding solutions Ψ and Ψ_N of the linear systems are also simply related. Starting with some Bonnet surface (see (80))

$$\alpha = \frac{(\mu+1)^2}{2}, \quad \beta = -\frac{\mu^2}{2}, \quad \gamma = 0, \quad \delta = \frac{1}{2},$$

with a proper choice of parameters via iteration of the transformation (86) one can obtain

$$\alpha_N = \frac{(\mu_N+1)^2}{2}, \quad \beta_N = -\frac{\mu_N^2}{2}, \quad \gamma_N = 0, \quad \delta_N = \frac{1}{2},$$

i.e. a new Bonnet surface. The geometrical interpretation of this transformation gives us a transformation of Bæcklung type. In particular the Schlesinger transformations for surfaces of type B were formulated in a recent paper [Eit96].

The Bonnet surface of type B presented in Fig. 1 looks very similar to the Mr. Bubble surfaces with 3 legs [T-P-F], [D-P-W] which is a CMC plane with intrinsic rotational symmetry and an umbilic point at the origin. The reason for this is that in Fig. 1 we see the immersion of a neighborhood of the point $t = \infty$. The analysis of the corresponding differential equation shows that $H(t)$ converges very fast to a fixed value

$$\lim_{t \to \infty} H(t) = H_0 .$$

In a natural local variable z at this point the Hopf differential $Q(z, \bar{z})$ has a zero of the order ω. If $\omega \in \mathbb{Z}$ then the surface does not ramify at the point $t = \infty$, which is an umbilic point of order ω. In a big neighborhood of this point the surface looks like the Mr. Bubble surface with $\omega + 2$ legs. The global behavior of this surface can be calculated explicitly as it has been done for the Mr. Bubble surface in [B91].

Fig. 1 shows the two distinct surfaces of a family of type A. In particular the left picture in Fig. 1 suggests that these surfaces become "closed" to helicoidal surfaces. Indeed their asymptotes, the exceptional value surfaces, are known to be helicoidal surfaces (see Theorem 5). Along the tubed curves the mean curvature and the metric are constant, i.e., the surfaces are intrinsically isometric and their principle curvatures are preserved along these curves.

Appendix A. Bonnet surfaces of type C with $\mu = 0$

Here we integrate explicitly the equation for the moving frame for the Bonnet surfaces of type C in the case $\mu = 0$ (see section 3). In this case equation (40) with (39) implies

$$H(w, \bar{w}) = \frac{a_0}{w + \bar{w}}, \quad e^{u(w,\bar{w})} \equiv \frac{2}{a_0}, \quad Q(w, \bar{w}) = -\frac{\bar{w}}{w(w+\bar{w})} . \tag{87}$$

Now let us pass to isothermic coordinates as in (16)–(19)

$$z = \log(w), \quad z = x + iy .$$

In the isothermic coordinates the fundamental functions are:

$$H(z,\bar{z}) = \frac{a_0}{2e^x \cos(y)}, \quad e^{u(z,\bar{z})} = \frac{2}{a_0} e^{2x}, \quad Q(z,\bar{z}) = -\frac{e^x}{2\cos(y)},$$

and the frame equations are of the following form:

$$\begin{gathered} F_{xx} = F_x, \quad F_{xy} = F_y, \quad F_{yy} = -F_x + \frac{2e^x}{\cos(y)} N, \\ N_x = 0, \quad N_y = -\frac{a_0}{e^x \cos(y)} F_y. \end{gathered} \tag{88}$$

One can integrate (88)

$$F(z,\bar{z}) = F_x(z,\bar{z}) = F_z(z,\bar{z}) + F_{\bar{z}}(z,\bar{z}) = wF_w(w,\bar{w}) + \bar{w}F_{\bar{w}}(w,\bar{w})$$

and so

$$F(w,\bar{w}) = -\frac{i}{\omega} \Phi^{-1} \begin{pmatrix} 0 & \bar{w} \\ w & 0 \end{pmatrix} \Phi = -\frac{it}{\omega} \Phi^{-1}(\lambda) \begin{pmatrix} 0 & 1-\lambda \\ \lambda & 0 \end{pmatrix} \Phi(\lambda), \quad \omega = \sqrt{\frac{a_0}{2}}, \tag{89}$$

where λ and t are defined as in (37). Here Φ depends on λ only and solves (see (38)) the ordinary differential equation

$$\Phi_\lambda(\lambda)\Phi^{-1}(\lambda) = \omega \begin{pmatrix} 0 & \dfrac{1}{\lambda} \\ \dfrac{1}{1-\lambda} & 0 \end{pmatrix}. \tag{90}$$

Let us recall that Φ is a quaternion

$$\Phi(\lambda) = \begin{pmatrix} \alpha(\lambda) & -\bar{\beta}(\bar{\lambda}) \\ \beta(\lambda) & \bar{\alpha}(\bar{\lambda}) \end{pmatrix}$$

with $|\alpha|^2 + |\beta|^2 \equiv \text{const} \neq 0$ and $\bar{\lambda} = 1-\lambda$.

In terms of these functions (90) gives:

$$\alpha'(\lambda) = \frac{\omega}{\lambda}\beta(\lambda), \quad \beta'(\lambda) = \frac{\omega}{1-\lambda}\alpha(\lambda)$$

where $'$ denotes the derivative with respect to λ. This system can be formulated as differential equation of second order for α only:

$$\lambda(1-\lambda)\alpha''(\lambda) + (1-\lambda)\alpha'(\lambda) - \omega^2\alpha(\lambda) = 0.$$

This is a hypergeometric equation which can be integrated in terms of special functions. The function ${}_2F_1(a,b,c,\lambda)$ is a solution of the hypergeometric equation

$$\lambda(\lambda-1)\alpha''(\lambda) + \big((a+b+1)\lambda - c\big)\alpha'(\lambda) + ab\alpha(\lambda) = 0 \tag{91}$$

which is regular at $\lambda = 0$, and which has, for $|\lambda| < 1$, the expansion

$$
{}_2F_1(a, b, c, \lambda) = \sum_{k=0}^{\infty} \frac{\Gamma(c)\,\Gamma(a+k)\,\Gamma(b+k)}{k!\,\Gamma(a)\,\Gamma(b)\,\Gamma(c+k)} \lambda^k . \tag{92}
$$

In our case

$$
a = \pm i\omega, \quad b = \mp i\omega, \quad c = 1 .
$$

The general solution of (90) is given by

$$
\alpha(\lambda) = C_1\, {}_2F_1(i\omega, -i\omega, 1; \lambda) + C_2 (1-\lambda)\, {}_2F_1(1+i\omega, 1-i\omega, 2; 1-\lambda) \tag{93}
$$

and finally

$$
\Phi(\lambda) = \frac{1}{\omega} \begin{pmatrix} \omega\alpha(\lambda) & -(1-\lambda)\alpha'(1-\lambda) \\ \lambda\alpha'(\lambda) & \omega\alpha(1-\lambda) \end{pmatrix} \tag{94}
$$

which is for geometrical λ in $\mathbb{R}\,\mathrm{SU}(2)$ if the constants $C_1, C_2 \in \mathbb{R}$. We can choose $C_2 = 0$ because any other solution of (90) differs from this special one only by a multiplication on the right by a quaternion independent of w and $\bar{w}$. This multiplication however is a rotation of the surface (89) as a whole.

Theorem 10. *In the isothermic coordinates (x, y) the Bonnet cones, which are the surfaces of type* C *with $\mu = 0$, are given by the formulas*

$$
F(x, y) = \frac{t}{\omega\big(\omega^2|\alpha(\lambda)|^2 + |\lambda|^2|\alpha'(\lambda)|^2\big)} \begin{pmatrix} \mathrm{Re}\big(\lambda\big(\omega^2\alpha(\lambda)^2 - |\lambda|^2\alpha'^2(\lambda)\big)\big) \\ \mathrm{Im}\big(\lambda\big(\omega^2\alpha(\lambda)^2 - |\lambda|^2\alpha'^2(\lambda)\big)\big) \\ 2\omega|\lambda|^2\,\mathrm{Re}\big(\alpha'(\lambda)\alpha(1-\lambda)\big) \end{pmatrix} \tag{95}
$$

where

$$
\begin{aligned}
&\alpha(\lambda) = C_1\, {}_2F_1(i\omega, -i\omega, 1; \lambda), \quad \alpha'(\lambda) = \omega^2 C_1\, {}_2F_1(1+i\omega, 1-i\omega, 2; \lambda), \\
&\lambda = \frac{1}{2}\big(1 + i\tan(y)\big), \quad t = 2e^x\cos(y)
\end{aligned} \tag{96}
$$

with some arbitary positive number ω.

For the proof one should use formula (89) and the isomorphism (7) between $\mathbb{R}^3$ and $\mathrm{su}(2)$. One should also note, that

$$
\left(\frac{d}{d\lambda}\right)^n {}_2F_1(a, b, c; \lambda) = \frac{\Gamma(a+n)\,\Gamma(b+n)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c+n)}\, {}_2F_1(n+a, n+b, n+c; \lambda) .
$$

Remark 16. Since α solves equation (91) for λ of the form (96),

$$
\omega\big(\omega^2|\alpha(\lambda)|^2 + |\lambda|^2|\alpha'(\lambda)|^2\big)
$$

is constant. Therefore this factor can be ignored.

A plot of such a surface which $\omega = 1$ is presented in Figure 3.

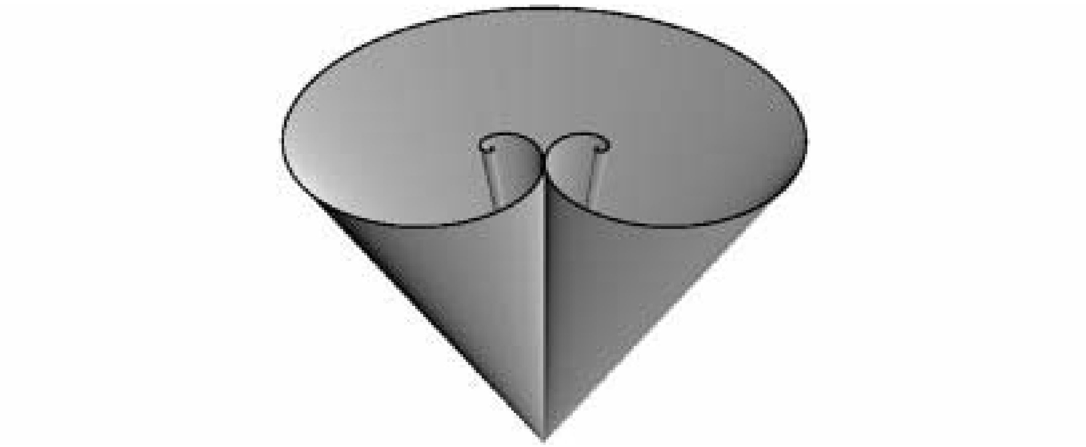

Figure 3. A Bonnet C surface with $\mu = 0$

The expectional value surfaces (asymptotes) of these surfaces are cylinders generated by logarithmic spirals as shown in [C-K].

Appendix B. A geometrical meaning of the first integral μ in the case A

As stated in Theorem 5 we know that each family of type A contains four different surfaces, two of them – the exceptional value surfaces ($T = \pm 1/2$) – are helicoidal surfaces. In this appendix we will prove this. Moreover we will find that surprisingly just these surfaces give us a meaning of the first integral μ in (72). To see this, let us collect what we know about these exceptional value surfaces (asymptotes). The coefficients of their fundamental forms are

$$
\begin{aligned}
Q_{T=\pm\frac{1}{2}}(w,\bar{w}) &= 2\big(\cot(2t) \pm i\big) = Q_{T=\pm\frac{1}{2}}(x) = 2\big(\cot(4x) \pm i\big)\,, \\
e^{u(w,\bar{w})} &= -\frac{8}{\sin^2(2t)H'(t)} = e^{u(x)} = -\frac{16}{\sin^2(4x)H'(x)}\,,
\end{aligned} \tag{97}
$$

with $t = w + \bar{w} = 2x$ and the mean curvature $H = H(t) = H(x)$ (we use the same symbol although the functions for t and x are different). H is a solution of (25) with $|Q|^2 = 4/\sin^2(2t)$. Let us denote by

$$
\gamma_x(y) = e^{-u(w,\bar{w})/2}F(x,y)\,, \quad F \text{ the immersion} \tag{98}
$$

the y-curve on the surface immersed by $F(x,y)$ for a fixed x. It is not hard to find that its curvature κ and torsion τ are

$$
\begin{aligned}
\kappa^2 &= \|\gamma_x(y)\|^2 = \left(\frac{u'(x)}{2}\right)^2 + e^{-u(x)}\big(H(x)e^{u(x)} - 4\cot(4x)\big)^2\,, \\
\tau^2 &= \frac{\det\big(\gamma_x'(y), \gamma_x''(y), \gamma_x'''(y)\big)}{\kappa^2} = 16e^{-u(x)}\,.
\end{aligned} \tag{99}
$$

It is a simple calculation to show that

$$\kappa^2 + \tau^2 \equiv C, \quad C \in \mathbb{R}^+, \quad x \text{ and } y \text{ independent}, \tag{100}$$

or in another form

$$\left(\frac{u'(x)}{2}\right)^2 + e^{-u(x)}\big(H(x)\,e^{u(x)} - 4\cot(4x)\big)^2 + 16\,e^{-u(x)} = C\,. \tag{101}$$

Lemma 5. *Equation* (101) *is, up to a constant factor, the first integral equation* (72). *The constants μ and C are related by*

$$C = -16\,\mu^2\,. \tag{102}$$

Proof. Substituting $H(x) = B(s)$, $s = e^{8ix}$ into (97) one finds

$$e^{u(x)} = -\frac{8i}{(s-1)^2 B'(s)}, \quad \text{and so} \quad u'(x) = -8is\left(\frac{B''(s)}{B'(s)} + \frac{2}{s-1}\right). \tag{103}$$

Now inserting these identities into (101) gives the left hand side of (72) up to a factor. Thus both constants, C and μ, are related by this constant factor which gives (102).

Lemma 6. *If $\kappa \neq 0$ then the curves $\gamma_x(y)$ are circular helices and $\sqrt{C}$ is the angular velocity.*

Proof. First we see that the curvature and the torsion of these curves are constant (they depend only on x). It is an elementary geometrical statement that these curves are circular helices and the velocity of the revolution is $\sqrt{\kappa^2 + \tau^2}$. See for instance [Sto].

Remark 17. The interpretation of $\sqrt{C}$ as the angular velocity of the "screwing" of the helices $\gamma_x(y)$ might suggest that this parameter can be omitted. However, we know that H as well as the metric e^u depend in a hidden way on C. Thus this parameter seems to be essential for the surfaces.

Theorem 11. *Let $F(x, y)$ be an immersion of an asymptote of a Bonnet surface of type* A. *Then the curvature of the curves $\gamma_x(y) = e^{-u(x)/2} F(x, y)$, x fixed, never vanishes.*

Proof. From (99) we see that κ can only vanish if both $u'(x)$ and

$$H(x)\,e^{u(x)} - 4\cot(4x)$$

are zero at the same time. $H(x) = \omega\cot(4x)$ satisfies both conditions for a proper ω. Since the solutions of (101) depend uniquely on the initial conditions $H(x)$ must be of this form. All solutions of (25) are solutions of (101), so this must be a solution of (25) which contradicts to the statement of Remark 14. Thus κ never vanishes. Finally we have the following

Theorem 12. *The exceptional value surfaces* (*asymptotes*) *in a Bonnet family of type* A *are helicoidale surfaces.*

Proof. A surface is a helicoidal surface if it is generated by a rigid motion acting on a plane curve such that the trace of any point of the plane curve under this action is a circular helix. In our case the curves $F(x = \text{const})$ are circular helices. But the curves $F(y = \text{const})$ are not plane curves. By introducing new coordinates

$$t = x, \quad s = y + \varphi(x)\,, \tag{104}$$

and the similarity condition $\hat{F}(t,s) = F(x,y)$ we find that the curves $\hat{F}(t = \text{const})$ are still circular helices. Moreover we can choose $\varphi(x)$ in such a way that the curves $\hat{F}(s = \text{const})$ are plane curves for all s. To see this, we compute the partial derivatives $\hat{F}_t(t,s_0)$, $\hat{F}_{tt}(t,s_0)$ and $\hat{F}_{ttt}(t,s_0)$ and express them in terms of the moving frame F_x, F_y, N, where F_x and F_y are the partial derivatives of $F(x,y)$ and N is the normal vector field. The coefficients involve just $H(x)$, $Q(x)$ and $e^{u(x)}$ as given in (97) and the first derivative of $u(x)$.

The curve $\hat{F}(t,s_0)$ is a plane curve, if

$$0 = \det(\hat{F}_t, \hat{F}_{tt}, \hat{F}_{ttt}) = \mathcal{G}\big(u(x), u'(x), H(x), \varphi'(x), \varphi''(x)\big)\det(F_x, F_y, N)\,. \tag{105}$$

The latter determinant is just e^u which is a function of x only. Thus (105) is an ordinary differential equation of second order in φ which we can solve for this function. The curves $\hat{F}(s = \text{const})$ are now plane curves and the trace of any point on this is just a circular helix, i.e. the trace of a rigid motion applied to the point.

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