Constant mean curvature surfaces and integrable equations

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CONTENTS

Introduction

Let us consider a smooth orientable surface $\mathscr{F}$ in three-dimensional Euclidean space and fix a direction of normal field N on it. To each point $P \in \mathscr{F}$ let us put into correspondence the set of spheres $S(k)$ of curvature $|k|$ tangent to $\mathscr{F}$ at P. We put $k > 0$ if the centre of $S(k)$ lies on that side of the surface where the normal field is directed, and $k < 0$ otherwise. We consider the tangent plane $S(0)$ as a particular case of the sphere. In the case of large $|k|$ the sphere $S(k)$ lies on one side of $\mathscr{D}_P$, where $\mathscr{D}_P \subset \mathscr{F}$ is some neighbourhood of P. The *principal curvatures at P* are the quantities

$$k_1 = \inf \{k\colon \mathcal{D}_P \cap S(k) \neq P\},$$
$$k_2 = \sup \{k\colon \mathcal{D}_P \cap S(k) \neq P\},$$

and the corresponding spheres $S(k_{1,2})$ are called the spheres of principal curvatures. There is exactly one sphere S, tangent to $S(k_1)$, $S(k_2)$ at P, the inversion with respect to which exchanges $S(k_1)$ and $S(k_2)$. The *mean curvature of a surface at P* is the curvature of S, which is equal to

$$H = \frac{k_1 + k_2}{2}.$$

Surfaces for all points of which H is the same are called *constant mean curvature* (CMC) *surfaces.*

Compact CMC embeddings may be characterized as extremal values of the area functional under the variations that preserve the volume of the figure contained inside the surface. This definition is applicable to general immersions if the components of the volume are calculated with suitable coefficients [26]. The simplest physical model of a CMC surface is a soap film in equilibrium between two regions of different gas pressure—inside and outside. Everyday experience convinces us that all bubbles are spheres.

The Hopf problem. *Are there compact CMC surfaces that are different from the standard sphere?*

A negative answer to this question has been obtained under the additional suppositions:

1) simple-connectedness—by Hopf [26],

2) embedding—by Aleksandrov [1],

3) existence of a local minimum of the area functional with respect to all volume-preserving variations—by Barbosa and do Carmo [13].

The result of Wente, who showed the existence of CMC tori, was quite unexpected; the evidence of this fact is the title of his work [44], as well as the series of publications that followed right away [11], [12], [41], [45]. In these works the Wente tori were studied in detail and some new examples were constructed. In particular, Abresch [11] characterized Wente tori as those that have one family of planar curvature lines and described them in elliptic integrals. Walter gave a more detailed integration [41].

The equation of Gauss–Peterson–Codazzi (GPC) for CMC tori

$$u_{z\bar{z}} + \sinh u = 0 \tag{0.1}$$

is integrable and is one of the possible real variants of the sine-Gordon equation, which is well known in the theory of solitons. This equation has an infinite series of integrals of motion, which in turn define commutative higher flows that are the symmetries of the equation. However, the problem of constructing CMC tori has been connected for the first time with the fact of integrability of equation (0.1) only in the recent works of Hitchin [25] and Pinkall and Sterling [37]. In [37], which is devoted to a classification of CMC tori, there is implicitly the important theorem that all doubly periodic solutions of equation (0.1) are stationary with respect to any one of the higher flows (see §5). The finiteness of the genus of the corresponding spectral curve has also been proved in another way in [25]. In this work one considers the similar problem of a harmonic map (including a minimal one with the same GPC equation (0.1), see §17) from a torus into a 3-dimensional sphere. Besides this, Hitchin characterized spectral curves corresponding to minimal tori in S^3.

The theorem mentioned above points out that this problem can be resolved within the framework of the theory of finite-zone (or finite-gap) integration, created by Novikov, Dubrovin, Matveev, Its, Krichever and others in the seventies [6], [7]. In full strength this theory has been applied to the description of CMC tori in the work [15] of the author. All the tori were described explicitly: in terms of the Baker–Akhiezer function a formula was obtained from immersion and periodicity conditions, singling out spectral curves relative to tori.

The present survey pursues a double objective. On the one hand, it is to draw the attention of specialists in the theory of integrable equations to a great number of geometric problems, to the solution of which the methods of the theory of solitons can be successfully applied. On the other hand, it is to demonstrate these methods on a concrete example to specialists in differential geometry. We will try to avoid an appeal to analogous standard results obtained in the theory of solitons for some other equations, in order to make the account self-contained as far as possible.

In the first chapter with the help of an analytic approach we study the problem of constructing CMC surfaces in a general formulation. We construct a representation of zero curvature with spectral parameter for the GPC equation of a generic CMC surface. The variables in the equation vary on a Riemann surface. In terms of the solution of the corresponding linear system the formula for immersion is obtained, and some geometric properties of the surface are defined, for example, its type with respect to regular homotopies. In the framework of the analytic approach the problem of constructing CMC surfaces leads to the theory of integrable equations on a Riemann surface. The development of this theory seems to be an interesting problem.

While the classification of simply-connected compact CMC surfaces (§2) is rather trivial, and the problem of constructing surfaces of genus $G \geqslant 2$ has more questions than answers (§4), the tori ($G = 1$) can be described just by using in full measure the apparatus of the theory of finite-zone integration. The second chapter, representing the greatest part of the survey, is devoted to a detailed exposition of the corresponding results.

The GPC equation for compact CMC surfaces in S^3 and H^3 has the same form as for Euclidean space. The results of the first and second chapters are carried over to this case in the third chapter. We discuss the connection between minimal surfaces in S^3 and Willmore surfaces, and give a summary of the results obtained, as well as a series of conjectures that are still waiting for their solutions.

In the appendix we use another known solution of equation (0.1), depending only on $|z|$. In this case the equation (0.1) is reduced to the third Painlevé equation, and with the help of the results of Its and Novokshenov [27] we describe asymptotically proper immersions of a CMC plane.

Let us conclude this introduction with some brief commentary on the literature. Some very similar problems have already been discussed in classical works. The case of a constant negative Gaussian curvature $K = -1$ has been intensively studied; for this case the corresponding non-linear equation is the standard real-valued form of the sine-Gordon equation $u_{xy} = \sin u$. In particular, there was given a classification of surfaces with one family of planar or spherical curvature lines, which turn out to be tori analogous to Wente tori [17], [19], [20]. The methods of constructing these examples for $K = -1$ and $H = 1/2$ are identical.

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CHAPTER I

COMPACT SURFACES OF CONSTANT MEAN CURVATURE

1. Differential equations of constant mean curvature surfaces

Let $\mathcal{F}$ be a smooth surface in a 3-dimensional Euclidean space. The Euclidean metric induces a metric Ω on this surface, which in turn generates a complex structure on a Riemann surface. The surface is covered by domains $\mathcal{D}_i$, $\bigcup_i \mathcal{D}_i = \mathcal{F}_i$, and in each of these there is defined a local coordinate $z_i : \mathcal{D}_i \to \mathcal{U}_i \subset \mathbb{C}$. If the intersection $\mathcal{D}_i \cap \mathcal{D}_j \neq \emptyset$ is non-empty, the sewing functions $z_i \circ z_j^{-1}$ are holomorphic. Under a such parametrization, which is called *conformal*, the surface is given by means of the vector-functions

$$F = (F_1, F_2, F_3)\,(z_i, \bar{z}_i):\; U_i \subset \mathbb{C} \to \mathbb{R}^3,$$

and the metric is diagonal: $\Omega = 4e^{u_i} dz_i d\bar{z}_i$. In the sequel we shall suppose that $\mathcal{F}$ is sufficiently smooth: $F \in C^3$. All our considerations in this section are carried out on a local level, so we shall restrict our attention to one domain $\mathcal{D}_i$ and omit the superscript i of the local coordinate z.

The diagonal form of the metric gives the following normalization of the function $F(z, \bar{z})$:

$$(1.1) \qquad \langle F_z, F_z \rangle = \langle F_{\bar{z}}, F_{\bar{z}} \rangle = 0, \quad \langle F_z, F_{\bar{z}} \rangle = 2e^u,$$

where the brackets mean the scalar product

$$\langle a, b\rangle = a_1 b_1 + a_2 b_2 + a_3 b_3,$$

and F_z and $F_{\bar z}$ are the partial derivatives

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The vectors F_z, $F_{\bar z}$, as well as the normal N,

$$\langle F_z, N\rangle = \langle F_{\bar z}, N\rangle = 0, \quad \langle N, N\rangle = 1, \tag{1.2}$$

define a moving basis on the surface, which, as follows from (1.1), (1.2), satisfies the following equations of Gauss and Weingarten:

$$\sigma_z = \mathcal{U}\sigma, \quad \sigma_{\bar z} = \mathcal{V}\sigma, \quad \sigma = (F_z, F_{\bar z}, N)^{\mathrm{T}}, \tag{1.3}$$

$$\mathcal{U} = \begin{pmatrix} u_z & 0 & Q \\ 0 & 0 & B \\ -\frac{e^{-u}}{2}B & -\frac{e^{-u}}{2}Q & 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} 0 & 0 & B \\ 0 & u_{\bar z} & \bar Q \\ -\frac{e^{-u}}{2}\bar Q & -\frac{e^{-u}}{2}B & 0 \end{pmatrix},$$

where

$$Q = \langle F_{zz}, N\rangle, \quad B = \langle F_{z\bar z}, N\rangle. \tag{1.4}$$

The first and the second quadratic forms

$$\langle dF, dF\rangle = \left\langle \mathrm{I}\begin{pmatrix} dx \\ dy\end{pmatrix}, \begin{pmatrix} dx \\ dy\end{pmatrix}\right\rangle, \quad z = x + iy,$$

$$-\langle dF, dN\rangle = \left\langle \mathrm{II}\begin{pmatrix} dx \\ dy\end{pmatrix}, \begin{pmatrix} dx \\ dy\end{pmatrix}\right\rangle$$

are given by the matrices

$$\mathrm{I} = 4e^u\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \quad \mathrm{II} = \begin{pmatrix} Q + \bar Q + 2B & i(Q - \bar Q) \\ i(Q - \bar Q) & -Q - \bar Q + 2B\end{pmatrix}.$$

The principal curvatures k_1 and k_2 are the eigenvalues of the matrix $\mathrm{II}\,\mathrm{I}^{-1}$, which gives the following expressions for the mean and the Gaussian curvatures:

$$\begin{cases} H = \dfrac{k_1 + k_2}{2} = \dfrac{1}{2}\operatorname{tr}(\mathrm{II}\,\mathrm{I}^{-1}) = \dfrac{1}{2}Be^{-u}, \\ K = k_1 k_2 = \det(\mathrm{II}\,\mathrm{I}^{-1}) = \dfrac{1}{4}(B^2 - Q\bar Q)e^{-2u}. \end{cases} \tag{1.5}$$

The Gauss–Peterson–Codazzi equations (GPC), which are the compatibility conditions of equations (1.3),

$$\mathcal{U}_{\bar z} - \mathcal{V}_z + [\mathcal{U}, \mathcal{V}] = 0,$$

have the following form:

$$u_{z\bar z} + \frac{1}{2}(B^2 - Q\bar Q)e^{-u} = 0,$$

$$Q_{\bar z} + u_z B - B_z = 0,$$

$$\bar Q_z + u_{\bar z} B - B_{\bar z} = 0.$$

For CMC surfaces we have $Q_{\bar{z}} = 0$. Taking into consideration the form (1.4) of the transformation of Q under an analytic change of coordinates, we deduce that $Q(dz)^2$ is a holomorphic quadratic differential, which is called the *Hopf differential.* We do not consider minimal surfaces, that is, $H = \text{const} \neq 0$. A change of scale in Euclidean space gives us the possibility, without loss of generality, of confining ourselves to the case $H = 1/2$.

***Theorem* 1.1.** *The surface of CMC $H = 1/2$ under conformal parametrization generates the homolorphic quadratic differential*

$$Q\,(dz)^2 = \langle F_{zz}, N\rangle\,(dz)^2,$$

and also the solution of the GPC equation

$$u_{z\bar{z}} + \frac{1}{2}e^u - \frac{1}{2}Q\bar{Q}e^{-u} = 0. \tag{1.6}$$

Now let us discuss the inverse problem of describing a surface from known Q and $u(z, \bar{z})$. Equation (1.6) may be represented as the compatibility condition

$$U_{\bar{z}} - V_z + [U, V] = 0 \tag{1.7}$$

of the system of equations

$$\Phi_z = U\Phi, \quad \Phi_{\bar{z}} = V\Phi, \tag{1.8}$$

where U and V are matrices dependent on an additional parameter λ, which in the theory of integrable equations is called *spectral*:

$$U = \frac{1}{2}\begin{pmatrix} 0 & -\lambda e^{u/2} \\ Qe^{-u/2} & u_z \end{pmatrix}, \quad V = \frac{1}{2}\begin{pmatrix} u_{\bar{z}} & -\bar{Q}e^{-u/2} \\ \lambda^{-1}e^{u/2} & 0 \end{pmatrix}. \tag{1.9}$$

The matrices (1.9) satisfy the reduction

$$\overline{U\,(\bar{\lambda}^{-1})} = \sigma_2 V\,(\lambda)\,\sigma_2,$$

which is carried over to the solution of the system (1.8)

$$\Phi\,(\lambda) = \sigma_2\overline{\Phi\,(\bar{\lambda}^{-1})}\,M\,(\lambda)$$

with some matrix $M(\lambda)$ not depending on $z, \bar{z}$. Here and below we use the standard notation for Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For λ lying on the unit circle,

$$\lambda = e^{2i\gamma},$$

we can choose a solution from the group of quartenions $\Phi \in \mathbb{R}_+SU(2)$ with $\det \Phi$ independent of γ.

Let us put into correspondence a 3-dimensional vector X with coordinates X_1, X_2, X_3 and the matrix

$$X = \sum_{k=1}^{3} X_k \frac{\sigma_k}{2i} \leftrightarrow X = (X_1, X_2, X_3). \tag{1.10}$$

We also denote by F and N the matrices obtained in this way from the vectors F and N.

***Theorem* 1.2.** *Let $u(z, \bar{z})$ be a solution of equation* (1.6), *where $Q(dz)^2$ is a holomorphic quadratic differential, and let $\Phi(z, \bar{z}, \lambda = e^{2i\gamma})$ be a solution of the system* (1.8), (1.9) *belonging to the group $\mathbb{R}_+ SU(2)$, where* $\det \Phi$ *does not depend on λ. Then F and N, defined by the formulae*

$$\begin{cases} F = \Phi^{-1} \dfrac{\partial}{\partial \gamma} \Phi \Big|_{\gamma=0}, \\ N = \Phi^{-1} \dfrac{\sigma_3}{2i} \Phi \Big|_{\gamma=0}, \end{cases} \tag{1.11}$$

satisfy equations (1.3) *with $B = e^u$ and describe a CMC surface.*

Proof. First of all, it is obvious that the matrices F and N given by formulae (1.11) belong to the algebra $su(2)$, and therefore with the help of the correspondence (1.10) define a surface. To prove the truth of equations (1.3) we shall use a convenient expression for the scalar product of vectors in terms of matrices (1.10):

$$\langle X, Y \rangle = -2 \operatorname{tr} (XY).$$

The normalizations (1.1) and (1.2) are a simple consequence of formulae (1.10) and

$$\begin{cases} F_z = -\Phi^{-1}\Phi_z\Phi^{-1}\Phi_\gamma + \Phi^{-1}(U\Phi)_\gamma = \Phi^{-1}U_\gamma\Phi, \\ F_{\bar{z}} = \Phi^{-1}V_\gamma\Phi. \end{cases} \tag{1.12}$$

The equalities

$$Q = -2 \operatorname{tr} (F_{zz}N), \quad B = e^u = -2 \operatorname{tr} (F_{z\bar{z}}N)$$

are also verified by simple computation. Equations (1.3) are equivalent to the union of all these equalities.

Remark. The fact that Φ is not uniquely fixed, which is connected with multiplication on the right by a matrix belonging to $SU(2)$, leads to a general Euclidean spatial motion according to formulae (1.11). Let us note that a formula similar to the first of the expressions (1.11) was first found by Sym [39] while describing surfaces of constant negative Gaussian curvature $K = -1$.

A point at which the principal curvatures coincide: $k_1 = k_2$, is called *umbilical* (*non-umbilical* if $k_1 \neq k_2$). Formulae (1.5) show that umbilical points are zeros of the differential Q:

$$(k_1 - k_2)^2 = Q\bar{Q}e^{-2u}.$$

The order n of a zero of $Q(dz)^2 \cong z^n(dz)^2$ is called the *order of the umbilical point*. In a neighbourhood of a non-umbilical point we can choose the variable z in such a way that the Hopf differential has the form $(dz)^2$. In such a variable the GPC equation is the elliptic sine-Gordon equation

$$u_{z\bar{z}} + \sinh u = 0. \tag{1.13}$$

2. Sphere. Hopf theorem

Now let us proceed to the problem of describing compact CMC surfaces. First of all, let us note that the surface must be orientable, because when $H \neq 0$ there is the greatest of the principal curvatures, and, hence, a selected orthogonal direction inside the sphere of the largest principal curvature. In the case of genus zero the answer is well known.

***Theorem* 2.1** (Hopf [26]). *The unique structure of CMC $H = 1/2$, topologically equivalent to a sphere, is the standard sphere of radius 2.*

Proof. On a Riemann surface of genus zero there are no quadratic differentials not equal to zero: $Q \equiv 0$, that is, all points are umbilical. The third rows of equations (1.3) with $B = e^u$ give

$$N_z + \frac{1}{2} F_z = N_{\bar{z}} + \frac{1}{2} F_{\bar{z}} = 0,$$

from which $F = -2N + C$, where C = const. This is the sphere

$$\langle F - C, F - C \rangle = 4$$

of radius 2 and with centre C.

3. Torus. Analytic formulation of the problem

In case of the torus the problem is noticeably simplified, and analytic tools enable us to achieve success. The reason for simplification is the fact that on a torus it is easy to introduce a complex global coordinate. Any Riemann surface of genus one is conformally equivalent to the quoteint $\mathbb{C}/\Lambda$ of the complex plane by a lattice Λ, and one can choose generators of the lattice of the form 1 and τ, where τ belongs to the fundamental domain of the modular group. It will be convenient to fix a scale of the plane in another way, so we suppose that the lattice is generated by generators of generic form:

$$Z_1 = X_1 + iY_1, \quad Z_2 = X_2 + iY_2.$$

Thus, z is the global coordinate, and $u(z, \bar{z})$ and $F(z, \bar{z})$ are doubly periodic functions with respect to the lattice Λ.

***Proposition* 3.1.** *CMC-tori do not have umbilical points.*

Proof. The differential $Q(z)(dz)^2$ is holomorphic, that is, $Q(z)$ is an elliptic function without singularities. Therefore, $Q(z)$ =const. This constant cannot be equal to zero, otherwise, as follows from the proof of the Hopf theorem, the surface is a sphere. Finally, $Q \equiv \text{const} \neq 0$, which proves the proposition.

Now we fix the coordiante z in such a way that

$$Q = \langle F_{zz}, N\rangle = 1.$$

Under such a parametrization the lines x = const, y = const give a net of lines of principal curvature on a torus, and the GPC equation is the elliptic sine-Gordon equation (1.13).

Remark. For the torus it is more convenient to use instead of Φ the function $\Phi_0 \in SU(2)$:

$$\Phi_0 = e^{-u/4}\Phi, \tag{3.1}$$

distinguished from the first one by a multiplier. This function satisfies equations (1.8) with the matrices

$$U_0 = \frac{1}{2}\begin{pmatrix} -\frac{u_z}{2} & -\lambda e^{u/2} \\ e^{-u/2} & \frac{u_z}{2} \end{pmatrix}, \quad V_0 = \frac{1}{2}\begin{pmatrix} \frac{u_{\bar z}}{2} & -e^{-u/2} \\ \lambda^{-1}e^{u/2} & -\frac{u_{\bar z}}{2} \end{pmatrix}. \tag{3.2}$$

Formulae (1.11) are invariant with respect to the transformation (3.1).

The problem of describing all CMC tori analytically is far more complicated than in the case of the sphere. However, this problem can be completely solved with the help of the theory of integrable equations. The solution to this problem is given in the second chapter. We succeed in describing all doubly periodic solutions of equation (1.13), and then, by using the formula (1.11), constructing all doubly periodic $F(z, \bar z)$.

4. Surfaces of higher genus

As a matter of fact, we have already formulated analytically the problem of constructing CMC surfaces of genus $G \geqslant 2$. First of all we need to construct globally on a Riemann surface $\mathcal{R}$ of genus G the metric $e^u dz d\bar z$ and the holomorphic quadratic differential $Q(dz)^2$, which satisfy equation (1.6) in local coordinates. Abresch showed (a private communication) that for any holomorphic quadratic differential there is a smooth metric satisfying (1.6). The set of parameters $\mathcal{P}$ of such a problem is the fibre bundle over the moduli space of Riemann surfaces with the linear space of holomorphic quadratic differentials as fibre. The number of real-valued parameters is equal to the sum of dimensions $2 \times 6(G-1)$. Then with the help of the solution Φ of equations (1.8), (1.9) we should construct the immersion $F(z, \bar z)$, uniquely defined on the same Riemann surface $\mathcal{R}$. There is a natural map from the homotopy group of $\mathscr{R}$ into the monodromy group of (1.8). Round a cycle c the function Φ has a monodromy

$$\Phi \to \Phi M_c, \quad M_c \in SU(2),$$

and for F this is equivalent to motions in the space. Every such motion is given by 6 parameters (3 rotation, 3 translation). The triviality of the monodromy of F leads to $12G-6$ conditions, which corresponds to $2G$ generators and one relation of the homotopy group. Taking into consideration the degree of freedom, mentioned in the Remark of §1, in the choice of Φ (additionally 6 parameters of general Euclidean spatial motion) we obtain the equality between the numbers of parameters and equations.

Question. *What kind of subset of $\mathscr{P}$ corresponds to CMC surfaces?*

The calculation of parameters given above shows that most likely this is a discrete one. Recently serious advances have been obtained in studying CMC surfaces. Kapouleas proposed a general method, enabling one to attach together such surfaces [28]. In particular, in this way he constructed some examples of compact CMC surfaces with $G \geqslant 3$, as well as properly embedded non-compact surfaces. A structure theory explaining the geometry of such surfaces was elaborated in [30]. At the same time, no analytic description is known of any example with $G \geqslant 2$. Besides this, there remains open the problem whether there are CMC surfaces of genus $G = 2$.

Now we shall define the dependence of Φ on the holomorphic coordinate z. Equations (1.8), (1.9) show that the following object is invariant under analytic changes of z:

$$\begin{pmatrix} \sqrt{d\bar z} & 0 \\ 0 & \sqrt{dz} \end{pmatrix} \Phi, \tag{4.1}$$

which gives a spinor structure on $\mathscr{R}$. Let a_n, b_n, $n = 1, \ldots, G$, be a canonical basis of cycles on $\mathscr{R}$; Φ may change sign round these cycles. Hence, to Φ there corresponds the theta-characteristic $[\alpha, \beta]$

$$\alpha = (\alpha_1, \ldots, \alpha_G), \quad \beta = (\beta_1, \ldots, \beta_G), \tag{4.2}$$

where $\alpha_n, \beta_n \in \{0, 1\}$ are determined from the condition for the following indeterminacy of Φ on going round the cycles a_n, b_n respectively:

$$\Phi \to (-1)^{\alpha_n+1}\Phi, \quad \Phi \to (-1)^{\beta_n+1}\Phi \tag{4.3}$$

This spinor sructure has a very simple geometric sense. To clarify it, let us consider the uniformization of $\mathscr{R}$. Let A_n, B_n be the closed geodesics in the metric of uniformization that correspond to a_n, b_n (of constant negative curvature if $G \geqslant 2$ and planar if $G = 1$). All these closed geodesics are simple loops without self-intersections. We consider one of these: γ. As a consequence of the uniformization there is also the fact that in its neighbourhood one can globally choose the complex coordinate $z = x+iy$ in such a way that the geodesic itself is given by the line $y = 0$, $0 \leqslant x \leqslant 1$, and its neighbourhood by a strip along this line, the ends of which are identified by the linear-fractional transformation $z \to \sigma z$ where $\sigma 0 = 1$. Let us write formulae (1.11), (1.12) for a moving frame on a CMC surface in this coordinate:

$$\left\{\begin{aligned} &F_z = -ie^{u/2}\Phi^{-1}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\Phi,\\ &F_{\bar z} = -ie^{u/2}\Phi^{-1}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\Phi,\\ &N = \Phi^{-1}\sigma_3\Phi/2i, \quad \Phi \in SU(2). \end{aligned}\right. \tag{4.4}$$

We consider a closed contour $F(\gamma)$ in $\mathbb{R}^3$. In the chosen coordinate F_x is the tangent field, and F_y and N the normal fields to the contour. $F(\gamma)$ together with the normal fields $N(\gamma)$ defines a closed orientable "band" in the space. Let N_γ be the number of twistors of this "band", or equivalently the interfacing index of the contours $F(\gamma)$ and $F(\gamma)+\varepsilon N$, where ε is small. By studying the simplest examples and taking into consideration the continuity, we prove the following lemma.

***Lemma* 4.1.** *Φ in (4.4) has the following indeterminacy under the bypass round γ:*

$$\Phi \to (-1)^{N_\gamma+1}\Phi.$$

This enables us to formulate the following proposition.

***Proposition* 4.2.** *The numbers α_n, β_n in (4.3) give the parity of the number of twistors of the "bands" corresponding to the cycles a_n, b_n respectively.*

The theta-charactristic (4.2) depends on the choice of basis of cycles on $\mathcal{R}$. Its parity is an invariant which is defined as the parity of the number

$$\langle \alpha, \beta \rangle = \sum_{i=1}^{G} \alpha_i \beta_i.$$

This parity completely classifies compact orientable surfaces with respect to regular homotopies. That is, there is a smooth homotopy F_t, $t \in [0, 1]$, of two immersions, F_0 and F_1, of a surface of genus G which at each moment remains an immersion if and only if the parities of the theta-characteristics described above, corresponding to F_0 and F_1, coincide [36].

And finally the last remark about surfaces of higher genus. Their characteristic property is the existence of umbilical points. The holomorphic quadratic differential of a surface of genus G has $4G-4$ zeros, which are umbilical points. Planes with one umbilical point of arbitrary order are constructed in the Appendix.

CHAPTER II

CONSTANT MEAN CURVATURE TORI

5. Doubly periodic solutions of the equation $u_{z\bar{z}} + \sinh u = 0$

It turns out that all doubly periodic non-singular solutions of equation (1.13) can be described by explicit fromulae. In this section we shall prove the central theorem explaining the reason for this unexpected phenomenon. First of all, let us note that due to the fact that (1.13) is elliptic and a torus is smooth, $u \in C^\infty$.

Equation (1.13) can be represented [10] as the compatibility condition (1.7) of the system

$$\Psi_z = U_1 \Psi, \qquad \Psi_{\bar{z}} = V_1 \Psi, \tag{5.1}$$

$$U_1 = \frac{1}{2}\begin{pmatrix} -u_z & -i\nu \\ -i\nu & u_z \end{pmatrix}, \quad V_1 = \frac{1}{2i\nu}\begin{pmatrix} 0 & e^{-u} \\ e^{u} & 0 \end{pmatrix}. \tag{5.2}$$

This pair U, V satisfies the following reductions:

$$\Psi(\nu) \to \sigma_3 \Psi(-\nu), \tag{5.3}$$

$$\Psi(\nu) \to R\overline{\Psi(\bar{\nu}^{-1})}, \quad R = \begin{pmatrix} 0 & -e^{-u/2} \\ e^{u/2} & 0 \end{pmatrix}, \tag{5.4}$$

where the arrow means that both sides are solutions of the system (5.1), (5.2) with the same u.

Equation (1.3) is totally integrable. It has an infinite series of integrals of motion, which in turn define commutative higher flows. We put $z = z_1$ and introduce an infinite series of new variables $z_2, \ldots, z_n, \ldots$. In the standard way [10] one can prove the following theorem.

***Theorem* 5.1.** *For any doubly periodic smooth solution $u(z, \bar{z})$ of* (1.13) *there is a real-valued function $u(z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n, \ldots)$, real-analytic in all variables, such that*

$$u(z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots)|_{z_n=0,\ n\geq 2} = u(z, \bar{z}).$$

Its dependence on additional variables is defined by the condition that the system (5.1), (5.2) *is compatible with the additional equations*

$$\Psi_{z_n} = U_n \Psi, \quad \Psi_{\bar{z}_n} = V_n \Psi,$$

moreover, the reductions (5.3), (5.4) *are preserved. The matrix U_n is a polynomial of degree $2n-1$ in* ν, *the coefficients of which are some polynomials in $u_z, \ldots, u_z^{(n)}$, and $V_n(\nu = \infty) = 0$.*

We shall not give the exact form of U_n, which is not important later on. For a matrix U that is gauge equivalent to (5.2), the formula (I.III.3.31) of the book [10] gives the generating function for all U_n. The real reduction (5.4) gives

$$U_n(\nu) = R_{z_n} R^{-1} + R\overline{V_n(\bar{\nu}^{-1})}\, R^{-1}.$$

Setting $\nu = \infty$, we obtain the higher sinh-Gordon equations:

$$u_{z_n} = P_n\,(u_z, \ldots, u_z^{(n)}),$$

where P_n is the coefficient of zero degree in U_n $(\nu = 0) = -P_n\sigma_3/2$, which proves that u is real-analytic. All P_n have been calculated in [37], where the authors proposed a geometric interpretation of higher flows as Jacobi fields.

Let t_1, t_2, ... denote an infinite series of real higher times $z_n = t_{2n-1} + it_{2n}$, $\bar{z}_n = t_{2n-1} - it_{2n}$. Because all higher flows commute, the set of solutions stationary with respect to some flow $u_{t_i} = 0$ is invariant with respect to all other flows, including the flows along $t_1 = x$ and $t_2 = y$. Such solutions in the soliton theory are called *finite-zone* (or *algebraic-geometric, polyphase, theta-functional*) [6], [7].

***Theorem* 5.2.** *Any non-singular doubly periodic solution of* (1.13) *is stationary with respect to any higher flow.*

Proof. All partial derivatives $v_i = \partial u/\partial t_i$ satisfy the equation

$$(\partial_z\partial_{\bar{z}} + \cosh u)\, v_i = 0. \tag{5.5}$$

The spectrum of the operator (5.5) on the torus $\mathbb{R}^2/\Lambda$ is discrete, so the v_i are linearly dependent. Hence, there is a higher time with respect to which u is stationary: $u_t = 0$.

6. The Baker–Akhiezer function. Analytic properties

Equation (1.13) is one of the real versions of the sine-Gordon equation, whose finite-zone solutions were first constructed by Kozel and Kotlyarov in [9]. Analogous results have been obtained by McKean [35]. The scheme of Krichever was applied to this equation by Its, who constructed the Baker–Akhiezer function [34] (see also [18]). Real doubly periodic solutions of (1.18) were constructed by the author in [15].

Let $u(z, \bar{z})$ be a solution of (1.13), invariant with respect to some higher flow t. The evolution Ψ along this flow is defined by a matrix that is a polynomial of degree $2n-1$ in ν as well as in ν^{-1}. The corresponding stationary equations look like this:

$$W_z = [U_1, W], \quad W_{\bar{z}} = [V_1, W]. \tag{6.1}$$

Without loss of generality, in these equations one can put

$$\operatorname{tr} W = 0, \quad W_{2n-1} = \sigma_1.$$

The change

$$W(\nu) \to \frac{1}{2}\,(W(\nu) + \sigma_3 W(-\nu)\,\sigma_3)$$

enables us to assume that the following reduction always holds:

$$\sigma_3 W(-\nu)\,\sigma_3 = W(\nu). \tag{6.2}$$

Equations (6.1) show that the eigenvalues of W are integrals of motion. The characteristic polynomial

$$\det\,(W(\nu) - \mu I) = 0 \tag{6.3}$$

defines an algebraic curve $\hat{X}$, which is called *spectral*.

Let $\hat{\Psi}(\nu, z, \bar{z})$ be the matrix solution of (5.1) with the normalization $\hat{\Psi}(\nu, 0, 0) = I$. This is an entire function of ν. Let P denote a point of the spectral curve with coordinates (ν, μ), and $H(P, z, \bar{z})$ an eigenvector of W

$$W(\nu, z, \bar{z})\, H(P, z, \bar{z}) = \mu H(P, z, \bar{z})$$

with the normalized first component $H_1 = 1$. This is a meromorphic function of the point P.

Definition. The vector-function

$$\psi(P, z, \bar{z}) = \hat{\Psi}(\nu, z, \bar{z})\, H(P, 0, 0) \tag{6.4}$$

is called the *Baker–Akhiezer function* (BA).

It is an analytic function on $\hat{X}$, satisfying the equations

$$\psi_z = U_1\psi, \quad \psi_{\bar{z}} = V_1\psi, \quad W\psi = \mu\psi.$$

Perhaps only the latter needs some comments. This equation is a consequence of the identity

$$W(\nu, z, \bar{z})\, \hat{\Psi}(\nu, z, \bar{z}) = \hat{\Psi}(\nu, z, \bar{z})\, W(\nu, 0, 0). \tag{6.5}$$

To prove it, let us note that both sides of (6.5) satisfy the system (5.1) with the same initial conditions when $z = \bar{z} = 0$. The uniqueness of the solution proves (6.5).

Firstly, we shall deduce the analytic properties of BA in the general complex case, when z and $\bar{z}$ are regarded as two independent complex variables. The spectral curve $\hat{X}$

$$\mu^2 = -\det W(\nu)$$

by virtue of (6.2) possesses the involution

$$\pi\colon\ (\nu, \mu) \to (-\nu, \mu). \tag{6.6}$$

The equation of the quotient $X = \hat{X}/\pi$ has the following form:

$$\mu^2 = \lambda^{-2n+1} \prod_{i=1}^{4n-2} (\lambda - \lambda_i), \quad \lambda = \nu^2.$$

It is a hyperelliptic curve of genus $g = 2n-1$, where $\lambda = 0, \infty$ are branch points.

The eigenvector H is equal to

$$H = \begin{pmatrix} 1 \\ \dfrac{\mu - A(\nu)}{B(\nu)} \end{pmatrix}, \tag{6.7}$$

where we have used the following notation for the elements of W:

$$W(\nu) = \begin{pmatrix} A(\nu) & B(\nu) \\ C(\nu) & -A(\nu) \end{pmatrix}.$$

Let us note that due to (6.2)

$$A(\nu) = A(-\nu), \quad B(\nu) = -B(-\nu).$$

After taking account of the normalization of $\hat{\Psi}(\nu)$ and the reduction (5.3), we obtain

$$\sigma_3 \hat{\Psi}(-\nu)\, \sigma_3 = \hat{\Psi}(\nu)$$

and the following transformation law of the ψ-function under the involution (6.6):

$$\sigma_3 \psi(P^\pi, z, \bar{z}) = \sigma_3 \hat{\Psi}(-\nu) \begin{pmatrix} 1 \\ \dfrac{\mu - A(-\nu)}{B(-\nu)} \end{pmatrix} = \psi(P, z, \bar{z}). \tag{6.8}$$

The polar divisor $\hat{\mathscr{D}}$ of ψ is defined by the vector $H(P, 0, 0)$, and consequently does not depend on z, $\bar{z}$. Moreover, it is invariant (6.8) with respect to π. The poles of $H(P, 0, 0)$ are situated at the $2(2n-1)$ zeros of $B(\nu)$. Over each zero of $B(\nu)$ there are situated two points of the curve $\hat{X}$, moreover, for one of these two points the numerator $\mu - A(\nu)$ in (6.7) vanishes too. Thus, we obtain the non-special polar divisor $\hat{\mathscr{D}}$ of degree $2(2n-1)$. Projecting $\hat{\mathscr{D}}$ on X, we get the non-special polar divisor $\mathscr{D}$ of degree $(2n-1)$, coinciding with the genus of X.

Remark. The fact that the genus $g = 2n-1$ of the curve X is odd is of no importance. We consider a singular spectral curve with a singularity at the point $\nu = \nu_0$ (and hence at $\nu = -\nu_0$)

$$\mu = (\nu^2 - \nu_0^2)\, \mu',$$

and let $\pm\nu_0$ be also the points of the polar divisor ψ, that is, $B(\pm\nu_0, 0, 0) = 0$. Because $A^2 = \mu^2 - BC$, $A(\nu)$ reduces to zero at these points too, and consequently

$$H = \begin{pmatrix} 1 \\ \dfrac{\mu' - A'}{B'} \end{pmatrix},$$

where $A = (\nu^2 - \nu_0^2)A'$, $B = (\nu^2 - \nu_0^2)B'$. As a result we obtain the BA function on $\hat{X}'$—the non-singular compactification of the curve $\hat{X}$ given by μ'.

The projection of the polar divisor on the curve $X' = \hat{X}'/\pi$ of genus $2n-2$ is a non-special divisor of degree $2n-2$. All analytic properties, including the transformation law (6.8), are preserved, so for uniformity we shall call a spectral curve *non-singular* in this case too. Other BA functions of singular curves do not lead to CMC tori. They are described in §13.

Let $\infty^{\pm}$ denote the points of the surface $\hat{X}$ with $\mu \to \pm\nu^{2n-1}$, $\nu \to \infty$, and $O^{\pm}$ the two points with $\nu = 0$. The involution π interchanges them:

$$(6.9) \qquad \infty^+ \overset{\pi}{\longleftrightarrow} \infty^-, \quad O^+ \overset{\pi}{\longleftrightarrow} O^-.$$

The function ψ has essential singularities at these points. To clarify their exact form, we shall consider the matrix function

$$\Psi(\nu, z, \bar{z}) = (\psi(P^+, z, \bar{z}), \psi(P^-, z, \bar{z})).$$

It is well defined on the union of neighbourhoods of the points $\nu = \infty$ and $\nu = 0$ in such a way that

$$P^{\pm} \to \infty^{\pm}, \ \nu \to \infty; \ P^{\pm} \to O^{\pm}, \ \nu \to 0.$$

On the other hand, because ψ and H are eigenvectors of W,

$$\psi(P, z, \bar{z}) = H(P, z, \bar{z})\, d(P, z, \bar{z}),$$

where $d(P, z, \bar{z})$ is some scalar function. The formula (6.7) gives the following asymptotics:

$$\Psi(\nu, z, \bar{z}) \to \left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + O(\nu^{-1})\right)\begin{pmatrix} d^+(\nu, z, \bar{z}) & 0 \\ 0 & d^-(\nu, z, \bar{z}) \end{pmatrix}, \quad \nu \to \infty.$$

Substituting it in (5.1), (5.2), we obtain

$$\hat{d}_z = -\frac{i}{2}\sigma_3\nu\hat{d}, \quad \hat{d}_{\bar{z}} = 0, \quad \hat{d} = \begin{pmatrix} d^+ & 0 \\ 0 & d^- \end{pmatrix}.$$

Together with the normalization $d^{\pm}(\nu, 0, 0) \to 1$, $\nu \to \infty$, which follows from definition (6.4), it yields

$$(6.10) \quad \Psi(\nu, z, \bar{z}) \to \left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + B\nu^{-1} + O(\nu^{-2})\right)\exp\left(-\frac{i}{2}\nu z\sigma_3\right), \quad \nu \to \infty.$$

Similarly we obtain

$$\Psi(\nu, z, \bar{z}) \to (Q + O(\nu))\exp\left(-\frac{i}{2\nu}\bar{z}\sigma_3\right), \quad \nu \to 0.$$

The symmetry (6.8) enables us to formulate analytic properties of the BA function as a function on a Riemann surface X that is important for constructing ψ explicitly. We fix that sheet of the covering $\hat{X} \to X$ that contains ∞^+ and O^+. This sheet can be identified with X, cut along the contour $\mathscr{L}$. One should consider that on $X \backslash \mathscr{L}$ there is singled out a one-valued branch of the function ν, two-valued on X, and local parameters $\sqrt{\lambda}$ at points $\lambda = 0, \infty \in X$ are chosen so that $\sqrt{\lambda} = \nu$, $\lambda \to 0, \infty$.

***Theorem* 6.1.** *The stationary solution of the higher sinh-Gordon equation generates a Riemann surface X (spectral curve)*

$$(6.11)\quad \mu^2 = \lambda^{-2n+1} \prod_{i=1}^{2g} (\lambda - \lambda_i), \quad g = 2n-1 \quad or \quad g = 2n-2,$$

of genus g with contour $\mathscr{L}$, *fixing the branch* $\sqrt{\lambda}$ *on* $X \setminus \mathscr{L}$. *The BA function* $\psi = (\psi_1, \psi_2)^T$ *is analytic on X, and*

1) *the functions* ψ_1 *and* $\sqrt{\lambda}\psi_2$ *are one-valued on X,*

2) ψ *has essential singularities of the kind*

$$\psi_{1,2} = (1 + O(1/\sqrt{\lambda})) \exp\left(-\frac{i}{2} z \sqrt{\lambda}\right), \quad \lambda \to \infty,$$
$$\psi_{1,2} = O(1) \exp\left(-\frac{i}{2\sqrt{\lambda}} \bar{z}\right), \quad \lambda \to 0,$$

3) ψ *is meromorphic on* $X \setminus \{\lambda = 0, \infty\}$, *its polar divisor is non-special of degree g and does not depend on* z, $\bar{z}$.

7. The Baker–Akhiezer function. Explicit formulae

The function ψ is defined uniquely by its analytic properties and can be explicitly expressed in terms of Riemann theta functions and Abelian integrals [5], [6], [18]. Let X be a hyperelliptic Riemann surface of genus g (6.11) with branch points $\lambda = 0, \infty$, and let $\mathscr{L}$ be a contour defining a one-valued branch of the function $\sqrt{\lambda}$ on $X \setminus \mathscr{L}$. Let a canonical basis of cycles a_n, b_n, $n = 1, \ldots, g$, be chosen so that

$$\mathscr{L} = a_1 + \ldots + a_g.$$

The normalized ($\int_{a_m} du_n = 2\pi i \delta_{mn}$) holomorphic Abelian differentials du_n define the period matrix $B_{mn} = \int_{b_m} du_n$, with the help of which the Riemann theta function is defined:

$$\theta(z) = \sum_{m \in \mathbb{Z}^g} \exp\left(\frac{1}{2}\langle Bm, m\rangle + \langle z, m\rangle\right), \quad z \in \mathbb{C}^g.$$

This function is periodic with periods $2\pi i \mathbb{Z}^g$:

$$(7.1)\qquad \theta(z + 2\pi i N) = \theta(z).$$

Let us also introduce two normalized ($\int_{a_m} d\Omega_i = 0$) differentials of the second kind with singularities of the following form:

$$(7.2)\qquad d\Omega_1 \to d\sqrt{\lambda}, \quad \lambda \to \infty, \; d\Omega_2 \to -\frac{d\sqrt{\lambda}}{\lambda}, \; \lambda \to 0.$$

We denote their periods by

$$U_n = \int_{b_n} d\Omega_1, \quad V_n = \int_{b_n} d\Omega_2.$$

***Theorem* 7.1.** *The BA function is given by the following formulae*:

$$(7.3)\qquad \begin{cases} \psi_1 = \dfrac{\theta(u+\Omega)\,\theta(D)}{\theta(u+D)\,\theta(\Omega)} \exp\left\{-\dfrac{i}{2}(\Omega_1 z + \Omega_2 \bar z)\right\}, \\ \psi_2 = \dfrac{\theta(u+\Omega+\Delta)\,\theta(D)}{\theta(u+D)\,\theta(\Omega+\Delta)} \exp\left\{-\dfrac{i}{2}(\Omega_1 z + \Omega_2 \bar z)\right\}, \\ U = (U_1, \ldots, U_g),\ V = (V_1, \ldots, V_g),\quad \Delta = \pi i\,(1, \ldots, 1). \end{cases}$$

Here $\Omega = -(i/2)(Uz + V\bar z) + D$, the Abelian map

$$u = \int_\infty^P du, \quad du = (du_1, \ldots, du_g), \quad P = (\lambda, \mu) \in X,$$

D is a vector such that $\theta(u+D)$ has $\mathscr{D}$ as a null divisor. It is arbitrary. The paths of integration in u and Ω_i coincide.

Suppose that $\mathscr{L}$ does not separate the points $\lambda = 0$ and $\lambda = \infty$. Then, substituting (7.3) in the second equation of (5.1) and taking into consideration that

$$\int_\infty^0 du = \Delta, \quad \int_\infty^0 d\Omega_i = 0,$$

we obtain the following theorem.

***Theorem* 7.2.** *The corresponding solution of* (1.13) *is given by the formula*

$$(7.4)\qquad u(z, \bar z) = 2 \ln \frac{\theta(\Omega)}{\theta(\Omega + \Delta)}\,.$$

The choice of the contour $\mathscr{L}$ separating the points 0 and ∞ is equivalent to the change $O^+ \leftrightarrow O^-$ (see §6) and leads to the transformation of (7.4):

$$\bar z \to -\bar z, \quad e^u \to -e^u.$$

It is convenient to represent X with the contour $\mathscr{L}$ in the following way. We shall make the cuts $[\lambda_{2n-1}, \lambda_{2n}]$, and let $\mathscr{L}$ be a union of g contours, each of which coincides with the corresponding cut $[\lambda_{2n-1}, \lambda_{2n}]$ and is invariant with respect to the involution of changing the sheets of X. Then to each value ν there corresponds one point on $X \backslash \mathscr{L}$. Let l be a path on X that does not intersect $\mathscr{L}$ and joins the points ∞ and (ν, μ), and l^* the analogous contour on the other sheet, joining ∞ and $(-\nu, -\mu)$. Let f_l denote the analytic continuation of f along the contour L. Then obviously $\nu_l = \pi^* \nu_{l^*}$, and hence ψ_l and $\sigma_3 \psi_{l^*}$ (see (6.8)) correspond to the same value ν. The matrix-valued function thus defined

$$\Psi(\nu) = (\psi_l, \ \sigma_3 \psi_{l^*})$$

satisfies equations (5.1). This function is analytic on the Riemann surface $\nu = \sqrt{\lambda}$ with the cuts $[\lambda_{2n-1}, \lambda_{2n}]$, on which there is a jump $\Psi_+ = \Psi_- \sigma_1$. For this function the reduction

$$\Psi(-\nu) = \sigma_3 \Psi(\nu)\, \sigma_1$$

holds and the following expression is true:

$$(7.5)\quad \Psi(\nu)=\begin{pmatrix} \dfrac{\theta(\Omega+u)}{\theta(\Omega)} & \dfrac{\theta(\Omega-u)}{\theta(\Omega)} \\ \dfrac{\theta(\Omega+u+\Delta)}{\theta(\Omega+\Delta)} & -\dfrac{\theta(\Omega+u-\Delta)}{\theta(\Omega+\Delta)} \end{pmatrix} \begin{pmatrix} \dfrac{\theta(D)}{\theta(D+u)}e^{\omega} & 0 \\ 0 & \dfrac{\theta(D)}{\theta(D-u)}e^{-\omega} \end{pmatrix},$$

$$\omega = -\frac{i}{2}(\Omega_1 z + \Omega_2 \bar{z}), \quad u = \int_l du.$$

Finally we give also a useful formula for

$$d = \det \Psi(\nu).$$

Studying its analytic properties, we obtain [15]:

$$(7.6)\qquad d = -2\frac{\theta^2(D)}{\theta(0)\,\theta(\Delta)}\,\frac{\theta(u)\,\theta(u+\Delta)}{\theta(u-D)\,\theta(u+D)}.$$

8. Reality

Now we consider z and $\bar{z}$ as complex-conjugate quantities. We shall obtain restrictions on the parameters that ensure the reality of $u(z, \bar{z})$. First of all, the curve possesses the antiholomorphic involution

$$(8.1)\qquad \tau\colon \lambda \to \bar{\lambda}^{-1}.$$

All branch points are situated in pairs ($|\lambda_i| \neq 1$)

$$(8.2)\qquad \bar{\lambda}_{2n-1}^{-1} = \lambda_{2n}.$$

The basis of cycles can be chosen so that τ acts on it in the following way:

$$\tau a_n = -a_n, \quad \tau b_n = b_n - a_n + \sum_{i=1}^{g} a_i.$$

Then the period matrix is equal to

$$B = B_R + \pi i\,(I - 1),$$

where B_R is real-valued, and $1_{mn} = 1$, $I_{mn} = \delta_{mn}$. The theta function is conjugated in the simplest way:

$$\overline{\theta(z)} = \theta(\bar{z}).$$

For differentials of the second kind we have

$$\tau^* d\Omega_1 = \overline{d\Omega_2},$$

whence $\bar{U} = V$, and hence the vector Ω is pure imaginary.

***Theorem* 8.1.** *All real-valued finite-zone solutions (corresdponding to non-singular spectral curves) are given by formula* (7.3), *where X is the real curve* (6.11), *and D is an arbitrary pure imaginary vector.*

This theorem may be proved by the standard technique of [8], and at the same time it can be proved that all solutions are non-singular. All real-valued finite-zone solutions of other real specializations of the sine-Gordon equation were obtained in [2].

The fixed points τ, $|\lambda| = 1$ form one (if g is even) or two (if g is odd) real ovals. The conjugation law of holomorphic differentials $\tau^* du = \overline{du}$ gives for points on the ovals $|\nu| = 1$

$$\overline{\int_\infty^\nu du} = \int_\infty^\nu du - \Delta \pmod{2\pi i\, \mathbb{Z}^g}.$$

The function (7.5) satisfies the reduction

$$(8.3)\quad \Psi = \begin{pmatrix} 0 & -e^{-u/2} \\ e^{u/2} & 0 \end{pmatrix} \overline{\Psi} \begin{pmatrix} 0 & -\alpha^{-1} \\ \bar{\alpha} & 0 \end{pmatrix}, \quad \alpha = \frac{\theta(u-D)}{\theta(u-D+\Delta)}, \quad u = \int_l du.$$

9. Formula for immersion

For constructing formulae for an immersion F and a normal field N, according to the representation (1.11) we need to construct the solution $\Phi_0(\lambda, z, \bar{z}) \in SU(2)$ of equations (1.8) with matrices (3.2). This is easy to do, using the obtained expression (7.5). In fact, the representations (3.2) and (5.2) are gauge equivalent, and

$$(9.1)\qquad \begin{pmatrix} (-i\nu)^{1/2} e^{u/4} & 0 \\ 0 & (-i\nu)^{-1/2} e^{-u/4} \end{pmatrix} \Psi(\nu)$$

is the solution of (1.8) with the required matrices (3.2) and $\lambda = \nu^2$. For $\Phi_0(\nu)$ we have

$$(9.2)\quad \Phi_0 = $$
$$= \begin{pmatrix} (-i\nu)^{1/2} & 0 \\ 0 & (-i\nu)^{-1/2} \end{pmatrix} \begin{pmatrix} i\theta(\Omega+u) & i\theta(\Omega-u) \\ i\theta(\Omega+\Delta+u) & -i\theta(\Omega+\Delta-u) \end{pmatrix} \begin{pmatrix} e^{\omega} & 0 \\ 0 & e^{-\omega} \end{pmatrix} M(\nu),$$

where $u = \int_\infty^\nu du$ is the Abelian map of the point ν (by the chosen contour $\mathcal{L}$ the points of the surface X are in a one-to-one corespondence with the values of ν), and $M(\nu)$ is some matrix depending only on ν. When $\nu = e^{i\gamma}$, the matrices in (9.2) belong to $\mathbb{R}_+ SU(2)$. The determinant (9.2) can be normalized to one with the help of the addition theorem (7.6):

$$\theta(\Omega+u)\theta(\Omega+\Delta-u) + \theta(\Omega-u)\theta(\Omega+\Delta-u) =$$
$$= 2\frac{\theta(u)\theta(u+\Delta)}{\theta(0)\theta(\Delta)}\theta(\Omega)\theta(\Omega+\Delta).$$

***Theorem* 9.1.** *The function*

$$(9.3)\quad \Phi_0 = in \exp\left(i\sigma_3\left(\frac{\gamma}{2}-\frac{\pi}{4}\right)\right) \begin{pmatrix} \theta(\Omega+u) & \theta(\Omega-u) \\ \theta(\Omega+\Delta+u) & -\theta(\Omega+\Delta-u) \end{pmatrix} \exp(\sigma_3\omega),$$
$$n = \left(\frac{\theta(0)\theta(\Delta)}{2\theta(u)\theta(u+\Delta)\theta(\Omega)\theta(\Omega+\Delta)}\right)^{1/2},$$

is a solution of equations (1.8), (3.2) *with* $\lambda = e^{2i\gamma}$, *and* $\Phi_0 \in SU(2)$. *With the help of formulae* (1.11) *it defines a CMC surface.*

A slightly different formula for F_3 was constructed in [15].

10. Periodicity conditions

We shall introduce the notations for real and imaginary parts of the following quantites:

$$U_n = U_n^R + iU_n^I, \quad \int_{\infty}^{\nu=1} d\Omega_1 = c^R + ic^I.$$

***Theorem* 10.1.** *All CMC tori are described by formulae* (1.11), (9.3) *when the immersion F is doubly periodic. The immersion F with the period lattice* Λ *generated by basis vectors* (X_1, Y_1), (X_2, Y_2) *is doubly periodic if and only if*

$$\frac{1}{2\pi}\begin{pmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{pmatrix}\begin{pmatrix} 2c^R & U_1^R \dots & U_g^R \\ -2c^I & -U_1^I \dots & -U_g^I \end{pmatrix} \tag{10.1}$$

is an integer matrix, and the differential $d\Omega_1$ *vanishes at the point* $\lambda = 1$:

$$d\Omega_1\,(\lambda = 1) = 0. \tag{10.2}$$

The numbers

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\pi}\begin{pmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{pmatrix}\begin{pmatrix} c^R \\ -c^I \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \pmod 2 \tag{10.3}$$

determine the parity of the number of "twistors" (§4) *relative to the cycles of the torus, defined by the basis vectors of the lattice.*

Proof. The trivial solution $u = 0$ generates a cylinder, so to prove the first statement it remains to show that singular curves do not lead to CMC tori. We shall do this in §13. Condition (10.1) follows from the periodicity condition (7.1) of the theta function and the fact that Φ_0 may change sign. These changes of sign (see (4.2)) according to §4 determine the parity of the number of "twistors". Finally, we observe that the differentiation of theta functions with respect to γ also gives functions with the lattice Λ, but on differentiating the exponential factor in F there arises the component

$$(rz - \bar{r}\bar{z})\,\sigma_3,$$

where $d\Omega_1 = r d\lambda$ when $\lambda = 1$. The condition that it vanishes on two linearly independent vectors of the lattice is just (10.2).

The periodicity conditions are conditions only on a spectral curve, moreover for the $2g+4$ parameters (branch points of X and basis vectors of Λ) we have $2g+4$ conditions (10.1), (10.2). It is impossible to satisfy (10.2) when $g = 1$, so the simplest CMC tori correspond to the case when $g = 2$. If $g \geqslant 2$ there is a discrete set of spectral curves generating tori.

In [15] we give a non-strict reasoning that verifies this fact. In the recent work [21] Ercolani, Knorrer, and Trubowitz have proved the existence of spectral curves that are strictly of even genus.

The vector D remains arbitrary. Its variation on the plane spanned by the vectors of the b-periods U^R and U^I corresponds to a simple reparametrization of the torus, but the variation in directions transversal to this plane changes the torus.

***Theorem* 10.2.** *The tori constructed from a spectral curve of genus g have $g-2$ commuting deformation flows preserving their area and the CMC property.*

We shall prove the preservation of area under variation of the vector D by calculating it in the next section.

11. Area

Let Π be the fundamental domain of the lattice Λ. The normalization (1.1) enables us to express the area of the CMC torus in the following way:

$$S=\int_{\Pi} |F_x||F_y|\,dxdy = 4\int_{\Pi} e^u dxdy,$$

where Π is the integration domain.

We can compute this integral explicitly. Substituting the asymptotic form (6.10) in the equation $\Psi_{\bar z} = V_1\Psi$, we get for the matrix element B_{21}

$$B_{21\bar z} = e^u/2i.$$

On the other hand, a straightforward calculation gives

$$(11.1)\quad B_{21\bar z} = \frac{\partial}{\partial \bar z}\frac{\partial}{\partial(1/\sqrt{\lambda})}\left[\frac{\theta(\Omega+\Delta+u)\,\theta(D)}{\theta(\Omega+\Delta)\,\theta(D+u)}e^{\omega}\right]\Bigg|_{\lambda=\infty} =$$
$$= \frac{\partial}{\partial \bar z}\frac{\partial}{\partial(1/\sqrt{\lambda})}[\theta(\Omega+\Delta+u)e^{\omega}] = -2i\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z}\ln\theta(\Omega+\Delta) - \frac{i}{2}k,$$

where k is the value of $d\Omega_2$ at $\lambda = \infty : d\Omega_2 = kd(1/\sqrt{\lambda})$. In deducing (11.1) we used the fact that the decompositon of the Abelian map at $\lambda = \infty$ has the form

$$u = -U/\sqrt{\lambda},$$

which is proved in the ordinary way [5], [22] with the help of the reciprocity law of Abelian differentials. The same law enables us to define in a more convenient form the constant

$$(11.2)\qquad d\Omega_1 = kd\sqrt{\lambda}, \quad \lambda \to \infty.$$

Applying Stokes's formula to (11.1), we arrive at the following statement.

***Theorem* 11.1.** *The area of the CMC torus is equal to*

$$S = 4kS\,(\Pi), \tag{11.3}$$

where $S(\Pi)$ is the area of the corresponding fundamental parallelogram Π.

The area, like the periodicity condition, is expressed in terms of the spectral curve.

12. Wente tori

Let us point out the position in the general picture of some examples known earlier. Suppose that the curve X (6.11), along with the hyperelliptic $h : (\lambda, \mu) \to (\lambda, -\mu)$ and the anti-holomorphic τ (8.1), possesses one more additional involution

$$i\colon\ (\lambda,\ \mu) \to (\lambda^{-1},\ \mu\lambda^{2n-1-g}). \tag{12.1}$$

The curve X covers two curves $X' = X/ih$ and $X'' = X/i$ of genus $n-1$ and $g-(n-1)$ respectively. The involution (12.1) transforms the differentials of the second kind into one another:

$$i^* d\Omega_1 = d\Omega_2,$$

so they are combinations of normalized differenitals of the second kind $d\omega'$ and $d\omega''$, defined on the curves X', X'':

$$d\Omega_1 = \frac{1}{2}(d\omega'' + d\omega'), \quad d\Omega_2 = \frac{1}{2}(d\omega'' - d\omega').$$

The dependence on x and y is given by

$$d\Omega_1 z + d\Omega_2 \bar{z} = d\omega'' x + i d\omega' y,$$

so the dynamics with respect to x and y happens on the Jacobians of two different curves, Jac(X'') and Jac(X'). As a consequence, the periodicity conditions with respect to x and y are separated. We deduce that the curvature lines x = const and y = const are closed, and Π is a rectangle ($Y_1 = X_2 = 0$). We note that the same curves define all the solutions of the Dirichlet and Neumann problems for (1.13) on a rectangle [4].

Detailed computations for the case $g = 2$ when both curves X', X'' are elliptic were given in [15]. As a result the simplest tori have been obtained, which were first found by Wente [44] and were then described in terms of elliptic functions by Abresch [11] and Walter [41]. In our approach the formulae are obtained in terms of Jacobi's theta functions with the help of the reduction theory of theta functions of curves with symmetries [3]. Wente tori have a beautiful geometric description: the curvature lines y = const are planar and look like a figure 8. The orthogonal curvature lines x = const lie on spheres. Starting from a geometric property, namely, from the existence of planar curvature lines, these tori were analytically described for the first time in [11].

The problem of studying tori that have only one family of curvature lines lying on spheres leads to the case of a curve X of genus $g = 3$ and symmetry (12.1). Such tori, permitting deformations (§10), are described in [46].

Let us observe that surfaces with such special (planar and spherical) curvature lines were studied by classical authors who knew a lot of analogous results, in particular, the similar case $K = -1$ (see the Introduction).

13. Singular spectral curves

We shall show that singular curves do not lead to doubly periodic immersions. Let the spectral curve $\hat{X}$ be (ν, μ)-singular:

$$(13.1) \qquad \mu = (\lambda - \lambda_1) \ldots (\lambda - \lambda_k)\, \mu', \qquad \lambda = \nu^2,\ \lambda_i = \nu_i^2.$$

At all points $\nu = \pm\nu_i$ the function μ vanishes, and according to (6.7) the values of ψ coincide on the two sheets of the covering $\hat{X} \to \nu$:

$$(13.2) \qquad \psi(\nu_i, \mu_i) = \psi(\nu_i, -\mu_i).$$

In the same manner as before, the analytic properties of the BA function are projected onto the curve $X(\lambda, \mu)$. They can be formulated as analytic properties of ψ on the non-singular curve X' determined by μ' (13.1). We fix a sheet $\sqrt{\lambda}$ on X' as before. The analytic properties 1), 2), 3) of Theorem 6.1 are preserved with the amendment that the degree of the polar divisor $\mathcal{D}$ of ψ is equal to the arithmetic genus of X, namely $g'+k$, where g' is the genus of X'. It follows from (13.2) that the additional property 4) is formulated in terms of Ψ in the following way:

$$(13.3) \qquad \Psi(\nu_i)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$

or, which comes to the same thing,

$$(13.4) \qquad \psi^{(\mathrm{I})}(\lambda_i) = \sigma_3 \psi^{(\mathrm{II})}(\lambda_i),$$

where the values of ψ on the two sheets of the covering $X' \to \lambda$ are denoted by Roman numerals.

***Lemma* 13.1.** *If $u(z, \bar{z})$ is real, then $|\lambda_i| \neq 1$.*

Proof. Suppose that $|\lambda_i| = 1$. This is a fixed point τ, and according to (5.4) for $\Psi(\nu_i)$ we have

$$(13.5) \qquad \Psi = \begin{pmatrix} 0 & -e^{-u/2} \\ e^{u/2} & 0 \end{pmatrix} \overline{\Psi} M,$$

from which it follows that

$$(13.6) \qquad M\overline{M} = -I.$$

The reduction (13.5) shows that $\overline{M}\begin{pmatrix}1\\-1\end{pmatrix}$ is also an eigenvector of $\Psi(\nu_i)$ with zero eigenvalue, hence we have the proportionality

$$M\begin{pmatrix}1\\-1\end{pmatrix}=m\begin{pmatrix}1\\-1\end{pmatrix}.$$

Applying $M\overline{M}$, we obtain

$$M\overline{M}\begin{pmatrix}1\\-1\end{pmatrix}=|m|^2\begin{pmatrix}1\\-1\end{pmatrix},$$

which contradicts (13.6).

The linear space of functions satisfying properties 1), 2), 3) is $k+1$-dimensional. Let ψ_0, ψ_1, ..., ψ_k denote a basis in it. For such a basis we can choose BA functions with different divisors $\mathscr{D}_i' \subset \mathscr{D}$ of degree g'. They are given by the same formulae (7.3) with the same exponential factors, differing only in the values of the free vectors D_i.

***Lemma* 13.2.** *The BA function of the spectral curve* (13.1) *is equal to*

$$\psi=(1-\alpha_1-\ldots-\alpha_k)\,\psi_0+\alpha_1\psi_1+\ldots+\alpha_k\psi_k, \tag{13.7}$$

where α_n are rational functions of $\psi_j^{\mathrm{I}}(\lambda_i)$, $\psi_j^{\mathrm{II}}(\lambda_i)$.

To prove the lemma, we note that (13.4) represents an inhomogeneous linear system on α_n.

Thus, in addition to the dependence on z, $\bar{z}$ in terms of theta functions and the Abelian integral

$$\exp\left[-\frac{i}{2}\int_{\infty}^{\nu=1}(d\Omega_1 z+d\Omega_2\bar{z})\right], \tag{13.8}$$

the function (13.7) of the spectral curve depends rationally also on

$$\exp\left[-\frac{i}{2}\int_{\infty}^{\lambda=\lambda_i}(d\Omega_1 z+d\Omega_2\bar{z})\right].$$

By virtue of Lemma 1, $k = 2s$, and the singularities are situated in pairs $\lambda_{2n-1} = \bar{\lambda}_{2n}^{-1}$.

***Theorem* 13.3.** *The condition that the immersion with period lattice Λ given by* (13.1) *is doubly periodic is the union of the conditions of Theorem* 10.1 *on its non-singular part X' and the condition* (13.8) *of double periodicity of the exponents, $i = 1$, ..., s, with the same lattice Λ.*

In examining the non-singular case we have seen that the periodicity conditions single out a discrete set of spectral curves and fix a period lattice for them. So the conditions (13.8) of double periodicity of exponents with a period lattice defined by X' represent $4s$ real-valued conditions on s complex unknowns λ_1, ..., λ_s. Of course, just as in the non-singular case, this computation of parameters is not a rigorous mathematical proof, but we hope it points out in a convincing manner that there are no tori corresponding to singular curves. This argument holds for the case of CMC tori in S^3, which is discussed in the next chapter.

CHAPTER III

CONSTANT MEAN CURVATURE SURFACES IN S^3 AND H^3

The results of the first two chapters carry over to the case of CMC surfaces in S^3 and H^3. The changes are not of principle, so in this chapter we shall choose a slightly more economic style of exposition, referring the reader to the more detailed reasoning of the previous chapters.

14. Equations of constant mean curvature surfaces in S^3 and H^3

Let $\mathscr{F}$ be a smooth CMC surface in S^3, and

$$F:\ \mathscr{R} \to S^3 \subset \mathbb{R}^4$$

its conformal parametrization. Here $\mathscr{R}$ is a Riemann surface with a complex structure, generated by the metric of $S^3 \subset \mathbb{R}^4$ on $\mathscr{F}$, and F is a unit vector

$$\langle F,\ F\rangle = 1, \tag{14.1}$$

where $\langle a, b\rangle = \sum_{k=0}^{3} a_k b_k$. Let w denote a local complex coordinate, and N a normal vector to the surface

$$\langle F_w,\ N\rangle = \langle F_{\bar{w}},\ N\rangle = \langle F,\ N\rangle = 0, \qquad \langle N,\ N\rangle = 1. \tag{14.2}$$

The conformality of the parametrization gives

$$\langle F_w,\ F_w\rangle = \langle F_{\bar{w}},\ F_{\bar{w}}\rangle = 0,\ \langle F_w,\ F_{\bar{w}}\rangle = 2e^u. \tag{14.3}$$

Introducing the notations

$$A = \langle F_{ww},\ N\rangle,\ \langle F_{w\bar{w}},\ N\rangle = 2He^u, \tag{14.4}$$

we obtain the following Gauss—Weingarten equations:

$$\sigma_w = \mathscr{U}\sigma,\quad \sigma_{\bar{w}} = \mathscr{V}\sigma,\ \sigma = (F,\ F_w,\ F_{\bar{w}},\ N)^{\mathsf{T}}, \tag{14.5}$$

$$\mathscr{U} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & u_w & 0 & A \\ -2e^u & 0 & 0 & 2He^u \\ 0 & -H & -Ae^{-u}/2 & 0 \end{pmatrix},\quad \mathscr{V} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -2e^u & 0 & 0 & 2He^u \\ 0 & 0 & u_{\bar{w}} & \bar{A} \\ 0 & -\bar{A}e^{-u}/2 & -H & 0 \end{pmatrix}.$$

The quantity H in (14.4) is called the *mean curvature*. The condition for it to be constant

$$H = \mathrm{const}$$

gives the following GPC equations:

$$(14.6)\qquad u_{w\bar{w}} + 2(1+H^2)e^u - A\bar{A}e^{-u}/2 = 0, \quad A_{\bar{w}} = 0.$$

In new variables

$$(14.7)\qquad z = \delta_S w,\ A = \delta_S e^{2i\varphi} Q, \quad \delta_S = 2\sqrt{1+H^2}$$

(with any constant φ, which we introduce according to technical considerations) (14.6) has the form (1.6), and we obtain a result completely analogous to Theorem 1.1.

There is an analogue of Theorem 1.2 too. To formulate it we shall identify the linear space $\mathbb{R}^4$ with the linear space $\mathbb{R}SU(2)$ of (2×2)-matrices satisfying the condition

$$(14.8)\qquad X = \sigma_2 \bar{X} \sigma_2$$

in the following way:

$$(14.9)\qquad X = X_0 I + i\sum_{k=1}^{3} X_k\sigma_k \leftrightarrow X = (X_0, X_1, X_2, X_3) \in \mathbb{R}^4.$$

We denote by F and N the matrices obtained in this way from the vectors F and N.

***Theorem* 14.1.** *Let $u(z, \bar{z})$ be a solution of* (1.6) *and let*

$$(14.10)\qquad \Phi_1 = \Phi(z, \bar{z}, \lambda = e^{2i\gamma_1}), \quad \Phi_2 = \Phi(z, \bar{z}, \lambda = e^{2i\gamma_2})$$

be two solutions of the system (1.8) (1.9), *belonging to the group* $\mathbb{R}_+SU(2)$ *with coinciding determinants* $\det\Phi_1 = \det\Phi_2$. *Then F and N, defined by the formulae*

$$(14.11)\qquad F = \Phi_1^{-1}\Phi_2, \qquad N = i\Phi_1^{-1}\sigma_3\Phi_2$$

in the variables (14.7), $2\varphi = \gamma_1 + \gamma_2$, *satisfy* (14.5) *and describe in S^3 the CMC surface*

$$(14.12)\qquad H = \cot p,\ p = \gamma_1 - \gamma_2.$$

Proof. For F and N defined by means of (14.11), (14.8) obviously holds. To prove (14.5) it is convenient to define the scalar product in $\mathbb{R}^4$ on the matrices (14.9):

$$(14.13)\qquad \langle X, Y\rangle = \frac{1}{2}\operatorname{tr}(XY^*),\ Y^* = \sigma_2 Y^{\top}\sigma_2.$$

We shall verify, for example, that the surface (14.11) lies in S^3:

$$\langle F, F\rangle = \frac{1}{2}\operatorname{tr}(\Phi_1^{-1}\Phi_2\Phi_2^*(\Phi_1^*)^{-1}) = 1,$$

where we have used the equality of determinants and the fact that $\Phi^* = \Phi^{-1}$ for $\Phi \in U(2)$. It is also easy to establish the truth of (14.2), (14.3), in particular,

$$\langle F_z, F_{\bar{z}}\rangle = \frac{1}{2}\operatorname{tr}((U_2 - U_1)\sigma_2(V_2 - V_1)^{\top}\sigma_2) = \frac{e^u}{2}\sin^2(\gamma_1 - \gamma_2),$$

$$U_k = U(\lambda = e^{2i\gamma_k}), \qquad V_k = V(\lambda = e^{2i\gamma_k}).$$

A straightforward calculation also gives

$$\langle F_{zz}, N\rangle = \frac{1}{2} Qe^{2i\varphi} \sin p,$$

$$F_{z\bar{z}} = -\frac{1}{2} e^u \sin^2 p\, F + \frac{1}{2} \sin p \cos p\, e^u N.$$

In the variables (14.7) this is equivalent to the statement of the theorem if H is given by (14.12).

The hyperbolic space H^3

$$\{F, F\} = -1, \tag{14.14}$$

is embedded in the Lorentz space $\mathbb{R}^{3,1}$ with the metric

$$\{a, b\} = a_1b_1 + a_2b_2 + a_3b_3 - a_0b_0,$$

which is positive on H^3. We normalize the complex coordinate w and the basis in the same way as for S^3:

$$\begin{aligned} &\{N, F\} = \{N, F_w\} = \{N, F_{\bar{w}}\} = \{F_w, F_w\} = \{F_{\bar{w}}, F_{\bar{w}}\} = 0, \\ &\{N, N\} = 1, \quad \{F_w, F_{\bar{w}}\} = 2e^u. \end{aligned} \tag{14.15}$$

Equations (14.5), imposed on the basis, are slightly different:

$$\mathcal{U} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & u_w & 0 & A \\ 2e^u & 0 & 0 & 2He^u \\ 0 & -H & -Ae^{-u}/2 & 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 2e^u & 0 & 0 & 2He^u \\ 0 & 0 & u_{\bar{w}} & \bar{A} \\ 0 & -\bar{A}e^{-u}/2 & -H & 0 \end{pmatrix},$$

$$A = \{F_{ww}, N\}, \; 2He^u = \{F_{w\bar{w}}, N\}. \tag{14.16}$$

The condition for the mean curvature to be constant, H = const, leads to the following GPC equatons:

$$u_{w\bar{w}} + 2\,(H^2 - 1)\, e^u - A\bar{A}e^{-u}/2 = 0, \qquad A_{\bar{w}} = 0.$$

According to the maximum principle there do not exist compact CMC surfaces with $|H| \leqslant 1$, so we shall restrict our attention to the case $|H| > 1$. In the variables

$$z = \delta_H w, \; A = \delta_H e^{2i\varphi} Q, \; \delta_H = 2\sqrt{H^2 - 1} \tag{14.17}$$

we again obtain (1.6).

***Theorem* 14.2.** *A CMC surface in S^3 or in H^3 (with $|H| > 1$) under the conformal parametrization generates the holomorphic quadratic differential $Q(dz)^2$ and a solution of* (1.6).

We shall identify the Lorentz space $\mathbb{R}^{3,1}$ with the space of (2×2) self-adjoint matrices $\overline{X}^{\mathsf{T}} = X$

$$X = X_0 I + \sum_{k=1}^{3} X_k \sigma_k \leftrightarrow X = (X_0, X_1, X_2, X_3) \in \mathbb{R}^{3,1} \tag{14.18}$$

with the scalar product

$$\{X,\ Y\} = -\frac{1}{2}\operatorname{tr}(X\sigma_2 Y^{\mathsf T}\sigma_2).$$

As before, we shall use the notations F and N for describing surfaces in the matrix representation.

***Theorem* 14.3.** *Let* $u(z, \bar z)$ *be a solution of* (1.6) *and let*

$$\Phi_0\,(z,\ \bar z,\ \lambda = e^q e^{2i\varphi}), \qquad q \in \mathbb{R}, \tag{14.19}$$

be some solution of (1.8) *with real determinant. Then*

$$\begin{cases} F = \Phi_0^{-1}\sigma_2\overline{\Phi}_0\sigma_2, \\ N = \Phi_0^{-1}\sigma_3\sigma_2\overline{\Phi}_0\sigma_2 \end{cases} \tag{14.20}$$

in the variables (14.17) *satisfy the equations on a moving basis with matrices* (14.16) *and describe the CMC surface in* H^3

$$H = \coth q. \tag{14.21}$$

Proof. The formulae (14.20) are completely similar to (14.11) if we take into account the adjointness reduction for Φ (§1) and use the notations

$$\Phi_1 = \Phi_0, \qquad \Phi_2 = \sigma_2\overline{\Phi}_0\sigma_2 = \Phi\,(z,\ \bar z,\ \lambda = e^{-q}e^{2i\varphi}).$$

Applying the equality $\sigma_2 X^{\mathsf T}\sigma_2 = X^{-1}\det X$ for invertible matrices, we easily obtain all the equalities (14.15). In particular,

$$\{F_z, F_{\bar z}\} = -\frac{1}{2}\operatorname{tr}((U_2 - U_1)\,\sigma_2\,(V_2 - V_1)^{\mathsf T}\sigma_2) = \frac{e^u}{2}\sinh^2 q,$$

$$U_k = U\,(\lambda = \exp\,[(-1)^{k+1}q + 2i\varphi]), \quad V_k = V\,(\lambda = \exp\,[(-1)^{k+1}q + 2i\varphi]).$$

A straightforward calculation gives

$$\{F_{zz}, N\} = \frac{1}{2}Qe^{2i\varphi}\sinh q,$$

$$F_{z\bar z} = \frac{1}{2}e^u\sinh^2 q\,F + \frac{1}{2}\sinh q\cosh q\,e^u N.$$

The change of variables (14.17) completes the proof.

15. Constant mean curvature spheres in S^3 and H^3

A direct analogue of Hopf's theorem (§2) holds.

***Theorem* 15.1.** *All surfaces of CMC* H *in* S^3 *that are topologically equivalent to a sphere are the intersection of* S^3 (14.1) *by the hyperplanes*

$$\langle F,\ C\rangle = 1, \qquad \langle C,\ C\rangle = H^{-2} + 1. \tag{15.1}$$

All surfaces of CMC $H(|H| > 1)$ *in* H^3 *that are topologically equivalent to a sphere are the intersection of* H^3 (14.14) *by the hyperplanes*

$$\{F,\ C\} = -1,\ \{C,\ C\} = H^{-2} - 1. \tag{15.2}$$

Proof. The holomorphic quadratic differential on the sphere $\mathcal{R}$ is equal to zero: $A = 0$. The last rows of (14.5) give

$$N + HF = HC = \text{const}, \tag{15.3}$$

from which we obtain (15.1),

$$1 = \langle N, N \rangle = H^2 \langle F - C, F - C \rangle \tag{15.4}$$

and the equalities (15.2) for H^3. Spheres of CMC H in S^3 may be represented just like the intersection of S^3 with the sphere (15.4) of radius H^{-1}, which is orthogonal to (14.1).

16. Constant mean curvature tori in S^3

Let

$$W_1 = X_1 + iY_1, \quad W_2 = X_2 + iY_2 \tag{16.1}$$

be two vectors on the plane w, generating the basis of the lattice Λ of periods of the immersion $F(w, \bar{w})$. We normalize the coordinate w in the same way as in (1.6): $Q = 1$.

We consider separately the zero solution of (1.13), which is, of course, doubly periodic too. To this solution there corresponds

$$\Phi(z, \bar{z}, \lambda = e^{2i\gamma}) = \frac{i}{\sqrt{2}} \exp\left[i\sigma_3\left(\frac{\gamma}{2} - \frac{\pi}{4}\right)\right]\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \exp\left[-\frac{i\sigma_3}{2}(e^{i\gamma}z + e^{-i\gamma}\bar{z})\right].$$

According to (14.10), (14.11) the corresponding surface is described by the expression

$$F(w, \bar{w}) = \exp\left[\frac{i\delta_S}{2}\sigma_3(e^{i\gamma_1}w + e^{-i\gamma_1}\bar{w})\right]\begin{pmatrix} \cos\frac{p}{2} & i\sin\frac{p}{2} \\ i\sin\frac{p}{2} & \cos\frac{p}{2} \end{pmatrix} \times$$
$$\times \exp\left[-\frac{i\delta_S}{2}\sigma_3(e^{i\gamma_2}w + e^{-i\gamma_2}\bar{w})\right], \tag{16.2}$$

where, as later on, $\gamma_1 = \varphi + p/2$, $\gamma_2 = \varphi - p/2$. Clearly, (16.2) is a torus. We denote

$$\begin{cases} u_k = \displaystyle\int_\infty^{\nu_k = \exp(i\gamma_k)} du, \quad \omega_k = -\frac{i\delta_S}{2}((c_k^R + ic_k^I)w + (c_k^R - ic_k^I)\bar{w}), \\ \Omega_1(\nu = e^{i\gamma_k}) = c_k^R + ic_k^I, \quad \Omega_2(\nu = e^{i\gamma_k}) = c_k^R - ic_k^I. \end{cases} \tag{16.3}$$

In a completely similar manner as in the case of Euclidean space, combining (14.10), (14.11), (9.3), (14.12), (14.7), we can prove the following theorem.

***Theorem* 16.1.** *All CMC tori in* S^3 *are described by* (16.2) *and also the doubly periodic matrices* (14.9)

$$(16.4)\quad F = m \exp(-\sigma_3\omega_1)\begin{pmatrix}\theta(\Omega+\Delta-u_1) & \theta(\Omega-u_1)\\ \theta(\Omega+\Delta+u_1) & -\theta(\Omega+u_1)\end{pmatrix}\times$$
$$\times\exp\left(-\frac{ip}{2}\sigma_3\right)\begin{pmatrix}\theta(\Omega+u_2) & \theta(\Omega-u_2)\\ \theta(\Omega+\Delta+u_2) & -\theta(\Omega+\Delta-u_2)\end{pmatrix}\exp(\sigma_3\omega_2),$$

$$m = \frac{\theta(0)\,\theta(\Delta)}{2\theta(\Omega)\,\theta(\Omega+\Delta)\,[\theta(u_1)\,\theta(u_2)\,\theta(u_1+\Delta)\,\theta(u_2+\Delta)]^{1/2}},$$

$$\Omega = -\frac{i}{2}\delta_S\,(Uw+\overline{U}\overline{w})+D$$

and by the analogous formula for N, in which the central matrix should be replaced by

$$\sigma_3\exp\left(-\frac{ip}{2}\sigma_3\right).$$

The immersion (16.4) *is doubly periodic if and only if there is a basis* (16.1) *of the lattice* Λ *on the plane* w *such that*

$$(16.5)\quad \frac{\delta_S}{2\pi}\begin{pmatrix}X_1 & Y_1\\ X_2 & Y_2\end{pmatrix}\begin{pmatrix}c_1^R+c_2^R & c_1^R-c_2^R & U_1^R\dots & U_g^R\\ -c_1^I-c_2^I & -c_1^I+c_2^I & -U_1^I\dots & -U_g^I\end{pmatrix}$$

is an integer matrix.

The numbers

$$(16.6)\quad \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{\delta_S}{\pi}\begin{pmatrix}X_1 & Y_1\\ X_2 & Y_2\end{pmatrix}\begin{pmatrix}c_1^R\\ -c_1^I\end{pmatrix}+\begin{pmatrix}1\\ 1\end{pmatrix}\pmod 2$$

determine the parity of the number of "twistors" corresponding to the cycles of the torus given by the basis vectors of the lattice. The area of the torus is

$$(16.7)\quad S = \frac{4k}{\delta_S^2}\begin{vmatrix}X_1 & Y_1\\ X_2 & Y_2\end{vmatrix}.$$

The periodicity condition (16.5) is the condition on a spectral curve that singles out a set of the type of the set of rational points. Apparently, in a small neighbourhood of any spectral curve there is a spectral curve satisfying the periodicity conditions (16.5). Thanks to the free vector D there are $g-2$ commuting deformation flows that preserve the area.

17. Minimal tori in S^3 and Willmore tori

A particular and most interesting case of CMC surfaces in S^3 are minimal surfaces. All minimal tori are described by the formulae of §16, where we should put $H = 0$. This implies that

$$\delta_S = 2,\quad p = \frac{\pi}{2},\quad \gamma_1 = \varphi+\frac{\pi}{4},\quad \gamma_2 = \varphi-\frac{\pi}{4},$$

that is, as points of the spectral parameter $\lambda_k = e^{2i\gamma_k}$ we should take two opposite points on the unit circle.

The simplest of the minimal tori is the Clifford torus T_{Cl}

$$F_0^2 + F_3^2 = F_1^2 + F_2^2 = \frac{1}{2}.$$

Setting $p = \pi/2$, $\varphi = 0$ in (16.2), we obtain

$$F_0 + iF_3 = \frac{1}{\sqrt{2}} e^{-\sqrt{2}(w-\bar{w})}, \quad F_1 + iF_2 = \frac{1}{\sqrt{2}} e^{-i\sqrt{2}(w+\bar{w})}.$$

The area of this torus is equal to

$$S(T_{Cl}) = 2\pi^2.$$

Minimal surfaces in S^3 are connected with Willmore surfaces in three-dimensional Euclidean space [47]. Let $F : M^2 \to \mathbb{R}^3$ be an immersion of an abstract surface of the given topology. Willmore surfaces are defined as extremals of the functional

$$W = \int_{M^2} H^2 \, dS, \tag{17.1}$$

where H is the mean curvature, and dS is the area functional. By virtue of the Gauss–Bonnet theorem, the functional (17.1) is equivalent to the functional

$$\int_{M^2} (k_1^2 + k_2^2) \, dS,$$

hence the most flattened surfaces of the given topology are global minima of the Willmore functional.

To each minimal surface in S^3 there corresponds a Willmore surface.

***Theorem* 17.1** [43]. *Let R be a minimal surface in S^3, and $\sigma : S^3 \to \mathbb{R}^3$ a stereographic projection. Then $\sigma(R)$ is a Willmore surface, and*

$$W(\sigma(R)) = S(R),$$

where $S(R)$ is the area of a minimal surface in S^3.

There are a lot of interesting results on Willmore surfaces. First of all, for any compact surface $W \geqslant 4\pi$, and equality holds only for the standard sphere [47]. For the Clifford torus we obviously have $W(\sigma(T_{Cl})) = 2\pi^2$.

Willmore's conjecture. *For tori $W \geqslant 2\pi^2$.*

Though this conjecture has recently attracted a great deal of attention, it remains unproven. It has been verified for tori generated by the motion of the centre of a ball of varying radius along a closed curve [24], and also for some conformal types of tori [33]. Simon in [38] proved the existence of a torus minimizing W. The absence of self-intersections have been proved by Li and Yau [33]. Their result is the following: if a surface possesses a self-intersection point of order n (n different pre-images), then $W \geqslant 4\pi n$. As a consequence, all surfaces with $W < 8\pi$ are necessarily embeddings, and there are similar results on areas of minimal surfaces in S^3.

The first results in the direction of proving Willmore's conjecture with the help of the analytic approach described in this survey were obtained by Ferus, Pedit, Pinkall, and Sterling [23]. Their results enable us to classify Willmore umbilic-free tori with the help of a special reduction of the two-dimensional Toda chain. Perhaps it will be easier to prove the following particular case of Willmore's conjecture.

Conjecture. *The area of any minimal torus in S^3 is greater than or equal to $2\pi^2$.*

This conjecture, according to the above-mentioned result of Li and Yau, is a consequence of the following conjecture.

Conjecture of Hsiang and Lawson. *The Clifford torus is the only embedded minimal torus in S^3.*

In this connection we note the result of [16], where the upper bound 16π for the area of an embedded minimal torus in S^3 is obtained.

In a proof of the last conjecture by topological considerations one may reject straightaway one quarter of minimal tori, which certainly have self-intersections. The classification of immersions of tori in S^3 with respect to regular homotopies is exactly the same as in the Euclidean case (§4). There are two types: "standard" and "knotted" tori. The latter must have self-intersections. It is characterized by the property that both "bands" of the normal corresponding to the cycle basis have an odd number of "twistors" $\alpha = \beta = 1 \pmod 2$ (16.6). In this case the functions Φ_1 and Φ_2 are doubly periodic. Expressions convenient for calculations for all the quantities contained in the formulae given above are obtained with the help of the Schottky uniformization in [15].

18. Constant mean curvature tori in H^3

In the same way as in the case of S^3 we normalize the coordinate w so that the GPC equation, which has the form (1.6) in the variables (14.17), reduces to (1.13), that is, we set $Q = 1$. According to (14.21) we have

$$\lambda_H = e^q e^{2i\varphi}, \quad e^q = \sqrt{\frac{H+1}{H-1}}, \quad \nu_H = e^{q/2} e^{i\varphi}.$$

To the trivial solution $u = 0$ there corresponds the expression

$$\Phi_0 = \frac{i}{2} \exp\left[\sigma_3\left(\frac{q}{4} + \frac{i\varphi}{2} - \frac{i\pi}{4}\right)\right] \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \exp\left[-\frac{i\delta_H}{2}\sigma_3(\nu_H w + \nu_H^{-1}\bar{w})\right].$$

The corresponding immersion is not doubly periodic. We introduce the notations

$$u = \int_{\infty}^{\nu_H} du, \quad \omega = -\frac{i\delta_H}{2}(\Omega_1(\nu_H)w + \Omega_2(\nu_H)\overline{w}),$$

$$\Omega = -\frac{i\delta_H}{2}(Uw + \overline{U}\overline{w}) + D, \quad \Omega_1(\nu_H) = c^R + ic^I.$$

***Theorem* 18.1.** *All CMC tori* ($|H| > 1$) *in* H^3 *are described by the doubly periodic matrices* (14.20), *where*

$$(18.1) \quad \Phi_0 = in \exp\left[\sigma_3\left(\frac{q}{4} + \frac{i\varphi}{2} - \frac{i\pi}{4}\right)\right] \times \\ \times \begin{pmatrix} \theta(\Omega+u) & \theta(\Omega-u) \\ \theta(\Omega+\Delta+u) & -\theta(\Omega+\Delta-u) \end{pmatrix} \exp(\sigma_3\omega)$$

is a matrix of $SL(2, \mathbb{C})$, *and n is given by* (9.3). *The immersion* (14.20), (18.1) *is doubly periodic with the period lattice* Λ *if and only if there is a basis* (16.1) *of* Λ *on the plane w such that*

$$(18.2) \qquad \frac{\delta_H}{2\pi}\begin{pmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{pmatrix}\begin{pmatrix} 2c^R & U_1^R \ldots U_g^R \\ -2c^I & -U_1^I \ldots -U_g^I \end{pmatrix}$$

is an integer matrix, and

$$(18.3) \qquad \int_{\overline{\nu}_H^{-1}}^{\nu_H} d\Omega_1 = 0.$$

Proof. Condition (18.3) is the consequence of the argument of the exponent on the lattice being pure imaginary and the fact that $\Omega_2(\nu_H) = \overline{\Omega}_1(\overline{\nu}_H^{-1})$. For a complete proof of the theorem it remains to show that singular spectral curves do not generate tori. The proof given in §13 is also true for the case of H^3, excluding spectral curves with singularities at the points ν_H and $\overline{\nu}_H^{-1}$, because the corresponding exponents (13.8) in the last case are doubly periodic (18.2). However, according to (13.3), both the columns of the matrix Ψ, and according to (9.1) of Φ, coincide in this case. In the neighbourhood $\nu \sim \nu_H$, Ψ has the representation

$$(18.4) \qquad \Psi = \Psi_0\begin{pmatrix} \nu-\nu_H & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

where Ψ_0 is invertible. In this case the matrix Φ is constructed in accordance with (9.1), replacing Ψ by Ψ_0. From (18.4) we get the following expression for Ψ_0:

$$(18.5) \qquad \Psi_0 = \left(\frac{\partial\psi}{\partial\nu}, \psi\right)\Big|_{\nu=\nu_H},$$

where ψ is the BA function (13.7). The first column, besides the usual theta functions and exponential factors, contains a term linearly dependent on w, $\bar{w}$ that has the values of differentials in ν_H as the coefficients:

$$-\frac{i}{2}(d\Omega_1(\nu_H)\, w + d\Omega_2(\nu_H)\, \bar{w}).$$

If we require this term to be doubly periodic, then

$$d\Omega_1(\nu_H) = d\Omega_2(\nu_H) = 0. \tag{18.6}$$

The additional condition (18.6) shows that doubly periodic immersions do not exist in the case under consideration.

CMC-tori in S^3 and H^3, similar to Wente tori, were studied in [15], [42]. These tori exist for any H and have spherical curvature lines.

19. Minimal surfaces of higher genus in S^3

All the considerations of §4 carry over directly to the case of minimal surfaces of higher genus $G \geqslant 2$ in S^3. In (14.11) Φ_1 and Φ_2 are the (1/2)-differentials (4.1), and due to the fact that $\Phi_1^{-1}\Phi_2$ is one-valued on $\mathcal{R}$, they define the same spinor structure $[\alpha, \beta]$ on $\mathcal{R}$. This structure, in turn, is defined by the monodromy Φ_i round the canonical basis of the cycles a_n, b_n:

$$\Phi_i \to (-1)^{\alpha_n+1}\Phi_i, \qquad \Phi_i \to (-1)^{\beta_n+1}\Phi_i.$$

The numbers α_n and β_n determine the parity of the number of "twistors" of the corresponding "bands" of the normal, and the parity of the theta-characteristic $[\alpha, \beta]$ determines the class of immersion with respect to regular homotopies.

The condition for the monodromy F to be trivial, given by (14.11), represents $12(G-1)$ conditions on the set of parameters $\mathcal{P}$, singling out in it a discrete set, apparently.

Quite a lot of examples of minimal surfaces of higher genus in S^3 have been constructed [29], [32]. All of them have large groups of symmetries. In particular, Lawson constructed a sequence of minimal embeddings $\xi_{g,1}$ of any genus g, including the Clifford torus as $\xi_{1,1}$.

Kusner's conjecture. *Willmore surfaces derived from $\xi_{g,1}$ under the stereographic projection $\sigma : S^3 \to \mathbb{R}^3$ are global minima of the functional W (17.1) among surfaces of genus g.*

It would be very interesting to obtain a systematic analytic description of at least one of these minimal surfaces.

APPENDIX

PAINLEVÉ PLANES OF CONSANT MEAN CURVATURE

Modern advances in the theory of integrable equations enable us to construct a sequence of CMC planes, including the ones with an umbilical point of arbitrary order. The metric of these planes depends only on $|z|$, and equation (1.6) reduces to the third Painlevé equation. The solutions of this equation in which we are interested, together with the corresponding functions, were described by Its and Novokshenov [27]. Their results enable us to obtain the asymptotic description of the planes. The problem of constructing planes of CMC was raised and discussed in a series of works [30], [40]. In particular, in [40] it was proved that the immersions of planes described by us in this appendix, with intrinsic symmetry with respect to rotations, are proper ones, however, the detailed classification of the planes has not been obtained.

Let $z = 0$ be an umbilical point of order m (including the case $m = 0$, when there are no umbilical points). We select the coordinate z so that

$$Q = z^m.$$

The GPC equation

$$u_{z\bar{z}} + \frac{1}{2}(e^u - |z|^{2m} e^{-u}) = 0 \tag{A.1}$$

in this case has smooth solutions, depending only on $|z|$.

Let us describe these solutions. After the change

$$e^u = e^v \left| \frac{m+2}{2} w \right|^{\frac{2m}{m+2}}, \quad z = \left(\frac{m+2}{2} w \right)^{\frac{2}{m+2}} \tag{A.2}$$

(A.1) reduces to the elliptic sinh-Gordon equation

$$v_{w\bar{w}} + \sinh v = 0. \tag{A.3}$$

The solutions, depending only on $|w|$:

$$w = \frac{1}{2}\rho e^{i\beta}, \tag{A.4}$$

satisfy the third Painlevé equation

$$v_{\rho\rho} + \frac{1}{\rho} v_\rho + \sinh v = 0. \tag{A.5}$$

In accordance with (A.2), (A.4) we are interested in the solutions $v(\rho)$ with the the following singularity:

$$v = -\frac{2m}{m+2} \ln \rho + \frac{2m}{m+2} \ln \frac{4}{m+2} + u(0), \quad \rho \to 0. \tag{A.6}$$

By the conditon $Q(dz)^2 = z^m(dz)^2$ the system of coordinates on the z-plane is fixed to within the rotation $z \to ze^{2\pi i/(m+2)}$. This shows that the plane is symmetric with respect to the rotation through the angle $2\pi/(m+2)$.

Let us give some necessary results from the theory of the third Painlevé equation, which were proved in [27]. (A.5) is the compatibility condition of the system

$$\text{(A.7)}\qquad \begin{cases} \Psi_\Lambda = A\Psi, \\ A = -\dfrac{i\rho^2}{16}\sigma_3 - \dfrac{1}{2\Lambda}\dfrac{\rho v_\rho}{2}\sigma_1 + \dfrac{1}{\Lambda^2}(i\cosh v\sigma_3 - \sinh v\sigma_2), \\ \Psi_\rho = \left\{-\dfrac{i\rho}{8}\sigma_3\Lambda - \dfrac{v_\rho}{2}\sigma_1\right\}\Psi. \end{cases}$$

Let us define the following regions on the plane Λ:

$$\hat\Omega_1^{(\infty)} = \{\Lambda:\ -\pi < \arg\Lambda < \pi,\ |\Lambda| > R > 0\},$$
$$\hat\Omega_1^{(0)} = \{\Lambda:\ -\pi < \arg\Lambda < \pi,\ 0 \neq |\Lambda| < R > 0\},$$
$$\hat\Omega_+ = \{\Lambda:\ 0 < \arg\Lambda < \pi,\ \Lambda \neq 0\},$$

as well as the canonical solutions $\Psi_1^{(\infty)}$, $\Psi_1^{(0)}$, Ψ_+, fixed by their asymptotics in these regions:

$$\text{(A.8)}\qquad \begin{cases} \Psi_1^{(\infty)} = (I + O(\Lambda^{-1}))\exp\left\{-\dfrac{i\rho^2}{16}\sigma_3\Lambda\right\}, & \Lambda\in\hat\Omega_1^{(\infty)}, \\ \Psi_1^{(0)} = B^{(0)}(I + O(\Lambda))\exp\{-i\sigma_3/\Lambda\}, & \Lambda\in\hat\Omega_1^{(0)}, \\ \Psi_+ = (I + O(\Lambda^{-1}))\exp\left\{-\dfrac{i\rho^2}{16}\sigma_3\Lambda\right\}, & \Lambda\in\hat\Omega_+, \\ \Psi_+ = B_+(I + O(\Lambda))\exp\{-i\sigma_3/\Lambda\}, & \Lambda\in\hat\Omega_+, \\ B^{(0)} = \begin{pmatrix} \cosh\frac{\nu}{2} & -\sinh\frac{\nu}{2} \\ -\sinh\frac{\nu}{2} & \cosh\frac{\nu}{2}\end{pmatrix}. \end{cases}$$

The connection matrix

$$\text{(A.9)}\qquad Q = [\Psi_1^{(0)}]^{-1}\Psi_1^{(\infty)}$$

does not depend on ρ. For real υ the matrices Q, B_+, $[\Psi_1^{(0)}]^{-1}\Psi_+$ have the following structure:

$$\text{(A.10)}\qquad \begin{cases} Q = \dfrac{1}{\sqrt{1-|p|^2}}\begin{pmatrix}1 & p\\ \bar p & 1\end{pmatrix}, \quad T = \begin{pmatrix}(1-|p|^2)^{-1/4} & 0 \\ 0 & (1-|p|^2)^{1/4}\end{pmatrix}, \\ B^{(0)} = B_+T^2, \quad [\Psi_1^{(0)}]^{-1}\Psi_+ = \begin{pmatrix}1 & p\\ 0 & 1\end{pmatrix}T^{-2}. \end{cases}$$

Let us note that the asymptotics (A.8) fix uniquely the corresponding solutions of the system (A.7). Henceforth, for us the solution

$$\Psi_+(\rho,\ \Lambda)$$

will be central, because the following theorem is true.

***Theorem* A.1.** *The matrix*

$$\text{(A.11)}\quad \Phi(w, \bar{w}, \lambda = e^{2i\gamma}) = = \frac{i}{\sqrt{2}} \exp\left[i\sigma_3\left(\frac{\gamma}{2} - \frac{\pi}{4}\right)\right] \begin{pmatrix} e^{v/4} & 0 \\ 0 & e^{-v/4} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Psi_+\left(\rho, \frac{4}{\rho} e^{i(\beta+\gamma)}\right) TS,$$

where $S \in SU(2)$ is an arbitrary constant matrix, is a solution of the equations

$$\Phi_w = \begin{pmatrix} -\frac{v_w}{4} & -\frac{\lambda}{2} e^{v/2} \\ \frac{1}{2} e^{-v/2} & \frac{v_w}{4} \end{pmatrix} \Phi, \quad \Phi_{\bar{w}} = \begin{pmatrix} \frac{v_{\bar{w}}}{4} & -\frac{1}{2} e^{-v/2} \\ \frac{1}{2\lambda} e^{v/2} & -\frac{v_{\bar{w}}}{4} \end{pmatrix} \Phi,$$

belonging to the group $SU(2)$. Formulae (1.11)

$$\text{(A.12)}\qquad F = \Phi^{-1}\Phi_\gamma |_{\gamma=0}, \quad N = \Phi^{-1} \frac{\sigma_3}{2i} \Phi |_{\gamma=0}$$

define a CMC surface.

The proof is given by a straightforward calculation. The reduction

$$\sigma_2 \bar{\Phi} \sigma_2 = \Phi$$

is deduced from the corresponding relation between the functions $\Psi_+(\rho, \Lambda)$ and $\Psi_+\left(\rho, \left(\frac{4}{\rho}\right)^2 \bar{\Lambda}^{-1}\right)$:

$$\overline{\Psi_+(\rho, \Lambda)}\, T^2 = B_0 \sigma_2 \Psi_+\left(\rho, \left(\frac{4}{\rho}\right)^2 \bar{\Lambda}^{-1}\right) \sigma_2.$$

The last equality is a consequence of the fact that both its sides satisfy the system (A.7) and the solution is uniquely fixed by the asymptotics (A.8). Theorem 1.2 (including the modified matrices (3.2)) does not depend on the choice of local variable on a Riemann surface, which proves (A.12).

The symmetry

$$\text{(A.13)}\qquad \overline{\Psi_+(\rho, \Lambda)} = \Psi_+(\rho, -\bar{\Lambda}),$$

which also follows from (A.7) and (A.8), shows that besides the symmetry mentioned above with respect to the rotation through the angle $2\pi/(m+2)$, the surface is symmetric with respect to the reflection

$$\beta \mapsto \pi - \beta.$$

Thus, there are $m+2$ planes of symmetry, intersecting along one axis. This enables us to restrict our attention to constructing the fundamental domain

$$\text{(A.14)}\qquad 0 \leqslant \beta \leqslant \pi/2,$$

from which the whole surface is reconstructed with the help of reflections with respect to planes of symmetry.

The solutions of the Painlevé equation are parametrized by the values of the monodromy data—the parameter p, where $|p| < 1$. The asymptotics of the solution as $\rho \to \infty$ and as $\rho \to 0$ are expressed by means of the same p, which establishes a connection between them [27].

***Theorem* A.2.** *Let p be the monodromy data for the solution of* (A.5). *The following asymptotics hold as* $\rho \to 0$:

$$v = r_0 \ln \rho + s_0 + O(\rho^{2-|r_0|}), \tag{A.15}$$

$$A = 2^{3r_0/2} e^{s_0/2} \Gamma^2\left(\frac{1}{2} + \frac{r_0}{4}\right), \quad B = 2^{-3r_0/2} e^{-s_0/2} \Gamma^2\left(\frac{1}{2} - \frac{r_0}{4}\right),$$

$$p = \frac{A e^{i\pi r_0/4} - B e^{-i\pi r_0/4}}{A + B},$$

and if moreover $\rho^2\Lambda \to 0$, *then*

$$\Psi_1^{(0)}(\rho, \Lambda) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\, e^{i\pi r_0/8} \Lambda^{-1/2} (1 + O(\rho^2\Lambda + \rho^{2-|r_0|})) \times \tag{A.16}$$

$$\times \left\{ e^{v/2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} H^{(2)}_{\delta-1} & 0 \\ 0 & -iH^{(1)}_{-\delta+1} \end{pmatrix} + e^{-v/2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -iH^{(2)}_{\delta} & 0 \\ 0 & H^{(1)}_{-\delta} \end{pmatrix} \right\},$$

where $H_\delta^{(i)} = H_\delta^{(i)}(\Lambda^{-1})$ *are Bessel functions* [14] *and* $\delta = \frac{1}{2} - \frac{r_0}{4}$.

In the book [27] there is a misprint in the sign of δ.

***Theorem* A.3.** *Let p be the monodromy data for the solution of* (A.5). *The following asymptotics are true as* $\rho \to \infty$:

$$\begin{cases} v = -\dfrac{\alpha}{\sqrt{\rho}} \sin \xi + o(\rho^{-1/2}), \quad \xi = \rho + \dfrac{\alpha^2}{16} \ln \rho + \varphi_0, \\ \alpha^2 = -\dfrac{8}{\pi} \ln (1 - |p|^2), \\ \varphi_0 = \dfrac{\alpha^2}{8} \ln 2 - \arg \Gamma\left(\dfrac{i\alpha^2}{16}\right) - \arg p + \dfrac{3\pi}{4}. \end{cases} \tag{A.17}$$

In the first quadrant $\operatorname{Re} t \geqslant 0$, $\operatorname{Im} t \geqslant 0$ *of the variable* $t = \rho\Lambda$ *there are two regions with different asymptotics* Ψ_+:

(1) $$\mathcal{D}_{WKB} = \{t:\ |t - 4|\sqrt{\rho} \to \infty\},$$

$$\Psi_+ = \left[I + O\left(\frac{1}{\sqrt{\rho}(t-4)}\right)\right] \times \tag{A.18}$$

$$\times \exp\left\{ i\sigma_3 \left[-\rho\left(\frac{t}{16} + \frac{1}{t}\right) + \frac{\alpha^2}{16} \ln \frac{t-4}{t+4} \right] \right\},$$

(2) $$\mathcal{D}_+^\varepsilon = \left\{ t:\ |t - 4| < \rho^{-\frac{1}{2} + \varepsilon}, \quad 0 < \varepsilon < \frac{1}{6} \right\},$$

$$\Psi_+ = \Psi_0 \begin{pmatrix} e^{\frac{i\pi}{2} s} & 0 \\ -i\dfrac{\sqrt{2\pi}}{\Gamma(s)} & -\dfrac{\alpha}{4} e^{-i\xi} e^{i\frac{\pi}{4}} \end{pmatrix} (I + o(1)) \times \tag{A.19}$$

$$\times \exp\left\{ \sigma_3 \left(-\frac{i\rho}{2} - \frac{s}{2} i\pi + \frac{s}{2} \ln \rho + s \ln 2 \right) \right\},$$

$$\Psi_0 = \begin{pmatrix} D_{-s}(i\zeta) & D_{s-1}(\zeta) \\ \dot{D}_{-s}(i\zeta) & \dot{D}_{s-1}(\zeta) \end{pmatrix},$$

$$s = -\frac{i\alpha^2}{16}, \quad \zeta = \frac{\sqrt{\rho}}{4} e^{-i\frac{\pi}{4}} (t - 4), \quad \dot{f}(\zeta) = \left[f_\zeta - \frac{\zeta}{2} f \right] \frac{4}{\alpha} e^{-i\frac{\pi}{4}} e^{i\xi}.$$

Here $\Gamma(x)$ and $D_s(x)$ are the gamma-function and the function of a parabolic cylinder [14].

Comparing (A.6) and (A.15), we obtain

$$r_0 = -\frac{2m}{m+2}, \quad s_0 = u(0) + \frac{2m}{m+2} \ln \frac{4}{m+2}. \tag{A.20}$$

The purely imaginary part of the monodromy parameter is fixed:

$$\operatorname{Im} p = \sin \frac{\pi r_0}{4}.$$

For a fixed order of the umbilical point we get a one-parameter family of surfaces, and the value of the metric at the midpoint is taken as a parameter.

The derivative Φ_γ may be computed with the help of the first equation of (A.7). The consequences of Theorem A.1 are the following formulae:

$$\begin{cases} F = R^{-1}\left[-\frac{\sigma_1}{2i} + \frac{4ie^{i\beta}}{\rho} A(\Lambda)\right] R, \\ N = R^{-1}\frac{\sigma_3}{2i} R, \\ F_w = \frac{1}{2} R^{-1}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} R, \\ F_{\bar{w}} = \frac{e^v}{2} R^{-1}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} R, \quad R = \Psi_+(\rho, \Lambda) TS, \end{cases} \tag{A.21}$$

where we should put

$$\Lambda = 4e^{i\beta}/\rho.$$

The substitution of the asymptotics of Theorems A.2 and A.3 into these expressions enables us to obtain the asymptotic description of the surface.

The matrix S in (A.11) corresponds to the general rotation of the surface. In order that the symmetry axis should coincide with the third basis vector of $\mathbb{R}^3$, let us choose S to be equal to

$$S = \frac{1}{\sqrt{A+B}}\begin{pmatrix} \sqrt{A} & \sqrt{B} \\ -\sqrt{B} & \sqrt{A} \end{pmatrix}. \tag{A.22}$$

The centre of the surface $\rho = 0$.

Let us substitute the asymptotics (A.15), (A.16) into expresisons (A.21), making use of the relation (A.10) between different Ψ-functions

$$R = \Psi_+ TS = \Psi_1^{(0)}\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} T^{-1} S.$$

Then, direct but rather tiresome computations connected with the decomposition of Bessel functions at zero (we need to take into consideration only the first terms) give

$$\begin{cases} F(\rho = 0) = \left(1 - \frac{r_0}{2}\right)\frac{\sigma_3}{2i}, \\ N(\rho = 0) = -\frac{\sigma_3}{2i}. \end{cases} \tag{A.23}$$

Asymptotics at infinity. The "foot": $\rho \to \infty$, $\sqrt{\rho}\beta \to \infty$.

This is the domain where WKB-asymptotics (A.18) holds. Taking into account only the first term, we have

$$\text{(A.24)} \qquad \begin{cases} \dfrac{4i}{\rho} e^{i\beta} A(\Lambda) \to -\rho \sin\beta \dfrac{\sigma_3}{2i}, \\ F \to S^{-1}\left(-\rho\sin\beta\dfrac{\sigma_3}{2i}\right)S. \end{cases}$$

Similarly,

$$N = \frac{1}{2i} S^{-1} \begin{pmatrix} 0 & e^{i\eta} \\ e^{-i\eta} & 0 \end{pmatrix} S + o(1),$$

$$F_w = S^{-1}\left(\frac{\sigma_3}{2} + \frac{1}{2}\begin{pmatrix} 0 & -e^{i\eta} \\ e^{-i\eta} & 0 \end{pmatrix}\right) S + o(1),$$

$$F_{\bar{w}} = S^{-1}\left(-\frac{\sigma_3}{2} + \frac{1}{2}\begin{pmatrix} 0 & -e^{i\eta} \\ e^{-i\eta} & 0 \end{pmatrix}\right) S + o(1),$$

$$\eta = \rho\cos\beta + \frac{1}{\pi}\ln\left(\tan\frac{\beta}{2}\right)\ln(1-|p|^2).$$

This enables us to conclude that

$$\text{(A.25)} \quad F = S^{-1}\left(-\rho\sin\beta\frac{\sigma_3}{2i} + \sin\eta\frac{\sigma_2}{2i} - \cos\eta\frac{\sigma_1}{2i}\right)S + F_0,$$
$$F_0 = o(\rho), \ F_{0\varphi} = o(\rho), \ F_{0\rho} = o(1).$$

The asymptotics (A.25) describes a cylinder with an axis whose direction is given by the vector

$$\text{(A.26)} \qquad \left(-\frac{2\sqrt{AB}}{A+B}, 0, \frac{B-A}{A+B}\right).$$

The symmetry (A.25) of the surface shows that the curve $\beta = \pi/2$ lies on the plane spanned by the basis vectors 1, 3. In the first approximation it is a straight line.

Asymptotics at infinity. Cone.

All the remaining domain is covered by the curves

$$\text{(A.27)} \qquad \sqrt{\rho}\beta = r = \text{const}.$$

We consider one such curve (A.27) with a fixed r. Let us show that to within the first order this curve is mapped into a straight ray in Euclidean space.

The asymptotics (A.17) leads to the next main term:

$$\frac{4ie^{i\beta}}{\rho} A(\Lambda) \to \sqrt{\rho}\left\{-r\frac{\sigma_3}{2i} + \frac{i\alpha}{4}\begin{pmatrix} 0 & e^{-i\xi} \\ e^{i\xi} & 0 \end{pmatrix}\right\}.$$

To evaluate R we should make use of the asymptotics (A.19), which holds in (A.27). We can simplify the formulae a little. Along the curve (A.27)

$$\zeta \to re^{i\pi/4}.$$

Bearing in mind that $\zeta \to$ const, we obtain from (A.19)

$$R = \Psi_+ TS \to \exp\left(-\frac{i\xi}{2}\sigma_3\right)\tilde{P}S,$$

$$\tilde{P} = \begin{pmatrix} D_{-s}(i\zeta)\, e^{\pi\alpha^2/32} - \frac{i\sqrt{2\pi}}{\Gamma(s)} D_{s-1}(\zeta) & -D_{s-1}(\zeta)\frac{\alpha}{4} e^{i\pi/4} \\ \frac{4e^{-i\pi/4}}{\alpha}\left\{\dot{D}_{-s}(i\zeta)\, e^{\pi\alpha^2/32} - \frac{i\sqrt{2\pi}}{\Gamma(s)}\dot{D}_{s-1}(\zeta)\right\} & -\dot{D}_{s-1}(\zeta) \end{pmatrix} \times$$

$$\times \exp\left[\sigma_3\left(-\frac{i\alpha^2}{16}\ln 2 + \frac{\pi\alpha^2}{64} + \frac{i\varphi_0}{2}\right)\right]\Bigg|_{\zeta = re^{i\pi/4}} = e^{-\pi\alpha^2/64}\begin{pmatrix} 1 & 0 \\ -\frac{2r}{\alpha} & 1 \end{pmatrix} P,$$

$$\text{(A.28)} \quad P = \begin{pmatrix} D_{-s}(re^{-i\pi/4}) & -\frac{\alpha}{4} e^{i\pi/4} D_{s-1}(re^{i\pi/4}) \\ -\frac{4e^{i\pi/4}}{\alpha} D'_{-s}(re^{-i\pi/4}) & -D'_{s-1}(re^{i\pi/4}) \end{pmatrix} \times$$

$$\times \exp\left[\sigma_3\left(-\frac{i}{2}\arg\Gamma(-s) - \frac{i}{2}\arg p + \frac{3\pi i}{8}\right)\right],$$

where $D'_s(z) = \frac{d}{dz}D_s(z)$.

In deducing (A.28) we used the identity [14]

$$D_{-s}(-z)\, e^{i\pi s/2} - \frac{i\sqrt{2\pi}}{\Gamma(s)} D_{s-1}(iz) = e^{-i\pi s/2} D_{-s}(z).$$

Finally the immersion is described by the fomula

$$\text{(A.29)} \qquad F = \frac{i\sqrt{\rho}}{4} S^{-1}P^{-1}\begin{pmatrix} 0 & \alpha \\ \frac{4r^2}{\alpha} + \alpha & 0 \end{pmatrix} PS + o(\sqrt{\rho}).$$

In the first approximation this is a ray. As r varies from 0 to ∞ it covers part of the cone, which generates all the cone with the help of reflections in the planes of symmetry. The asymptotics (A.29) goes over into (A.24) as $r \to \infty$.

***Theorem* A.4.** *For any $m \in \{0, \mathbb{N}\}$ there is a one-parameter family of proper immersions $\Pi_m(s_0)$ of the CMC plane $\mathbb{C} \ni z$, possessing the following properties:*

(1) *The metric $e^{u(z,\bar{z})}$ induced by this immersion depends only on $|z|$.*

(2) *The surface $\Pi_m(s_0)$ has $m+2$ planes of symmetry, intersecting along one axis l.*

(3) *The midpoint ($z = 0$) lies on l and is umbilical of order m (non-umbilical if $m = 0$).*

(4) *Asymptotically $\Pi_m(s_0)$ is the cone* (A.29).

(5) *The surface $\Pi_m(s_0)$ has $m+2$ "feet"* (A.25), *and their axes lie on planes of symmetry.*

(6) *The value of the metric at the midpoint $u(0)$ is taken as the parameter s_0.*

It is easier to imagine the "fine" structure of these surfaces by using the example with two "feet": $m = 0$. We wind the plane z into an infinite-sheeted covering of the cylinder. Let y be the coordinate in $\mathbb{R}^3$ along the axis of the cylinder. We cut it by the orthogonal plane $y = 0$. We take the sheets $-\infty$ and $+\infty$ of the covering at the points of the plane $y = 0$ and pull the covering along some straight line lying in the plane $y = 0$. Instead of the initial circle in the section $y = 0$ we obtain a curve of cycloid type. The greater y is, the less is the "splitting" of sheets of the initial covering, and in the limit as $y \to \infty$ we get a "foot". The immersion is a proper one. We need only give to the constructed surface the form of the cone (A.29) to the first order, and on this form we impose the described "fine" structure.

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