Projective Geometry

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Draft from December 12, 2019

Preliminary version. Partially extended and partially incomplete. Based on the lecture Geometry I (Winter Semester 2016 TU Berlin).

Written by Thilo Rörig based on the Geometry I course and the Lecture Notes of Boris Springborn from WS 2007/08.

Acknoledgements: We thank Alina Hinzmann and Jan Techter for the help with preparation of these notes.

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Chapter 1

Introduction

It seems natural that a course entitled *Geometry* should begin with the question:

What is geometry?

Right now, I would like to answer this question in the form of a short historic overview of the subject. Geometry is, after all, something that people have been doing for a very long time. The following brief history of geometry will be incomplete, inaccurate (true history is much more complicated) and biased (we will ignore what happened in India or China, for example). It is a shortened, smoothed out version of history that is meant only as a rough explanation of how the material that will be covered in this course came into being.

The word *geometry* comes from the Greek word $\gamma\varepsilon\omega\mu\varepsilon\tau\rho\iota\alpha$ which is a composite of the words for *earth* and *measure*. Geometry began as the science of measuring the earth, or surveying, and it began $\sim$2000 BC in Egypt and Mesopotamia (Babylon, in today's Iraq). These were among the first great civilizations and they depended on agriculture along the rivers Nile in Egypt, and Tigris and Euphrates in Mesopotamia. These rivers would periodically inundate and fertilize the surrounding land, which made periodic surveying necessary to delimit the fields. The science of geometry developed from this, with applications also in construction and astronomy. The Egyptians and Babylonians could compute areas and volumes of simple geometric figures, they had some approximations for π, and they already knew Pythagoras' theorem. Strangely though, no records of general theorems or proofs have survived from this period. Egyptian papyri and Babylonian clay tables with their cuneiform script contain only worked exercises. Maybe they did not state general theorems, maybe they just did not write them down, or maybe they did but these documents did not survive. Basically we have no idea how they conducted their research.

This changed with the period of Greek geometry (Thales $\sim$600 BC to Euclid $\sim$300 BC). They clearly stated general theorems for which they gave proofs. That is, they deduced more complicated statements from simpler ones by logical reasoning. This suggests putting all statements in order so that each statement is proved using only statements that have previously been proved. By necessity, one must begin with a few (as few as possible) hopefully very simple statements that are accepted without proof. In Euclid's Elements, geometry (most or even all of what was known at the time) is presented in this form. It begins with a few definitions and postulates (today we say axioms) from which all theorems are deduced one by one. These postulates were simple statements like "there is a unique straight line through two points" and "two lines intersect in a unique point or they are parallel". But one of the postulates was considered more complicated and less obvious than the others, the parallel postulate: "Given a line and a point not on the line, there is a unique parallel to the line through the point." For centuries to come, people tried to prove this one postulate using the other, simpler ones, so that it could be eliminated from the unproved postulates. One way people tried to prove the parallel postulate was to assume instead that there are many parallels and derive a contradiction. But even though some strange theorems could be deduced from this alternative parallel postulate (like that there is an upper bound for the area of triangles) no true contradiction would appear. This finally lead to the realization that the alternative parallel postulate did not contradict the other postulates. Instead it leads to a logically equally valid version of geometry which is now called *hyperbolic geometry* (Lobachevsky 1829, Bólyai 1831). Later it was realized that one may also assume that there are no parallels (the other postulates also have to be changed a little for this), and the resulting geometry is called *elliptic geometry*. This is simply the geometry on the sphere, where pairs of opposite points are considered as *one* point, and lines are great circles. Both hyperbolic and elliptic geometry are called *non-Euclidean geometries*, because their axioms are different from Euclid's.

Another important development in geometry was the introduction of coordinates by Descartes and Fermat in the first half of the 17th century. One could then describe geometric figures and prove theorems using numbers. This way of doing geometry was called *analytic geometry*, as opposed to the old way beginning with geometric axioms, which was called *synthetic geometry*. By the late 19th/early 20th century, it was proved that both approaches are in fact equivalent: One can either start with axioms for numbers and use them to define the objects of geometry, or one can start with axioms of geometry and define numbers geometrically, one gets the same theorems.

The study of the rules of perspective in painting (da Vinci & Dürer $\sim$1500) lead to the development of *projective geometry* (Poncelet, 1822), dealing with the question: Which properties of geometric figures do not change under projections? For example, straight lines remain straight lines, but parallels do not remain parallel.

Another type of geometry is *Möbius geometry*, which deals with properties that remain unchanged under transformations mapping circles to circles (such as inversion on a circle). Then there is also *Lie geometry* (about which I will say nothing now) and there are other types of geometry. *Klein's Erlangen Program* (1871) provided a systematic treatment of all these different kinds of geometry and their interrelationships. It also provided a comprehensive and maybe surprising answer to the question: What is geometry? We will come back to this.

Chapter 2

Projective geometry

2.1 Introduction

We start with an example which is at the origin of projective geometry in the renaissance. When a painter wanted to paint a real scene onto canvas he was facing the problem of projecting a plane in the scene, e.g., the floor of a room, onto his canvas. In mathematical terms this can be rephrased as follows:

Consider projecting a plane E in $\mathbb{R}^3$ to another plane E' from a point P not on E or E', so the image A' of a point $A \in E$ is the intersection of the line PA with E'. Every point in E has an image in E' except points on the *vanishing line* ℓ of E, which is the intersection of E with the plane parallel to E' through P. Conversely, every point in E' has a preimage in E except points on the vanishing line ℓ' of E', which is the intersection of E' with the plane parallel to E through P as shown in Fig 2.1. So the projection is not bijective.

The projection maps lines to lines. A family of parallel lines in E is mapped to a family of lines in E' which intersect in a point on the vanishing line.

Idea: Introduce, in a addition to the ordinary points of E, new points which correspond to points on the vanishing line of E'. In the same way, introduce new points of E' which are images of the vanishing line of E. These new points are called *points at infinity*, and the extended planes are called *projective planes*. The projection becomes a bijection between projective planes. Parallel lines in E intersect in a point at infinity. The points at infinity of E form a line called the *line at infinity* which corresponds to the vanishing line of E'.

Drawing a floor tiled with square tiles

Suppose you have already drawn the first tile. (We don't want to go into the details of how one can construct the image of the first tile, even though that is interesting and not difficult.) The figure shows how the other tiles can then be constructed: The first square has two pairs of parallel sides, so the point of intersection of these two pairs define the *horizon*. But a square tiling defines two more sets of parallel lines, namely, the diagonals. Since the corresponding diagonals of the square are also parallel, they have to intersect in one point

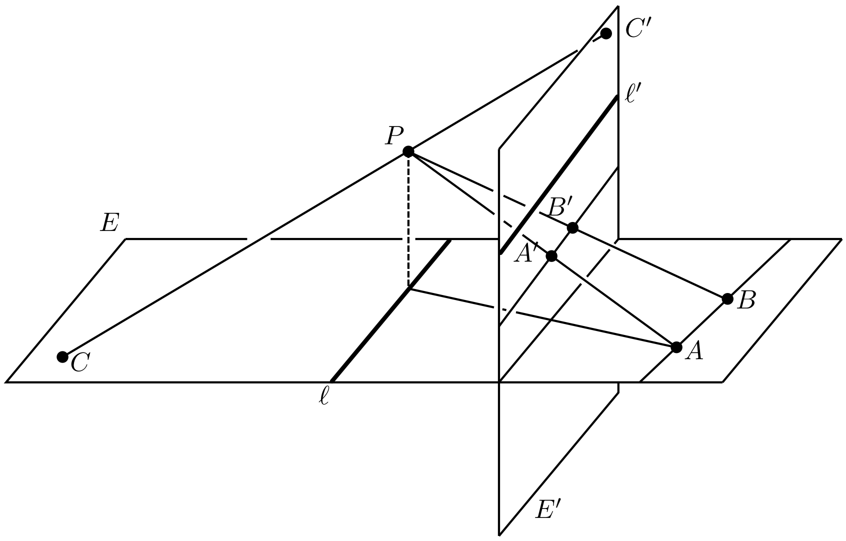


Figure 2.1: Central projection of two planes in $\mathbb{R}^3$ onto each other.

and this point has to lie on the horizon as well. So from the initial square we can construct the diagonals of the neighboring squares and hence the neighboring squares.

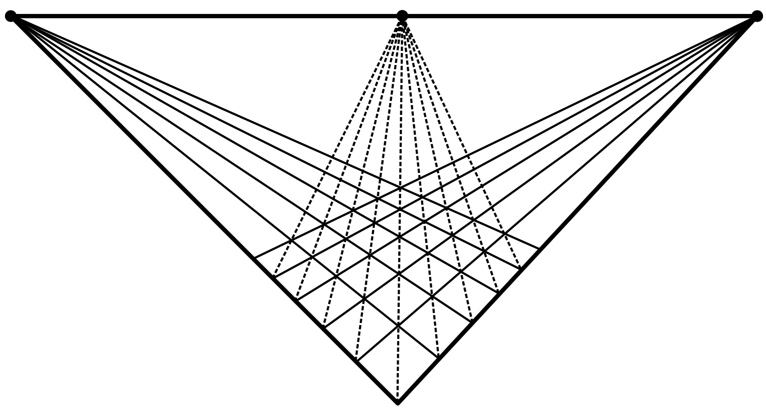


Figure 2.2: The three sets of parallel lines in the square tiling each intersect in a unique point on the horizon.

Projective geometry

Projective geometry deals with the properties of figures that remain unchanged under projections. An example for a theorem of projective geometry is Pappus' theorem. It is concerned with points, lines, and the incidence relation between points and lines.

We will see that a curve being a conic section is a projective property. But the distinction between circles, ellipses, parabolas and hyperbolas is not, as shown in Fig. 2.3.

Here the circular opening of the flash light is projected from the pointed light source to a hyperbola on the wall plane.

Also the distinction between ordinary points and points at infinity is not a projective property, because as we have seen, a projection can map ordinary points to points at infinity and vice versa. So from the point of view of projective geometry, points at infinity of a projective plane are not distinguished from ordinary points and the line at infinity is a line like any other.

Figure 2.3: The photograph shows the central projection of a circle onto a hyperbola.

2.2 Projective spaces

We start with the definition of the projective space of a general vector space over an arbitrary field, but for most parts of the book we will be considering real and complex projective spaces only.

Definition 2.1. Let V be a vector space over a field F. The *projective space* of V is the set $\mathrm{P}(V)$ of 1-dimensional subspaces of V. If the dimension of V is $n+1$, then the *dimension of the projective space* $\mathrm{P}(V)$ is n.

If the vector space has dimension 2, then the corresponding 1-dimensional projective space is called a *projective line*. In case of a 3-dimensional vector space, the 2-dimensional projective space is called a *projective plane*. An element of $\mathrm{P}(V)$ (that is, a 1-dimensional subspace of V) is called a *point* of the projective space.

If $v \in V \setminus \{0\}$, then we write $[v] := \mathrm{span}\{v\}$ for the 1-dimensional subspace spanned by v. So $[v]$ is a point in $\mathrm{P}(V)$, and v is called a *representative vector* for this point. If

$\lambda \neq 0$ then $[\lambda v] = [v]$ in $\mathrm{P}(V)$ and λv is another representative vector for the same point. This defines an equivalence relation on $V_* = V \setminus \{0\}$: For $v, w \in V_*$ define:

$$v \sim w \;:\Longleftrightarrow\; \exists \lambda \neq 0 : v = \lambda w.$$

From this point of view the projective space is the set of equivalence classes with respect to the equivalence relation $\sim$, i.e., the quotient space $V_*/\sim$.

2.2.1 Projective subspaces

Definition 2.2. A *projective subspace* of the projective space $\mathrm{P}(V)$ is a projective space $\mathrm{P}(U)$, where U is a vector subspace of V:

$$\mathrm{P}(U) = \{[v] \in \mathrm{P}(V) \mid v \in U\}.$$

If k is the dimension of $\mathrm{P}(U)$ (that is, $k+1$ is the dimension of U), then $\mathrm{P}(U)$ is called a k*-plane* in $\mathrm{P}(V)$.

A projective subspace of dimension one is called a projective *line*, of dimension two a projective *plane* and of dimension $n-1$ a projective *hyperplane* in $\mathrm{P}(V)$. When it is clear from the context we will omit the prefix "projective".

We do not have arithmetic operations for points in projective space, but we may use operations on linear subspaces to combine projective subspaces. If $\mathrm{P}(U_1)$ and $\mathrm{P}(U_2)$ are two projective subspaces of $\mathrm{P}(V)$, then the intersection $\mathrm{P}(U_1) \cap \mathrm{P}(U_2)$ is the projective subspace $\mathrm{P}(U_1 \cap U_2)$. The *projective span* or *join* of $\mathrm{P}(U_1)$ and $\mathrm{P}(U_2)$ is the projective subspace $\mathrm{P}(U_1 + U_2)$ (compare Ex. 2.2).

Proposition 2.3. *Through any two distinct points in a projective space there passes a projective line.*

In contrast to Euclidean geometry, the following proposition declines the existence of parallel lines.

Proposition 2.4. *Two distinct lines in a projective plane intersect in a unique point.*

The proofs of these two propositions can easily be performed by linear algebra. We will illustrate the methods on a slightly more general statement.

Proposition 2.5. *Let $H = \mathrm{P}(U)$ be a projective hyperplane in $\mathrm{P}(V)$ ($\dim \mathrm{P}(V) \geq 2$) and $\ell = \mathrm{P}(U')$ a projective line which is not contained in $\mathrm{P}(U)$. Then ℓ and H intersect in a unique point.*

Proof. Let $\dim V = n+1$. We have $\dim U = n$ and $\dim U' = 2$. The dimension of the sum $U + U' \subseteq V$ is at most $n+1$. Hence the dimension formula from linear algebra yields

$$\dim U \cap U' = \dim U + \dim U' - \dim(U + U') \geq n + 2 - (n+1) = 1.$$

Additionally, the line ℓ is not contained in H, hence $\dim U \cap U' \leq 1$. So $\dim U \cap U' = 1$ and H and ℓ intersect in a point. $\square$

2.2.2 Homogeneous and affine coordinates

The above definition of a projective space and projective subspaces is canonical in the sense that it does not depend on a choice of basis. Suppose we have a basis $v_1, \ldots, v_{n+1}$ of V. This gives an identification of V with F^{n+1} and of $\mathrm{P}(V)$ with $\mathrm{P}(F^{n+1})$. A vector $v \in V$ can be represented by

$$v = \sum_{j=1}^{n+1} x_j v_j \ ,$$

where $x = (x_1, \ldots, x_{n+1}) \in F^{n+1}$ are the coordinates of v with respect to the basis $v_1, \ldots, v_{n+1}$. For a point $[v] \in \mathrm{P}(V)$ the coordinates of a representative vector $v \in V$ are called *homogeneous coordinates*. If $\lambda \neq 0$, then $\lambda x_1, \ldots, \lambda x_{n+1}$ are also homogeneous coordinates of $[v]$. So for every point in projective space and every choice of basis we obtain homogeneous coordinates that are unique up to non-zero scalar multiples.

Let $\mathcal{U} \subset \mathrm{P}(V)$ be the subset of points for which a particular linear functional φ on F^{n+1}, say $\varphi(x) = x_{n+1}$, does not vanish:

$$\mathcal{U} = \big\{[v] \in \mathrm{P}(V) \,\big|\, v = \sum_{j=1}^{n+1} x_j v_j \text{ with } \varphi(x) = x_{n+1} \neq 0\big\}.$$

Then $\mathcal{U}$ is in bijection with F^n via the map

$$[v] \longmapsto \begin{pmatrix} x_1/x_{n+1} \\ x_2/x_{n+1} \\ \vdots \\ x_n/x_{n+1} \end{pmatrix} =: \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \ .$$

It's inverse map is given by

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \longmapsto [\sum_{j=1}^{n} u_j v_j + v_{n+1}].$$

The unique numbers $u_1, \ldots, u_n$ are called *affine coordinates* of $[v] \in \mathcal{U}$.

Remark 2.6. *The homogeneous coordinates depend on a choice of basis and affine coordinates depend on a choice of a basis* and *an affine hyperplane given by a linear form on the vector space.*

The projective space $\mathrm{P}(F^{n+1})$ can be decomposed into the following two subsets:

$$\mathcal{U} = \big\{[x] \in \mathrm{P}(F^{n+1}) \,\big|\, x_{n+1} \neq 0\big\} \qquad \text{and} \qquad \tilde{\mathcal{U}} = \big\{[v] \in \mathrm{P}(F^{n+1}) \,\big|\, x_{n+1} = 0\big\} \ .$$

As we have seen above $\mathcal{U} \simeq F^n$ and $\tilde{\mathcal{U}}$ is just $\mathrm{P}(F^n)$. So

$$\mathrm{P}(F^{n+1}) \simeq F^n \cup \mathrm{P}(F^n)\,.$$

The points in the $\mathrm{P}(F^n)$ part of $\mathrm{P}(F^{n+1})$ are often called the *points at infinity*. We summarize the above discussion in the following Lemma.

Lemma 2.7. *Let V be an $(n+1)$-dimensional vector space and $\varphi : V \to F$ a non-zero linear functional on V. Then we can decompose* $\mathrm{P}(V)$ *into*

$$\begin{aligned} \mathrm{P}(V) &\simeq U_{\text{aff}} \cup U_\infty \textit{ with} \\ U_{\text{aff}} &= \{[v] \in \mathrm{P}(V) \,|\, \varphi(v) \neq 0\}, \textit{ and} \\ U_\infty &= \{[v] \in \mathrm{P}(V) \,|\, \varphi(v) = 0\}. \end{aligned}$$

The affine part U_{aff} is isomorphic to F^n and the part at infinity U_∞ is isomorphic to $\mathrm{P}(F^n)$.

Remark 2.8. *In the following sections and proofs we will only "finite" picture since we may always choose the hyperplane at infinity, such that it does not contain any of the points.*

We will mostly consider the case where the base field F of the vector space is the field of real numbers $\mathbb{R}$. In this case, the concepts of a point, line or curve, etc., have their intuitively geometric meaning. But many theorems of projective geometry hold for arbitrary base fields. In particular, when dealing with curves and surfaces defined by algebraic equations, it is natural to use the base field $\mathbb{C}$. Finite fields are used in elliptic curve cryptography.

One usually writes $\mathbb{R}\mathrm{P}^n$ for $\mathrm{P}(\mathbb{R}^{n+1})$ and $\mathbb{C}\mathrm{P}^n$ for $\mathrm{P}(\mathbb{C}^{n+1})$. More generally, if V is any finite dimensional real or complex vector space, then $\mathrm{P}(V)$ is called an $\mathbb{R}\mathrm{P}^n$ or $\mathbb{C}\mathrm{P}^n$, respectively.

2.2.3 Models of real projective spaces

Every point in real projective space $\mathbb{R}\mathrm{P}^n$ corresponds to a 1-dimensional subspace U in $\mathbb{R}^{n+1}$, which is generated by a non-zero vector $v \in U \setminus \{0\}$. Since $\mathbb{R}^{n+1}$ comes with the Euclidean norm, we may choose representative vectors of length one, i.e., vectors on the unit sphere $S^n \in \mathbb{R}^{n+1}$. So every point $P \in \mathbb{R}\mathrm{P}^n$ has two representative vectors on the unit sphere $P = [v] = [-v]$. This defines an equivalence relation $\sim$ on the sphere S^n and hence $\mathbb{R}\mathrm{P}^n = S^n/\sim$ as shown in Fig 2.4 (left). We may also discard one hemisphere and consider the upper hemisphere where opposite points on the equator are identified (see Fig. 2.4 (right)).

The open hemisphere in S^n is homeomorphic to $\mathbb{R}^n$ by projecting the hemisphere from the origin onto the tangent plane at the north pole. Further the equator is a sphere S^{n-1}. Hence we obtain a decomposition of $\mathbb{R}\mathrm{P}^n$ into $\mathbb{R}^n$ and $\mathbb{R}\mathrm{P}^{n-1}$ which we have already discussed in Sect. 2.2.2.

Examples

Let us have a look at the real projective spaces of dimensions zero, one, and two in the different models. The real line $\mathbb{R}^1$ has only one 1-dimensional subspace. So $\mathbb{R}\mathrm{P}^0$ consists of a single point.

The real projective line The points of $\mathbb{RP}^1$, the 1-*dimensional real projective space* or the *real projective line* are the 1-dimensional subspaces

$$\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] = \mathbb{R} \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \subset \mathbb{R}^2, \qquad \text{where } \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \neq 0.$$

The points with homogeneous coordinate $x_2 \neq 0$ are described by one affine coordinate $x_1/x_2 \in \mathbb{R}$. On the other hand, all representative vectors $(x_1, 0) \in \mathbb{R}^2 \setminus \{0\}$ represent the same point $[(1,0)] \in \mathbb{RP}^1$. So one can think of $\mathbb{RP}^1$ as $\mathbb{R}$ plus one additional point (which is usually denoted by ∞). This is illustrated in Fig. 2.5 (left). But the point at infinity is only special in this representation of $\mathbb{RP}^1$. If we choose a different linear functional to define an affine coordinate, a different point will become the "point at infinity".

From the sphere model we see, that all points are "equal", i.e., $\mathbb{RP}^1$ is a homogeneous space. The identification of opposite points of the circle S^1 is a double cover of a circle (see Fig. 2.6, left). That $\mathbb{RP}^1$ is a circle is also obvious from the hemisphere model: the 1-dimensional hemisphere is just a line segment whose end points (equator) have to be identified (see Fig. 2.6, right).

The real projective plane The points of $\mathbb{RP}^2$, the *two-dimensional real projective space* or the *real projective plane* are the 1-dimensional subspaces

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbb{R} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \subset \mathbb{R}^3, \qquad \text{where } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \neq 0.$$

The points with homogeneous coordinate $x_3 \neq 0$ are described by two affine coordinates $u_1 = x_1/x_3$, $u_2 = x_2/x_3$. This corresponds to normalizing the representative vectors to have $x_3 = 1$. Hence these vectors can be identified with the plane $x_3 = 1$ in $\mathbb{R}^3$ as shown in Fig. 2.5 (right). On the other hand, the points $[(x_1, x_2, 0)]$ form the 1-dimensional real projective space $\mathrm{P}(U)$, where $U \subset \mathbb{R}^3$ is the subspace $U = \{(x_1, x_2, 0) \in \mathbb{R}^3\}$. So $\mathbb{RP}^2$ is $\mathbb{R}^2$ with a projective line at infinity.

In the sphere model, the projective lines in $\mathbb{RP}^2$ correspond to great circles. Since two distinct great circles intersect in a pair of antipodal points, this is another proof of

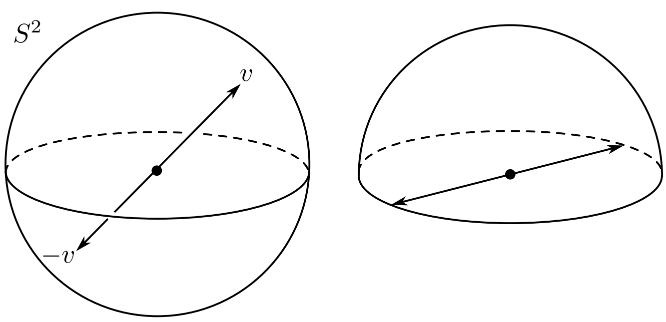


Figure 2.4: The real projective plane may be obtain by identifying opposite points on the unit 2-sphere S^2 (left) or from a hemisphere by identifying points on the equator.

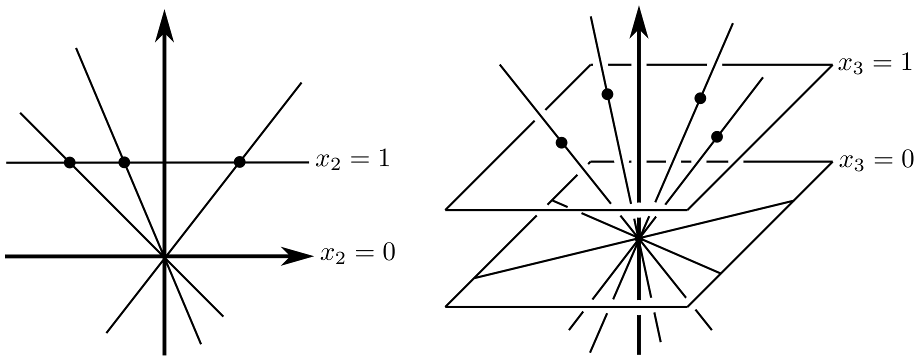


Figure 2.5: Affine coordinates for $\mathbb{R}\mathrm{P}^1$ and $\mathbb{R}\mathrm{P}^2$.

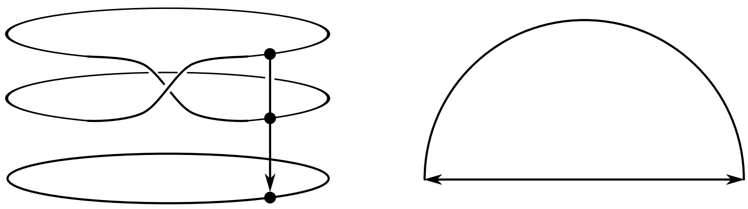


Figure 2.6: The real projective line $\mathbb{R}\mathrm{P}^1$: The identification of opposite points of the unit circle yields a double cover of the circle. The identification of opposite points on the equator of the hemisphere (i.e., hemicircle) yields a circle as well.

Prop. 2.4. We can also easily see, that $\mathbb{R}\mathrm{P}^2$ is non-orientable since it contains a Möbius strip: Consider the hemisphere model with identifications on the equator and cut a strip out of the hemisphere connecting two opposite segments on the equator. If we identify the opposite segments we obtain a Möbius strip as shown in Figure 2.7.

Complex projective line The decomposition of a projective space into an affine part and a part at infinity also works for complex projective spaces. So the *complex projective line* $\mathbb{C}\mathrm{P}^1$ is the union of $\mathbb{C}$ and $\mathbb{C}\mathrm{P}^0$, i.e., one additional point which is usually denoted ∞. In complex analysis, $\mathbb{C}\mathrm{P}^1 = \mathbb{C} \cup \{\infty\}$ is called the *extended complex plane* or the *Riemann sphere* and denoted by $\widehat{\mathbb{C}}$.

2.2.4 Projection of two planes onto each other

In this section we will look back at the example in the introduction and use coordinates to describe the projection shown in Fig. 2.1. We will see that the central projection can be written as an invertible linear map.

Suppose E is the u_1u_2-plane in $\mathbb{R}^3$, E' is the u_2u_3-plane, and $P = (-1, 0, 1)$. A point $A = (u_1, u_2, 0) \in E$ is mapped to a point $A' = (0, v_1, v_2) \in E'$, and by solving $A' = P + t(A - P)$ for t one finds that $v_1 = \frac{u_2}{u_1+1}$ and $v_2 = \frac{u_1}{u_1+1}$. So in terms of the

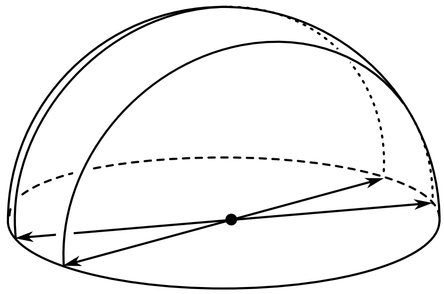


Figure 2.7: A Möbius strip in the hemisphere model of the real projective plane shows that it is not orientable.

coordinates u_1, u_2 of plane E and v_1, v_2 of plane E', the projection is the function

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = f\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \frac{1}{u_1+1}\begin{pmatrix} u_2 \\ u_1 \end{pmatrix}.$$

The vanishing line of E is the line $u_1 = -1$, and the vanishing line of E' is the line $v_2 = 1$. We introduce *homogeneous coordinates*: Instead of using two numbers (u_1, u_2) to describe a point in E, use three numbers (x_1, x_2, x_3) such that $u_1 = \frac{x_1}{x_3}$ and $u_2 = \frac{x_2}{x_3}$. The homogeneous coordinates for a point are not unique: $(u_1, u_2, 1)$ are homogeneous coordinates for the point (u_1, u_2), but for any $\lambda \neq 0$, $(\lambda u_1, \lambda u_2, \lambda)$ are homogeneous coordinates for the same point. In the same way, we use homogeneous coordinates (y_1, y_2, y_3) with $v_1 = \frac{y_1}{y_3}$ and $v_2 = \frac{y_2}{y_3}$ to describe a point $(v_1, v_2) \in E'$. Let us write the projection f in terms of homogeneous coordinates. Let (x_1, x_2, x_3) be homogeneous coordinates for (u_1, u_2) and let (y_1, y_2, y_3) be homogeneous coordinates for $(v_1, v_2) = f(u_1, u_2)$. Then

$$\begin{aligned} \frac{y_1}{y_3} &= v_1 = \frac{u_2}{u_1+1} = \frac{\frac{x_2}{x_3}}{\frac{x_1}{x_3}+1} = \frac{x_2}{x_1+x_3}, \\ \frac{y_2}{y_3} &= \ldots = \frac{x_1}{x_1+x_3}, \end{aligned}$$

so we may choose

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \\ x_1+x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =: F \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Using homogeneous coordinates, the projection may thus be written as a linear map $F : \mathbb{R}^3 \to \mathbb{R}^3$. Moreover, F *is bijective*! The points on the vanishing line $u_1 = -1$ of E have homogeneous coordinates (x_1, x_2, x_3) with $x_1 + x_3 = 0$, and F maps these to vectors $(y_1, y_2, 0)$, which are not homogeneous coordinates for any point in E'. But we can interpret them as homogeneous coordinates for a point at infinity of the extended plane.

Thus, each non-zero vector $x \in \mathbb{R}^3$ represents a point in a projective plane (of which it is the vector of homogeneous coordinates) and two vectors x, x' represent the same point if and only if $x' = \lambda x$ for some $\lambda \neq 0$.

2.2.5 Points in general position

The concept of linear independence is very important in linear algebra. A similar notion exists in projective spaces and is called general position.

Definition 2.9. Let $\mathrm{P}(V)$ be an n-dimensional projective space, then $n+2$ points in $\mathrm{P}(V)$ are said to be *in general position* if no $n+1$ of the points are contained in an $(n-1)$-dimensional projective subspace. Equivalently, any $n+1$ of the points have linearly independent representative vectors.

So three points on a line are in general position if they are distinct, four points in a plane are in general position if no three of them lie on a line, and five points in a 3-dimensional projective space are in general position if no four of them lie in a plane. For points in general position there exists a very useful normalization of the representative vectors given in the next lemma.

Lemma 2.10. *Let $\mathrm{P}(V)$ be an n-dimensional projective space and $P_1, \ldots, P_{n+2}$ in $\mathbb{R}\mathrm{P}^n$ in general position. Then representative vectors $p_1, \ldots, p_{n+1} \in V$ may be chosen so that*

$$p_1 + p_2 + \cdots + p_{n+1} + p_{n+2} = 0.$$

This choice is unique up to a common factor. That is, if $\tilde{p}_1, \ldots, \tilde{p}_{n+1}$ is another choice of representative vectors with $\tilde{p}_1 + \tilde{p}_2 + \cdots + \tilde{p}_{n+1} + \tilde{p}_{n+2} = 0$, then $\tilde{p}_k = \lambda p_k$ for all $k = 1, \ldots, n+2$ and some $\lambda \neq 0$.

Proof. Let $v_1, \ldots, v_{n+2}$ be any representative vectors for the points $P_1, \ldots, P_{n+2}$. They are linearly dependent because $\dim V = n+1$. So

$$\sum_{j=1}^{n+2} a_j v_j = 0$$

for some a_j which are not all zero. In fact, no a_k can be zero, because that would mean that there are $n+1$ among the v_j which are linearly dependent, but then the P_i were not in general position. Hence we may choose

$$p_1 = a_1 v_1, \quad p_2 = a_2 v_2, \quad \ldots \quad p_{n+1} = a_{n+1} v_{n+1}, \quad p_{n+2} = a_{n+2} v_{n+2}.$$

To see the uniqueness up to a common factor, suppose $\lambda_1 p_1, \ldots, \lambda_{n+2} p_{n+2}$ is another choice of representative vectors with $\sum_{k=1}^{n+2} \lambda_k p_k = 0$. This amounts to a homogeneous system of equations of rank $n+1$ for the $n+2$ variables λ_k. So the solution space is 1-dimensional and hence $\lambda_1 = \lambda_2 = \ldots = \lambda_{n+2}$. □

Points in general position in projective spaces may be considered analogs to linear independent vectors in vector spaces. Hence the above lemma is often used to normalize configurations in a way that calculations become easier.

Given an n-dimensional vector space V together with a basis $\mathcal{B} = \{v_1, \ldots, v_{n+1}\}$. Then the points $\{[v_1], \ldots, [v_{n+1}], -\sum_{i=1}^{n+1} v_i\}$ are in general position in $\mathrm{P}(V)$. Conversely, if we have $n+2$ points $[p_1], \ldots, [p_{n+2}]$ in general position in $\mathrm{P}(V)$ normalized as in the above Lemma, then $\{p_1, \ldots, p_{n+1}\}$ is a basis of V. Furthermore, the homogeneous coordinates of the points with respect to this basis are $p_i = (\delta_{ij})_{j=1}^{n+1}$ for $i = 1, \ldots, n+1$, i.e., the i-th standard basis vector, and $p_{n+2} = (-1, \ldots, -1)$.

2.3 Desargues' Theorem

Theorem 2.11 (Desargues)**.** *Let A, A', B, B', C, C' be points in a projective plane such that the lines AA', BB' and CC' intersect in one point P.*

Then the intersection points $C'' = AB \cap A'B'$, $A'' = BC \cap B'C'$ and $B'' = AC \cap A'C'$ lie on one line.

As stated in Remark 2.8, we may always choose the line at infinity such that all considered points are finite and the claim may be formulated in affine terms.

Proof. If A, A', P are not distinct, the statement of the theorem is obvious. (Check this.) So we may assume that A, A', P are distinct and also B, B', P and C, C', P. But then A, A', P are three points on a line in general position. So by Lemma 2.10 we may choose representative vectors $a, a', p \in V$ with $a + a' = p$. For the same reason we may also choose representative vectors b, b' and c, c' so that $b + b' = p$ and $c + c' = p$. Then

$$a + a' = b + b' = c + c'.$$

This implies $a - b = b' - a'$. Obviously, the vector $a - b = b' - a'$ is in the span of a and b and also in the span of a' and b'. So the point $[a - b] = [b' - a'] \in \mathrm{P}(V)$ lies on the line AB and on the line $A'B'$, hence it is the point of intersection C''. Similarly,

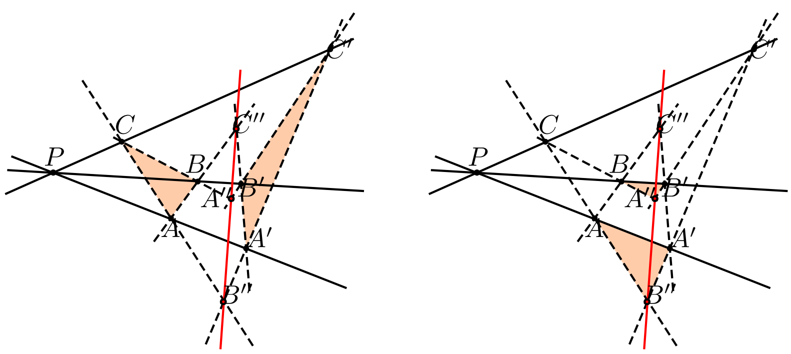


Figure 2.8: Desargues configuration with triangles in perspective shaded (left). The converse of Desargues theorem can be proved using a different pair of triangles (right).

$A'' = [b - c] = [c' - b']$, and $B'' = [c - a] = [a' - c']$. But

$$(a - b) + (b - c) + (c - a) = 0,$$

which means that vectors $(a - b), (b - c), (c - a)$ are linearly dependent and so they span a subspace of dimension at most 2. Therefore, C'', A'' and B'' lie on a line. $\square$

In other words, Desargues' theorem says: If the lines joining corresponding points of two triangles meet in one point, then the intersections of corresponding sides lie on one line. The converse is also true: If the intersections of corresponding sides of two triangles lie on one line, then the lines joining corresponding points meet in one point. Surprisingly, the converse statement is in fact equivalent to the original statement after a permutation of the point labels, see Fig. 2.8 (right).

Desargues' theorem also holds for triangles in two different planes of a 3-dimensional projective space $\mathrm{P}(V)$. In this case, it can be proved without any calculations: The intersection points of corresponding sides lie on the line in which the planes of the two triangles intersect.

The planar version of Desargues' theorem can also be proved without any calculations if the third dimension is used:

3D proof of Desargues' theorem. Let E be the plane of the two triangles ABC, $A'B'C'$ and the point P. Choose a line through P which is not in E and two points X and Y on it. The lines XA and YA' lie in one plane, so they intersect in a point $\tilde{A}$. Similarly, let $\tilde{B} = XB \cap YB'$ and $\tilde{C} = XC \cap YC'$. Now the intersection of the line $\tilde{A}\tilde{B}$ and the plane E lies on the line AB, because the plane $X\tilde{A}\tilde{B}$ intersects E in AB. Similarly, $\tilde{A}\tilde{B} \cap E$ also lies on the line $A'B'$, so $\tilde{A}\tilde{B} \cap E = C''$. In the same way, $\tilde{B}\tilde{C} \cap E = A''$ and $\tilde{C}\tilde{A} \cap E = B''$. Hence A'', B'' and C'' lie on the line where E intersects the plane $\tilde{A}\tilde{B}\tilde{C}$. $\square$

Combinatorial symmetry of Desargues' configuration The preceding proof also suggests the following 3-dimensional way to generate any planar Desargues configuration. This construction also reflects the high degree of combinatorial symmetry of the configuration. Let P_1, P_2, P_3, P_4, P_5 be five points in general position in a 3-dimensional projective space, and let E be a plane that contains none of these points. Let $l_{ij} = l_{ji}$ be the 10 lines joining P_i and P_j $(i \neq j)$. The 10 points P_{ij} where these lines intersect E form a Desargues configuration. If (i, j, k, r, s) is any permutation of $(1, 2, 3, 4, 5)$, then the points P_{ij}, P_{jk}, P_{ki} always lie on a line (the intersection of the plane $P_iP_jP_k$ with E), which we denote by g_{rs}. Any one of the points P_{ij} lies on the line g_{rs} if the four indices $ijrs$ are different. So there are three lines through each point and three points on each line. Corresponding points of the triangles P_{ir}, P_{jr}, P_{kr} and P_{is}, P_{js}, P_{ks} are joined by the lines g_{jk}, g_{ki}, g_{ij}, which all pass trough P_{rs}. The intersection points of corresponding sides all lie on the line g_{rs}. The same Desargues figure therefore contains $5 \cdot 4/2 = 10$ pairs of triangles satisfying the condition of Desargues' theorem.

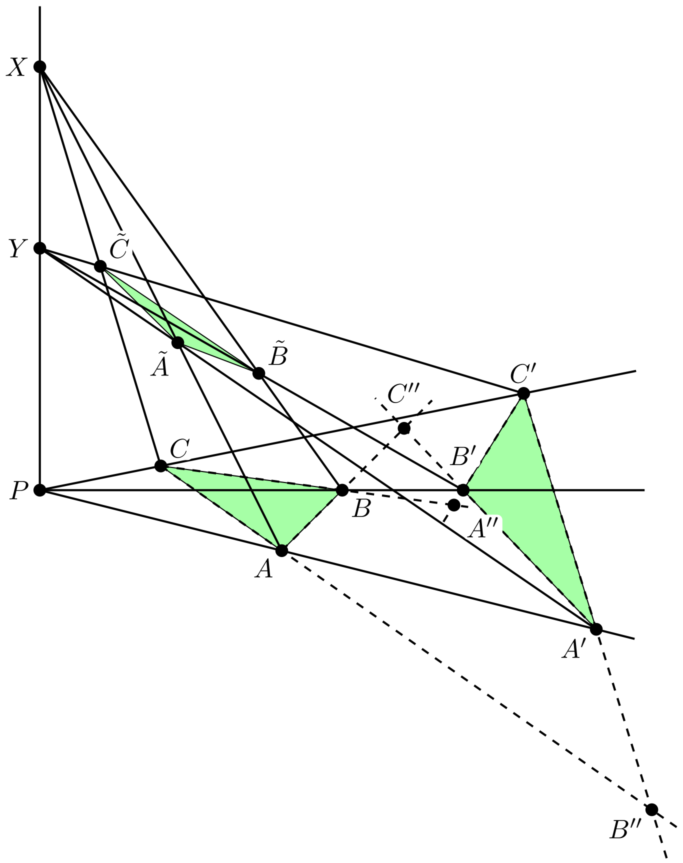


Figure 2.9: 3-dimensional construction to proof Desargues' theorem.

2.4 Projective transformations

Let V, W be $(n+1)$-dimensional vector spaces, and let $f : V \to W$ be an invertible linear map. Since f maps any 1-dimensional subspace $[v] \subseteq V$ to a 1-dimensional subspace $[f(v)] \subseteq W$, it defines an invertible map $\mathrm{P}(V) \to \mathrm{P}(W)$. A map between projective spaces which arises in this way is called a projective transformation:

Definition 2.12. A map $f : \mathrm{P}(V) \to \mathrm{P}(W)$ is a *projective transformation* if there is an invertible linear map $F : V \to W$ such that $f([v]) = [F(v)]$ for all $[v] \in \mathrm{P}(V)$.

A projective transformation maps lines in $\mathrm{P}(V)$ to lines in $\mathrm{P}(W)$ and more generally k-planes to k-planes.

In homogeneous coordinates, a projective transformation is represented by matrix multiplication: A point in $\mathrm{P}(V)$ with homogeneous coordinates $x = (x_1, \dots, x_{n+1})$ is mapped to the point in $\mathrm{P}(W)$ with homogeneous coordinates $y = Ax$ for some invertible $(n+1) \times (n+1)$ matrix A. In affine coordinates $u_i = x_i/x_{n+1}$, $w_i = y_i/y_{n+1}$ $(i = 1, \dots, n)$

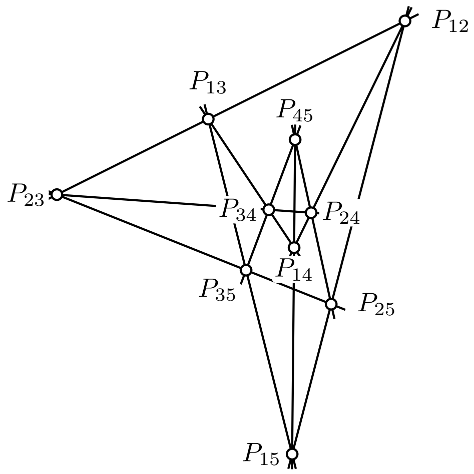


Figure 2.10: Combinatorics of Desargues' configuration: the indices ij of the points P_{ij} are indicated.

the map is a so-called fractional linear transformation:

$$w_i = \frac{\sum_{j=1}^{n} a_{ij}u_j + a_{i,n+1}}{\sum_{j=1}^{n} a_{n+1,j}u_j + a_{n+1,n+1}}.$$

Each w_i is the quotient of two affine linear functions of the u_j, where the denominator is the same for all i.

Proposition 2.13. *Two invertible linear maps $F, G : V \to W$ give rise to the same projective transformation $\mathrm{P}(V) \to \mathrm{P}(W)$ if and only if $G = \lambda F$ for some scalar $\lambda \neq 0$.*

Proof. If $G = \lambda F$, then $[G(v)] = [\lambda F(v)] = [F(v)]$. Conversely, suppose $[G(v)] = [F(v)]$ for all $v \in V \setminus \{0\}$. This implies $G(v) = \lambda(v)F(v)$ for some non-zero scalar $\lambda(v)$ which may *a priori* depend on v. We have to show that it does not. So suppose $v, w \in V \setminus \{0\}$. If v, w are linearly dependent, then it is obvious from the definition of $\lambda(v)$ that $\lambda(v) = \lambda(w)$. So assume v, w are linearly independent. Now

$$G(v + w) = G(v) + G(w) = \lambda(v)F(v) + \lambda(w)F(w)$$

but also

$$G(v + w) = \lambda(v + w)F(v + w) = \lambda(v + w)(F(v) + F(w)).$$

Since $F(v)$ and $F(w)$ are linearly independent we obtain $\lambda(v) = \lambda(v + w) = \lambda(w)$. $\square$

Similar to the linear transformations $V \to V$ which form the *general linear group*

$$\mathrm{GL}(V) = \{F : V \to V \mid F \text{ invertible}\}$$

we define the group of projective transformations as follows.

Definition 2.14. The projective transformations $\mathrm{P}(V) \to \mathrm{P}(V)$ form a group called the *projective linear group* $\mathrm{PGL}(V)$. It is the quotient of the general linear group $\mathrm{GL}(V)$ by the normal subgroup of non-zero multiples of the identity:

$$\mathrm{PGL}(V) = \mathrm{GL}(V)/\{\lambda I\}_{\lambda \neq 0}.$$

The group of projective transformations of $\mathbb{R}\mathrm{P}^n$ is given by the quotient of the general linear group $\mathrm{GL}(n+1,\mathbb{R})$ and hence denoted by $\mathrm{PGL}(n+1,\mathbb{R})$.

Projective transformations of $\mathbb{R}\mathrm{P}^1$ Projective transformations $f : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ are defined by linear isomorphisms $F : \mathbb{R}^2 \to \mathbb{R}^2$. Given a basis of $\mathbb{R}^2$ the map F corresponds to a matrix $A \in \mathrm{GL}(2,\mathbb{R})$, so

$$\begin{aligned} f : \mathbb{R}\mathrm{P}^1 &\longrightarrow \mathbb{R}\mathrm{P}^1, \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &\longmapsto \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right] = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} \end{aligned}$$

If we decompose $\mathbb{R}\mathrm{P}^1 = \mathbb{R} \cup \{\infty\}$ where $\mathbb{R} = \{[x_1, x_2] \,|\, x_2 \neq 0\}$ and $\infty = [x_1, 0] = [1, 0]$, then we can write the above transformation as follows:

$$\begin{aligned} f([\begin{smallmatrix} u \\ 1 \end{smallmatrix}]) &= \begin{bmatrix} au + b \\ cu + d \end{bmatrix} = \begin{cases} \begin{bmatrix} \frac{au+b}{cu+d} \\ 1 \end{bmatrix} & \text{, if } cu + d \neq 0 \\ [\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}] & \text{, if } cu + d = 0 \end{cases} \\ \text{and} \quad f([\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]) &= \begin{bmatrix} a \\ c \end{bmatrix}. \end{aligned}$$

If we introduce the affine coordinate $u = \frac{x_1}{x_2}$ then the linear map turns into a fractional linear transformation on $\mathbb{R} \cup \{\infty\}$ given by:

$$f(u) = \frac{au+b}{cu+d}, \text{ for } u \neq -\frac{d}{c}, \qquad f(-\tfrac{d}{c}) = \infty, \qquad \text{and} \qquad f(\infty) = \frac{a}{c},$$

where $ad - bc \neq 0$.

Projective transformations of $\mathbb{C}\mathrm{P}^1$ are described exactly in the same way. They can be identified with fractional linear transformations $f(z) = \frac{az+b}{cz+d}$, where $ad - bc \neq 0$. The action extends to the whole Riemann complex sphere $\mathbb{C} \cup \{\infty\} \simeq \mathbb{C}\mathrm{P}^1$ by $f(\infty) = \frac{a}{c}$ and $f(-\frac{d}{c}) = \infty$. They build the group of conformal automorphisms of the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

Affine transformations Consider an affine transformation $\mathbb{R}^n \to \mathbb{R}^n$ given by $u \mapsto Mu + b$ with $M \in \mathrm{GL}(n,\mathbb{R})$ and $b \in \mathbb{R}^n$. In homogeneous coordinates $x_1, \ldots, x_{n+1}$ with $u_i = \frac{x_i}{x_{n+1}}$, this map is given by

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \end{pmatrix} \longmapsto \underbrace{\left(\begin{array}{ccc|c} & M & & b \\ & & & \\ \hline 0 & \cdots & 0 & 1 \end{array}\right)}_{A} \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \end{pmatrix}.$$

The affine map $u \mapsto Mu + b$ can be extended to the projective transformation

$$\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n, \quad [x] \mapsto [Ax].$$

It maps the plane at infinity given by the points in $\mathbb{R}\mathrm{P}^n$ with $x_{n+1} = 0$ to the plane at infinity. Conversely, any projective transformation which maps the plane $x_{n+1} = 0$ to itself corresponds to an affine map $\mathbb{R}^n \to \mathbb{R}^n$ in the affine coordinates u.

The following theorem shows the analogy between bases of vector spaces for linear isomorphisms and points in general position in projective spaces for projective maps.

Theorem 2.15. *Let* $\mathrm{P}(V)$ *and* $\mathrm{P}(W)$ *be two* n*-dimensional projective spaces and suppose*

$$A_1, \dots, A_{n+2} \in \mathrm{P}(V) \quad \textit{and} \quad B_1, \dots, B_{n+2} \in \mathrm{P}(W)$$

are points in general position. Then there exists a unique projective transformation f : $\mathrm{P}(V) \to \mathrm{P}(W)$ *with* $f(A_i) = B_i$ *for* $i = 1, \dots, n+2$.

Proof. We will first prove existence of the projective map and then the uniqueness of the map.

Existence: By Lemma 2.10 on points in general position, we may choose representative vectors $a_1, \dots a_{n+2}$ for $A_1, \dots, A_{n+2}$ and $b_1, \dots, b_{n+2}$ for $B_1, \dots, B_{n+2}$ such that $\sum_{i=1}^{n+2} a_i = 0$ and $\sum_{i=1}^{n+2} b_i = 0$. Also by the general position assumption, $a_1, \dots, a_{n+1}$ and $b_1, \dots, b_{n+1}$ are bases of V and W, respectively. Hence there is an invertible linear map $F : V \to W$ with $F(a_i) = b_i$ for $i = 1, \dots, n+1$. But then also

$$F(-a_{n+2}) = F\Big(\sum_{i=1}^{n+1} a_i\Big) = \sum_{i=1}^{n+1} F(a_i) = \sum_{i=1}^{n+1} b_i = -b_{n+2}.$$

So F maps the 1-dimensional subspaces $A_i = [a_i] \subseteq V$ to $B_i = [b_i] \subseteq W$ for $i = 1, \dots, n+2$.

Uniqueness: Let $\tilde{F} : V \to W$ be another invertible linear map with $[\tilde{F}(a_i)] = B_i$ for $i = 1, \dots, n+2$. Then $\tilde{b}_i = \tilde{F}(a_i)$ is another set of representative vectors for the B_i with

$$\sum_{i=1}^{n+2} \tilde{b}_i = \sum_{i=1}^{n+2} \tilde{F}(a_i) = \tilde{F}\Big(\sum_{i=1}^{n+2} a_i\Big) = \tilde{F}(0) = 0$$

By the uniqueness part of the Lemma 2.10 this implies $\tilde{b}_i = \lambda b_i$ for some $\lambda \neq 0$. Hence $\tilde{F} = \lambda F$, and $\tilde{F}$ and F induce the same projective transformation $\mathrm{P}(V) \to \mathrm{P}(W)$. $\square$

One can classify projective transformations $f : \mathrm{P}(V) \to \mathrm{P}(V)$ according to the normal forms of the corresponding linear maps $F : V \to V$. Note that fixed points of f correspond to 1-dimensional eigenspaces of F.

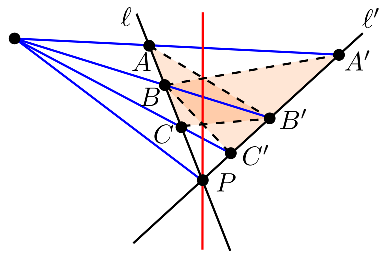
2.4.1 Central projections and Pappus' Theorem

In this section we come back to our initial motivating example and show that central projections and their generalizations are projective transformations. In contrast to the above definition, the central projections map proper projective subspaces in a given projective space onto each other.

Axis of a projective map and Pappus' Theorem We start with projective transformations between two lines in the projective plane. By Theorem 2.15 a projective transformation on lines is defined by the images of three points. So consider a projective transformation f from ℓ to $\ell' \neq \ell$. We denote the intersection point of the two lines by $P = \ell \cap \ell'$. Choose three distinct points A, B, and C on ℓ and their resp. images A', B', and C' on ℓ'. We may assume, that $A \neq P$. We may construct the map f in the following way:

Consider the *axis of the projective transformation* g through $AC' \cap A'C$ and $BC' \cap B'C$ and the two central projections $f_1 : \ell \to g$ from C' and $f_2 : g \to \ell$ from C. Then $f = f_2 \circ f_1$. We will show that the axis does not depend on the choice of the three points A, B, and C.The intersection point $\ell \cap g$ is mapped to P and P is mapped to $g \cap \ell'$. In other words, the line g passes through the points $f(P)$ and $f^{-1}(P)$.

- If $f(P) \neq f^{-1}(P)$ then the line g passes through $f(P)$ and $f^{-1}(P)$ and is uniquely determined by the projective transformation. In particular, g does not depend on A, B, and C.
- If $f(P) = f^{-1}(P)$ then the projective map f has a fixed point $P = f(P) = f^{-1}(P)$ and coincides with the central projection with center $AA' \cap BB'$. Using Desargues' Theorem we see that the axis g passes through arbitrary intersection points $(A\, f(A)) \cap (B\, f(B))$ as shown in Fig. 2.11.



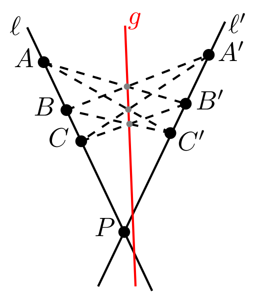


Figure 2.11: The axis of a projective map from a line to a line in $\mathbb{R}\mathrm{P}^2$ if the map has a fixed point (left) and if it doesn't (right).

Proposition 2.16. *Let ℓ_1, ℓ_2 be two different lines in a projective plane $\mathrm{P}(V)$. A projective transformation $\ell_1 \to \ell_2$ is a central projection if and only if it maps the intersection $\ell_1 \cap \ell_2$ to itself. Otherwise it is the composition of two central projections $\ell_1 \to \ell \to \ell_2$ with an intermediate line ℓ.*

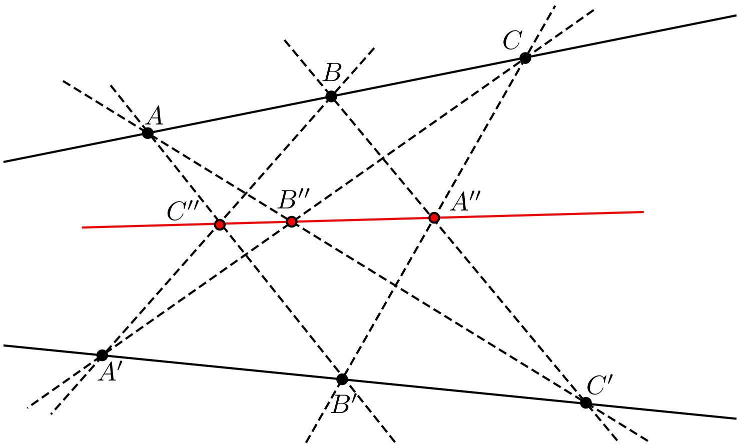


Figure 2.12: Pappus' configuration.

Proof. The proof follows from the construction shown in Fig. 2.11 and the fact that a projective transformation is uniquely determined by its action on three points. □

The independence of the axis of a projective map between two lines in the projective plane $\mathbb{R}\mathrm{P}^2$ from the choice of points and their images can be used to prove Pappus' Theorem.

Theorem 2.17 (Pappus)**.** *Let A, B, C be points on one line ℓ in a projective plane* $\mathrm{P}(V)$*, and let A', B', C' be points on another line ℓ'. Then the points*

$$C'' = AB' \cap A'B, \quad A'' = BC' \cap B'C, \quad B'' = CA' \cap C'A$$

lie on a line.

Proof. Consider the projective map $f : \ell \to \ell'$ given by $f(A) = A'$, $f(B) = B'$, and $f(C) = C'$. Then the axis of the projective map f is exactly the line through the points A'', B'', and C''. □

Central projections Let $H_1 = \mathrm{P}(U_1)$ and $H_2 = \mathrm{P}(U_2)$ be two hyperplanes in a projective space $\mathrm{P}(V)$ and let $C \in \mathrm{P}(V)$ be a point not in H_1 or H_2. Then the *central projection of H_1 onto H_2 from C* is the map $f : H_1 \to H_2$ that maps a point $A \in H_1$ to the intersection of the line CA with H_2. This intersection exists by Prop. 2.5.

Theorem 2.18. *Let H_1 and H_2 be two hyperplanes and C a point not on the two hyperplanes. Then the central projection f of H_1 to H_2 with center C is a projective transformation.*

Proof. We have to show that f is induced by an invertible linear map $F : U_1 \to U_2$, where $H_i = \mathrm{P}(U_i)$. The point C, as point in $\mathrm{P}(V)$, corresponds to a 1-dimensional subspace W of V. Since it does not lie in H_2, $W \cap U_2 = \{0\}$. This means that V is the direct sum

$$V = W \oplus U_2,$$

and there are two linear maps $\pi_W : V \to W$ and $\pi_{U_2} : V \to U_2$ (the projections onto W and U_2) such that for any $v \in V$, $\pi_W(v)$ and $\pi_{U_2}(v)$ are the unique vectors in W and U_2 such that $v = \pi_W(v) + \pi_{U_2}(v)$.

Claim: The central projection f is induced by the linear map $\pi_{U_2}|_{U_1}$, the restriction of π_{U_2} to U_1.

To see this, let $a \in U_1$ be a representative vector of $A \in H_1$. Then $\pi_{U_2}(a) \neq 0$, because $\pi_{U_2}(a) = 0$ would mean $a \in W$, but $U_1 \cap W = \{0\}$ because by assumption H_1 does not contain C. This shows that $\pi_{U_2}|_{U_1}$ is invertible, because it is an injective linear map $U_1 \to U_2$ and $\dim U_1 = \dim U_2$. Now $\pi_{U_2}(a) \in U_2$, so $[\pi_{U_2}(a)] \in H_2$. Also $a = \pi_W(a) + \pi_{U_2}(a)$, or

$$\pi_{U_2}(a) = a - \pi_W(a),$$

so $\pi_{U_2}(a) \in [a]+W$, which means that $[\pi_{U_2}(a)]$ is on the (projective) line through $A \in \mathrm{P}(V)$ and $C \in \mathrm{P}(V)$. Hence $[\pi_{U_2}(a)]$ is the intersection of H_2 with the line through C and A, so it is the image of A under the central projection. $\square$

Generalized central projections One can also consider more general types of projections. For example, let ℓ_1 and ℓ_2 be two lines in a 3-dimensional projective space $\mathrm{P}(V)$, and let ℓ_0 be a line that does not intersect ℓ_1 or ℓ_2. Then the projection $\ell_1 \to \ell_2$ with the line ℓ_0 as center of projection is defined as follows: A point $A \in \ell_1$ is mapped to the intersection of ℓ_2 with the plane spanned by ℓ_0 and A. This map $\ell_1 \to \ell_2$ is also a projective transformation, and the proof is the same (apart from obvious modifications).

Most generally, in an n-dimensional projective space $\mathrm{P}(V)$, one can project one k-plane onto another k-plane from any disjoint $(n-k-1)$-plane as center of projection; this is also projective transformation.

2.5 The cross-ratio

Consider the projective line $\mathbb{R}\mathrm{P}^1$ with its projective transformation group $\mathrm{PGL}(2,\mathbb{R})$. In Prop. 2.13 we have shown that any three distinct points on the real projective line can be mapped to any other three distinct points by a projective transformation. Hence a projective invariant of three points on a projective line cannot exist. So we need to consider at least four points.

Definition 2.19. Let P_1, P_2, P_3, and P_4 be four distinct points on the projective line $\mathbb{R}\mathrm{P}^1$ (or $\mathbb{C}\mathrm{P}^1$) with homogeneous coordinates $P_i = [v_i] = [x_i, y_i]$. The *cross-ratio* is

$$\mathrm{cr}(P_1, P_2, P_3, P_4) := \frac{\det\begin{pmatrix} v_1 & v_2 \end{pmatrix} \det\begin{pmatrix} v_3 & v_4 \end{pmatrix}}{\det\begin{pmatrix} v_2 & v_3 \end{pmatrix} \det\begin{pmatrix} v_4 & v_1 \end{pmatrix}} = \frac{(x_1y_2 - x_2y_1)(x_3y_4 - x_4y_3)}{(x_2y_3 - x_3y_2)(x_4y_1 - x_1y_4)}\,.$$

Remark 2.20. *All this works not only for the real and complex projective line but also for other projective spaces* $\mathrm{P}(V)$ *of* 2*-dimensional vector spaces* V *over any field.*

Since each representative vector occurs once in the numerator and once in the denominator, the value of the cross-ratio does not depend on the choices of representative vectors (x_i, y_i) but only on the points P_i.

To derive an expression for the cross-ratio in terms of the affine coordinate $u = \frac{x}{y}$, we assume first that no y_i is 0, i.e., no u_i is ∞:

$$\begin{aligned}\operatorname{cr}(P_1, P_2, P_3, P_4) &= \frac{y_1 y_2 \left(\frac{x_1}{y_1} - \frac{x_2}{y_2}\right) y_3 y_4 \left(\frac{x_3}{y_3} - \frac{x_4}{y_4}\right)}{y_2 y_3 \left(\frac{x_2}{y_2} - \frac{x_3}{y_3}\right) y_4 y_1 \left(\frac{x_4}{y_4} - \frac{x_1}{y_1}\right)} \\ &= \frac{(u_1 - u_2)(u_3 - u_4)}{(u_2 - u_3)(u_4 - u_1)} =: \operatorname{cr}(u_1, u_2, u_3, u_4).\end{aligned}$$

Even if one of the u_i is ∞, the previous formula yields the correct results in the limit. For example, if $y_1 = 0$ so that $u_1 = \infty$, we obtain

$$\operatorname{cr}(P_1, P_2, P_3, P_4) = \frac{x_1 y_2 (x_3 y_4 - x_4 y_3)}{(x_2 y_3 - x_3 y_3)(-x_1 y_4)} = \frac{x_1 y_2 y_3 y_4 \left(\frac{x_3}{y_3} - \frac{x_4}{y_4}\right)}{y_2 y_3 (-x_1 y_4)\left(\frac{x_2}{y_2} - \frac{x_3}{y_3}\right)} = -\frac{u_3 - u_4}{u_2 - u_3},$$

so the following calculation gives the correct result:

$$\operatorname{cr}(\infty, u_2, u_3, u_4) = \lim_{u_1 \to \infty} \frac{(u_1 - u_2)(u_3 - u_4)}{(u_2 - u_3)(u_4 - u_1)} = -\frac{u_3 - u_4}{u_2 - u_3}.$$

The homogeneous coordinates as well as the affine coordinates depend on a particular choice of basis, but the following lemma shows that the cross-ratio is a projective quantity.

Lemma 2.21. *The definition of the cross-ratio is independent of the choice of the basis used for the homogeneous coordinates of* $P_i = [x_i, y_i]$, $i = 1, 2, 3, 4$.

Proof. A change of basis is given by a matrix $A \in \mathrm{GL}(2, \mathbb{R})$. Suppose the new coordinates are given by

$$\begin{pmatrix} \tilde{x}_i \\ \tilde{y}_i \end{pmatrix} = A \begin{pmatrix} x_i \\ y_i \end{pmatrix}.$$

Then for determinants in the cross-ratio computation this implies

$$\det \begin{pmatrix} \tilde{x}_i & \tilde{x}_j \\ \tilde{y}_i & \tilde{y}_j \end{pmatrix} = \det \left(A \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix} \right) = \det(A) \cdot \det \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}$$

and the factors $\det(A)$ in the numerator and denominator cancel. □

Since every linear isomorphism can be represented by a matrix this implies the following for projective maps $\mathbb{RP}^1$ to $\mathbb{RP}^1$.

Corollary 2.22. *If* $f : \mathbb{RP}^1 \to \mathbb{RP}^1$ *is a projective map, then*

$$\operatorname{cr}(f(P_1), f(P_2), f(P_3), f(P_4)) = \operatorname{cr}(P_1, P_2, P_3, P_4)$$

for four points P_1, P_2, P_3, P_4.

Theorem 2.23. *Let* $\mathrm{P}(V)$ *be a projective space of dimension* n *and* $f : \mathrm{P}(V) \to \mathrm{P}(V)$ *a projective transformation. Then* f *maps lines to lines and preserves the cross-ratio of any four distinct points lying on a line.*

Proof. The projective transformation f maps lines to lines since the corresponding linear isomorphism $F : V \to V$ maps 2-dimensional vector subspaces to 2-dimensional subspaces.

The cross-ratio of four points on a line is preserved by Corollary 2.22, since the restriction of f to the line containing the points is a projective map. □

Proposition 2.24. *The cross-ratio* $\mathrm{cr}(P_1, P_2, P_3, P_4)$ *is the affine coordinate of the image of* P_1 *under the projective transformation that maps* P_2, P_3, P_4 *to the points with affine coordinates* $0, 1, \infty$.

Proof. Projective transformations preserve the cross-ratio and may be calculated using affine coordinates. Hence

$$\begin{aligned} \mathrm{cr}(P_1, P_2, P_3, P_4) &= \mathrm{cr}(f(P_1), f(P_2), f(P_3), f(P_4)) \\ &= \mathrm{cr}(f(P_1), 0, 1, \infty) = f(P_1). \end{aligned}$$

□

Proposition 2.25. *The cross ratio of four distinct points can take all values except* $0, 1, \infty$. *Moreover,*

$$\begin{aligned} \mathrm{cr}(P_1, P_2, P_3, P_4) = 0 \quad &\Leftrightarrow \quad P_1 = P_2 \text{ or } P_3 = P_4 \\ \mathrm{cr}(P_1, P_2, P_3, P_4) = 1 \quad &\Leftrightarrow \quad P_1 = P_2 \text{ or } P_3 = P_4 \\ \mathrm{cr}(P_1, P_2, P_3, P_4) = \infty \quad &\Leftrightarrow \quad P_1 = P_2 \text{ or } P_3 = P_4 \end{aligned}$$

Proof. The first claim follows from Prop. 2.24. The rest can be checked by direct computation. □

In Theorem 2.15 we have shown that we can map three distinct points on a projective line to arbitrary three distinct points on another line. The next theorem characterizes projective transformations of lines that map four distinct points.

Theorem 2.26. *There exists a projective transformation that maps four distinct points* P_1, P_2, P_3, P_4 *of a line to four distinct points* Q_1, Q_2, Q_3, Q_4 *on the same or another line if and only if*

$$\mathrm{cr}(P_1, P_2, P_3, P_4) = \mathrm{cr}(Q_1, Q_2, Q_3, Q_4).$$

Proof. If we are given a projective transformation that maps P_i to Q_i for $i = 1, 2, 3, 4$, then by Thm. 2.23 the cross-ratio is invariant. Conversely, consider the projective map f defined by $f(P_i) = Q_i$ for $i = 2, 3, 4$. We have

$$\mathrm{cr}(Q_1, Q_2, Q_3, Q_4) = \mathrm{cr}(P_1, P_2, P_3, P_4) = \mathrm{cr}(f(P_1), Q_2, Q_3, Q_4)$$

which implies $f(P_1) = Q_1$. □

The cross ratio depends on the order of the points. How does it change if the points are permuted? The cross ratio does not change if we simultaneously interchange two disjoint pairs of the points:

$$\operatorname{cr}(u_1,u_2,u_3,u_4)=\operatorname{cr}(u_2,u_1,u_4,u_3)=\operatorname{cr}(u_3,u_4,u_1,u_2)=\operatorname{cr}(u_4,u_3,u_2,u_1).$$

This is easy to see from the equation for the cross ratio in terms of the u_i. Of the 24 permutations of u_1,u_2,u_3,u_4, we need therefore only consider the six which fix u_1 and permute u_2, u_3, and u_4. If $\operatorname{cr}(u_1,u_2,u_3,u_4)=q=\operatorname{cr}(q,0,1,\infty)$, then

$$\begin{aligned}
\operatorname{cr}(u_1,u_3,u_2,u_4)&=\operatorname{cr}(q,1,0,\infty)=\frac{(q-1)(0-\infty)}{(1-0)(\infty-q)}=1-q,\\
\operatorname{cr}(u_1,u_2,u_4,u_3)&=\operatorname{cr}(q,0,\infty,1)=\frac{(q-0)(\infty-1)}{(0-\infty)(1-q)}=\frac{q}{q-1},\\
\operatorname{cr}(u_1,u_4,u_3,u_2)&=\operatorname{cr}(q,\infty,1,0)=\frac{(q-\infty)(1-0)}{(\infty-1)(0-q)}=\frac{1}{q},\\
\operatorname{cr}(u_1,u_3,u_4,u_2)&=\operatorname{cr}(q,1,\infty,0)=\frac{(q-1)(\infty-0)}{(1-\infty)(0-q)}=\frac{q-1}{q}=1-\frac{1}{q},\\
\operatorname{cr}(u_1,u_4,u_2,u_3)&=\operatorname{cr}(q,\infty,0,1)=\frac{(q-\infty)(0-1)}{(\infty-0)(1-q)}=\frac{1}{1-q}\,.
\end{aligned}$$

To obtain an invariant of the four points independent of their order, one introduces the function

$$\mathcal{I}(q)=q^2+\frac{1}{q^2}+(1-q)^2+\frac{1}{(1-q)^2}+\left(\frac{q}{q-1}\right)^2+\left(\frac{q-1}{q}\right)^2,$$

where $q=\operatorname{cr}(u_1,u_2,u_3,u_4)$, or closely related

$$\begin{aligned}
\mathcal{I}_2(q)&=\frac{\mathcal{I}(q)+3}{2}=\frac{(q^2-q+1)^3}{q^2(1-q)^2},\\
\mathcal{I}_3(q)&=\mathcal{I}_2-\frac{27}{4}=\left(\frac{(q+1)(q-2)(q-\frac{1}{2})}{q(1-q)}\right)^2.
\end{aligned}$$

We can also define the cross-ratio for four concurrent lines in the following way.

Definition 2.27. Let $\ell_1,\ell_2,\ell_3,\ell_4$ be four lines through a point P and g a line the does not contain P in $\mathbb{R}\mathrm{P}^2$. The cross-ratio of the lines is

$$\operatorname{cr}(\ell_1,\ell_2,\ell_3,\ell_4)=\operatorname{cr}(P_1,P_2,P_3,P_4)\quad\text{with } P_i=\ell_i\cap g.$$

This definition does not depend on the choice of the line g: Let $\tilde{g}$ be another line not containing P. Then the intersection points of $P_i=\ell_i\cap g$ and $\tilde{P}_i=\ell_i\cap\tilde{g}$ are related by a central projection from P. Hence

$$\operatorname{cr}(P_1,P_2,P_3,P_4)=\operatorname{cr}(\tilde{P}_1,\tilde{P}_2,\tilde{P}_3,\tilde{P}_4)$$

2.5.1 Projective involutions of the real projective line

For any four points $A, B, C, D \in \mathbb{R}\mathrm{P}^1$ there is a unique projective transformation $f : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ with $f(A) = B$, $f(B) = A$, $f(C) = D$, $f(D) = C$, because

$$\mathrm{cr}(A, B, C, D) = \mathrm{cr}(B, A, D, C).$$

The transformation f is an *involution*, that is, $f \neq id$ but $f \circ f = id$.

A pair of points $\{A, B\} \subset \mathbb{R}\mathrm{P}^1$ *separates* another pair $\{C, D\} \subset \mathbb{R}\mathrm{P}^1$ if C and D are in different connected components of $\mathbb{R}\mathrm{P}^1 \setminus \{A, B\}$.

$$\{A, B\} \text{ separates } \{C, D\} \qquad \Longleftrightarrow \qquad \mathrm{cr}(A, C, B, D) < 0.$$

The involution f has no fixed points if $\{A, B\}$ separates $\{C, D\}$, otherwise it has two fixed points. If f has two fixed points P and Q, then for all $X \in \mathbb{R}\mathrm{P}^1$, $\mathrm{cr}(X, P, f(X), Q) = -1$.

For any two points $P, Q \in \mathbb{R}\mathrm{P}^1$ there is a unique projective involution of $\mathbb{R}\mathrm{P}^1$ that fixes P and Q. This involution can be defined by the above cross-ratio equation.

If A, B, P, Q are four points in $\mathbb{R}\mathrm{P}^1$, then one says *the pair* $\{A, B\}$ *separates the pair* $\{P, Q\}$ *harmonically*, if $\mathrm{cr}(A, P, B, Q) = -1$.

2.6 Complete quadrilateral and quadrangle

The complete quadrilateral and complete quadrangle are basic configurations in the projective plane. They will be a key ingredient to the prove of the fundamental theorem of projective geometry in Section 2.7.

Definition 2.28. A configuration consisting of four lines in the projective plane, no three through one point, and the six intersection points, one for each pair of lines, form a *complete quadrilateral*.

Let A, B, C and D be four points in general position in a projective plane. The *complete quadrilateral* consists of these four points and six lines - one for each pair of points.

The points of the complete quadrilateral come in pairs of opposite points shown in different colors in Fig. 2.13 (left). As for the complete quadrilateral, the six lines come in three pairs of opposite lines, each generate by a decomposition of the four points into two disjoint pairs. These pairs are shown with the same color in Fig. 2.13 (right).

Definition 2.29. Let A, B, C, and D be four points in general position defining a complete quadrangle and ℓ be a line not through any of the points. We obtain three pairs of intersection points P_i, Q_i for $i = 1, 2, 3$ with the three pairs of opposite lines. The set $\big\{\{P_1, Q_1\}, \{P_2, Q_2\}, \{P_3, Q_3\}\big\}$ is called a *quadrangular set* (see Fig. 2.14).

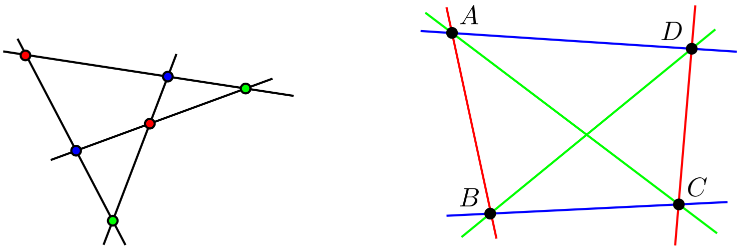


Figure 2.13: Left: A complete quadrilateral consisting of four lines an three pairs of opposite points. Right: A complete quadrangle consisting of four points and three pairs of opposite lines.

Remark 2.30. *A quadrangular set is a set of (unordered) pairs and we do not want to distinguish the order of the points and to which of the opposite lines an intersection point belongs.*

As a configuration, one does not distinguish the points of the pairs due to the symmetry of the configuration with respect to the exchange of P_i and Q_i and the permutation of the indices 1, 2, 3.

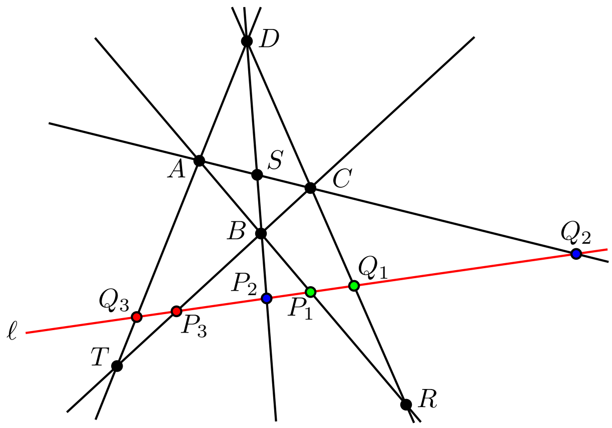


Figure 2.14: A quadrangular set defined by a complete quadrangle.

Similar to the cross-ratio we define a projective invariant for six points on a projective line.

Definition 2.31. Let $P_1, P_2, \dots, P_6$ be six distinct points on a projective line with representative vectors $P_i = [v_i] = [x_i, y_i]$ for $i = 1, \dots, 6$. The *multi-ratio* of these six points is:

$$\begin{aligned}\mathrm{mr}(P_1, P_2, P_3, P_4, P_5, P_6) &= \frac{\det(v_1 v_2)}{\det(v_2 v_3)} \frac{\det(v_3 v_4)}{\det(v_4 v_5)} \frac{\det(v_5 v_6)}{\det(v_6 v_1)} \\ &= \frac{(x_1 y_2 - x_2 y_1)(x_3 y_4 - x_4 y_3)(x_5 y_6 - x_6 y_5)}{(x_2 y_3 - x_3 y_2)(x_4 y_5 - x_5 y_4)(x_6 y_1 - x_1 y_6)}.\end{aligned}$$

The following properties of the multi-ratio are analogous to the properties of the cross-ratio:

- The multi-ratio is independent of the choice of homogeneous coordinates.
- The multi-ratio is a projective invariant.
- For the calculation we may use affine coordinates $u_i = x_i/y_i$ for $P_i = [x_i, y_i]$:

  $$\mathrm{mr}(P_1, P_2, P_3, P_4, P_5, P_6) = \frac{(u_1 - u_2)(u_3 - u_4)(u_5 - u_6)}{(u_2 - u_3)(u_4 - u_5)(u_6 - u_1)}$$

  If one of the coordinates is infinite, we consider the limit as we did in the calculation of the cross-ratio.

Theorem 2.32. *A quadrangular set $\big\{\{P_1, Q_1\}, \{P_2, Q_2\}, \{P_3, Q_3\}\big\}$ is characterized by the relation*

$$\mathrm{mr}(P_1, P_2, P_3, Q_1, Q_2, Q_3) = -1. \tag{$*$}$$

Remark 2.33. *Note that although the multi-ratio changes under permutations of its arguments, the condition $\mathrm{mr}(P_1, P_2, P_3, Q_1, Q_2, Q_3) = -1$ is invariant under permutation symmetry of the quadrangular set from Remark 2.30.*

Remark 2.34. *Any five points of a quadrangular set determine the sixth point uniquely. Indeed using affine coordinates one can easily see that the relation $\mathrm{mr}(P_1, P_2, P_3, Q_1, Q_2, Q_3) = -1$ determines the affine coordinate of the sixth point, and hence the point itself.*

Proof. We will use the following relation between cross-ratio and multi-ratio

$$\mathrm{mr}(P_1, P_2, P_3, Q_1, Q_2, Q_3) = (-1)\,\mathrm{cr}(P_1, P_2, P_3, Q_1)\,\mathrm{cr}(P_1, Q_1, Q_2, Q_3).$$

The central projections of the line ℓ to the line CD from B and A preserve the cross-ratio and yield the following two identities (see Fig. 2.14 for the labels):

$$\begin{aligned}\ell \xrightarrow{B} CD\colon\ \mathrm{cr}(P_1, P_2, P_3, Q_1) &= \mathrm{cr}(R, D, C, Q_1) \\ \ell \xrightarrow{A} CD\colon\ \mathrm{cr}(P_1, Q_1, Q_2, Q_3) &= \mathrm{cr}(R, Q_1, C, D) \\ &= \frac{1}{\mathrm{cr}(R, D, C, Q_1)}.\end{aligned}$$

This implies $(*)$. □

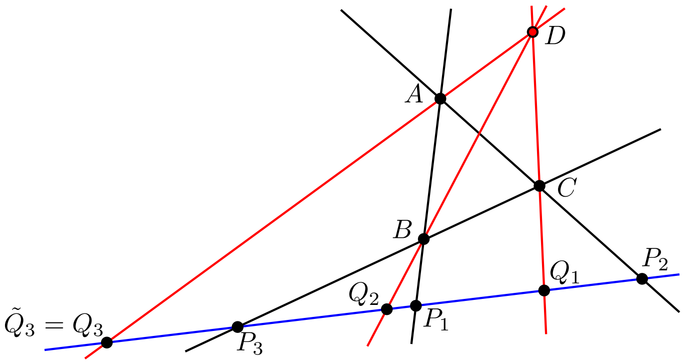


Figure 2.15: Points with multi-ratio -1 as a quadrangular set.

The converse statement of the above theorem is formulated in the following lemma.

Lemma 2.35. *Six points $P_1, P_2, P_3, Q_1, Q_2, Q_3$ satisfying*

$$\mathrm{mr}(P_1, P_2, P_3, Q_1, Q_2, Q_3) = -1$$

build a quadrangular set $\{P_1, Q_1; P_2, Q_2; P_3, Q_3\}$.

Proof. Let A, B, and C be three distinct points such that ABP_1, ACP_2, and BCP_3 are collinear. Further, let $D = Q_1C \cap Q_2B$ and $\tilde{Q}_3 = AD \cap \ell$. Then $\{P_1, Q_1; P_2, Q_2; P_3, \tilde{Q}_3\}$ is a quadrangular sets. Thus $\{P_1, Q_1; P_2, Q_2; P_3, \tilde{Q}_3\}$ and $\{P_1, Q_1; P_2, Q_2; P_3, Q_3\}$ satisfy equation $(*)$. Due to Remark 2.34, five points determine the sixth one, and thus $\tilde{Q}_3 = Q_3$. Hence the points A, D, and Q_3 lie on one line, which completes the complete quadrangle. $\square$

Theorem 2.36 (Complete quadrilateral)**.** *Let ℓ_1, ℓ_2, ℓ_3, and ℓ_4 be four lines in the projective plane such that no three go through one point. Further let $A = \ell_1 \cap \ell_2$, $B = \ell_2 \cap \ell_3$, $C = \ell_3 \cap \ell_4$, $D = \ell_4 \cap \ell_1$, and $P = \ell_1 \cap \ell_3$ and $Q = \ell_2 \cap \ell_4$ be the six points of the complete quadrilateral. Define $\ell = PQ$ and $X = AC \cap \ell$ and $Y = BD \cap \ell$. Then*

$$\mathrm{cr}(P, X, Q, Y) = -1.$$

In the following we will give two proofs of the theorem.

quadrangular sets. If we relabel and indentify the points in the Theorem on quadrangular sets as follows:

$$P := P_1 = Q_1, \quad Q := P_3 = Q_3, \quad X := Q_2, \quad Y := P_2.$$

Then Theorem 2.32 implies:

$$\mathrm{mr}(P, Y, Q, P, X, Q) = -1 \;\Leftrightarrow\; \mathrm{cr}(P, Y, Q, X) = -1.$$

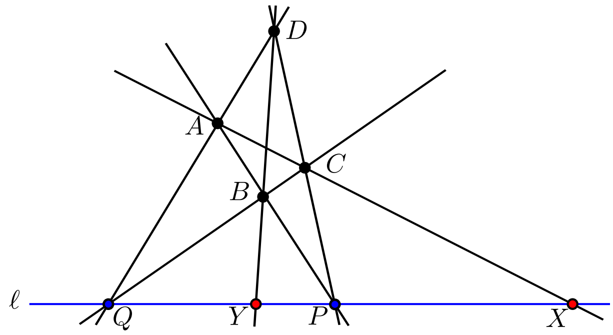


Figure 2.16: Complete quadrilateral and harmonic points.

After cyclic permutation and permutation of pairs we get

$$\mathrm{cr}(P, X, Q, Y) = -1.$$

□

using a projective involution of $\mathbb{R}\mathrm{P}^2$. Since A, B, C, D are in general position, there is a projective transformation of the plane $\mathbb{R}\mathrm{P}^2$ that maps $A \mapsto B$, $B \mapsto A$, $C \mapsto D$, $D \mapsto C$. It is an involution of $\mathbb{R}\mathrm{P}^2$ which maps the lines AB and CD onto themselves. It maps the line AD to BC and vice versa. Hence, the points P and Q are fixed, and the line ℓ is mapped to itself. Since the line AC is mapped onto BD and vice versa, X is mapped to Y and Y to X. Thus, the restriction to ℓ is an involution of ℓ with fixed points P, Q and interchanging X, Y. So $\mathrm{cr}(P, X, Q, Y) = -1$. □

Definition 2.37. The pair $\{P, Q\}$ *separates* the pair $\{X, Y\}$ *harmonically* or Y is the *harmonic conjugate* of X with respect to $\{P, Q\}$ if

$$\mathrm{cr}(P, X, Q, Y) = -1.$$

2.6.1 Möbius tetrahedra and Koenigs cubes

Consider a pair of complete quadrangles with a common quadrangular set as shown in Figure 2.17. The figure has an interpretation in 3-dimensional space as well. Consider two planes in 3-space containing the two complete quadrangles that intersect in the line with the common quadrangular set. Then a pair of lines that meets in a point of the quadrangular set defines a plane through four points – two of each quadrangular set. Now we add labels of a combinatorial cube to the vertices of the two complete quadrangles as shown in Figure 2.18. Then every vertex of the cube lies in one plane with its neighbors in the cube, for example, the vertex P_{12} lies in one plane with the vertices P_1, P_2, and P_{123}. This plane corresponds to a pair of lines that intersect in a point of the quadrangular set. We introduce labels for

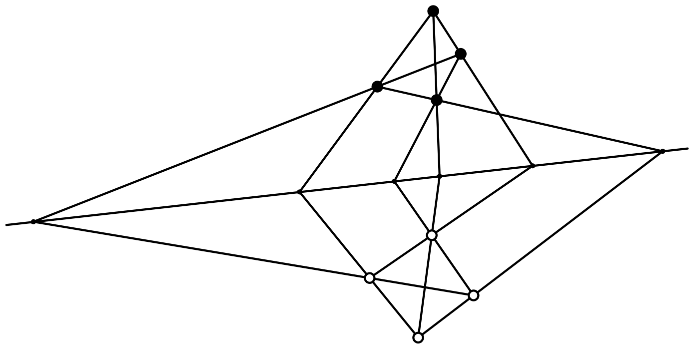


Figure 2.17: Two complete quadrangles sharing a common quadrangular set.

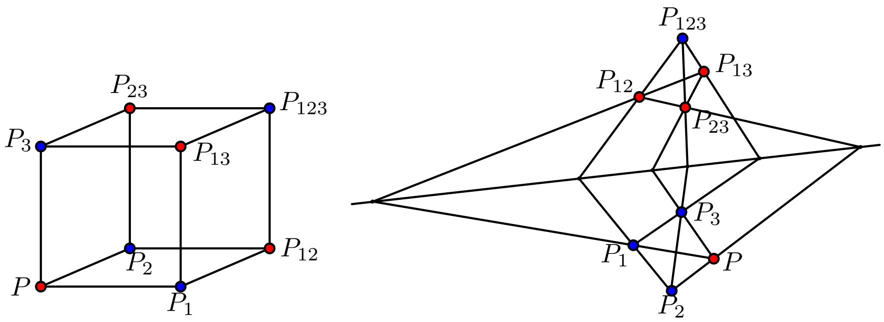


Figure 2.18: Möbius pair of tetrahedra and a shared quadrangular set

these planes in the following way: the plane A_i (resp. A_{ij}, A_{123}, and A) contains the vertex P_i and all its neighboring vertices in the combinatorial cube. The configuration with these eight vertices and eight planes is a Möbius pair of tetrahedra.

Definition 2.38. A pair of tetrahedra $(P, P_{12}, P_{13}, P_{23})$ and (P_1, P_2, P_3, P_{123}) is a *Möbius pair of tetrahedra* if each vertex of each tetrahedron lies in the corresponding face plane of the other tetrahedron, i.e.,

$$\begin{aligned} P &\in (P_1, P_2, P_3) =: A & \qquad P_{123} &\in (P_{12}, P_{13}, P_{23}) =: A_{123} \\ P_1 &\in (P, P_{12}, P_{13}) =: A_1 & P_{12} &\in (P_1, P_2, P_{123}) =: A_{12} \\ P_2 &\in (P, P_{12}, P_{23}) =: A_2 & P_{13} &\in (P_1, P_3, P_{123}) =: A_{13} \\ P_3 &\in (P, P_{13}, P_{23}) =: A_3 & P_{23} &\in (P_2, P_3, P_{123}) =: A_{23} \end{aligned}$$

Theorem 2.39. *Consider two tetrahedra in $\mathbb{RP}^3$. If the four vertices of the first tetrahedron lie in the four face planes of the second and three (of four) vertices of the second lie in three face planes of the first, then the fourth vertex of the second tetrahedron lies in the remaining face plane of the first.*

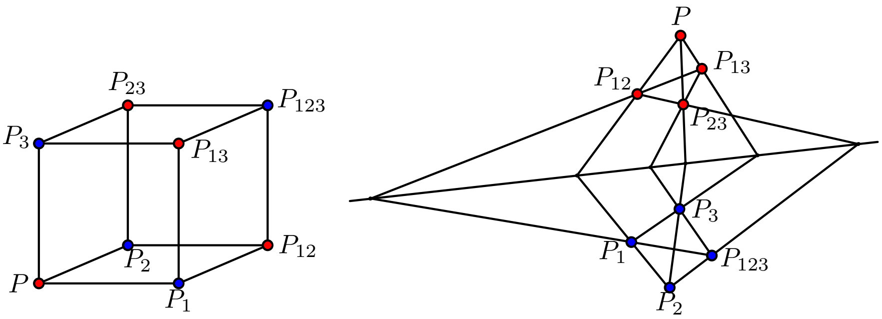


Figure 2.19: Koenigs cube and and a shared quadrangular set

Proof. Follows from Lemma 2.35 on quadrangular sets. □

Another labelling of the two complete quadrangles with a common quadrangular set gives a combinatorial cube with planar faces. This labelling is shown in Fig. 2.19. This configuration is called a Koenigs cube.

Definition 2.40. A cube with vertices P, P_i, P_{ij}, and P_{123} is a *Koenigs cube* if all its faces (P, P_i, P_j, P_{ij}) and $(P_i, P_{ij}, P_{ik}, P_{123})$ as well as the black $(P, P_{12}, P_{13}, P_{23})$ and white (P_1, P_2, P_3, P_{123}) vertices are planar.

As in the case of the Möbius pair seven incidences determine the eigth.

Theorem 2.41. *Consider a three dimensional cube with planar faces and vertices P, P_i, P_{ij}, and P_{123}. Then the white vertices (P_1, P_2, P_3, P_{123}) lie in one plane if and only if the black vertices $(P, P_{12}, P_{13}, P_{23})$ lie in one plane.*

The proof is again an application of Lemma 2.35 on quadrangular sets.

2.6.2 Projective involutions of the real projective plane

Suppose $f : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2$ is a projective involution of the real projective plane. Let $A \in \mathbb{R}\mathrm{P}^2$ be a point which is not a fixed point, and $A' = f(A)$. Then also $A = f(A')$ and hence the line AA' is mapped to itself. Let $B \in \mathbb{R}\mathrm{P}^2$ be a point not on this line which is also not a fixed point, and $B' = f(B)$. Then the line BB' is also mapped to itself, so $P = AA' \cap BB'$ is a fixed point of f.

The restriction of f to the line AA' is an involution of AA' with a fixed point P, so it has another fixed point Q, and this is the point such that $\{P, Q\}$ separates $\{A, A'\}$ harmonically. Equally, the restriction of f to the line BB' is an involution of BB' with fixed points P and R such that $\{P, R\}$ separates $\{B, B'\}$ harmonically. Now f fixes every point on the line $\ell = QR$. (Can you see why?) Thus:

Any projective involution of $\mathbb{R}\mathrm{P}^2$ has a whole line ℓ of fixed points and another fixed point $P \notin \ell$.

Conversely, if ℓ is a line in $\mathbb{RP}^2$ and P is a point not on ℓ, then there is a unique projective involution f that fixes P and every point on ℓ. This is *the projective reflection on ℓ and P.*

Indeed if X, Y are any two points on ℓ, and any representative vectors of P, X, Y are chosen as basis of $\mathbb{R}^3$, then the matrix of f in this basis must be $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(What does this reflection look like in an affine chart in which ℓ is the line at infinity? What does it look like if P is a point at infinity?)

2.7 The fundamental theorem of real projective geometry

We know that projective transformations map lines to lines. For real projective spaces they are in fact *all* transformations that map lines to lines.

Theorem 2.42 (Fundamental theorem of real projective geometry)**.** *If a bijective map $\mathbb{RP}^n \to \mathbb{RP}^n$ $(n > 1)$ maps lines to lines, then it is a projective transformation.*

We will present a proof of the fundamental theorem only for the case $n = 2$ of the real projective plane. This already contains all the important ideas, so you can figure out for yourself how it works for $n > 2$. The proof depends on the following two lemmas.

Lemma 2.43. *Let $f : \mathbb{RP}^1 \to \mathbb{RP}^1$ be a bijective map such that if $A, B, C, D \in \mathbb{RP}^1$ are four points with $\operatorname{cr}(A, B, C, D) = -1$, then $\operatorname{cr}(f(A), f(B), f(C), f(D)) = -1$. Then f is a projective transformation.*

Proof. We will show that if f also fixes 0, 1, and ∞, it must be the identity. This implies the lemma: For general f let g be the projective transformation that maps $f(0) \mapsto 0$, $f(1) \mapsto 1$, $f(\infty) \mapsto \infty$. Then the composition $g \circ f$ satisfies the assumptions of the theorem and fixes $0, 1, \infty$. If it is the identity, then $f = g^{-1}$ is a projective transformation.

So assume in addition that f fixes 0, 1, ∞. Then for all $x, y \in \mathbb{R}$:

(1) $f(\frac{x+y}{2}) = \frac{f(x)+f(y)}{2}$, because $\operatorname{cr}(x, \frac{x+y}{2}, y, \infty) = -1$.

(2) $f(2x) = 2f(x)$, because $\operatorname{cr}(0, x, 2x, \infty) = -1$.

(3) $f(x + y) = f(x) + f(y)$. This follows from (1) and (2).

(4) $f(-x) = -f(x)$ because $0 = f(0) = f(x + (-x)) = f(x) + f(-x)$.

(5) $f(nx) = nf(x)$ for $n \in \mathbb{Z}$. This follows from (3) and (4).

(6) $f(qx) = qf(x)$ for $q \in \mathbb{Q}$. This follows from (5).

(7) $f(q) = q$ for $q \in \mathbb{Q}$ because $f(q) = f(q \cdot 1) = qf(1) = q \cdot 1$.

(8) $f(x^2) = f(x)^2$. This follows form (4) and $\operatorname{cr}(-x, 1, x, x^2) = -1$.

(9) $x > 0 \Rightarrow f(x) > 0$. This follows from (8) because the a real number is positive if and only if it is the square of a real number.

(10) f is increasing on $\mathbb{R}$. This follows from (3,4,9) because

$$0 < x - y \quad \Longrightarrow \quad 0 < f(x-y) = f(x) - f(y).$$

Finally: An increasing function on $\mathbb{R}$ which fixes the rationals is the identity: Assume there exists $x \in \mathbb{R} \setminus \mathbb{Q}$ with $f(x) = y \neq x$. Without loss of generality assume $x < y$. Then there exists a rational $q \in Q$ with $x < q < y$. But f fixes the rationals, but $x < q$ and $f(x) > q$ which contradicts (10). Thus f is the identity. □

Lemma 2.44. *Let $f : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2$ be a bijective map that maps lines to lines. Further let A, B, C, D be four distinct points on a line in $\mathbb{R}\mathrm{P}^2$ with $\mathrm{cr}(A, B, C, D) = -1$. Then* $\mathrm{cr}(f(A), f(B), f(C), f(C)) = -1$.

Proof. Use Theorem 2.36 on the complete quadrilateral. □

of the fundamental theorem, $n = 2$. We will show that if f also fixes the four points

$$P_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad P_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad P_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

it must be the identity. This implies the theorem: For general f let $g : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2$ be the projective transformation that maps $f(P_i)$ to P_i. Then the composition $g \circ f$ is bijective, maps lines to lines and fixes the points P_i. If it is the identity, then $f = g^{-1}$ is a projective transformation.

So assume that $f : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2$ is bijective, maps lines to lines and fixes P_1, P_2, P_3, P_4. Let $X \in \mathbb{R}\mathrm{P}^2$ be any point not on the line P_1P_2 (which we consider as the line at infinity). We will show that $f(X) = X$. The points and lines of the following construction are shown in Figure 2.20.

Let $\ell_1 = P_3P_1$ and $\ell_2 = P_3P_2$. Since f fixes these points, it maps ℓ_1 to ℓ_1 and ℓ_2 to ℓ_2. By the lemmas, the restrictions $f|_{\ell_i} : \ell_i \to \ell_i$ are projective transformations. But $f|_{\ell_1}$ fixes P_1, P_3, and $E_1 = P_2P_4 \cap \ell_1$, so it is the identity. Equally, $f|_{\ell_2}$ fixes P_2, P_3, and $E_2 = P_1P_4 \cap \ell_2$, so it is the identity. Hence f fixes also $X_1 = P_2X \cap \ell_1$ and $X_2 = P_1X \cap \ell_2$. Since f maps lines to lines, $X = X_1P_2 \cap X_2P_1$ implies $f(X) = f(X_1)f(P_2) \cap f(X_2)f(P_1) = X_1P_2 \cap X_2P_1 = X$.

We have shown that $f(X) = X$ for all X not on P_1P_2. But then it also fixes all points on P_1P_2. (Why?) Hence, f is the identity.

□

Note that it is not necessary to assume that the map is continuous. Further we could replace the preservation of lines by the preservation of arbitrary k-planes.

Lemma 2.45. *A bijective map $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$ maps lines to lines if and only if it maps k-planes to k-planes.*

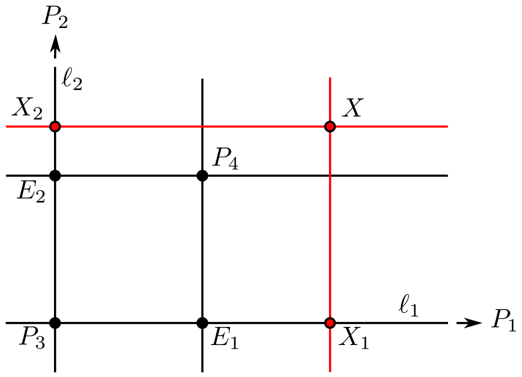


Figure 2.20: Proof of the fundamental theorem in dimension two.

Remark 2.46. *The corresponding statement for* $\mathbb{C}\mathrm{P}^n$ *is false. For example, the conjugation map* $\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$,

$$\begin{bmatrix} z_1 \\ \vdots \\ z_{n+1} \end{bmatrix} \longmapsto \begin{bmatrix} \bar{z}_1 \\ \vdots \\ \bar{z}_{n+1} \end{bmatrix}$$

is bijective and maps lines to lines, but it is not a projective transformation. The fundamental theorem for general projective spaces $\mathrm{P}(V)$ *of a vector space* V *over a field* F *says the following: If* $f : \mathrm{P}(V) \to \mathrm{P}(V)$ *is bijective and maps lines to lines, then* f *comes from an almost linear map* $\varphi : V \to V$. *A map* $\varphi : V \to W$ *between vector spaces over* F *is called* almost linear *if for all* $u, v \in V$, $\lambda \in F$,

$$\varphi(u + v) = \varphi(u) + \varphi(v) \qquad \text{and} \qquad \varphi(\lambda v) = \alpha(\lambda)\varphi(v),$$

where $\alpha : F \to F$ *is a field automorphism. (For example, complex conjugation is an automorphism of* $\mathbb{C}$. *The field* $\mathbb{R}$ *of real numbers has no automorphism except the identity.)*

Corollary 2.47. *A bijective map* $f : \mathbb{R}^n \to \mathbb{R}^n$ *which maps lines to lines is an affine transformation* $f(x) = Ax + b$ *for some* $A \in GL(n, \mathbb{R}), b \in \mathbb{R}^n$.

(Because any such map $\mathbb{R}^n \to \mathbb{R}^n$ can be extended to a bijective map $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$ which maps lines to lines and the hyperplane at infinity to itself. (How?))

Localized version of the fundamental theorem. *Let* U *be a subset of* $\mathbb{R}\mathrm{P}^n$ *that contains an open ball* $B \subset \mathbb{R}^n \subset \mathbb{R}\mathrm{P}^n$. *Suppose an injective map* $f : U \to \mathbb{R}\mathrm{P}^n$ *maps lines to lines in the following sense: If* ℓ *is a line in* $\mathbb{R}\mathrm{P}^n$ *which intersects* U, *then there is a line* ℓ' *such that* $f(\ell \cap U) = \ell' \cap f(U)$. *Then* f *is the restriction of a projective transformation of* $\mathbb{R}\mathrm{P}^n$ *to* U.

Again, for simplicity, I will present a proof for the case $n = 2$ only. This already contains all the important ideas so you can figure out for yourself how it works for $n > 2$.

Proof. (for $n = 2$) Define a map $\hat{f} : \mathbb{RP}^2 \to \mathbb{RP}^2$ as follows. For $X \in B$ let $\hat{f}(X) = f(X)$. If $X \notin B$, let ℓ_1, ℓ_2 be two lines through X that intersect B and let $f(X)$ be the intersection of the lines ℓ'_1 and ℓ'_2, the images of ℓ_1, ℓ_2 under f (in the sense explained in the theorem). This point is well defined because it does not depend on the choice of ℓ_1 and ℓ_2. To see this, use Desargues' theorem to show that if ℓ_1, ℓ_2, ℓ_3 are three lines that intersect B and all go through one point outside B, then their images under f intersect in one point. You have to convince yourself that you always have enough room in the open ball to construct (the relevant part of) a Desargues figure. (See left figure below.)

We have defined $\hat{f}$ using only information about f on B. In fact, $\hat{f}$ coincides with f on U. (Why?) Further, $\hat{f}$ maps lines to lines. To see this, use (the inverse) Desargues' theorem to show that $\hat{f}$ maps three points on a line to three points on a line. Again you have to convince yourself that you have enough room in B to construct (the relevant part of) a Desargues figure. (See right figure below.) Finally, by the fundamental theorem (global version), $\hat{f}$ is a projective transformation. $\square$

2.8 Duality

In homogeneous coordinates x_1, x_2, x_3, the equation for a line in a projective plane is

$$a_1x_1 + a_2x_2 + a_3x_3 = 0,$$

where not all coefficients a_i are zero. The coefficients a_1, a_2, a_3 can be seen as homogeneous coordinates for the line, because if we replace in the equation a_i by λa_i for some $\lambda \neq 0$ we get an equivalent equation for the same line. Thus, the set of lines in a projective plane is itself a projective plane, the *dual plane*. Points in the dual plane correspond to lines in the original plane. Moreover, if we consider in the above equation the x_i as fixed and the a_i as variables, we get an equation for a line in the dual plane. Points on this line correspond to lines in the original plane that contain $[x]$. Thus, a the points on a line in the dual plane correspond to lines in the original plane through a point.

It makes sense to look at this phenomenon in a basis independent way and for arbitrary dimension. It boils down to the duality of vector spaces.

Let V be a finite dimensional vector space over a field F.

The *dual vector space* V^* of V is the vector space of linear functions $V \to F$ (linear forms on V).

If $v_1, \ldots, v_n$ is a basis of V, the *dual basis* of V^* is $\varphi_1, \ldots, \varphi_n$ with $\varphi_i(v_j) = \delta_{ij}$. In particular $\dim V = \dim V^*$. But there is no natural way to identify V^* with V. ("Natural" means independent of any arbitrary choices. In this case: choice of a basis.)

There is, however, a natural identification of V with V^{**}: A vector $v \in V$ is identified with the linear form $V^* \to F$, $\varphi \mapsto \varphi(v)$. With this identification, V is also the dual vector space of V^*.

Let $f : V \to W$ be a linear map. The *dual linear map* $f^* : W^* \to V^*$ is defined by $f^*(\psi)(v) = \psi(f(v))$. Note that the dual map "goes in the opposite direction". If f is invertible, then $f^{*-1} = f^{-1*}$ is a map $V^* \to W^*$.

If $U \subseteq V$ is a linear subspace, the *annihilator* of U is the linear subspace

$$U^0 = \{\varphi \in V^* \mid \varphi(v) = 0 \text{ for all } v \in U\} \subseteq V^*$$

of linear forms that vanish on U.

This provides a correspondence between subspaces of V with subspaces of V^*.

The dimensions of U and U^0 are related by

$$\dim U + \dim U^0 = \dim V.$$

Indeed, let $v_1, \dots, v_k$ be a basis for U and extend it to a basis $v_1, \dots, v_n$ of V. Let $\varphi_1, \dots, \varphi_n$ be the dual basis of V^*. Then (one sees easily that) $\varphi_{k+1}, \dots, \varphi_n$ is a basis of U^0.

(In fact, the above dimension formula is just a coordinate free way of saying that each linearly independent homogeneous equation in the coordinates reduces the dimension of the solution space by 1.)

If U_1 and U_2 are subspaces of V, then

$$(U_1 \cap U_2)^0 = {U_1}^0 + {U_2}^0 \qquad \text{and} \qquad (U_1 + U_2)^0 = {U_1}^0 \cap {U_2}^0.$$

(Can you see this?)

Now let $\mathrm{P}(V)$ be the n-dimensional projective space of an $(n+1)$-dimensional vector space V. The *dual projective space* is $\mathrm{P}(V^*)$.

A point $[v] \in \mathrm{P}(V)$ corresponds to the hyperplane $\mathrm{P}([v]^0) \in \mathrm{P}(V^*)$, and a point $[\varphi] \in \mathrm{P}(V^*)$ corresponds to the hyperplane $\mathrm{P}([\varphi]^0)$ in $\mathrm{P}(V)$. Note that the points of the hyperplane $\mathrm{P}([\varphi]^0)$ correspond to the hyperplanes in $\mathrm{P}(V)$ that contain $[v]$.

In general, a k-plane $\mathrm{P}(U) \subseteq \mathrm{P}(V)$ corresponds to the plane $\mathrm{P}(U^0) \subseteq \mathrm{P}(V^*)$ of dimension

$$\dim U^0 - 1 = \dim V - \dim U - 1 = (n+1) - (k+1) - 1 = n - k - 1.$$

The points in $\mathrm{P}(U^0)$ correspond to the hyperplanes in $\mathrm{P}(V)$ that contain $\mathrm{P}(U)$.

Let us take another look at duality for projective planes. (Hyperplanes in a plane are lines.) To aid the imagination, let us focus on the real projective plane $\mathbb{R}\mathrm{P}^2 = \mathrm{P}(\mathbb{R}^3)$ and its dual plane $\mathrm{P}(\mathbb{R}^{3*})$ which we denote by $\mathbb{R}\mathrm{P}^{2*}$ (although everything holds in general).

So each point in $\mathbb{R}\mathrm{P}^2$ corresponds to a line in $\mathbb{R}\mathrm{P}^{2*}$ and vice versa. The points on a line in $\mathbb{R}\mathrm{P}^2$ correspond to the lines through the corresponding point in $\mathbb{R}\mathrm{P}^{2*}$. Lines through a point in $\mathbb{R}\mathrm{P}^2$ correspond to the points on the corresponding line in $\mathbb{R}\mathrm{P}^{2*}$.

Every theorem about $\mathbb{R}\mathrm{P}^2$ can also be read as a theorem about $\mathbb{R}\mathrm{P}^{2*}$. This leads to the following *duality principle*:

From every theorem that talks only about incidence relations between points and lines in a projective plane, one obtains another valid theorem by interchanging the words “point” and “line” (and the phrases “goes through” and “lies on”).

For example, the theorem that is obtained from the Desargues theorem in this way (the dual Desargues theorem) turns out to be the converse of Desargues’s theorem.

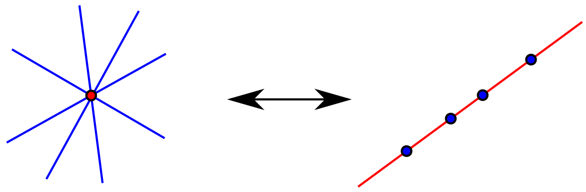


Figure 2.21: Duality in the real projective plane

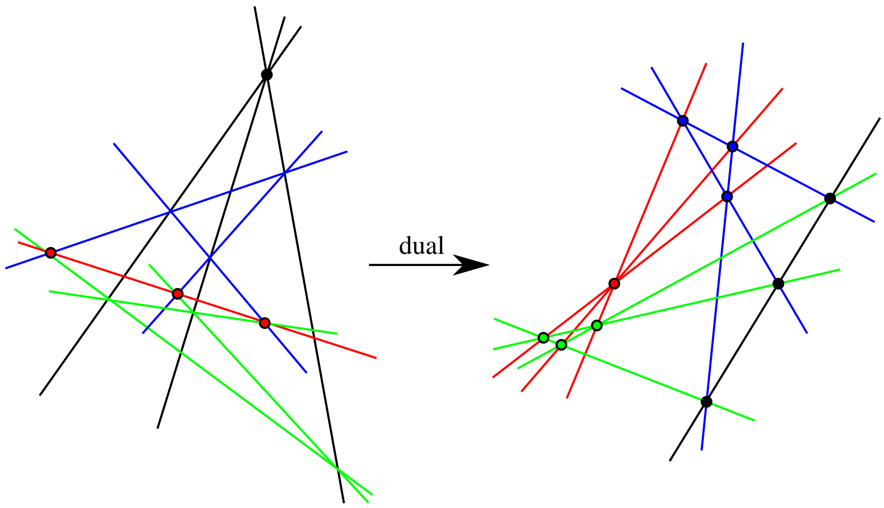


Figure 2.22: Desargues configuration and its dual

We had seen that the the converse of Desargues is equivalent to Desargues, so Desargues's theorem turns out to be *self-dual*. The same is true for Pappus's theorem. (Check it out.)

Note that four lines through a point in $\mathbb{R}\mathrm{P}^2$ correspond to four points on a line in $\mathbb{R}\mathrm{P}^{2*}$. But for four points on a line we had defined the cross ratio. Via duality this gives us a definition for the cross ratio of four lines through a point.

Proposition 2.48. *Let P be a point in $\mathbb{R}\mathrm{P}^2$ and let P^* be the corresponding line in $\mathbb{R}\mathrm{P}^{2*}$, so that each point of P^* corresponds to a line through P. Let ℓ be a line in $\mathbb{R}\mathrm{P}^2$ that does not contain P. Then the map $P^* \to \ell$ that maps a point of P^* to the intersection of the corresponding line with ℓ is a projective transformation.*

Proof. Let $P = [v_1]$, and let $[v_2], [v_3]$ be two points on ℓ. Then v_1, v_2, v_3 is a basis of $\mathbb{R}^3$. Let $\varphi_1, \varphi_2, \varphi_3$ be the dual basis of $\mathbb{R}^{3*}$. The line P^* is spanned by $[\varphi_2], [\varphi_3]$. Hence the points $[\varphi] \in P^*$ have representative vectors $\varphi = s\varphi_2 + t\varphi_3$, and s, t are homogeneous coordinates on P^*. The line in $\mathbb{R}\mathrm{P}^2$ corresponding to $[\varphi]$ intersects ℓ in a point $[v]$ such that

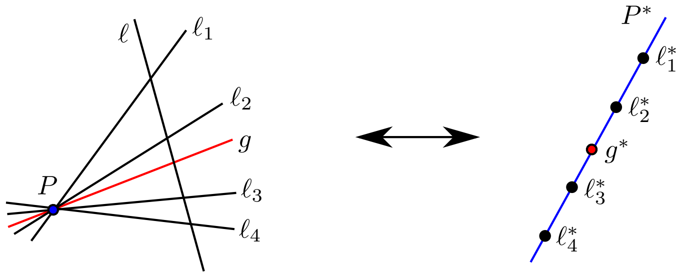


Figure 2.23: Cross-ratio of four lines through a point

$v = xv_2 + yv_3$ and

$$0 = \varphi(v) = (s\varphi_2 + t\varphi_3)(xv_2 + yv_3) = sx + ty.$$

This is the case for $x = t, y = -s$. So the map $P^* \to \ell$ in question comes from the linear map $s\varphi_2 + t\varphi_3 \mapsto tv_2 - sv_3$. $\square$

2.9 Conic sections – The Euclidean point of view

In this section we will study conic sections in $\mathbb{R}^2$. The properties studied are invariant under Euclidean transformations, i.e., the group transformations generated by reflections (rotations, and translations).

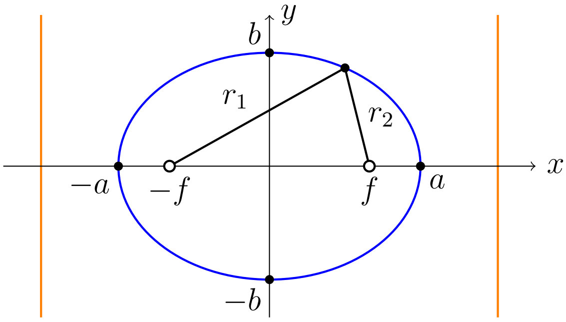
Definition 2.49. The set of solutions of any quadratic equation in two variables x, y,

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0 \tag{$*$}$$

is called a *conic section* or a *conic*.

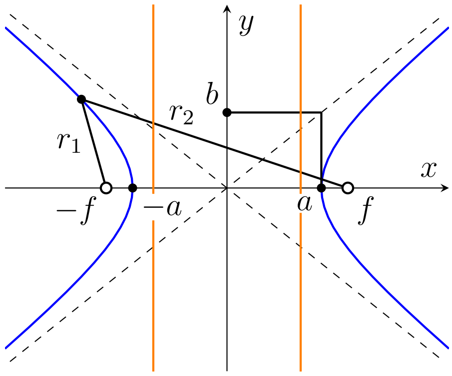
Theorem 2.50. *There is a change of coordinates* $\binom{u}{v} = A\binom{x}{y} + t$ *with* $A \in O(2)$, $t \in \mathbb{R}^2$ *which reduces* ($*$) *to one of the standard forms: Ellipses (including circles), parabolas, hyperbolas, and the degenerate cases of a pair of lines, which may degenerate further to one "double" line, and a single point, or the empty set.*

ellipse



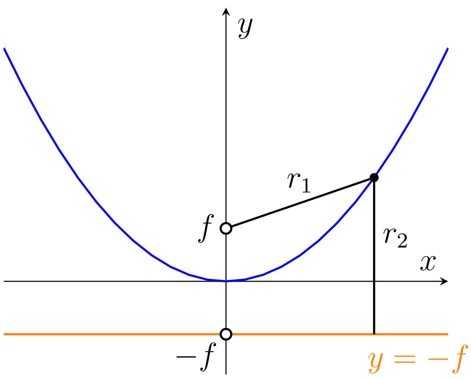
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$
$$e = \sqrt{1 - \frac{b^2}{a^2}} < 1$$
$$f = \sqrt{a^2 - b^2}$$

hyperbola



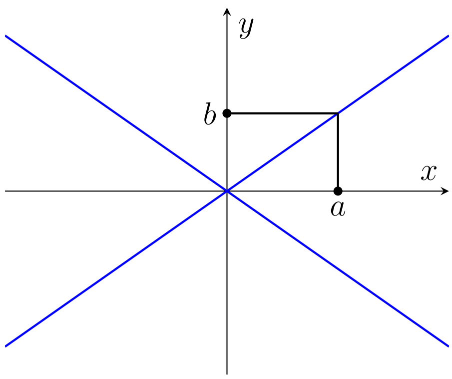
$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1.$$
$$|r_1 - r_2| = \mathit{const.}$$
$$e = \sqrt{1 + \frac{b^2}{a^2}} > 1$$
$$f = \sqrt{a^2 + b^2}$$

parabola



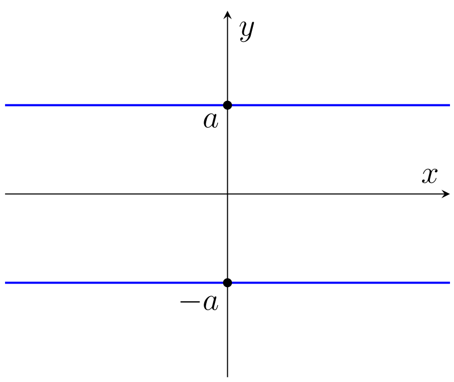
$$y = ax^2$$
$$r_1 = r_2$$
$$f = \frac{1}{4a}$$

two intersecting lines



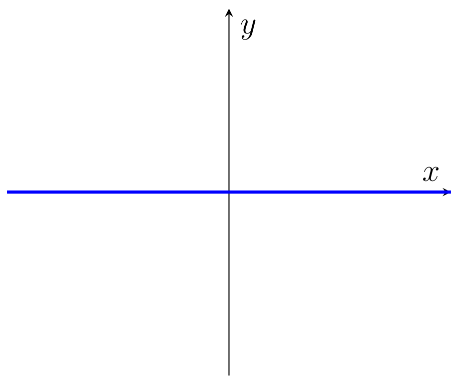
$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 0$$

two parallel lines



$y^2 = a$

one “double” line



$y^2 = 0$

Proof. The symmetric matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ can be diagonalized by the conjugation with an orthogonal matrix A. Subsequently, the conic can be centered by the corresponding translation $t \in \mathbb{R}^2$. $\square$

The following theorem can be proven by direct computation.

Theorem 2.51. *1. The sum of the distances from a point on an ellipse to its foci is constant. The difference of the distances from a point on a hyperbola to its foci is constant.*

2. The ratio of the distances from a point on an ellipse, hyperbola, or parabola to a focus and to the corresponding directrix is constant and called the eccentricity.

The name conic section comes from the fact that they arise as intersections of a plane with a cone (or cylinder in the case of two parallel lines).

Theorem 2.52. *Consider the right circular cone intersected by a plane not containing its vertex. The corresponding section curve is (see Fig. 2.24):*

- *an ellipse, if the plane intersects all generators (straight lines) of the cone in one sheet,*
- *a hyperbola, if the plane intersects both sheets, and*
- *a parabola, if the plane intersects all but one generator in one sheet.*

A proof of this fact for ellipses is presented in Fig. 2.25.

Proof. Let $\mathcal{Q}$ be the intersection of a round cone with a plane E as shown in Fig. 2.25. Consider two spheres inscribed in the cone and touching E from different sides. Let F_1 and F_2 be the corresponding touching points and C_1 and C_2 the corresponding touching circles on the cone. Consider a point $A \in \mathcal{Q}$ and the line ℓ on the cone passing through A.

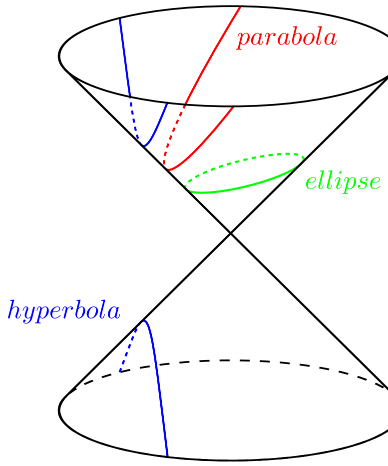


Figure 2.24: Intersection of cone of revolution with affine planes

Let $A_1 = \ell \cap C_1$ and $A_2 = \ell \cap C_2$ be the intersection points with the touching circles. Since all touching segments to a sphere from a point have equal lengths, we obtain that $|AF_1| + |AF_2| = |AA_1| + |AA_2| = |A_1A_2|$ is the same for all points on $\mathcal{Q}$. Thus $\mathcal{Q}$ is an ellipse with foci F_1 and F_2. □

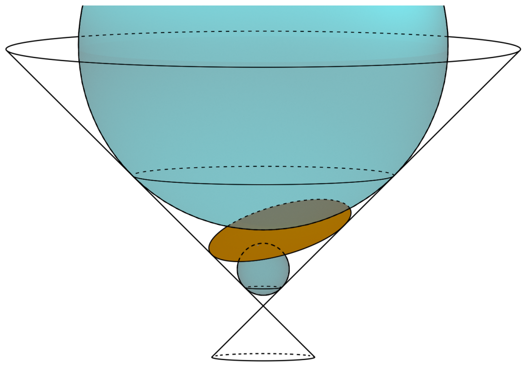


Figure 2.25: Illustration of Dandelin's proof that the conic section is an ellipse.

2.9.1 Optical properties of the conic sections

Theorem 2.53. *An elliptic mirror has the property that light beams emitted from one focus converge at the other focus (see Fig. 2.26, left). Light beams emitted from one focus of a hyperbolic mirror after reflection are emitted from the other focus (see Fig. 2.26, middle).*

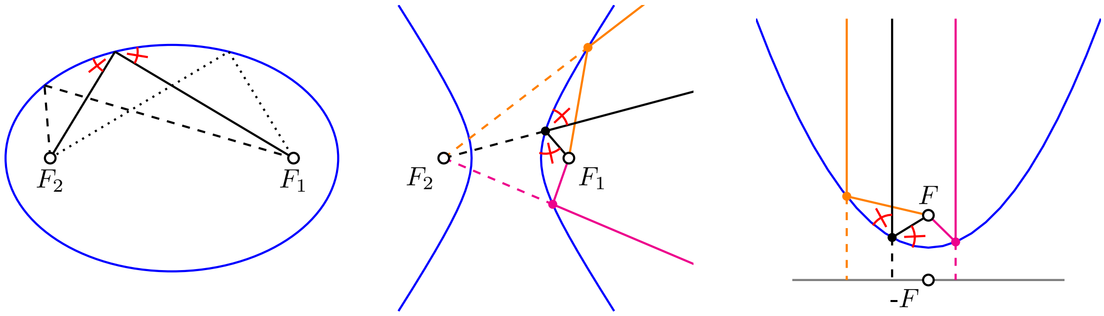


Figure 2.26: Reflection in conic mirrors

Light beams emitted from the focus of a parabolic mirror after reflection become parallel to the axis of the parabola (see Fig. 2.26, right).

Proof. We give a proof of the elliptic case only. The other two can be proved similarly.

Let P be a point on the ellipse with distances r_1 and r_2 from the foci F_1 and F_2, respectively. Extend the line segment F_2P a distance of r_1 beyond P. Call the new endpoint of the extended segment F_2'. The tangent of the ellipse at P is the perpendicular bisector ℓ of F_1F_2'. Indeed, P lies on ℓ because it has equal distance r_1 from F_1 and F_2'. Consider any other point $\tilde{P}$ on ℓ and let $\tilde{r}_1$ be its distance to both F_1 and F_2' and let $\tilde{r}_2$ be its distance to F_2. Then $\tilde{r}_1 + \tilde{r}_2 > r_1 + r_2$ so $\tilde{P}$ does not lie on the ellipse. Hence, ℓ intersects the ellipse in precisely one point, P, and thus is tangent to P. Now the equality of the angles follows easily. □

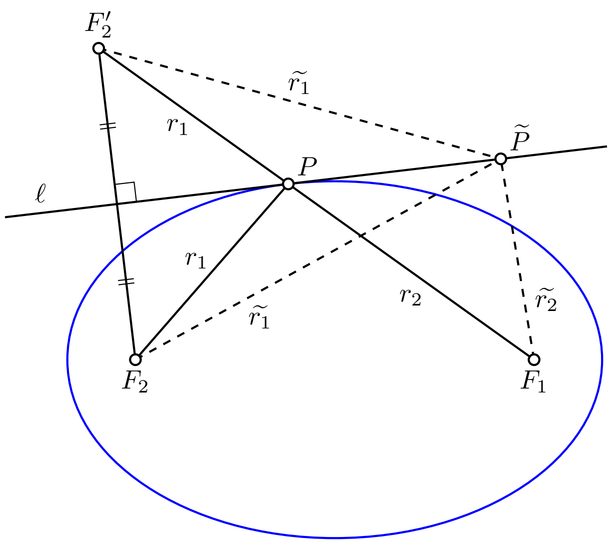


Figure 2.27: Illustration of the proof of the optical properties of an ellipse.

Remark 2.54. *This theorem describes also trajectories in elliptic billiards which are governed by the same reflection law (more on elliptic billiards in Sect. 2.14.*

The following theorem can be proven in the same way

Theorem 2.55. *Let c be a circle with center F_2 and let F_1 be a point inside c. The locus of the centers of all circles that go through F_1 and touch c is an ellipse with foci F_1 and F_2.*

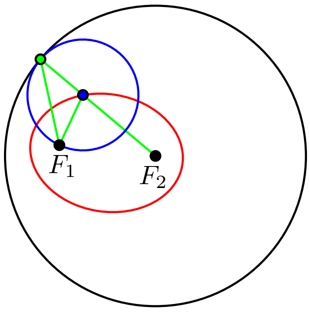


Figure 2.28: Construction of an ellipse from two focal points and two circles.

2.10 Conics – The projective point of view

Before we look at conics in the real projective plane, we need to recall some basic facts about quadratic forms.

Quadratic forms

Definition 2.56. Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A map $\mathrm{q}\colon V \times V \to \mathbb{R}$ is a *symmetric bilinear form*, if

$$\begin{aligned} \mathrm{q}(\alpha_1 v_1 + \alpha_2 v_2, w) &= \alpha_1\,\mathrm{q}(v_1, w) + \alpha_2\,\mathrm{q}(v_2, w) && \text{for } \alpha_1, \alpha_2 \in \mathbb{R},\ v_1, v_2, w \in V \\ \mathrm{q}(v, w) &= \mathrm{q}(w, v) && \text{for all } v, w \in V \end{aligned}$$

The bilinear form q is *non-degenerate*, if

$$\mathrm{q}(v, w) = 0 \quad \forall w \in V \quad \Rightarrow \quad v = 0\,.$$

The corresponding *quadratic form* is defined by $\mathrm{q}(v) = \mathrm{q}(v, v)$.

We denote the symmetric bilinear form and the corresponding quadratic form by the same letter, since they are equivalent via the following polarization identity:

$$\mathrm{q}(v, w) = \frac{1}{2}\big(\,\mathrm{q}(v + w) - \mathrm{q}(v) - \mathrm{q}(w)\big)\,.$$

If $\{b_1, \ldots, b_n\}$ is a basis of V, then we can associate a matrix to a bilinear form by

$$Q = (q_{ij})_{i,j=1,\ldots,n} \quad \text{with} \quad q_{ij} := \mathrm{q}(b_i, b_j)\,.$$

Hence the bilinear form may be evaluated in the following way:

$$\mathrm{q}(v,w) = \mathrm{q}\left(\sum_{i=1}^n x_i b_i\,, \sum_{j=1}^n y_j b_j\right) = \sum_{i=1}^n \sum_{j=1}^n \mathrm{q}(b_i, b_j) x_i y_j = \sum_{i=1}^n \sum_{j=1}^n q_{ij} x_i y_j = v^T Q w\,.$$

For the quadratic form this implies

$$\mathrm{q}(v) = v^T Q v = \sum_{i,j=1}^n q_{ij} x_i x_j\,.$$

Remark 2.57. *Quadratic forms correspond to homogeneous polynomials of degree two. With respect to affine coordinates in* $\mathbb{R}\mathrm{P}^2$ *these polynomials are of the form of Definition 2.49.*

Theorem 2.58. *For vector spaces over* $\mathbb{R}$*, for a given quadratic form* q *there exists a basis such that*

$$\mathrm{q}(v) = \sum_{i=1}^p x_i^2 - \sum_{i=p+1}^{p+q} x_i^2\,.$$

The triple $(p, q, n-p-q)$ *is the* signature *of the quadratic form* q*. The signature is invariant wrt. change of basis.* q *is non-degenerate, if and only if* $p + q = n$*. Over* $\mathbb{C}$ *there exists a basis such that*

$$\mathrm{q}(v) = \sum_{i=1}^p z_i^2\,.$$

For small p, q, n we will also use the alternative notation

$$(\underbrace{++\ldots+}_{p}\ \underbrace{--\ldots-}_{q}\ \underbrace{00\ldots0}_{n-p-q})$$

for the signature.

Definition 2.59. If q is a symmetric bilinear form on $\mathbb{R}^3$. Then

$$\mathcal{Q} = \{[x] \in \mathbb{R}\mathrm{P}^2 \mid \mathrm{q}(x,x) = 0\}$$

is a *conic* or a *conic section*.

This definition does not depend on a choice of basis. If we choose a basis b_1, b_2, b_3 of $\mathbb{R}^3$ then we can associate a symmetric 3×3-matrix Q with q such that

$$\mathrm{q}(v,w) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^t \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\,,$$

where $v = \sum_{i=1}^{3} x_i b_i$ and $w = \sum_{i=1}^{3} y_i b_i$. So a symmetric bilinear form is defined by 6 real values and can be identified with a point in $\mathbb{R}^6$. But q and $\lambda\,\mathrm{q}$ (with $\lambda \neq 0$) define the same conic $\mathcal{Q}$, so a conic corresponds to a point in $\mathbb{R}\mathrm{P}^5$.

According to the Sylvester Theorem there exists a basis, such that

$$\mathrm{q}(v) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^3 \quad \text{with } \lambda_i = -1, 0, 1\,.$$

With this normal form it is now easy to classify the conics in $\mathbb{R}\mathrm{P}^2$. The conic is determined by the signature of the bilinear form. Since the conic of a symmetric bilinear form is defined by a homogeneous equation flipping all the signs will not change the conic. So there exist only two different non-degenerate conics up to projective transformations

- $\mathrm{q}(v) = x_1^2 + x_2^2 + x_3^2$ with signature $(+{+}{+})$: The corresponding conic is empty. In the rest of the book we will only consider non-empty conics, if not stated otherwise.
- $\mathrm{q}(v) = x_1^2 + x_2^2 - x_3^2$ with signature $(+{+}{-})$: Depending on the choice of affine coordinate, this conic is an ellipse, a parabola or a hyperbola.

So in projective geometry the conics with signature $(+ + -)$ are the same, in particular, there exist an affine coordinate such that the conic is a circle. As the choice of affine coordinate corresponds to a projective transformation there exist projective transformations that map the circle to an ellipse, a parabola, or a hyperbola. The Euclidean shape of the conic depends on the line that is mapped to infinity by the projective transformation as shown in Figure 2.29. If the line is outside the circle we obtain an ellipse. If we choose a tangent line of the circle to be mapped to infinity, then we obtain a parabola. Hence we say, that the parabola is tangent to the line at infinity. If the line intersects the circle, then the circle is mapped onto a hyperbola. The two points where the line intersected the circle are mapped to line at infinity and correspond to the directions of the asymptotes of the hyperbola.

If we consider signatures with 0-entries, we obtain the degenerate conics as well.

- $\mathrm{q}(v) = x_1^2 + x_2^2$ with signature $(+{+}0)$ is a point in $\mathbb{R}\mathrm{P}^2$
- $\mathrm{q}(v) = x_1^2 - x_2^2$ with signature $(+{-}0)$ is a pair of lines in $\mathbb{R}P^2$ defined by $x_1 = x_2$, $x_1 = -x_2$. Depending of the affine coordinates the two lines may intersect or be parallel.
- $\mathrm{q}(v) = x_1^2$ with signature $(+00)$ is a line in $\mathbb{R}\mathrm{P}^2$

Theorem 2.60. *A non-degenerate conic determines its corresponding quadratic form up to a scalar factor.*

Proof. Let $\mathcal{Q}$ be a non-degenerate conic determined by the quadratic form q, and let e_1, e_2, e_3 be an orthonormal basis of q with signature $(+{+}{-})$. Let $\tilde{\mathrm{q}}$ be another quadratic form defining the conic,

$$\mathcal{Q} = \{[v] \in \mathbb{R}\mathrm{P}^2 \,|\, \tilde{\mathrm{q}}(v) = 0\}.$$

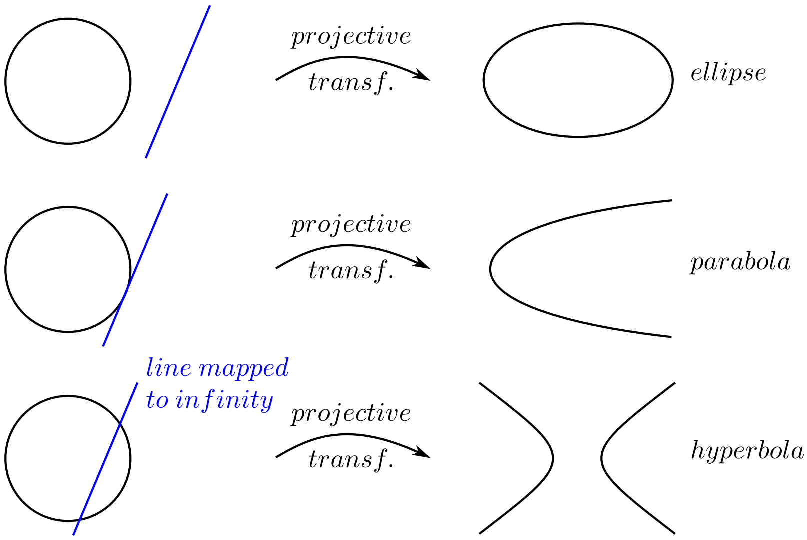


Figure 2.29: Projective transformations mapping a circle onto an ellipse, a parabola, or a hyperbola.

Then $\tilde{\mathrm{q}}(e_1 \pm e_3) = \mathrm{q}(e_2 \pm e_3) = 0$ since $[e_1 \pm e_3], [e_2 \pm e_3] \in \mathcal{Q}$. This implies $\tilde{\mathrm{q}}(e_1, e_3) = \tilde{\mathrm{q}}(e_2, e_3) = 0$ and $\lambda := \mathrm{q}(e_1) = \mathrm{q}(e_2) = -\mathrm{q}(e_3)$. Taking $[e_1 + e_2 + \sqrt{2}e_3] \in \mathcal{Q}$ we get $\tilde{\mathrm{q}}(e_1 + e_2 + \sqrt{2}e_3) = 0$ and finally $\tilde{\mathrm{q}}(e_1, e_2) = 0$. Thus $\tilde{\mathrm{q}} = \lambda\,\mathrm{q}$. $\square$

Theorem 2.61. *Let P_1, P_2, P_3, P_4, P_5 be five points in $\mathbb{R}P^2$, then there exists a conic through $P_1, \ldots, P_5$. Moreover*

- *If no four points lie on a line, the conic is unique.*
- *If no three points lie on a line, the conic is non-degenerate.*

We start with a simple but important observation.

Lemma 2.62. *If three collinear points are on a conic, then the conic contains the whole line.*

Proof. Let q be the symmetric bilinear defining the conic. Since we consider three distinct points on a projective line we may choose vectors v_1 and v_2 such that the three points are $A = [v_1]$, $B = [v_2]$, $C = [-v_1 - v_2]$, see Lemma 2.10. Since the points lie on the conic we have $\mathrm{q}(v_1, v_1) = \mathrm{q}(v_2, v_2) = 0$ and

$$0 = \mathrm{q}(-v_1 - v_2) = \mathrm{q}(-v_1 - v_2, -v_1 - v_2) = \mathrm{q}(v_1, v_1) + 2\,\mathrm{q}(v_1, v_2) + \mathrm{q}(v_2, v_2)$$
$$\Rightarrow \mathrm{q}(v_1, v_2) = 0\,.$$

So for an arbitrary point $X = [sv_1 + tv_2]$ on the line we obtain

$$\mathrm{q}(sv_1 + tv_2) = s^2\,\mathrm{q}(v_1, v_1) + 2st\,\mathrm{q}(v_1, v_2) + t^2\,\mathrm{q}(v_2, v_2) = 0$$

and X lies on the conic. $\square$

of Theorem 2.61. First we prove the existence of a conic: Let $P_i = [v_i]$ with $v_i \in \mathbb{R}^3$ with $i = 1, \ldots, 5$. Let $\mathrm{q} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be a symmetric bilinear form defining a conic $\mathcal{Q}$ represented by the symmetric matrix Q. Then an incidence $P_i \in \mathcal{Q}$ yields a homogeneous equation for the entries of Q: $\mathrm{q}(v_i, v_i) = 0 \Leftrightarrow v_i^t Q v_i = 0$. The system $\mathrm{q}(v_i, v_i) = 0$ for $i = 1, \ldots, 5$ is a system of five homogeneous linear equations for six variables q_{ij}, $1 \leq i \leq j \leq 3$. It has at least a one dimensional space of solutions. So there exists a quadratic form, such that the conic contains all P_i. If the linear equation are dependent we obtain a higher dimensional space of solutions.

Now we turn to the question of uniqueness: If four of the P_i lie on a line, then the conic contains a line and is degenerate. By the above classification of the degenerate conics we see, that there exists a one parameter family on conics as shown in Figure 2.30.

Let three points P_1, P_2 , P_3 lie on a line ℓ, but P_4 and P_5 do not. Then by Lemma 2.62 $\ell \subset \mathcal{Q}$ and $\mathcal{Q}$ is degenerate. So the conic consists of the lines ℓ and the line P_4P_5. In particular $\mathcal{Q}$ is unique (see Figure 2.30).

Now assume that no three of the five points are collinear and q_1 and q_2 are two symmetric bilinear forms with $\mathrm{q}_1(v_i, v_i) = \mathrm{q}_2(v_i, v_i) = 0$. Then the bilinear form $\mathrm{q} = \mathrm{q}_1 + \lambda\, \mathrm{q}_2$ satisfies $\mathrm{q}(v_i, v_i) = 0$ for all $i = 1, \ldots, 5$, as well. The determinant $\det(\mathrm{q}_1 + \lambda\, \mathrm{q}_2)$ is a polynomial in λ of degree three. So it has a real zero, i.e., there exists λ_0 such that $\det(\mathrm{q}_1 + \lambda_0\, \mathrm{q}_2) = 0$. The conic defined by $\mathrm{q}_1 + \lambda_0\, \mathrm{q}_2$ is degenerate. Thus three points must be collinear. This is a contradiction and hence $\mathrm{q}_1 = \mathrm{q}_2$ is unique and non-degenerate. $\square$

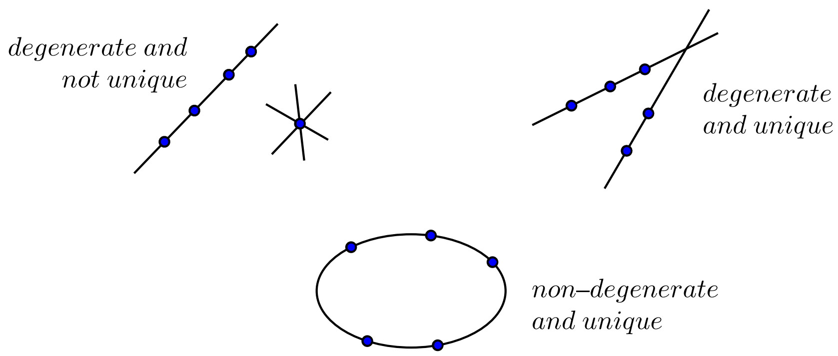


Figure 2.30: Conics through five points: There exist a one parameter family of degenerate conics through five points, if four of the points are collinear (top left). If only three are collinear, the conic is unique but degenerate (top right). If no three points are collinear, then the conic is unique and non-degenerate.

Another point of view on the above theorem is the following: A conic may be identified with a symmetric bilinear form q. As we have seen in Thm. 2.60 this bilinear form is unique up to non-zero multiples. So we may use the entries of the corresponding symmetric 3×3 matrix as homogeneous coordinates and obtain a point in $\mathbb{RP}^5$ for each conic. So with this interpretation $\mathbb{RP}^5$ becomes the space of conics. As we have seen in the proof of Theorem 2.61, a point that lies on a conic defines a homogeneous equation for the coordinates/matrix entries. In other words, all conics that pass through a point lie in a hyperplane in $\mathbb{R}P^5$. So all conics passing through five points lie in the intersection of the corresponding five hyperplanes. This intersection is generically a point and this point represents the conic through the five points.

We have found interpretations for points and some hyperplanes in $\mathbb{R}P^5$ in terms of conics. So the next object we study are lines in the space of conics.

2.11 Pencils of conics

Definition 2.63. A *pencil of conics* is a line in $\mathbb{R}P^5$, which is the space of conics in $\mathbb{R}P^2$.

Let $[\mathrm{q}_1]$ and $[\mathrm{q}_2]$ be two conics, then the conic $[\mathrm{q}]$ is in the pencil spanned by $[\mathrm{q}_1]$ and $[\mathrm{q}_2]$, if there exist homogeneous coordinates (λ_1, λ_2), such that $\mathrm{q} = \lambda_1 \mathrm{q}_1 + \lambda_2 \mathrm{q}_2$.

Proposition 2.64. *Let P_1, P_2, P_3, P_4 be four points in general position in $\mathbb{R}P^2$. Then the conics through these four points build a pencil. The pencil contains three degenerate conics determined by the quadratic forms: $\mathrm{q}_1 = l_{12}l_{34}$, $\mathrm{q}_2 = l_{13}l_{24}$, $\mathrm{q}_3 = l_{14}l_{23}$, where $l_{ij} : \mathbb{R}^3 \to \mathbb{R}$ is the linear function vanishing on the points P_i and P_j (see Figure 2.31).*

Proof. For an arbitrary fifth point $P_5 \neq P_i$ for $i = 1, \dots, 4$ there exists a unique conic through $P_1, \dots, P_5$ by Thm. 2.61. The conics of the pencil are given by homogeneous coordinates (λ_1, λ_2) with $\mathrm{q} = \lambda_1 \mathrm{q}_1 + \lambda_2 \mathrm{q}_2$. The fifth point $P_5 = [v_5]$ lies on the conic defined by q if

$$\mathrm{q}(v_5) = 0 \quad \Leftrightarrow \quad \lambda_1 \mathrm{q}_1(v_5) + \lambda_2 \mathrm{q}_2(v_5) = 0\,.$$

A solution to this equation is given by $\lambda_1 = \mathrm{q}_2(v_5)$ and $\lambda_2 = -\mathrm{q}_1(v_5)$. So the conic containing P_5 lies in the pencil and is given by $\mathrm{q} = \mathrm{q}_2(v_5)\,\mathrm{q}_1 - \mathrm{q}_1(v_5)\,\mathrm{q}_2$. $\square$

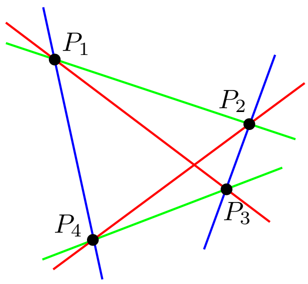


Figure 2.31: A pencil of conics defined by four points contains three degenerate conics.

The pencil of conics defined by four points is special, since not for all pencils there exist four points that lie on all conics of the pencil. With the argument from the proof of Theorem 2.61 we can deduce, that every pencil contains at least one degenerate conic.

Geometrically this can be expressed in the following way: The set of degenerate conics in $\mathbb{R}\mathrm{P}^5$ is given by the singular symmetric 3×3 matrices. These can be described by the determinant, i.e., $\det Q = 0$. This is a homogeneous polynomial of degree three so it describes a cubic hypersurface in $\mathbb{R}\mathrm{P}^5$. If we restrict the determinant to a pencil we obtain a real cubic polynomial and this has at least one real root, i.e., every line in $\mathbb{R}\mathrm{P}^5$ intersects the cubic hypersurface of singular conics. Further, if the cubic polynomial restricted to the pencil is not constantly zero, we may have either one, two or three distinct roots. In case of three distinct roots,

Pencils of conics are a nice tool to proof the following classical theorem by Pascal. The theorem can be seen as a generalization of Pappus' theorem from degenerate to non-degenerate conics.

Theorem 2.65 (Pascal's theorem)**.** *Let A, B, C, D, E, F be six points on a non-degenerate conic. Then the intersection points of opposite sides of the hexagon A, B, C, D, E, F are collinear, i.e., there exists a line (Pascal line) containing $G = AB \cap DE$, $I = BC \cap EF$, $H = CD \cap AF$.*

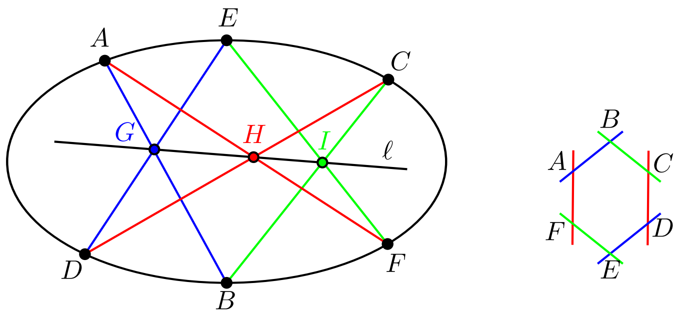


Figure 2.32: Pascal's theorem.

Proof. Consider the two pencils of the conics through A, B, C, D and A, D, E, F, respectively. Then both pencils contain the original conic defined by q and there exist λ_1, λ_2 and μ_1, μ_2 such that:

$$\begin{aligned} & \mathrm{q} = \lambda_1 \ell_{AB}\ell_{CD} + \lambda_2 \ell_{AD}\ell_{BC} = \mu_1 \ell_{AF}\ell_{DE} + \mu_2 \ell_{AD}\ell_{EF} \\ \Leftrightarrow \quad & \lambda_1 \ell_{AB}\ell_{CD} - \mu_1 \ell_{AF}\ell_{DE} = \ell_{AD}(\mu_2 \ell_{EF} - \lambda_2 \ell_{BC}) \end{aligned}$$

We will show, that the line given by $\mu_2 \ell_{EF} - \lambda_2 \ell_{BC}$ contains the intersection points $G = AB \cap DE$, $H = AF \cap CD$, and $I = BC \cap EF$ (see Fig. 2.32).

The point $G = [v_G]$ is on the line, since $\ell_{AD}(v_G) \neq 0$, but $\ell_{AB}(v_G) = 0$ and $\ell_{DE}(v_G) = 0$. Similarly, we obtain that H is on the line. Finally, $I = [v_I]$ is on the line, since $\ell_{EF}(v_I) = 0$ and $\ell_{BC}(v_I) = 0$. $\square$

2.12 Rational parametrizations of conics

Conics can be parametrized using the projection to a line from a point on a conic.

Theorem 2.66. *Let $\mathcal{Q}$ be a non-degenerate conic defined by the quadratic form* q*, $W = [w]$ a point on the $\mathcal{Q}$, and ℓ a line not containing W.*

1. *There exists a bijection $f : \mathcal{Q} \to \ell$ such that for any point $A = [a] \in \mathcal{Q}$ the points $A, W, f(A)$ are collinear.*
2. *The inverse mapping $f^{-1} : \ell \to \mathcal{Q}$ parametrizes the conic by quadratic polynomials:*

$$a = \mathrm{q}(x,x)w - 2\,\mathrm{q}(w,x)x\,, \tag{2.1}$$

   where $A = [a], W = [w], f(A) = [x]; a, w, x \in \mathbb{R}^3$.

3. *The projections to two different lines $f_1 : \mathcal{Q} \to \ell_1$ and $f_2 : \mathcal{Q} \to \ell_2$ differ by a projective transformation $f_2 \circ f_1^{-1} : \ell_1 \to \ell_2$.*
4. *Projections from two different points W_1 and W_2 on $\mathcal{Q}$ also differ by a projective transformation of ℓ.*

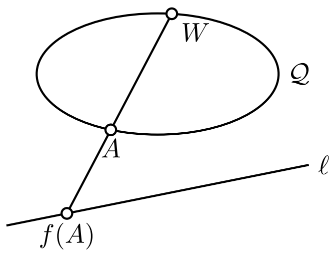


Figure 2.33: Projection of a point on a conic to a line.

Proof. 1. The line through W and A contains no further points of $\mathcal{Q}$ since $\mathcal{Q}$ is non-degenerate (see Lemma 2.45). Obviously f is injective since different lines through W intersect ℓ at different points. The surjectivity of f follows from the explicit formula for f^{-1} in 2.

2. Let $f(A) = [x]$. Then $A = [a] = [\lambda w + \mu x]$ with $\mathrm{q}(\lambda w + \mu x) = 0$. The last identity determines λ and μ:

$$0 = \mathrm{q}(\lambda w + \mu x, \lambda w + \mu x) = 2\lambda\mu\,\mathrm{q}(w,x) + \mu^2\,\mathrm{q}(x,x)\,.$$

   The case $\mu = 0$ is exceptional $A = W$, the corresponding line is tangent. If $\mu \neq 0$ then $2\lambda\,\mathrm{q}(w,x) + \mu\,\mathrm{q}(x,x) = 0$ implies 2.1. The right hand side is quadratic with respect to x.

3. The transformation $f_2 \circ f_1^{-1} : \ell_1 \to \ell_2$ is the central projection with the center W, which is a projective tranformation, see Proposition 2.16.

4. By a projective transformation any non-degenerate conic can normalized to a circle. Let $f_i : \mathcal{Q} \to \ell, i = 1, 2$ be the projections to ℓ from W_i. The map $f_2 \circ f_1^{-1} : \ell \to \ell$ is a projective transformation since it preserves the cross-ratios (see Theorem 2.26). Indeed, computing the areas of the corresponding triangles we obtain

$$\mathrm{cr}(f_i(A_1), f_i(A_2), f_i(A_3), f_i(A_4)) = -\frac{\sin\alpha \sin\gamma}{\sin\beta \sin(\alpha+\beta+\gamma)} .$$

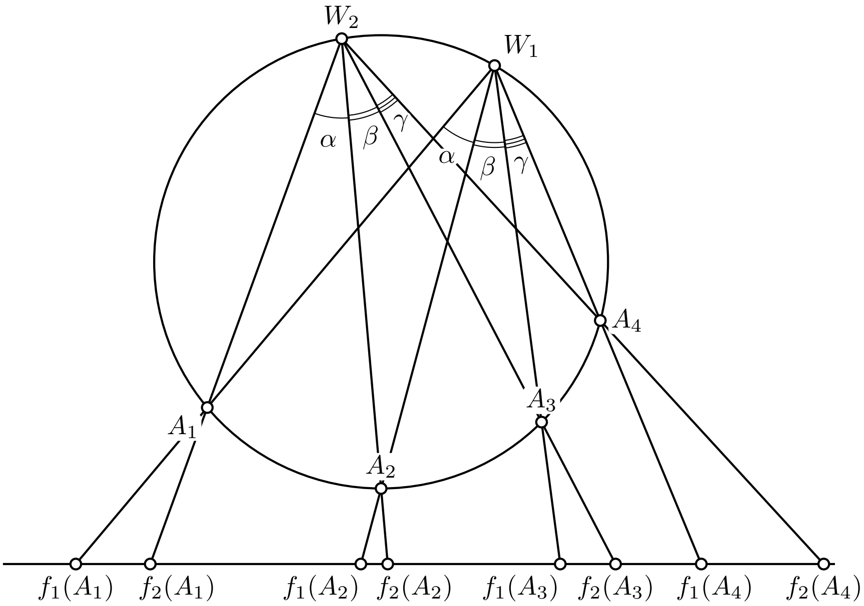


Figure 2.34: Projection from two different points on a circle. The corresponding angles coincide.

□

Definition 2.67. The *cross-ratio* of four points $A_i, i = 1, \dots, 4$ on a conic is defined as the cross-ratio of their projections $f(A_i), i = i = 1, \dots, 4$ to a line from a point on a conic or as the cross-ratio of lines $WA_i, i = 1, \dots, 4$ passing through A_i and a point W on the conic (see Definition 2.19)

$$\begin{aligned} \mathrm{cr}(A_1, A_2, A_3, A_4) :&= \mathrm{cr}(f(A_1), f(A_2), f(A_3), f(A_4)) \\ &= \mathrm{cr}(WA_1, WA_2, WA_3, WA_4) \end{aligned}$$

Corollary 2.68. *A non-degenerate Euclidean conic*

$$au^2 + 2buv + cv^2 + du + ev + f = 0$$

can be parametrized via quadratic polynomials p, q, r*:*

$$u = \frac{p(t)}{r(t)}; \quad v = \frac{q(t)}{r(t)}.$$

2.13 The pole-polar relationship, the dual conic and Brianchon's theorem

If we have a non-degenerate symmetric bilinear form q on $\mathbb{R}^n$ then we can define a duality of subspace with respect to the bilinear form similar to orthogonality in case of Euclidean vectorspaces.

Definition 2.69. Let $U \leq \mathbb{R}^n$ be a vector subspace. Then

$$U^\perp = \{v \in \mathbb{R}^n \mid \mathrm{q}(u, v) = 0, \forall u \in U\}.$$

If $U = \{u_1, \ldots, u_k\}$ then

$$U^\perp = (\text{span } U)^\perp = \{v \in V \mid \mathrm{q}(u_i, v) = 0, \forall i = 1, \ldots, k\}.$$

We call $U^\perp$ the *orthogonal complement* of U.

If the dimension of U is k then the dimension of the orthogonal complement $U^\perp$ is $n - k$.

For $\mathbb{R}^2$ we have two possible signatures for the non-degenerate symmetric bilinear form $(++)$ or $(+-)$. In the first case we have the Euclidean scalar product and we know what orthogonality means. So let us have a look at the orthogonal complement of a 1-dimensional subspace in case of the indefinite case $(+-)$ symmetric bilinear form:

$$\mathrm{q}\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = x_1 x_2 - y_1 y_2.$$

Then the orthogonal complement of a line spanned by the vector (x_1, y_1) is the line of all (x, y) with $0 = \mathrm{q}((x, y), (x_1, y_1)) = x_1 x - y_1 y$. So it is spanned by the vector (y_1, x_1). This is the image of (x_1, y_1) under the reflection in the line spanned by $(1, 1)$. In particular, the orthogonal complement of the subspace generated by $(1, 1)$ is the subspace itself.

If q is indefinite, i.e., the signature contains $+$'s and $-$'s, then there exist vectors v with $\mathrm{q}(v, v) = 0$. These are called *isotropic vectors*.

In the above example, if we restrict to the subspace $\{\lambda(1, 1) \mid \lambda \in \mathbb{R}\}$, the non-degenerate bilinear form restricted to this subspace is degenerate. We even have a basis of isotropic vectors $\{(1, 1), (-1, 1)\}$. Nevertheless, the matrix representing the bilinear form with respect to this basis is $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$, and in particular not singular.

Using the orthogonal complement, we define a map between the points and the lines of $\mathbb{R}\mathrm{P}^2$ in the following way:

$$\begin{array}{rcl} \text{point } A = [a] & \mapsto & \text{line } \ell = \mathrm{P}(\{a\}^\perp) \\ \text{line } \ell = \mathrm{P}(U) & \mapsto & \text{point } A = \mathrm{P}(U^\perp). \end{array}$$

Definition 2.70. The line $\mathrm{P}(\{a\}^\perp)$ corresponding to a point $[a]$ in $\mathbb{R}\mathrm{P}^2$ is the *polar* (or *polar line*) of $[a]$ and $[a]$ is the *pole*.

Every non-empty, non-degenerate conic defines an indefinite symmetric bilinear form and hence a *pole-polar relationship* or *polarity*, see Thm. 2.60.

Definition 2.71. A line in $\mathbb{R}\mathrm{P}^2$ is a *tangent* to a non-degenerate conic if it has one point in common with the conic.

Proposition 2.72. *The polar line of a point* on *a conic is a tangent at this point.*

Proof. Let $\mathcal{Q} = \{[x] \in \mathbb{R}P^2 \mid \mathrm{q}(x,x) = 0\}$. The polar line to $[a]$ is

$$\mathrm{P}(\{a\}^\perp) = \{[x] \in \mathbb{R}P^2 \mid \mathrm{q}(a,x) = 0\}.$$

The point $[a]$ lies on the polar, since $\mathrm{q}(a,a) = 0$. Let us show, that this is the only points of the polar on the conic. Indeed, assume $[b] \in \mathbb{R}\mathrm{P}^2$ is another point on the conic $\mathrm{q}(b,b) = 0$ on the polar $\mathrm{q}(a,b) = 0$. But then

$$\mathrm{q}(\lambda a + b, \lambda a + b) = \lambda^2\,\mathrm{q}(a,a) + 2\lambda\,\mathrm{q}(a,b) + \mathrm{q}(b,b) = \mathrm{q}(b,b) = 0,$$

and the whole line $[a][b]$ is in the conic and by Lemma 2.62 the conic is degenerate. This contradiction shows that the the polar of $[a]$ is a tangent. □

Proposition 2.73. *Let $A, B \in \mathbb{R}\mathrm{P}^2$ be two distinct points and $\mathcal{Q}$ a non-degenerate conic, then the polar lines of A and B intersect in the pole of the line AB.*

Proof. Let $A = [a], B = [b]$ and $P = \mathrm{P}(\{a\}^\perp) \cap \mathrm{P}(\{b\}^\perp) = [p]$. Then $\mathrm{q}(a,p) = 0$ and $\mathrm{q}(b,p) = 0$. Thus

$$\mathrm{q}(\lambda a + \mu b, p) = 0 \quad \text{for all } \lambda, \mu \in \mathbb{R}.$$

So the line AB is the polar line of P. □

Construction of polars. First let us note, that the inside and the outside of a non-degenerate conic can be defined in a projectively invariant way. A point lies *inside* a conic if any line through the point intersects the conic. A point lies *outside* a conic if there exists a line through the point which does not intersect the conic.

Consider a point P outside a (non-degenerate, non-empty) conic, then the polar line can be constructed using the two tangents touching the conic as shown in Fig. 2.35.

For a point P inside the conic, we can consider two arbitrary lines ℓ_1 and ℓ_2 through P. The polar line ℓ of P is the line through P_1 and P_2 that are the poles of ℓ_1 and ℓ_2 (see Fig. 2.35).

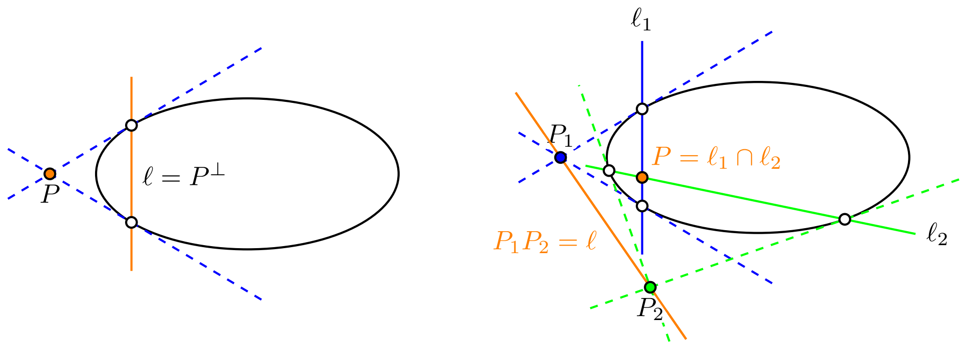


Figure 2.35: Construction of polar lines to points inside and outside the conic

Dual conics and Brianchon's Theorem

Theorem 2.74. *Let $\mathcal{Q}$ be a non-degenerate conic in $\mathbb{R}\mathrm{P}^2$. Then the set of tangents to $\mathcal{Q}$ forms a non-degenerate conic $\mathcal{Q}^*$ in the dual plane $(\mathbb{R}\mathrm{P}^2)^*$. This conic $\mathcal{Q}^*$ is called the* dual conic *of $\mathcal{Q}$.*

Proof. Let $\mathcal{Q} = \{[v] \in \mathbb{R}\mathrm{P}^2 \mid \mathrm{q}(v) = 0\}$, $v = \sum x_i e_i$, $\mathrm{q}(v) = \sum q_{ij} x_i x_j = x^t Q x$. Then the tangent line to the conic through $[x_0]$ is given by $x_0^t Q x = 0$. It is the element $[a] = [x_0^t Q] \in (\mathbb{R}\mathrm{P}^2)^*$ of the dual projective space. It belongs to the conic determined by the inverse matrix Q^{-1}:

$$aQ^{-1}a^t = x_0 Q Q^{-1} Q^t x_0 = x_0^t Q x_0 = 0$$

since $[x_0] \in \mathcal{Q}$. So the tangent lines to a conic can be identified with the points of the dual conic, and the tangent lines of the dual conic with the points of the original conic. $\square$

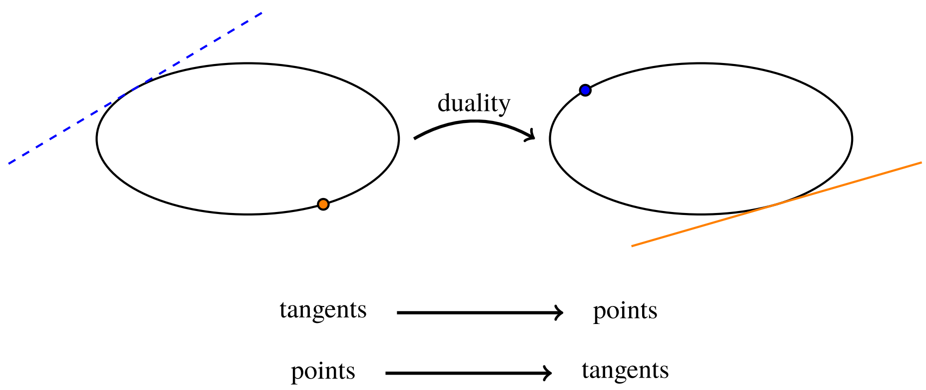


Figure 2.36: Dual of a conic yields lines enveloping a conic.

Theorem 2.75 (Brianchon)**.** *Let A, B, C, D, E, F be a hexagon circumscribed around the a conic (i.e. $AB, BC, \ldots$ are tangents), then the lines AD, BE, and CF connecting opposite points intersect in one point.*

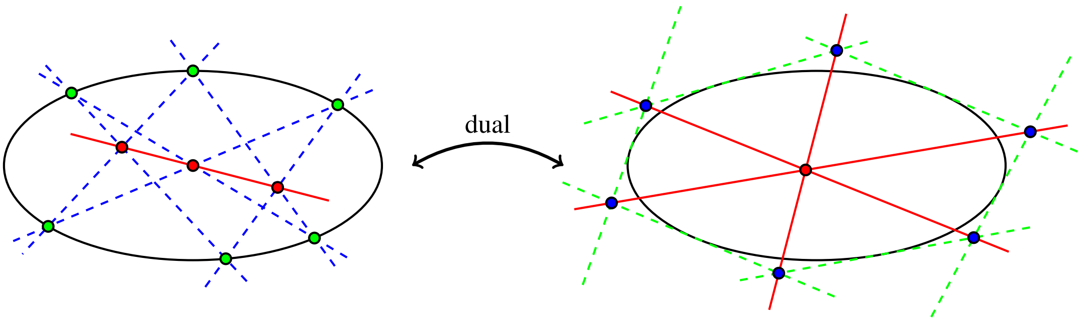


Figure 2.37: Duality of Pascal's and Brianchon's configurations

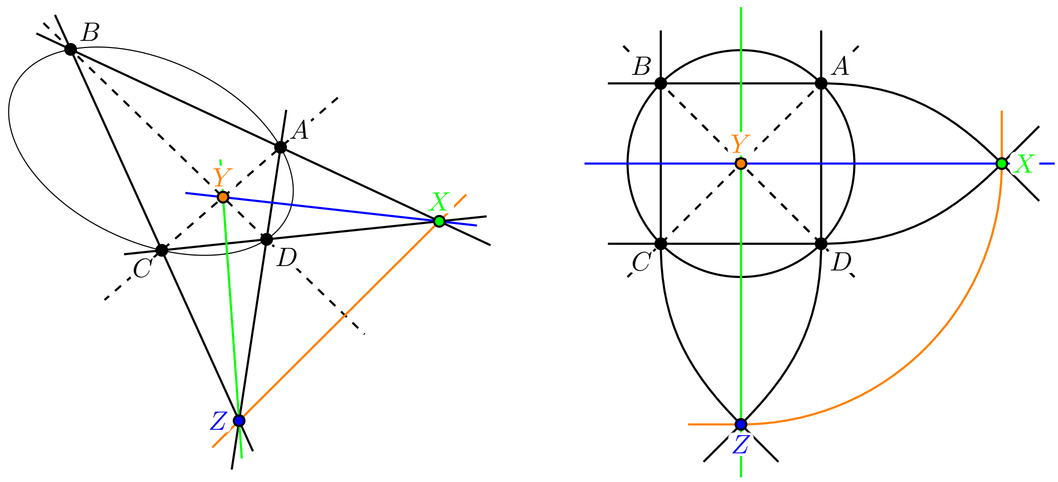


Figure 2.38: Four points on a conic with a polar triangle $\Delta(XYZ)$ (left). Normalized polar triangle configuration (right).

Proof. Dualize it and then use Pascal's theorem. Note that the cyclic order of the points on the conic is preserved. In the above picture this is not the case! The order of the points/tangents was intentionally changed to obtain a nice picture for both of the theorems. □

Theorem 2.76 (Polar triangle)**.** *Let $\mathcal{Q}$ be a non-degenerate conic in $\mathbb{RP}^2$ through four points $A, B, C, D \in \mathbb{RP}^2$. Let X, Y, Z be the intersection points of pairs of opposite sides of the complete quadrangle A, B, C, D. Then X, Y, and Z form a polar triangle, i.e. X is the pole of the line YZ, Y is the pole of the line XZ, and Z is the pole of the line XY.*

Proof. We prove the statement by calculation. So we start with a suitable normalization shown in Fig. 2.38 (right):

$$A = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ C = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \ D = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

Then

$$X = \begin{pmatrix}1\\0\\0\end{pmatrix}, \; Y = \begin{pmatrix}0\\0\\1\end{pmatrix}, \; Z = \begin{pmatrix}0\\1\\0\end{pmatrix}.$$

Let $Q = (q_{ij})_{1\le i\le j\le 3}$ be the symmetric matrix representing the conic $\mathcal{Q}$. Since the points A, B, C, and D are on the conic we obtain the following equations:

$$\begin{aligned}
q_{11} + q_{22} + q_{33} - 2q_{12} - 2q_{13} + 2q_{23} &= 0\\
q_{11} + q_{22} + q_{33} + 2q_{12} + 2q_{13} + 2q_{23} &= 0\\
q_{11} + q_{22} + q_{33} - 2q_{12} + 2q_{13} - 2q_{23} &= 0\\
q_{11} + q_{22} + q_{33} + 2q_{12} - 2q_{13} - 2q_{23} &= 0
\end{aligned}$$

By subtracting equations we obtain

$$q_{12} + q_{13} = 0, \quad q_{13} - q_{23} = 0, \quad q_{12} - q_{13} = 0.$$

This implies $q_{12} = q_{13} = q_{23} = 0$ and hence Q is diagonal

$$Q = \begin{pmatrix} q_{11} & 0 & 0\\ 0 & q_{22} & 0\\ 0 & 0 & q_{33}\end{pmatrix}$$

In the above calculation we only used that the points A, B, C, and D are on the conic. So X, Y, Z is a polar triangle for an arbitrary non-degenerate conic in the pencil. □

Corollary 2.77. *Consider a non-degenerate conic $\mathcal{Q}$, a point A not on $\mathcal{Q}$ and a line ℓ through A intersecting $\mathcal{Q}$ in two points X and Y. Let B be the intersection of ℓ with the polar line of P, then*

$$\operatorname{cr}(A, X, B, Y) = -1$$

Proof. Use the theorem on the polar triangle and the theorem on the complete quadrilateral as shown in Fig. 2.39 to obtain:

$$\operatorname{cr}(A, X', B', Y') = -1.$$

The central projection with center Z yields the desired result. □

2.14 Confocal conics and elliptic billiard

Consider a family of conics (λ-family)

$$\frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} = 1, \quad a > b\,.$$

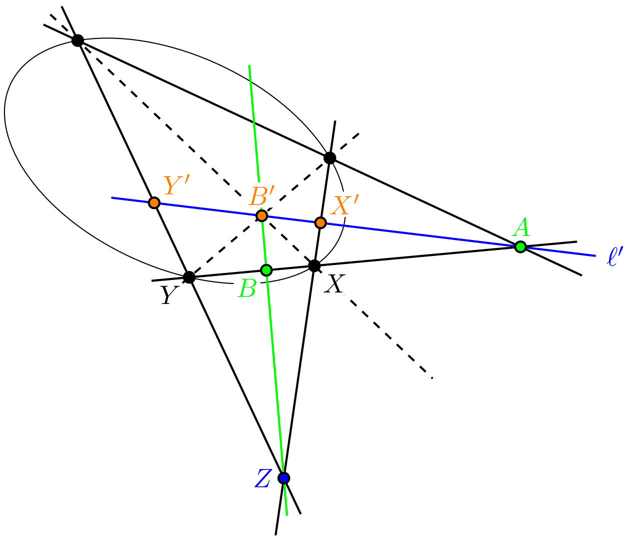


Figure 2.39: A polar triangle defines many quadruples of harmonically seperating points, e.g., the points A, X, B, Y.

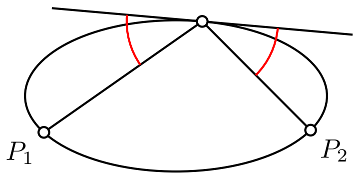


Figure 2.40: Billiard in an ellipse by equal reflection angle law.

It includes ellipses ($b > \lambda$) and hyperbolas ($b < \lambda < a$). It is easy to see that all conics of this λ-family have the same foci $(\pm f, 0) = (\pm\sqrt{a-b}, 0)$. This family of conics is called *confocal*.

Consider a billiard in an ellipse defined by the equal reflection angle law (see Figure 2.40).

Theorem 2.78. *A billiard trajectory inside an ellipse forever remains tangent to a fixed confocal conic.*

Proof. Let A_0A_1 and A_1A_2 be the two subsequentive segments of the trajectory, and assume $[A_0A_1] \cap [F_1F_2] = \varnothing$. From the optical properties of the ellipses (see Theorem 2.53) we have $\angle A_0A_1F_1 = \angle A_2A_1F_2$. Reflect F_1 and F_2 in the lines (A_0A_1) and (A_1A_2) respectively, we obtain F_1' and F_2' (see Figure 2.41). Define $B = (F_1'F_2) \cap (A_0A_1)$ and $C = (F_2'F_1) \cap (A_1A_2)$. Let Q_1 be the conic with foci F_1, F_2 (confocal) that is tangent to A_0A_1. From the optical properties of ellipses (equal reflection angles) we see that Q_1 touches (A_0A_1) at B.

Similarly the confocal conic Q_2 touches the line (A_1A_2) at C. To prove $Q_1 = Q_2$ it is enough to show that $|F_2F_1'| = |F_1F_2'|$. The triangles $\triangle\ F_1A_1F_2'$ and $\triangle\ F_1'A_1F_2$ are

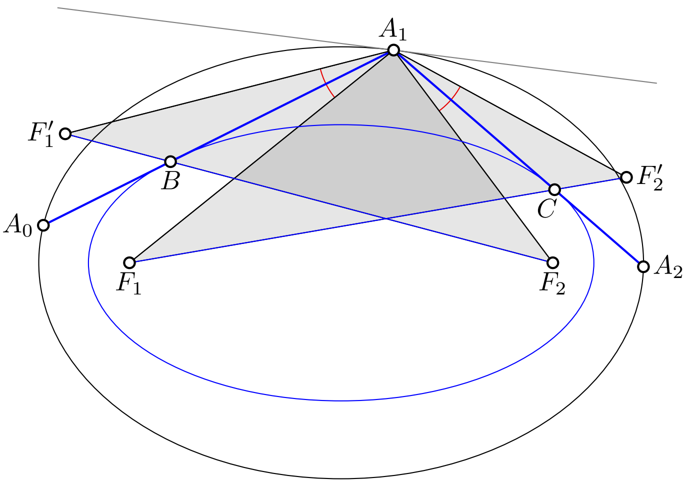


Figure 2.41: Billiard in an ellipse.

congruent, they have the same angle at A_1 and equal pairs of edges at this vertex. Their third edges must also coincide: $|F_2F_1'| = |F_1F_2'|$.

Thus, two consequentive and then all edges are tangent to the same confocal conic. □

Exercise 2.79. *Show* $\angle BAF_1 = \angle CAF_2$.

Proof. Consider the confocal conic through A and use the elliptic billiard. □

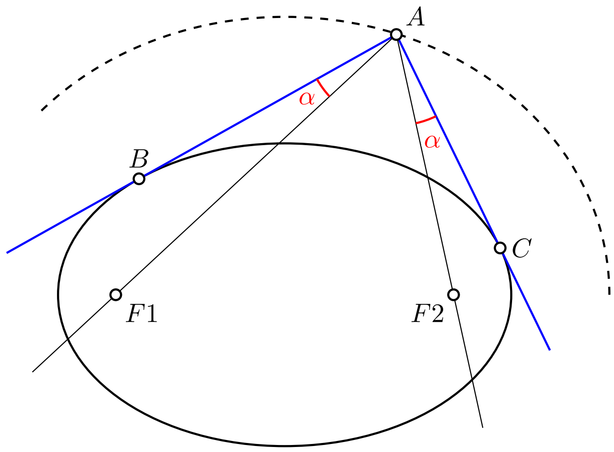


Figure 2.42: Elliptic billiard.

2.14.1 Circumscribable complete quadrilateral

Theorem 2.80 (characterization of incircles)**.** *Let* A, B, C, D *be a convex quadrilateral and* E, F *the points of intersection of opposite sides. Then the following are equivalent:*

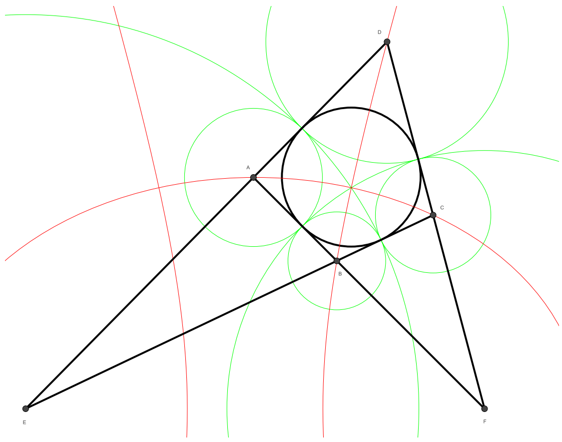


Figure 2.43: Characterization of circumscribable quadrilaterals

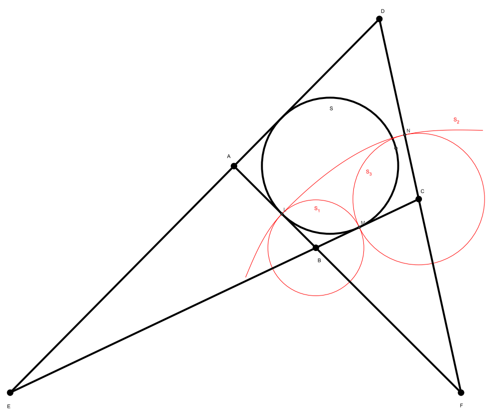


Figure 2.44: To the proof of Theorem 2.80

(i) There exists a circle inscribed into $ABCD$.

(ii) $|AB| + |CD| = |BC| + |AD|$.

(iii) $|EA| + |AF| = |EC| + |CF|$.

(iv) $|ED| - |DF| = |EB| - |BF|$.

Proof. The implications i $\Rightarrow$ ii, iii, iv are simple. Consider the corresponding touching circles with centers A, B, C, D (see Figure 2.43.

To prove the converse, iii $\Rightarrow$ i, we construct the central circle S touching DA, AB, and BC. We have to show that S touches DF.
Let S_1 and S_2 be the circles orthogonal to S with centers at B and F. Let L, M, N be

the points of intersection of S_1 and S_2 with the three line segments AB, BC, and CD respectively, as in Figure 2.44. With $|EK| = |EM|$ and $|AK| = |AL|$ we obtain

$$|EA| + |AF| = |EK| + |LF| = |EM| + |NF| \,. \tag{2.2}$$

Comparing (2.2) with iii we find $|MC| = |CN|$. Thus the circle S_3 centered at C and passing through M touches S_2 in N and intersects S orthogonally at M. We have

$$S_3 \parallel S_2 \qquad S_3 \perp S \qquad S_2 \perp S.$$

To intersect both S_2 and S_3 orthogonally the circle S must go through their touching point. Thus S touches FD in N.

The implications ii $\Rightarrow$ i and iv $\Rightarrow$ i can be proven in the same way. □

Conditions iii and iv mean that the pairs of points A, C and B, D lie on an ellipse and hyperbola with the same foci E, F respectively. Thus Theorem 2.80 is a limiting case of the Graves-Chasles theorem (which we give without proof):

Theorem 2.81 (Graves-Chasles)**.** *Let A, B, C, D be a convex quadrilateral such that all its sides touch a conic α Then the following three properties are equivalent:*

(i) There exists a circle inscribed into $ABCD$.

(ii) The points A and C lie on a conic confocal with α

(iii) The points B and D lie on a conic confocal with α

(iv) The points E and F lie on a conic confocal with α

Construction of incircular nets

Start with a circle S and four tangent lines. Let F_1 and F_2 be the intersection points of opposite tangents as in Figure 2.45. The circles S_1 and S_2 are uniquely determined. Draw the lines ℓ_1, ℓ_2 tagent to S_1 and S_2 respectively. We show that the quadrilateral $BKLM$ is circumscribed by applying Theorem 2.80 consequently:

Since G, B and B, H lie on confocal conics we also have that G, H lie on a confocal conic. Thus, $GLHD$ is circumscribed and D, L lie on a confocal conic. On the other hand, D, B lie on a confocal conic. Thus, B, L lie on a confocal conic. So, $KLMB$ is circumscribed.

Applying this construction further we get a circle pattern such that the combinatorially diagonal intersection points of straight lines lie on confocal conics.

We can use Theorem 2.81 to generalize this construction and obtain general incircular nets.

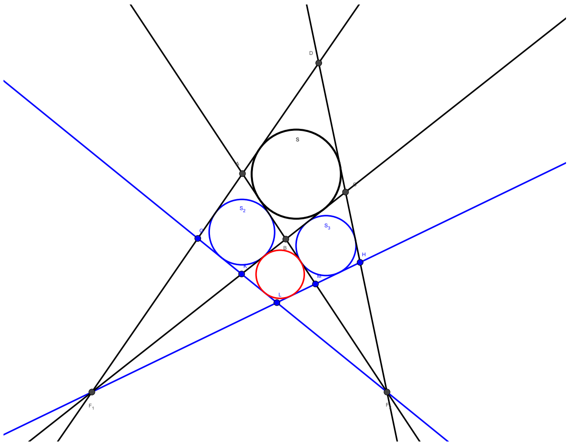


Figure 2.45: Construction of IC-nets from the degenerate Graves-Chasles theorem.

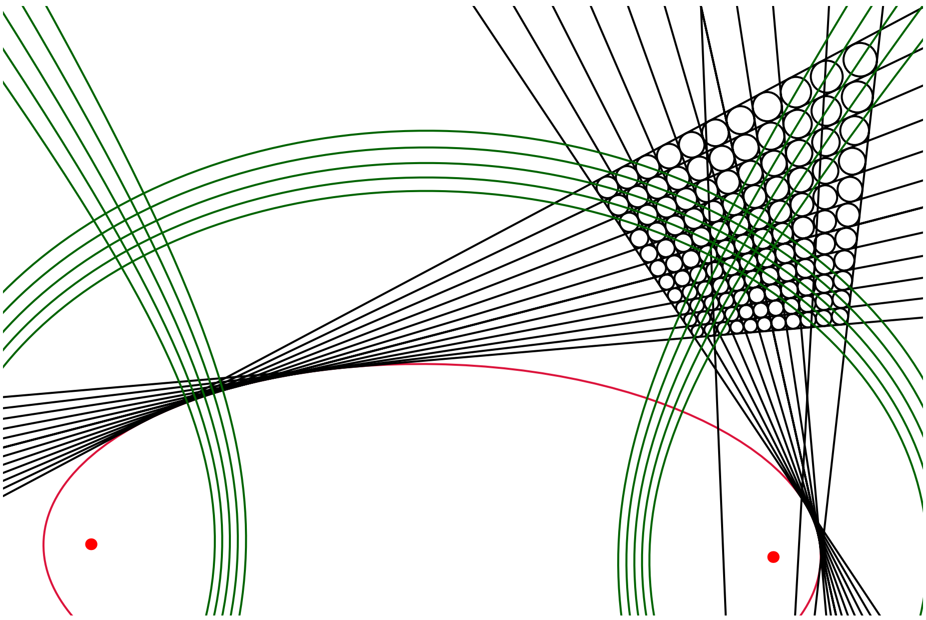


Figure 2.46: Construction of incircular nets with general Graves-Chasles theorem

2.15 Quadrics. The Euclidean point of view. Confocal quadrics (and orthogonal coordinate systems)

Quadrics are the generalization of conics to arbitrary dimension: They are the sets definded by one quadratic equation in the corrdinates. Conic sections are the special case of quadrics in the plane.

Definition 2.82. A *quadric* $\mathcal{Q}$ in the Euclidean space $\mathbb{R}^n$ is defined by a quadratic equation

$$\mathcal{Q} = \{x \in \mathbb{R}^n \mid x^T Bx + b^T x + c = 0\},$$

where B is a symmetric $n \times n$ matrix, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

It can be brought to normal form by an Euclidean motion $x \mapsto Ax + a$, where $a \in \mathbb{R}^n$ and $A \in O(n)$ diagonalizes the symmetric matrix B. The following theorem lists the cases that can occur in $\mathbb{R}^3$:

Theorem 2.83. *By an appropriate change of coordinates $x \mapsto Ax + a$, $A \in O(3)$, $a \in \mathbb{R}^3$, any quadric in $\mathbb{R}^3$ can be transformed to one of the following standard forms:*

- *ellipsoid:* $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$
- *elliptic paraboloid:* $z = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$
- *2-sheeted hyperboloid:* $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = -1$
- *1-sheeted hyperboloid:* $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 1$
- *hyperbolic paraboloid:* $z = \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$

and some degenerate cases:

- *cones and cylinders over a conic*
- *two planes*
- *one "double" plane*
- *one line*
- *one point*
- *the empty set .*

This theorem is similar to the corresponding theorem 2.50 for conics, and can be proven in the same way by diagonalizing the matrix B and subsequent centering by shifting.

We consider a special family of quadrics in $\mathbb{R}^n$ generalizing confocal conics.

Definition 2.84. Let $a_1 > a_2 > \ldots > a_n > 0$ be given. The one-parameter family of quadrics

$$\mathcal{Q}_\lambda = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^n \frac{x_i^2}{a_i + \lambda} = 1 \right\}, \quad \lambda \in \mathbb{R}$$

is called *confocal*.

Theorem 2.85. *Precisely n confocal quadrics pass through any point $x = (x_1, \ldots, x_n)$ with $x_1 \cdot \ \ldots \cdot x_n \neq 0$, and the quadrics passing through one point are orthogonal.*

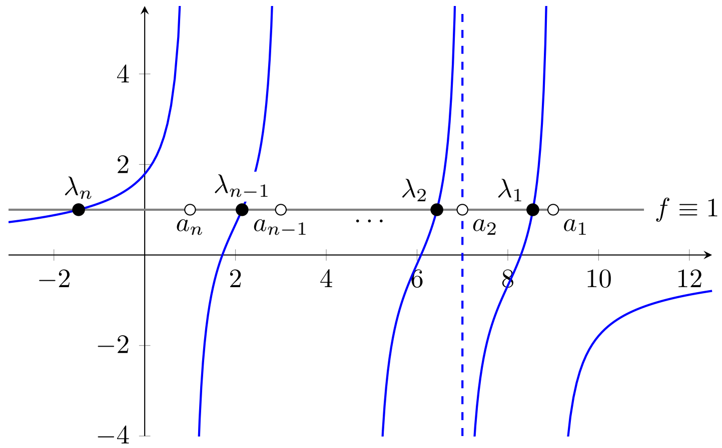


Figure 2.47: Plot of the function f.

Proof. For a given point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $x_1 \cdot \ldots \cdot x_n \neq 0$ the equation $\sum_{i=1}^n \frac{x_i^2}{a_i - \lambda} = 1$ is, after clearing the denominators, a polynomial equation of degree n in λ with n real roots $\lambda_1, \ldots, \lambda_n$ lying in the intervals $\lambda_n < a_n < \ldots < \lambda_2 < a_2 < \lambda_1 < a_1$. This fact follows immediatly from the plot of the function

$$f(\lambda) := \frac{x_1^2}{a_1 - \lambda} + \cdots + \frac{x_n^2}{a_n - \lambda}.$$

It has exactly n different intersection points with the horizontal line $f = 1$. Since the gradient of f at the point $w = (w_1, \ldots, w_n)$ equals

$$\operatorname{grad}_w f(\lambda) = 2 \left(\frac{w_1}{a_1 - \lambda}, \ldots, \frac{w_n}{a_n - \lambda} \right),$$

the tangent plane is given by

$$\sum_{i=1}^n \frac{x_i w_i}{a_i - \lambda} = 1.$$

Let $\mathcal{Q}_\lambda$, $\mathcal{Q}_\mu$ be two confocal quadrics passing through the point w, and consider their tangent hyperplanes $\sum_{i=1}^n \frac{x_i w_i}{a_i-\lambda} = 1, \sum_{i=1}^n \frac{x_i w_i}{a_i-\mu} = 1$. They are orthogonal since

$$\begin{aligned}\langle \operatorname{grad}_w f(\lambda), \operatorname{grad}_w f(\lambda)\rangle &= \sum_{i=1}^n \frac{w_i^2}{(a_i-\lambda)(a_i-\mu)} \\ &= \frac{1}{\lambda-\mu}\left(\sum_{i=1}^n \frac{w_i^2}{a_i-\lambda} - \sum_{i=1}^n \frac{w_i^2}{a_i-\mu}\right) = 0\,.\end{aligned}$$

Thus the n roots $\lambda_1, \ldots, \lambda_n$ of

$$\sum_{i=1}^n \frac{x_i^2}{a_i-\lambda} - 1 = -\frac{\prod_{i=1}^n(\lambda_i-\lambda)}{\prod_{i=1}^n(a_i-\lambda)} \tag{2.3}$$

correspond to n confocal quadrics $\mathcal{Q}_{\lambda_i}, i = 1, \ldots, n$ that intersect at the point x:

$$\sum_{i=1}^n \frac{x_i^2}{a_i-\lambda_k} = 1, k = 1, \ldots, n \quad \Leftrightarrow \quad x \in \bigcap_{i=1}^n \mathcal{Q}_{\lambda_i}\,.$$

Each of the quadrics is of a different signature. Calculating the residue of 2.3 at $\lambda = a_j$ we obtain the formula for x_j through $(\lambda_1, \ldots, \lambda_n)$:

$$x_j = \frac{\prod_{i=1}^n(a_j-\lambda_i)}{\prod_{i\neq j}(a_j-a_i)}\,, \quad j = 1, \ldots, n\,. \tag{2.4}$$

Thus for each point x of $\mathbb{R}_+^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > 0, \ldots, x_n > 0\}$ there is exactly one solution $(\lambda_1, \ldots, \lambda_n) \in \Lambda$ of 2.4, where

$$\Lambda = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \mid \lambda_n < a_n < \lambda_{n-1} < \ldots < \lambda_1 < a_1\}\,.$$

On the other hand, the formulas 2.4 are mirror symmetric with respect to the coordinate hyperplanes. $\square$

Definition 2.86. The coordinates $(\lambda_1, \ldots, \lambda_n) \in \Lambda$ of $\mathbb{R}_+^n$ given by 2.4 are called *confocal coordinates* (or elliptic coordinates, following Jacobi).

2.16 Quadrics. The projective point of view

Quadrics are the generalization of conic sections to arbitrary dimensions. They are the sets defined by one quadratic equation in the coordinates. Conic sections are the special case of quadrics in the plane.

Definition 2.87. If q is a quadratic form on $\mathbb{R}^{n+1}$ (or $\mathbb{C}^{n+1}$), $q \neq 0$. Then

$$\mathcal{Q} = \{[v] \in \mathbb{R}\mathrm{P}^n \mid \mathrm{q}(v) = 0\}$$

is called a *quadric* in $\mathbb{R}\mathrm{P}^n$ (or $\mathbb{C}\mathrm{P}^n$).

Non-degenerate quadrics in $\mathbb{R}\mathrm{P}^{n+1}$ can be classified by the signature of the corresponding quadratic from on $\mathbb{R}^{n+1}$. So the number of non-empty non-degenerate quadrics in $\mathbb{R}\mathrm{P}^n$ is $\lfloor \frac{n+1}{2} \rfloor$

In $\mathbb{R}\mathrm{P}^3$, there are only three non-degenerate cases depending on the signature of q:

(0) $(+ + ++)$ or $(- - --)$, $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$; The case of definite q leading to an empty quadric $Q = \emptyset$. We exclude this case from now on.

(1) $(+ + +-)$ or $(+ - --)$, $x_1^2 + x_2^2 + x_3^2 = x_4^2$; In an affine image of $\mathbb{R}\mathrm{P}^3$, $\mathcal{Q}$ looks like an ellipsoid, an elliptic paraboloid, or a 2-sheeted hyperboloid, depending on whether the plane at infinity does not intersect, is tangent to, or intersects the quadric (without being tangent).

(2) $(+ + --)$, $x_1^2 + x_2^2 = x_3^2 + x_4^2$; In an affine image of $\mathbb{R}\mathrm{P}^3$, $\mathcal{Q}$ looks like a 1-sheeted hyperboloid or a hyperbolic paraboloid, depending on whether the plane at infinity intersects (without being tangent) or is tangent to $\mathcal{Q}$. (In this case, any plane meets $\mathcal{Q}$.)

In $\mathbb{C}\mathrm{P}^n$, there is up to projective transformations only *one* non-degenerate quadric. There are n degenerate ones, depending on the rank of q (which can be $1, \dots, n$).

If q is a degenerate bilinear form, then the kernel of the bilinear form

$$\ker(\mathrm{q}) = \{u \in U \mid \mathrm{q}(u, v) = 0 \ \forall v \in V\}$$

is a subspace of V. Consider a subspace $U_1 \subset V$ such that $V = \ker(\mathrm{q}) \oplus U_1$, then $b\,|_{U_1}$ defines a non-degenerate quadric Q_1 in $\mathrm{P}(U_1)$. The quadric Q defined by q is the union of lines through points in the non-degenerate quadric Q_1 defined by $b|_{U_1}$ and points in $\mathrm{P}(\ker(\mathrm{q}))$ if $Q_1 \neq \emptyset$ (see Exercise 8.1).

Example 2.88. *Consider the following bilinear form in* $\mathbb{R}\mathrm{P}^2$*:*

$$\mathrm{q}(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}) = {x_1}^2 - {x_2}^2.$$

Then the kernel of the bilinear form is $\ker(\mathrm{q}) = \mathrm{span}\{e_3\}$ *and* $\mathbb{R}^3 = \ker(\mathrm{q}) \oplus U_1$ *with* $U_1 = \mathrm{span}\{e_1, e_2\}$*. Projectively,* $\mathrm{P}(\ker(\mathrm{q}))$ *is a point and the quadric defined by* $\mathrm{q}\,|_{U_1}$ *in* $\mathrm{P}(U_1)$ *consists of two points. So the degenerate/singular conic defined by* q *in* $\mathbb{R}\mathrm{P}^2$ *consists of two crossing lines.*

The following theorem is a generalization of Thm. 2.60 and can be proved in the same way.

Theorem 2.89. *A non-degenerate (non-empty) quadric* $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^n$ *determines the corresponding bilinear form up to a non-zero scalar multiple.*

2.16.1 Orthogonal Transformations

Definition 2.90. Let q be a non-degenerate symmetric bilinear form on $\mathbb{R}^{n+1}$. Then $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is *orthogonal* with respect to q if

$$\mathrm{q}(F(v), F(w)) = \mathrm{q}(v, w) \quad \text{for all } v, w \in \mathbb{R}^{n+1}.$$

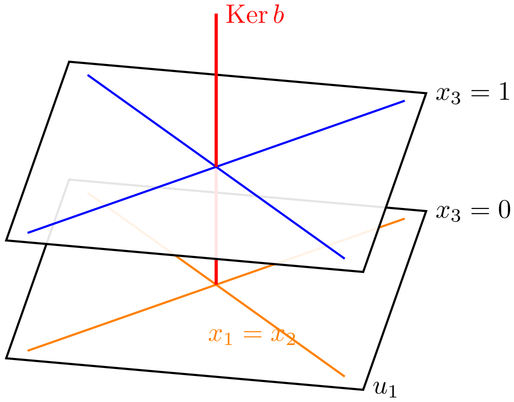


Figure 2.48: Degenerate quadric in $\mathbb{R}\mathrm{P}^1$

The *group of orthogonal transformations* for a bilinear form of signature (p, q) with $p+q = n+1$ is denoted by $O(p, q)$.

If $q = 0$ we obtain the "usual" group of orthogonal transformations $O(n+1) = O(n+1, 0)$.

If $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^n$ is a non-degenerate quadric defined by q and $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is an orthogonal transformation, then the map $f : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$ with $f([x]) = [F(x)]$ maps the quadric onto itself: $f(\mathcal{Q}) = \mathcal{Q}$.

Theorem 2.91. *If the signature of the bilinear form* q *is not neutral* $(p \neq q)$*, then any projective transformation that maps* $\mathcal{Q}$ *to* $\mathcal{Q}$ *comes from a linear map which is orthogonal with respect to* q.

Hence, under the assumption of non-neutral signature, the group of projective transformations mapping $\mathcal{Q}$ to $\mathcal{Q}$ is $\mathrm{PO}(k, n+1-k)$, the *projective orthogonal group* for signature $(k, n+1-k)$.

Proof. Suppose $[x] \mapsto [f(x)]$ maps $\mathcal{Q}$ to $\mathcal{Q}$. This means that the symmetric bilinear forms q and $\tilde{\mathrm{q}}$ defined by $\tilde{\mathrm{q}}(x, y) = \mathrm{q}(f(x), f(y))$ define the same quadric. Hence by Thm. 2.89 there exists $\lambda \in \mathbb{R} \setminus \{0\}$ with $\tilde{\mathrm{q}} = \lambda\,\mathrm{q}$. Hence $\mathrm{q}(f(x), f(y)) = \lambda\,\mathrm{q}(x, y)$ for all $x, y \in \mathbb{R}^{n+1}$. We will show that λ is positive. Then $\frac{1}{\sqrt{\lambda}} f$ defines the same projective transformation and is orthogonal with respect to q. Now to see that λ is positive, let $e_1, \ldots, e_{n+1}$ be an orthonormal basis with respect to q. Then $f(e_1), \ldots, f(e_{n+1})$ is still an ortho*gon*al basis. If λ were negative, it would contain $n+1-k$ spacelike and k timelike vectors. This cannot be, because every orthogonal basis contains k spacelike and $n+1-k$ timelike vectors. □

Remark 2.92. *In case of neutral signature, i.e.* $p = q$*, for example* $p = q = 2$ *for a quadric in* $\mathbb{R}\mathrm{P}^3$*, then there exists a projective transformation preserving the quadric not induced by an orthogonal transformation: Let* $\mathrm{q}(x, x) = {x_1}^2 + {x_2}^2 - {x_3}^2 - {x_4}^2$*. Then the map:*

$$f : \mathbb{R}\mathrm{P}^3 \to \mathbb{R}\mathrm{P}^3, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mapsto \begin{bmatrix} x_3 \\ x_4 \\ x_1 \\ x_2 \end{bmatrix}$$

yields $\mathrm{q}(F(v), F(w)) = -\mathrm{q}(x, x) = {x_3}^2 + {x_4}^2 - {x_1}^2 - {x_2}^2$. *So f preserves the quadric but it is not induced by an orthogonal transformation F.*

2.16.2 Lines in a quadric

A line intersects a quadric in $\mathbb{RP}^n$ either *not at all*, in *two points*, in *one point*, or it *lies entirely in the quadric*. In the last two cases, the line is called a *tangent*. In $\mathbb{RP}^3$, the only non-degenerate quadrics that contain lines are the ones with neutral signature $(++--)$.

Proposition 2.93. *Let $\mathcal{Q}$ be a quadric in $\mathbb{RP}^3$ with neutral signature. Then through any point in $\mathcal{Q}$ there are precisely two lines lying entirely in Q. Moreover, $\mathcal{Q}$ contains two families of pairwise skew lines, and each line of the first family intersects each line of the second family.*

Proof. To see that there are no more than two lines through a point $[p] \in \mathcal{Q}$ lying entirely in $\mathcal{Q}$, show that any such line must lie in the plane $q(p, \cdot\,) = 0$, and note that the intersection of $\mathcal{Q}$ with a plane is a conic section, so it cannot contain more that two lines.

To see that there are actually two such lines, we may assume (after a change of coordinates, if necessary) that $\mathcal{Q}$ is the quadric $x_1^2 + x_2^2 - x_3^2 - x_4^2 = 0$. This equation is equivalent to $(x_1 + x_3)(x_1 - x_3) + (x_2 + x_4)(x_2 - x_4) = 0$, and, after changing to new coordinates

$$y_1 = x_1 + x_3, \quad y_2 = x_1 - x_3, \quad y_3 = -(x_2 + x_4), \quad y_4 = x_2 - x_4,$$

to

$$y_1 y_2 - y_3 y_4 = 0.$$

Now the map

$$f: \quad \mathbb{RP}^1 \times \mathbb{RP}^1 \longrightarrow \mathcal{Q}, \quad \left(\left[\begin{smallmatrix} s_1 \\ t_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} s_2 \\ t_2 \end{smallmatrix}\right]\right) \longmapsto \left[\begin{smallmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{smallmatrix}\right] = \left[\begin{smallmatrix} s_1 s_2 \\ t_1 t_2 \\ s_1 t_2 \\ t_1 s_2 \end{smallmatrix}\right]$$

is actually a bijection $\mathbb{RP}^1 \times \mathbb{RP}^1 \leftrightarrow \mathcal{Q}$. Indeed, if $\left[\begin{smallmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{smallmatrix}\right] \in \mathcal{Q}$, then $\left[\begin{smallmatrix} s_1 \\ t_1 \end{smallmatrix}\right]$ is determined by $\frac{s_1}{t_1} = \frac{y_1}{y_4}$ or by $\frac{s_1}{t_1} = \frac{y_3}{y_2}$. (It can happen that one of the right hand sides is $\frac{0}{0}$, but not both. If neither is $\frac{0}{0}$, they are equal.) Similarly, $\left[\begin{smallmatrix} s_2 \\ t_2 \end{smallmatrix}\right]$ is determined by $\frac{s_2}{t_2} = \frac{y_1}{y_3}$ or by $\frac{s_2}{t_2} = \frac{y_4}{y_2}$. For any point $P = f(P_1, P_2) \in \mathcal{Q}$, the images of the functions $f(P_1, \cdot\,) : \mathbb{RP}^1 \to \mathcal{Q}$ and $f(\,\cdot\,, P_2) : \mathbb{RP}^1 \to \mathcal{Q}$ are two lines through P lying entirely in $\mathcal{Q}$. $\square$

In fact this proof also shows, since $\mathbb{RP}^1$ is homeomorphic to the circle S^1, $\mathcal{Q}$ is homeomorphic to $S^1 \times S^1$, so it is topologically a torus.

Proposition 2.94. *Given three pairwise skew lines in $\mathbb{RP}^3$ there exists a unique quadric containing these lines. It is a quadric in $\mathbb{RP}^3$ with neutral signature from Prop. 2.93.*

Proof. For each point on the line ℓ_1 there exists a unique line $\tilde{\ell}_1$ intersecting ℓ_2 and ℓ_3 (see Ex. 2.5). Take three such lines $\tilde{\ell}_1$, $\tilde{\ell}_2$, and $\tilde{\ell}_3$ and the nine intersection points of these lines with the lines ℓ_1, ℓ_2, and ℓ_3 (see Fig. 2.49).

There exists a quadric $\mathcal{Q}$ containing all these nine points. Indeed, we have 9 homogeneous linear equations for 10 coefficients of the quadratic form. This quadric contains all six lines ℓ_1, ℓ_2, ℓ_3, and $\tilde{\ell}_1, \tilde{\ell}_2, \tilde{\ell}_3$, since it contains the three intersection points on each of the lines (see Lemma 2.62).

The quadric $\mathcal{Q}$ is non-degenerate, since degenerate quadrics in $\mathbb{RP}^3$ (which are cone generated by conics) may contain up to two skew lines only. A non-degenerate quadric $\mathcal{Q}$ that contains lines has the signature $(++--)$. The rest follows from Prop. 2.93.

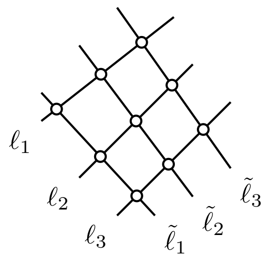


Figure 2.49: Three skew lines generate a 1-parameter family of lines intersecting all three of them.

$\square$

2.16.3 Brianchon hexagon

Definition 2.95. A hexagon $ABCDEF$ in $\mathbb{RP}^3$ is called a *Brianchon hexagon* if its diagonals AD, BE, CF meet at one point.

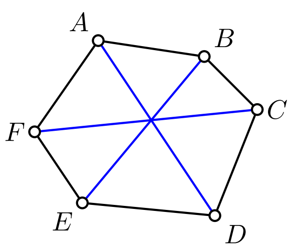


Figure 2.50: Brianchon hexagon with intersecting diagonals

Theorem 2.96. *A non-planar hexagon in $\mathbb{RP}^3$ is a Brianchon hexagon if and only if all its sides lie on a quadric.*

Proof. Assume that $ABCDEF$ is Brianchon, i.e., the diagonals meet at one point. Since the hexagon is not planar, the lines BC, DE, and FA are skew (why?). Consider the quadric generated by the lines BC, DE, and FA. Then the line AB intersects the generating lines BC, DE, and FA: Indeed AB obviously intersects FA and BC, but it also intersects DE, since the diagonals AD and BE intersect and hence AB and DE lie in one plane. Similarly, CD and EF intersect all three generating lines and thus the entire hexagon lies on the quadric.

On the other hand, suppose that AB, CD, and EF lie on a quadric containing BC, DE, and AF. Then the lines AB and ED intersect, which implies that BE and AD also intersect. In the same way we get that AD and CF, as well as, BE and CF intersect. Since the hexagon is not planar, the diagonals do not lie in one plane and hence have to intersect in one point.

$\square$

2.17 Polarity

A non-degenerate symmetric bilinear form q on a vector space V defines a relation between the points and hyperplanes of $\mathrm{P}(V)$: To each point $[v] \in \mathrm{P}(V)$ corresponds the *polar hyperplane*

$$\{[w] \in \mathrm{P}(V) \mid \mathrm{q}(v, w) = 0\},$$

and to each hyperplane there is a corresponding point, its *pole*. Note that

$$[x] \in \text{polar hyperplane of } [y] \quad \Longleftrightarrow \quad [y] \in \text{polar hyperplane of } [x] \quad \Longleftrightarrow \quad \mathrm{q}(x, y) = 0.$$

More generally, let $U \subseteq V$ be a $(k+1)$-dimensional linear subspace of V, and let $n + 1 = \dim V$. The *orthogonal subspace* of U (with respect to q) is

$$U^{\perp} = \{w \in V \mid \mathrm{q}(u, w) = 0 \text{ for all } u \in U\}.$$

The dimension of $U^{\perp}$ is $\dim V - \dim U = n - k$, and $U^{\perp\perp} = U$. The k-plane $\mathrm{P}(U)$ and the $(n-k-1)$-plane $\mathrm{P}(U)^{\perp}$ in $\mathrm{P}(V)$ are called *polar* to each other. Polarity (with respect to q) is therefore a one-to-one relation between k-planes and $(n-k-1)$-planes in the n-dimensional projective space $\mathrm{P}(V)$. In particular, if $n = 3$, polarity is a relation between points and planes and between lines and lines.

Proposition 2.97. *Let $\mathcal{Q}$ be a (non-empty) non-degenerate quadric in $\mathbb{R}\mathrm{P}^n$ $(\mathbb{C}\mathrm{P}^n)$ defined by the symmetric bilinear form q, and let $X \in \mathcal{Q}$, $Y \in \mathbb{R}\mathrm{P}^n$ $(\mathbb{C}\mathrm{P}^n)$. Then*

$$\textit{The line } XY \textit{ is tangent to } \mathcal{Q} \quad \Longleftrightarrow \quad X \textit{ is in the polar hyperplane of } Y.$$

Proof. Let $X = [x]$, $Y = [y]$. Then $\mathrm{q}(x, x) = 0$ because $X \in \mathcal{Q}$. The line XY is tangent to $\mathcal{Q}$ either if it intersects $\mathcal{Q}$ in no other point but X, or if it is contained entirely in $\mathcal{Q}$. The points on the line XY except X are parameterized by $[tx+y]$ with $t \in \mathbb{R}$ $(\mathbb{C})$. Such a point lies in $\mathcal{Q}$ if

$$0 = \mathrm{q}(tx+y, tx+y) = t^2\,\mathrm{q}(x,x) + 2t\,\mathrm{q}(x,y) + \mathrm{q}(y,y) = 2t\,\mathrm{q}(x,y) + \mathrm{q}(y,y).$$

This equation for t has one solution if $\mathrm{q}(x,y) \neq 0$, it has no solution if $\mathrm{q}(x,y) = 0$ and $\mathrm{q}(y,y) \neq 0$, and it is satisfied for all t if $\mathrm{q}(x,y) = \mathrm{q}(y,y) = 0$. So the line XY contains no other points of $\mathcal{Q}$ except X or lies entirely in $\mathcal{Q}$ precisely if $q(x,y) = 0$. □

This provides a simple geometric interpretation of the polarity relationship between points and hyperplanes in the case when the polar hyperplane intersects $\mathcal{Q}$: The tangents from a point to the quadric touch the quadric in the points in which the quadric intersects the polar hyperplane.

If a quadric in $\mathbb{R}\mathrm{P}^3$ is illuminated by a point light source outside the quadric (or by parallel light), the borderline between light and shadow on the quadric is a conic in the polar plane; and the shadow that the quadric throws on some other another plane is a projected image of this conic.

What about the polarity between lines in $\mathbb{R}\mathrm{P}^3$? If a point moves on a line, the polar planes rotate about a line, and these two lines are polar to each other.

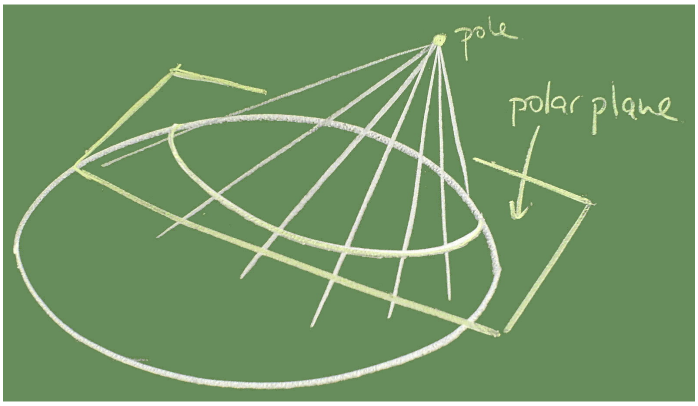


Figure 2.51: The silhouette of an ellipse is given by the intersection of the ellipse with the polar plane.

2.18 The synthetic approach to projective geometry

We have defined a projective space $\mathrm{P}(V)$ of a vector space V over a field as the set of 1-dimensional subspaces of V. Of course this definition is based on basic axioms of algebra: the field axioms and the vector space axioms. This section provides a rough outline of how the theory of projective spaces $\mathrm{P}(V)$ can be based on geometric axioms.

The following definition of a projective space in terms of geometric axioms (due to Oswald Veblen & John W. Young, 1908) is not equivalent to our definition of a projective space of a vector space over a field. (One obtains an equivalent definition if Pappus' theorem is added as an independent axiom; see the structure theorem below.)

A *projective space* $P = (\mathcal{P}, \mathcal{L})$ is a set $\mathcal{P}$, the elements of which are called *points*, together with a set $\mathcal{L}$ of subsets of P, which are called *lines*, such that the following axioms are satisfied.

AXIOM 1. For any two distinct points there exists a unique line which contains both points.

AXIOM 2. If $A, B, C \in P$ are three distinct points and $l \in \mathcal{L}$ is a line that intersects the lines AB and AC in distinct points, then l intersects the line BC.

AXIOM 3. Every line contains at least three points.

Axiom 1 implies that two lines intersect in at most one point. Axiom 2 is a clever way of saying that two lines in one plane always intersect without first defining what a plane is.

A *projective subspace* of P is a subset $\mathcal{U} \subseteq \mathcal{P}$ of points such that the line through any two points of $\mathcal{U}$ is contained in $\mathcal{U}$. Together with the subset of lines $\{l \in \mathcal{L} \mid l \subseteq U\}$, the subspace is a projective space in its own right. The intersection of two projective subspaces is a projective subspace.

If $S \subseteq \mathcal{P}$ is any set of points, then the *projective span* of S is the smallest projective subspace containing S, or equivalently, the intersection of all projective subspaces which contain S.

The *dimension* of the projective space P is the smallest number n for which there exist $n+1$ points $P_1, \ldots, P_{n+1} \in \mathcal{P}$ such that $\mathcal{P}$ is the projective span of $\{P_1, \ldots, P_{n+1}\}$.

The Axioms 1–3 together with the assertion that the dimension of P is 2 are equivalent to the following *axioms for a projective plane*. (Can you prove this equivalence? It is a little tricky.)

AXIOM P1. Same as Axiom 1.

AXIOM P2. Any two lines have non-empty intersection.

AXIOM P3. Same as Axiom 3.

AXIOM P4. There are at least two different lines.

If the dimension of P is at least 3, then Desargues' theorem can be deduced from Axioms 1–3. The 3D proof of the last lecture works in this setting, it uses only the incidence relations between points, lines and planes, and does not involve any calculations. However, there are projective planes in which Desargues' theorem does not hold. The purely 2-dimensional proof does not work here because it is based on calculations.

A projective plane in which Desargues' theorem holds is called a *Desarguesian plane*.

Theorem 2.98 (Veblen & Young). *Any projective space in which Desargues' theorem holds (that is, any projective space of dimension ≥ 3 and any Desarguesian plane) is isomorphic to a projective space $\mathrm{P}(V)$ of a vector space V over a skew field F. If Pappus' theorem also holds in P, then F is a field.*

(Two projective spaces are isomorphic if there is a bijection between their points that maps lines to lines. A skew field satisfies all field axioms except that the multiplication may not be commutative. You may check that our computational proof of Pappus' theorem does not work if multiplication is not commutative.)

In a projective plane, Desargues' theorem can be deduced from Pappus' theorem. (This was demonstrated by Hessenberg in 1905. It is not obvious at all.)

Thus, any theorem that holds in any projective space $\mathrm{P}(V)$ of a vector space V over a filed can also be deduced from Axioms 1–3 together with Pappus' theorem as independent axiom, and vice versa. Further axioms of order and of continuity have to be added to single out the real projective spaces $\mathbb{R}\mathrm{P}^n$ (just like further axioms have to be added to the general field axioms to single out field of reals).

Problems

Problem 2.1. *Let* $\mathrm{P}(V)$ *be a* 3*-dimensional projective space. Show:*
(a) Any two planes in $\mathrm{P}(V)$ *intersect in a line.*
(b) If A *is a plane in* $\mathrm{P}(V)$ *and* l *is a line that is not contained in* A*, then* A *and* l *intersect in exactly one point.*

Problem 2.2. *Let* U_1*,* U_2 *be vector subspaces of* V *with* $U_1, U_2 \neq \{0\}$*. Show that the projective subspace*

$$\mathrm{P}(U_1 + U_2) \subseteq \mathrm{P}(V)$$

is the set of points obtained by joining each $x \in P(U_1)$ *and* $y \in P(U_2)$ *by a projective line.*

Problem 2.3. *Into how many regions is the real projective plane separated by* n *lines in general position, i.e.,* n *lines such that no three pass through one point?*

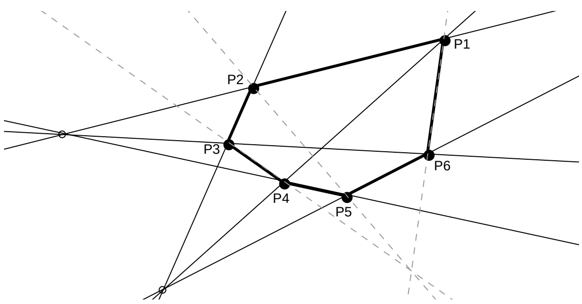
Problem 2.4. *Let* V *be the vector space of dimension* 3 *over the two element field* $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$*. Consider the projective space* $\mathrm{P}(V)$ *with* $V = (\mathbb{Z}_2)^3$*. How many points does it contain? How many lines? How many points lie on each line? How many lines pass through each point? Draw a schematic picture of the configuration.*

Problem 2.5. *i) Prove that, in general, two lines in* $\mathbb{R}\mathrm{P}^3$ *do not intersect. Such lines are called skew lines.*

ii) Given three lines which are pair-wise skew, prove that there are an infinite number of lines which intersect all three lines.

iii) In $\mathbb{R}^3$*, two lines are called skew if they do not intersect and are not parallel. Given two skew lines* l *and* m *in* $\mathbb{R}^3$*, show that there exists exactly one plane* L *containing* l *and one plane* M *containing* m *such that* L *is parallel to* M*.*

Problem 2.6. *Let* $P_1, P_2, P_3, P_4, P_5, P_6$ *be distinct points in the projective plane* $\mathbb{R}\mathrm{P}^2$*. Suppose that the three lines* P_1P_2*,* P_4P_5*,* P_3P_6*, as well as the three lines* P_2P_3*,* P_5P_6*,* P_4P_1 *intersect in one point. Show that the lines* P_3P_4*,* P_6P_1*,* P_5P_2 *also intersect in one point.*



Problem 2.7. *Give an analytic proof of Pappus Theorem 2.17 by choosing an appropriate basis and computing the linear dependence of the three points on the line.*

Problem 2.8. *Consider the decomposition of $\mathbb{RP}^n$ into an affine part $\mathbb{R}^n$ and a part at inifinity $\mathbb{RP}^{n-1}$. Show that two lines that are parallel in $\mathbb{R}^n$ intersect in a point at infinity. (It suffices to show the statement for $n = 2$. Why?).*

Problem 2.9. *Can a projective transformation $\mathbb{RP}^1 \to \mathbb{RP}^1$ which is not the identity have three fixed points? How many fixed points can a projective transformation $\mathbb{RP}^1 \to \mathbb{RP}^1$ have? Give an example for each possible case.*

Problem 2.10. *Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PGL(2, \mathbb{R})$, corresponding to a projective transformation $\mathbb{RP}^1 \to \mathbb{RP}^1$.*

1. *Given a point with homogeneous coordinates $(x_1, x_2) \in \mathbb{RP}^1$, with $x_2 \neq 0$, define affine coordinates for this point by $x = \frac{x_1}{x_2}$. Extend these coordinates to the point at infinity $(1, 0)$ by defining the symbol ∞ to represent this point. Then M acts on a point with coordinate x by $M(x) = \frac{ax+b}{cx+d}$.*

   Now given three points P, Q, R with affine coordinates $p \neq q \neq r \neq p$, find M so that $M(0) = P, M(1) = Q, M(\infty) = R$.

2. *Show that if $M \neq \mathrm{id}$ is an involution ($M^2 = \lambda\mathrm{id}$), it has either 0 or 2 fixed points.*

Problem 2.11. *Let $f : \mathbb{RP}^2 \to \mathbb{RP}^2$ be a projective map that sends the line at infinity $\{[x_0, x_1, x_2] \in \mathbb{RP}^2 \mid x_0 = 0\}$ to the line at infinity. Show that in affine coordinates $y_i = \frac{x_i}{x_0}$ a point $y = \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right)$ is mapped to $Ay + b$ for some $A \in GL(2, \mathbb{R})$, $b \in \mathbb{R}^2$.*

Problem 2.12. *Let ℓ, ℓ_1, and ℓ_2 be pairwise skew, i.e., non-intersecting lines in a 3-dimensional projective space $\mathrm{P}(V)$. Consider the map $f : \ell_1 \to \ell_2$ which assigns to each point $p \in \ell_1$ the point $q \in \ell_2$ that is the intersection of ℓ_2 with the plane spanned by p and ℓ. Show that f is a projective transformation.*

Problem 2.13. *Given linearly independent vectors A, B, C, $D \in \mathbb{R}^2$, with $C = A + \lambda B$ and $D = A + \mu B$. Show that $\mathrm{cr}([A], [C], [B], [D]) = \frac{\lambda}{\mu}$.*

Problem 2.14. *Let $A, B, C, P, Q \in \mathbb{RP}^1$ be five distinct points. Show that*

$$\mathrm{cr}(P, A, Q, B)\,\mathrm{cr}(P, B, Q, C) = \mathrm{cr}(P, A, Q, C).$$

Problem 2.15. *Let E_1, E_2, E_3, E_4 be four planes in $\mathbb{RP}^3$ which all pass through one line l. Show that the quantities c_1 and c_2 defined below are all equal: $c_1 = c_2$.*

(i) Let l' be a line that is not contained in any of the four planes and let P_i be the intersection of l' with E_i. Let $c_1 = \mathrm{cr}(P_1, P_2, P_3, P_4)$.

(ii) Let E' be a plane that does not contain the line l. Let l_i be the line of intersection of E_i and E'. Let $c_2 = \mathrm{cr}(l_1, l_2, l_3, l_4)$.

Problem 2.16. *Consider four lines $\ell_1, \ell_2, \ell_3, \ell_4$ through a point P in $\mathbb{R}^2$. Denote the oriented angle between the lines by $\alpha_{ij} = \angle(\ell_i, \ell_j)$. Show that*

$$\mathrm{cr}(\ell_1, \ell_2, \ell_3, \ell_4) = \frac{\sin\alpha_{12} \cdot \sin\alpha_{34}}{\sin\alpha_{23} \cdot \sin\alpha_{41}}.$$

Problem 2.17. *Draw a complete quadrilateral and find as many harmonically seperated pairs of points as possible. Give two alternative proofs of the theorem with the complete quadrilateral using normalization and calculation, and projective involutions.*

Problem 2.18. *Define the cross-ratio for the complex projective line. Consider a square with vertices A, B, C, and D in $\mathbb{C}\mathrm{P}^1$. What is the cross-ratio of the four points?*

Problem 2.19. *In the following exercise, $\langle x, y\rangle$ for $x, y \in \mathbb{R}\mathrm{P}^n$ should be interpreted as the inner product in the space $\mathbb{R}^{n+1}$ of homogeneous coordinates for x and y.*

Let $P = [p]$ be a point in $\mathbb{R}\mathrm{P}^n$ and $H = \{[v] \in \mathbb{R}\mathrm{P}^n \mid \langle v, n\rangle = 0\}$ a hyperplane *in $\mathbb{R}\mathrm{P}^n$ for some fixed $[n] \in \mathbb{R}\mathrm{P}^n$. The projective reflection with center p and axis H is the projective transformation f of $\mathbb{R}\mathrm{P}^n$ with fixed point set $F = \{P\} \cup H$, whose action on $X \notin F$ is defined as follows:*

Construct the line l joining P and $X = [x]$, and find its unique intersection Q with H. Then $f(X)$ is defined as the unique point on l which satisfies $\mathrm{cr}(f(X), P, X, Q) = -1$.

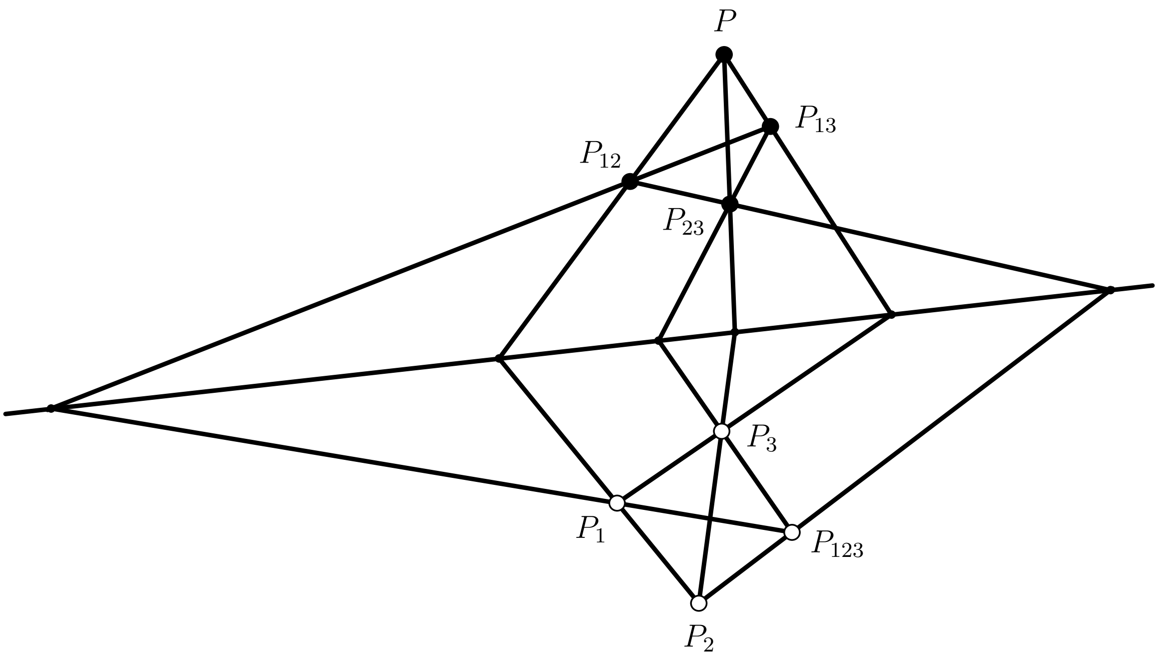
1. *Show that f is a well-defined projective transformation given by*

$$f([x]) = [x - 2\frac{\langle x, n\rangle}{\langle p, n\rangle}p].$$

2. *Calculate the element of $PGL(3, \mathbb{R})$ for the harmonic reflection of $\mathbb{R}\mathrm{P}^2$ with center $P = [1, 1, 1]$ and $H = \{[v] \in \mathbb{R}\mathrm{P}^2 | \langle v, [1, 1, -1]\rangle = 0\}$ (i.e. the line $x + y = 1$ in affine coordinates).*

Problem 2.20. *Given a projective transformation f of $\mathbb{R}\mathrm{P}^1$. Show that f can be factored as the product (concatenation) of three involutions.*

Problem 2.21. *Let P, P_i, P_{ij}, P_{123} be a 3-dimensional cube with planar faces. Then if its four black vertices P, P_{12}, P_{23}, and P_{13} are coplanar, then so are its four white vertices P_1, P_2, P_3, and P_{123}.*



Problem 2.22. *Let x, y be two distinct points on $\mathbb{R} \cup \{\infty\} = \mathbb{R}\mathrm{P}^1$, different from 0 and ∞. Construct the point $x + y$ geometrically using complete quadrilaterals.*

Problem 2.23. *Derive the following statement from the fundamental theorem of real projective geometry: A bijective map $f : \mathbb{R}^n \to \mathbb{R}^n$, $n > 1$, that maps lines to lines is an affine transformation $f(x) = Ax + b$ for some $A \in \mathrm{GL}(n, \mathbb{R})$, $b \in \mathbb{R}^n$.*

Problem 2.24. *Show that $\{P_1, Q_1; P_2, Q_2; P_3, Q_3\}$ is a quadrangular set of points if and only if there exists a projective involution which exchanges the pairs $P_i \leftrightarrow Q_i$ for $i = 1, 2, 3$.*

Duality

Problem 2.25. *Consider the following formulation of the Desargues Theorem as a theorem in $\mathbb{R}\mathrm{P}^3$:*

> *Given two triangles ΔABC and $\Delta A'B'C'$ which lie in the same plane. If joining lines of corresponding vertices meet in a point, then the interesections of corresponding sides lie on a line."*

Dualize this theorem in $\mathbb{R}\mathrm{P}^3$ (not in $\mathbb{R}\mathrm{P}^2$!). Use the letters α, β, γ to represent the dual elements to A, B, C, respectively. Sketch of the resulting configuration.

Example: *A triangular configuration in $\mathbb{R}\mathrm{P}^3$ can be precisely described as: "three distinct points lying in a plane, along with their three joining lines". The dual of this in $\mathbb{R}\mathrm{P}^3$ (called a trihedron) is: "three planes passing through a point, along with their three lines of intersection."*

Problem 2.26. *Given A_1, A_2, A_3, O, O' points in $\mathbb{R}\mathrm{P}^3$. Consider the intersection point of the lines A_iO and A_jO', and the intersection point of the lines A_jO and A_iO'. Define l_{ij} to be the joining line of these two points. Show that the lines l_{ij} have a common point. [Hint: duality]*

Problem 2.27. *Let $A, B, C \in \mathbb{R}\mathrm{P}^2$ be points with homogeneous coordinates relative to some basis by*

$$A = [2, 1, 0] \quad B = [0, 1, 1] \quad C = [-1.1, 2].$$

Find coordinates with respect to the dual basis of the three points in the dual space $\mathbb{R}\mathrm{P}^{2^*}$ that represent the three sides of the triangle. What are the coordinates of the lines joining these points in the dual space?

Problem 2.28. *Dualize the following construction by writing out the dual of each step, and provide a legible, labeled drawing of the dual construction and prove the claim.*

Note: The dual of the point P should be the line p, etc.

1. *On a given line l, choose three points P, Q, and X.*
2. *Choose line p passing through P, q passing through Q, and x passing through X such that the three lines do not pass through one point.*
3. *$A := pq$, $B := px$, $C := qx$*
4. *$s := PC$, and $r := QB$*
5. *$D := sr$*
6. *$y := AD$*
7. *$Y := yl$. Then $cr(Y, P, X, Q) = -1$.*

Problem 2.29. *Let U_1 and U_2 be two subspaces of a vectorspace V. Prove the following identities:*

1. *$(U_1 \cap U_2)^0 = U_1^0 + U_2^0$, and*
2. *$(U_1 + U_2)^0 = U_1^0 \cap U_2^0$.*

Quadrics and conics

Problem 2.30. *Let $B \in GL(3, \mathbb{R})$ be a non-singular symmetric matrix and $\mathcal{C} = \{[x] \in \mathbb{R}\mathrm{P}^2 | x^t Bx = 0\}$ be the corresponding non-degenerate conic. For every $P = [p] \in \mathcal{C}$ let $\ell_P = \{[x] \in \mathbb{R}\mathrm{P}^2 | p^t Ax = 0\}$ denote its tangent line and $P^* \in (\mathbb{R}\mathrm{P}^2)^*$ the dual point of ℓ_P.*

1. *Prove that $P^* = [Bx]$ with respect to the basis of $(\mathbb{R}^3)^*$ that is dual to the basis of the coordinates x.*

2. *Prove that $\mathcal{C}^* = \{P^* : P \in \mathcal{C}\} \subset (\mathbb{R}\mathrm{P}^2)^*$ is a non-degenerate conic.*

Problem 2.31. *Which of the following two quadratic forms defines a singular conic? If it does, write it as a product of two linear forms.*

- $x_0^2 - 2x_0x_1 + 4x_0x_2 - 8x_1^2 + 2x_1x_2 + 3x_2^2$

- $x_0^2 - 2x_0x_1 + x_1^2 - 2x_0x_2$

Problem 2.32. *Let $\mathcal{C} \subset \mathbb{R}\mathrm{P}^2$ be a non-degenerate conic through the vertices of a quadrilateral $ABCD$. Let ℓ be the line passing through the points $AC \cap BD$ and $AD \cap BC$. Prove that the tangents to $\mathcal{C}$ at A and B intersect at a point on ℓ.*

Problem 2.33. Definition. *Given a non-degenerate conic $\mathcal{C} \subset \mathbb{R}\mathrm{P}^2$. The cross-ratio of four points A, B, C, and D on $\mathcal{C}$ is defined by $cr(A, B, C, D) = cr(PA, PB, PC, PD)$, where P is another arbitrary point on $\mathcal{C}$.*

Let $\mathcal{C} \subset \mathbb{R}\mathrm{P}^2$ be a non-degenerate conic. Let P, Q, and R be such that $\mathcal{C}$ is tangent to PQ at Q and PR at R. Prove that for any $A, B \in \mathcal{C}$ the following formula holds:

$$(cr(Q, A, R, B))^2 = cr(PQ, PA, PR, PB).$$

Problem 2.34. *What is the general formula for the one-parameter family of conics through the points $[1, 1, 1], [1, -1, 1], [-1, -1, 1], [-1, 1, 1] \in \mathbb{R}\mathrm{P}^2$? Draw a picture of this family.*

What is the equation of the unique conic in this family that passes through the point $P = [2, 0, 1]$?

Problem 2.35.
- *Suppose $R = [u]$ and $S = [v]$ are two distinct points in $\mathbb{R}\mathrm{P}^2$ and consider the conic defined by the symmetric bilinear form B. Show that RS is tangent to the conic if and only if $B(u, u)B(v, v) - (B(u, v))^2 = 0$.*
- *Find the equation of the two tangent lines from the point $(1.5, -2, 1)$ to the conic given by $xy = z^2$.*

Problem 2.36. *Let $Q(x)$ be the quadratic form on $\mathbb{R}^3$ defined by $Q(x) = x_0x_1 + x_1x_2 + x_2x_0$ for $x = (x_0, x_1, x_2)$. Find a linear change of coordinates $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that with respect to the coordinates $y = T(x)$, the quadratic form is diagonal: $Q(y) = \sum \pm y_i^2$. Determine the rank and the signature of $Q(x)$.*

Problem 2.37. *This exercise is concerned not with projective, but with Euclidean geometry.*

1. *Let l be a line in the Euclidean plane and let P be a point not on l. Let e be a positive real number. Consider the set C_e of points X such that the ratio of distances from X to P and to l is equal to e:*
$$\frac{\mathrm{dist}(X, P)}{\mathrm{dist}(X, l)} = e.$$
Show that C_e is an ellipse if $e < 1$, a parabola if $e = 1$, and a hyperbola if $e > 1$.

2. *Let F_1 and F_2 be two points in $\mathbb{R}^2$ such that $d(F_1, F_2) = f$. Assume further that they are located on the x-axis symmetric with respect to the origin. Let $P = (x, y)$ be a third point such $d(F_1, P) + d(P, F_2) = l$, where $l > f$. Show that P satisfies an equation $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$ for suitable $a > 0$ and $b > 0$.*

Problem 2.38. *Let $\mathcal{C}$ be a non-degenerate conic in $\mathbb{R}\mathrm{P}^2$ and let P_1, P_2 be two points on $\mathcal{C}$. The conic $\mathcal{C}$ can be described in terms of lines through P_1, P_2. For each P_i there is a one-parameter family of lines passing through P_i, which is called a* pencil of lines. *Each pencil itself is a line $p_i \subset (\mathbb{R}\mathrm{P}^2)^*$. In this setting, the conic $\mathcal{C}$ corresponds to a projective map $f : p_1 \to p_2, L \mapsto G = f(L)$. Points $P \in \mathcal{C}$ are obtained as intersection points of corresponding lines, i.e., $P = l \cap g$.*

Denote by h the line spanned by P_1 and P_2, i.e., $H = p_1 \cap p_2$, and let $L_0 = f^{-1}(H)$ and $G_0 = f(H)$.

a) *What is the geometric meaning of l_0 and g_0 with respect to $\mathcal{C}$?*
   Draw a sketch of the Steiner construction, such that in the affine coordinates used for the sketch, $\mathcal{C}$ is an ellipse. Label the sketch according to the given description.
b) *Define $P_3 = l_0 \cap g_0$ and Let $P \neq P_1, P_2$ be a third point on $\mathcal{C}$. Why are the four points P, P_1, P_2, P_3 in general position? This allows to choose homogeneous coordinates, such that $P_1 = [1, 0, 0], P_2 = [0, 1, 0], P_3 = [0, 0, 1], P = [1, 1, 1]$. What is the equation describing $\mathcal{C}$ with respect to these coordinates?*

Problem 2.39. *Consider an ellipse in affine $\mathbb{R}^2$ that is intersected by a family of parallel lines, which gives a family of line segments that are contained in the interior of the ellipse. Show that the midpoints of those line segments are collinear.*

Quadrics

Problem 2.40. *Let q be a degenerate, but non-vanishing quadratic form on $\mathbb{R}^{n+1}$, which defines the quadric $Q \subset \mathbb{R}\mathrm{P}^n$. Let b be the corresponding symmetric bilinear form and denote $U_0 = \ker q = \{u \in \mathbb{R}^{n+1} \mid b(u, v) = 0 \; \forall v \in \mathbb{R}^{n+1}\}$. Consider any complementary subspace U_1 of U_0, such that $\mathbb{R}^{n+1} = U_0 \oplus U_1$.*

Denote $Q_1 \subset P(U_1)$ the non-degenerate quadric Q_1 that is defined by the restriction $q|_{U_1}$. Under the assumption $Q_1 \neq \emptyset$, show that Q is the union of all lines joining any point in $P(U_0)$ with any point on Q_1. Draw a sketch that illustrates this decomposition in $\mathbb{R}\mathrm{P}^3$. What happens, if Q_1 is empty?

Problem 2.41. *Consider an ellipsoid $\mathcal{E}$ in $\mathbb{R}^3$ together with a point P outside $\mathcal{E}$. Think of P as being a point light source, and take any plane Π such that the shadow of $\mathcal{E}$ on this plane is bounded.*

Show that the boundary of the shadow is a conic. What kind of conic?

Problem 2.42. *For fixed affine coordinates of $\mathbb{R}\mathrm{P}^3 = \mathbb{R}^3 \cup \mathbb{R}\mathrm{P}^2$, consider the quadric Q in $\mathbb{R}\mathrm{P}^3$ whose affine image is the unit sphere $S^2 \subset \mathbb{R}^3$. Polarity with respect to Q gives not only a relation between points and planes in $\mathbb{R}\mathrm{P}^3$, but also a relation between lines: for any line l one obtains the polar line $l^\perp$.*

What is the signature of the quadric Q? Find a way to construct the line $l^\perp$ from a given line l. Describe and prove the geometric relation between l and $l^\perp$ in $\mathbb{R}^3$ and draw a sketch.

Problem 2.43. *a) Prove Brianchon's theorem: The diagonals of a non-planar hexagon in* $\mathbb{R}\mathrm{P}^3$ *intersect in one point if and only if the extended edges are contained in a quadric of signature* $(++--)$*, i.e., the extended edges are so-called* rulings *of this quadric. The hexagon is then called a* Brianchon hexagon *and the intersection point of diagonals is called* Brianchon point.

b) Obtain Pascal's theorem about non-planar hexagons by dualizing Brianchon's theorem, where you may use that the dual of a quadric of signature $(++--)$ *also has signature* $(++--)$*. Show that a non-planar hexagon is a Brianchon hexagon if and only if it is a Pascal hexagon. What is the relation between the Brianchon point and the Pascal plane with respect to the quadric that contains the hexagon?*

Problem 2.44. *Let* B *be the symmetric bilinear form on* $\mathbb{R}^4$ *representing the surface* $x^2 + y^2 = z^2 + w^2$ *– a quadric of signature* $(++--)$*.*

- *Sketch the quadric surface* $P(B(u,u) = 0) \subset \mathbb{R}\mathrm{P}^3$.
- *Show that the polar plane of any point* P*, not lying on the quadric, cuts the quadric in a non-degenerate conic.*
- *Find and sketch the polar plane of the point* $S = (0, 0, 1, 0)$.
- *Show that* $T = (1, 0, 0, 1)$ *lies on the surface, and find the equations of the two lines lying on the surface which pass through* T.