Geometry II - Discrete Differential Geometry

Prof. Dr. Alexander Bobenko

Stand: May 31, 2007

Contents

1 The idea of DDG

Aim

DDG aims to develop discrete equivalents of the geometric notions and methods of classical differential geometry. The latter appears then as a limit of refinement of the the discretization.

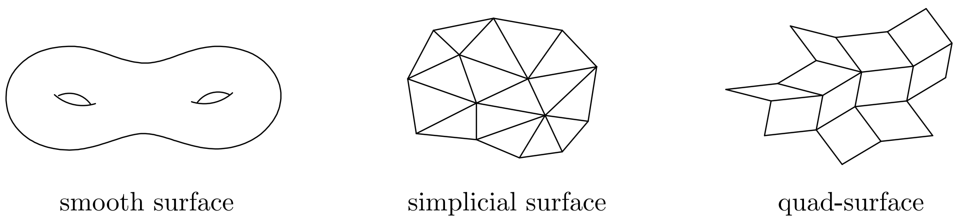


Figure 1: Different kinds of surfaces.

One might suggest many different reasonable discretizations (with the same smooth limit). Among these, which one is the best? DDG initially arose from the observation that when a notion from smooth geometry (such as the notion of a minimal surface) is discretized "properly", the discrete objects are not merely approximations of the smooth ones, but have special properties of their own, which make them form a coherent entity by themselves.

DDG versus Differential Geometry

In general

- DDG is more *fundamental*: The smooth theory can always be recovered as a limit, while it is a nontrivial problem to find out which discretization has the desired properties.

- DDG is *richer*: The discrete theory uses some structures (such as combinatorics of the mesh) which are missing in the smooth theory.
- DDG is *clarifying*: Often a discretization clarifies the structures of the smooth theory (for example unifies surfaces and their transformations, cp. Fig. 2).

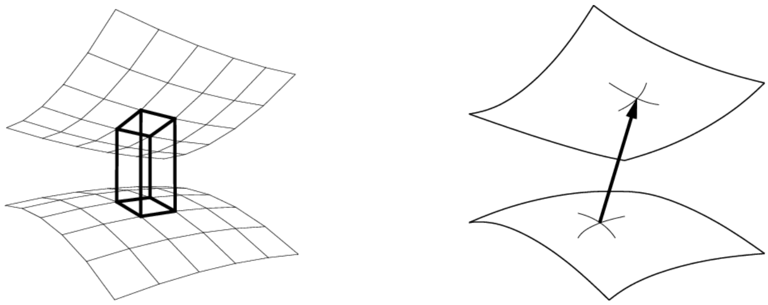


Figure 2: From the discrete master theory to the classical theory: surfaces and their transformations appear by refining two of three net directions.

- DDG is *simpler*: It uses difference equations and elementary geometry instead of calculus and analysis.
- DDG has (unexpected) *connections* to projective geometry and its subgeometries. In particular some theorems of differential geometry follow from incidence theorems of projective geometry.

Applications

Current interest in DDG derives not only from its importance in pure mathematics, but also from its applications in other fields including:

- *Computer graphics*. CG deals with discrete objects (surfaces and curves for instance) only.

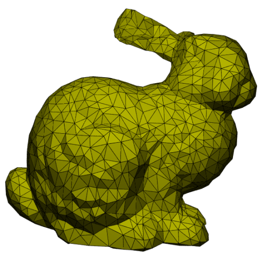


Figure 3: A simplicial rabbit.

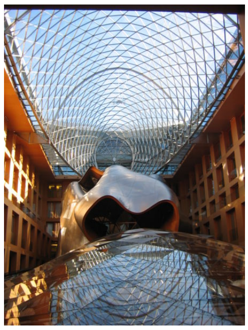
 

Figure 4: Two examples from Berlin: The “Philologische Bibliothek der FU Berlin” and an inside view of the DZ Bank at Pariser Platz.

- *Architecture.* Freeform architecture buildings have non-standard (curved) geometry but are made out of planar pieces. Common examples are glass an steel constructions.
- *Numerics.* “Proper” discretizations of differential equations are often geometric in order to preserve some important properties. There are many examples and also recent progress in hydrodynamics, electrodynamics, elasticity and so on.
- *Mathematical physics.* Discrete models are popular. DDG helps for example to clarify the phenomenon of integrability.

History

Three big names in (different branches of) DDG are:

- *R. Sauer* (starting 1930’s)

  Theory of quad-surfaces (build from quadrilateral) as an analogue of parametrized surfaces. Important difference equations (related to integrable systems), special classes of surfaces.
- *A.D. Alexandrov* (starting 1950’s)

  Metric geometry of discrete surfaces. Approximation of smooth surfaces by polyhedral surfaces.
- *W. Thurston* (1980’s)

  Developed Koebe’s ideas of discrete complex analysis based on circle patterns. Further development of this theory led in particular to construction of surfaces from circles.

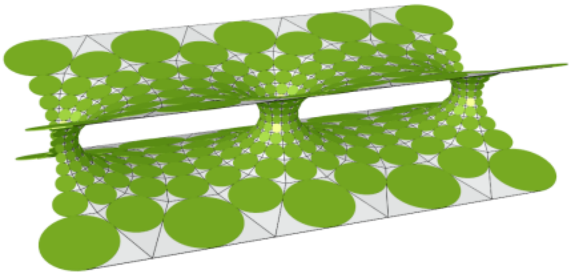


Figure 5: A discrete version of the Sherk tower, made out of touching discs.

1.1 Discretization principles

Which discretization ist the best?

- *Theoretical point of view.* The one which preserves all the fundamental properties of the smooth theory.
- *Practical point of view* (from Applications). The one which possesses good convergence properties and represents a smooth shape by a discrete shape with just a few elements very well.

Fortunately it turns out that in many cases a "natural" theoretical approximation possesses remarkable approximation properties.

Two principles of geometric discretization

1) *Transformation group principle*: Smooth geometric objects and their transformations should belong to the same geometry. In particular discretizations should be invariant with respect to the same transformation group as the smooth objects are (projective, hyperbolic, Möbius etc.). For example a discretization of a notion which belongs to Riemannian geometry should be given in metric terms only.

Another discretization principle is more special, deals with parametrized objects, and generalizes the observation from Fig. 2:

2) *Consistency principle*: Discretizations of surfaces, coordinate systems, and other smooth parametrized objects should allow to be extended to multidimensional consistent nets. All directions of such nets are indistinguishable.

2 Discrete Curves

Definition 2.1 (Discrete Curve). A *discrete curve* in $\mathbb{R}^N$ is a map $\gamma : I \to \mathbb{R}^N$ of an interval $I \subseteq \mathbb{Z}$. The interval I may be finite or infinite. A discrete curve is called *regular* if any three successive points are pairwise different. The *length* of a discrete curve is defined as

$$L(\gamma) = \sum_{k,k+1 \in I} \|\gamma_{k+1} - \gamma_k\|.$$

perhaps its better to define regularity as a requirement on the angles, since with this definition the angle might be zero (for non-arclength parametrized curves)

A discrete curve is *parametrized by arclength* if

$$\|\gamma_{k+1} - \gamma_k\| = 1 \quad \text{for all} \quad k, k+1 \in I.$$

The *tangent vector* of a regular curve is defined as

$$T_k := \frac{\gamma_{k+1} - \gamma_k}{\|\gamma_{k+1} - \gamma_k\|}.$$

For an arc length parametrized curve one has $T_k = \gamma_{k+1} - \gamma_k$.

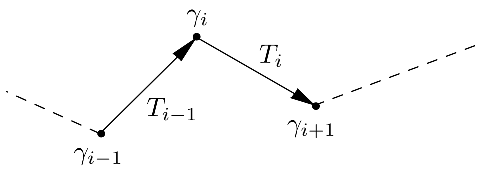


Figure 6: A part of a discrete curve.

Remarks. • We are mainly interested in the cases $N = 2$ of planar curves and $N = 3$ of space curves. In the case of planar curves it is convenient to identify $\mathbb{R}^2 \cong \mathbb{C}$.

• Even though the notation T_i suggests that the tangent vector is associated with the vertex γ_i, it should be thought of as belonging to the edge from γ_i to γ_{i+1}.

Notation. We write γ_k for $\gamma(k)$ and $\Delta\gamma_k$ for the forward difference $\gamma_{k+1}-\gamma_k$. (Like the tangent vector, $\Delta\gamma_k$ should be thought of as associated with an edge rather than a vertex.) Often we will drop the indices altogether and use the following notation: We will write

$$\gamma \text{ for } \gamma_k, \quad \gamma_1 \text{ for } \gamma_{k+1}, \quad \gamma_{\bar{1}} \text{ for } \gamma_{k-1}.$$

This notation may seem strange at first. It will become really useful when we will deal with discrete surfaces f_{jk}. Then we will write f_1 for $f_{j+1,k}$ and f_2 for $f_{j,k+1}$, etc.

Can one define a tangent vector at vertices? Yes, one can as we will see next.

2.1 Tangent flow for arclength parametrized discrete curves

A *flow* of a discrete curve $\gamma : \mathbb{Z} \to \mathbb{R}^N$ is a continuous deformation

$$\begin{aligned} \gamma : \mathbb{Z} \times \mathbb{R} \supseteq I \times J &\to \mathbb{R}^N \\ (k,t) &\mapsto \gamma(k,t), \end{aligned}$$

of the curve given by the evolution of the vertices

$$\gamma_t := \frac{\partial}{\partial t}\gamma = v,$$

where

$$v : I \times J \to \mathbb{R}^N$$

is a vector field along the curve.

If one is looking for flows on discrete arclength parametrized curves in "tangential" direction, it is natural to consider the flows of the form

$$v = \alpha(T_{\bar{1}}, T)(\gamma_1 - \gamma_{\bar{1}}),$$

where α is a real function defined on the vertices.

Definition 2.2 (Tangential flow)**.** A flow on a discrete arclength parametrized curve is called *tangential* if it is parallel to $\gamma_1 - \gamma_{\bar{1}}$ and preserves the arclength parametrization.

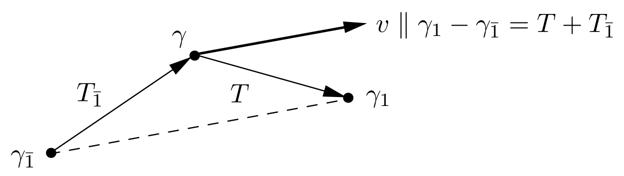


Figure 7: Tangential flow.

Remark. For smooth curves, tangent vector fields v lead to tangent flows which are curve reparametrizations. In particular, for an arclength parametrized curve one simply has $\gamma(s,t) = \gamma(s-t, 0)$ where s is the arclength parametrizia-tion, i.e. the curve stays arclength parametrized.

Proposition 2.3. *The tangential flow of an arclength parametrized discrete curve is unique up to a multiplicative constant and given by*

$$\gamma_t = \frac{T_{\bar{1}} + T}{1 + \langle T_{\bar{1}}, T \rangle}.$$

Proof. Let us compute α in $\gamma_t = \alpha(T_{\bar{1}}, T)(T_{\bar{1}} + T)$ from the condition that the arclength parametrization is preserved:

We have

$$T_t = (\gamma_1 - \gamma)_t = \alpha(T, T_1)(T + T_1) - \alpha(T_{\bar{1}}, T)(T_{\bar{1}} + T)$$

and therfore

$$\begin{array}{rcl} 0 & = & \langle T, T_t \rangle \\ \Leftrightarrow \quad \alpha(T, T_1)(1 + \langle T, T_1 \rangle) & = & \alpha(T_{\bar{1}}, T)(1 + \langle T_{\bar{1}}, T \rangle). \end{array}$$

Since the left hand side is independent of $T_{\bar{1}}$ and the right hand side is independent of T_1 we have, up to a constant,

$$\alpha(T, T_1) = \frac{1}{1 + \langle T, T_1 \rangle}.$$

□

Remark. In the smooth case the straight line can be characterized by the property that the tangential flow is a translation. In the discrete case this is different:

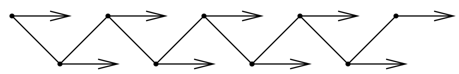
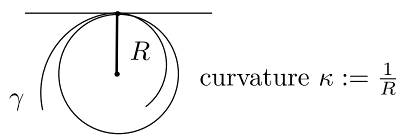


Figure 8: A dicrete zig-zag curve evolves by parallel translation under the tangential flow.

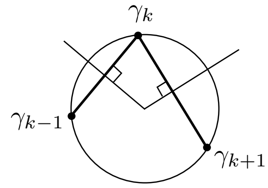
2.2 Curvature of planar discrete curves

In the smooth case, the curvature of a planar curve is defined as the inverse radius of the osculating circle.



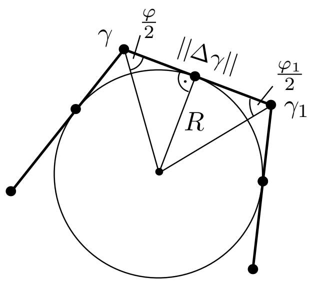
For discrete curves, the curvature may be defined in a similar way. We discuss three possibilities.

1. *Vertex osculating circle.* The circle through 3 consecutive vertices. Its center is the intersection of the perpendicular edge bisectors.



This definition has a big disadvantage: The curvature of an arclength parametrized curve cannot become larger than 2, which is strange.

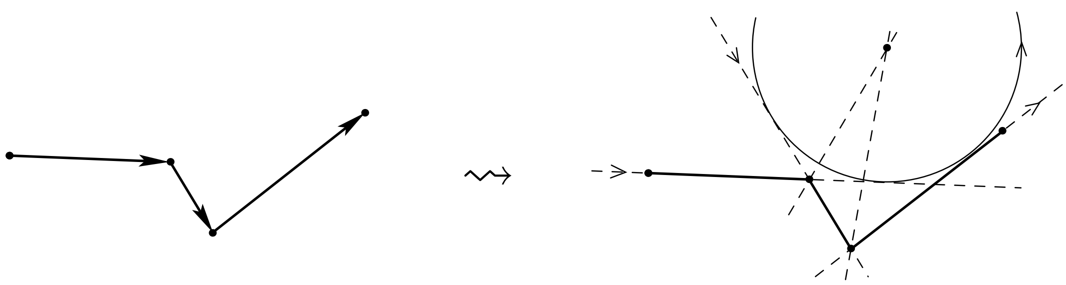
2. *Edge osculating circle.* The circle which touches 3 consecutiv edges. More precisely: The oriented circle that touches the lines through three consecutive edges (which are oriented by the direction of the curve) in such a way that at the points of contact the orientation of the circle coincides with the orientation of the lines.



For the curvature one obtains

$$\kappa = \frac{1}{R} = \frac{\cot\frac{\varphi}{2} + \cot\frac{\varphi_1}{2}}{\|\Delta\gamma\|}.$$

Note that the above definition always gives a unique osculating circle, even when the curve is non-convex.



3. *Edge osculating circle for arclength parametrized curves* (in general for curves with constant edge length). One may define the osculating circle to be the circle that touches two consecutive edges at their midpoints.

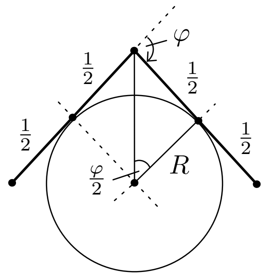


Figure 9: Osculating circle for discrete arclength parametrized curves.

With this definition of osculating circle, the curvature becomes

$$\kappa = \frac{1}{R} = 2\tan\frac{\varphi}{2}.$$

3 Tractrix and Darboux transform of a discrete curve

We start with the theory of smooth planar curves.

Assume that a point moves along a curve γ and pulls an interval $(\gamma, \widehat{\gamma})$ so that the distance $\|\widehat{\gamma} - \gamma\|$ ist constant, and the velocity vector $\widehat{\gamma}'$ is parallel to $\gamma - \widehat{\gamma}$. The curve $\widehat{\gamma}$ can be thought of as a trajectory of the second wheel of a bicycle whose first wheel moves along the curve γ.

Definition 3.1 (Smooth tractrix)**.** Let $\gamma : \mathbb{R} \supseteq I \to \mathbb{R}^2$ be a smooth planar curve. A curve $\widehat{\gamma} : I \to \mathbb{R}^2$ is called a *tractrix* of γ, if the difference $v := \widehat{\gamma} - \gamma$ satisfies

$$\|v\| = const. \qquad \text{and} \qquad \widehat{\gamma}' \parallel v.$$

Lemma 3.2. *Let γ be arclength parameterized, let $\widehat{\gamma}$ be a tractrix of γ, and let $v = \widehat{\gamma} - \gamma$. Then the curve $\widetilde{\gamma} := \gamma + 2v$ is also arclength parametrized and $\widehat{\gamma}$ is a tractrix of $\widetilde{\gamma}$ as well.*

Proof. We will show that $\|\widetilde{\gamma}'\|^2 - \|\gamma'\|^2 = 0$. Note that

$$\|\widetilde{\gamma}'\|^2 - \|\gamma'\|^2 = \langle \widetilde{\gamma}' + \gamma', \widetilde{\gamma}' - \gamma' \rangle = 2\langle \widetilde{\gamma}' + \gamma', v' \rangle.$$

But $\frac{1}{2}(\widetilde{\gamma} + \gamma) = \gamma + v$ is the tractrix of γ. The derivative $\widetilde{\gamma}' + \gamma'$ is therefore parallel to v. Now the claim follows from $0 = (\|v\|^2)' = 2\langle v, v' \rangle$. □

Definition 3.3 (Smooth Darboux transform)**.** Two arclength parameterized curves $\gamma, \widetilde{\gamma}$ are called *Darboux transforms* of each other if

$$\|\widetilde{\gamma}(s) - \gamma(s)\| = const.,$$

and $\widetilde{\gamma}$ is not just a translate of γ.

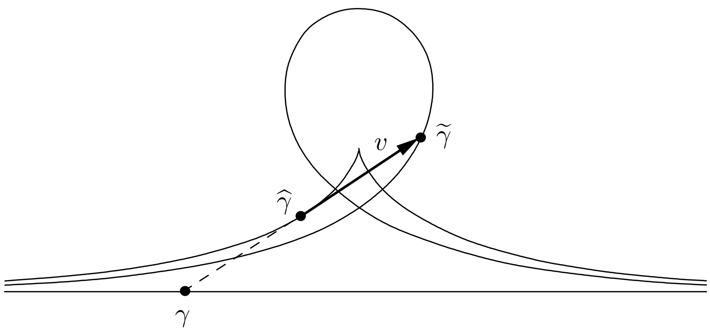


Figure 10: A Traktrix and the corresponding Darboux transform of γ.

Theorem 3.4. *Let $\gamma : I \to \mathbb{R}^2$ be an arclength parametrized curve. Then the following claims are equivalent:*

(i) $\widetilde{\gamma}$ is a Darboux transform of γ

(ii) $\widehat{\gamma} := \frac{1}{2}(\gamma + \widetilde{\gamma})$ is a tractrix of γ (and of $\widetilde{\gamma}$).

Proof. (ii) $\Rightarrow$ (i): This is the statement of Lemma 3.2.
(i) $\Rightarrow$ (ii): It is clear that $v := \frac{1}{2}(\widetilde{\gamma} - \gamma)$ is of constant length. It remains to show that $\widehat{\gamma}' \parallel v$ which is the same as $\widehat{\gamma}' \perp v'$ in this case. This is true since

$$\langle \widehat{\gamma}', v' \rangle = \langle \frac{1}{2}(\gamma' + \widetilde{\gamma}'), \frac{1}{2}(\widetilde{\gamma}' - \gamma') \rangle = \frac{1}{4}(1 - 1) = 0.$$

□

For discrete curves the definition is the same:

Definition 3.5 (Discrete Darboux transform)**.** Two discrete arclength parametrized curves $\gamma, \widetilde{\gamma} : I \to \mathbb{R}^2$ are called Darboux transforms of each other if their corresponding points are at constant distance, $\|\widetilde{\gamma}_k - \gamma_k\| = const$, and $\widetilde{\gamma}$ is not a parallel translation of γ.

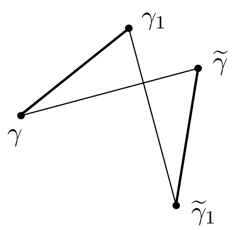


Figure 11: An elementary quadrilateral of the Darboux transformation ("Darboux butterfly").

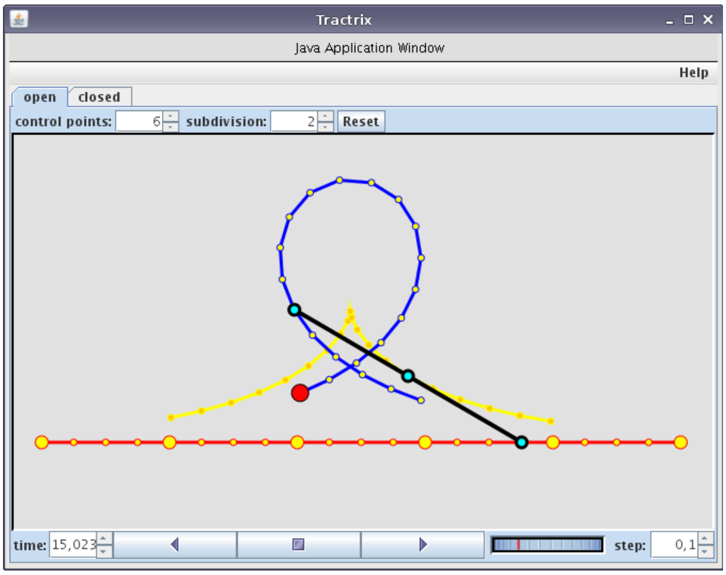


Figure 12: A pair of discrete curves in Darboux relation together with the curve traced out by the midpoint of the assembly. For this and other applications look at: http://www.math.tu-berlin.de/geometrie/lab/

There are two important generalizations of the Darboux transformation:

1. Möbius geometric (cross-ratio condition)

2. Space curves (non-commutative).

Next we will discuss the Möbius geometric generalization in detail.

3.1 Cross-ratio generalization and consistency

Definition 3.6 (Cross-ratio)**.** The *cross-ratio* of four points $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}} \cong \mathbb{CP}^1$ is defined as

$$\mathrm{cr}(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z1)}.$$

Important poperties:

1. The cross-ratio is preserved by *fractional linear transformations*

$$z \mapsto \frac{az+b}{cz+d}, \quad ad - bc \neq 0.$$

These are isomorphic to the group $\mathrm{PSL}(2, \mathbb{C})$:

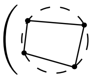
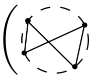
$$\frac{az+b}{cz+d} \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C}),$$

where

$$\mathrm{PSL}(2,\mathbb{C}) == \{A \in GL(2,\mathbb{C}) \mid \det(A) = 1\}/_{\{\pm I\}}.$$

Extended by the complex conjugation $z \mapsto \bar{z}$ these group expands to the group of *Möbius transformations* of the plane. The real part and the absolute value of the cross-ratio are preserved by Möbius transformations since the complex conjugation $z \mapsto \bar{z}$ induces $\mathrm{cr} \mapsto \overline{\mathrm{cr}}$.

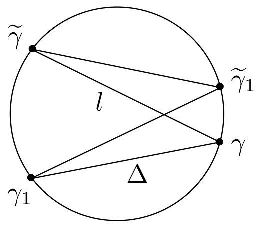
2. $\mathrm{cr}(z_1, z_2, z_3, z_4) \in \mathbb{R} \Leftrightarrow$ the points z_1, z_2, z_3, z_4 are concircular. Moreover cross-ratios of embedded circular quads are negative, and of non-embedded ones are positive.

cr () < 0, cr () > 0.

3. Möbius transformations map circles and straight lines to circles and straight lines

Lemma 3.7. *Assume $\gamma, \gamma_1, \widetilde{\gamma}$ satisfying $|\gamma_1 - \gamma| = \Delta$ and $|\widetilde{\gamma} - \gamma| = l$ are given. Let $\widetilde{\gamma}_1$ be the point determined from the condition*

$$cr(\gamma, \gamma_1, \widetilde{\gamma}_1, \widetilde{\gamma}) = \frac{\Delta^2}{l^2}.$$



Then $[\widetilde{\gamma}_1, \widetilde{\gamma}]$ is the Darboux transform of $[\gamma_1, \gamma]$.

Proof. The Darboux transform $\widetilde{\gamma}_1$ is geometrically uniquely determined by γ, γ_1 and $\widetilde{\gamma}$. From the definition of the cross-ratio follows

$$\mathrm{cr}(\gamma, \gamma_1, \widetilde{\gamma}_1, \widetilde{\gamma}) = \frac{\Delta^2}{l^2}$$

(the value of the cross-ratio is greater 0 since the quadrilateral is not embedded). Since the cross-ratio for three fixed points and one variable argument is a bijective function, the former equation also determines $\widetilde{\gamma}_1$ uniquely. □

This leads to the following Möbius generalization of the Darboux transformation.

Definition 3.8 (Möbius Darboux transform)**.** Let $\gamma : I \to \mathbb{C}$ be a discrete curve, and let $\alpha_i \in \mathbb{R}$ (or $\mathbb{C}$) associated to the edges $[\gamma_i, \gamma_{i+1}]$. A curve $\widetilde{\gamma} : I \to \mathbb{C}$ is called a *Möbius Darboux transform* of γ with parameter $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), if

$$\mathrm{cr}(\gamma_i, \gamma_{i+1}, \widetilde{\gamma}_{i+1}, \widetilde{\gamma}_i) = \frac{\alpha_i}{\lambda}.$$

Lemma 3.7 shows that the standard Euclidean definition 3.5 of the Darboux transformation corresponds to $\lambda = \frac{1}{l^2}$ and $\alpha_i = 1\ \forall i$ in the Möbius generalization, where $l = |\widetilde{\gamma} - \gamma|$ is the constant distance.

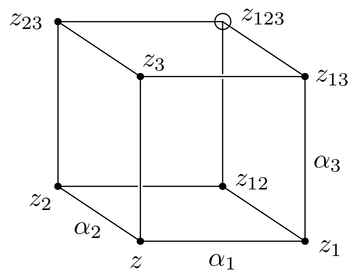
Note that the cross-ratio condition can also be applied to non-arclength parametrized curves.

Now consider a three dimensional combinatorial cube and assume that all edges of the cube parallel to the axis j carry a label α_j. Suppose that the values $z, z_1, z_2, z_3 \in \mathbb{C}$ are given at a vertex and its three neighbours. Then the *cross-ratio equation*

$$\mathrm{cr}(z, z_i, z_{ij}, z_j) = \frac{\alpha_i}{\alpha_j}$$

applied to the three faces intersecting at z uniquely determines the values z_{12}, z_{13}, z_{23}.

define all this in general for quad-equations?



After that the cross-ratio equation delivers three a priori different values for z_{123}, coming from the three different faces containing z_{123} on which the equation can be imposed.

In general, if for such a system these values, which can be computed in several ways, coincide for any choice of the initial data z, z_1, z_2, z_3, then the system is called *3D-consistent.*

It is not difficult to show that any 3D-consistent system is ND-consistent for any $N \geq 3$, thus can be consistently defined on a $\mathbb{Z}^N$-lattice.

Theorem 3.9. *The cross-ratio equation is 3D-consistent.*

Proof. $\mathrm{cr}(z, z_1, z_{13}, z_3) = \frac{\alpha_1}{\alpha_3}$ can be rewritten as

$$\begin{aligned}
& \frac{\alpha_1}{\alpha_3}\frac{z_{13} - z_1}{z_1 - z} = \frac{z_{13} - z_3}{z_3 - z} \\
\Leftrightarrow\quad & \frac{\alpha_1}{\alpha_3}(z_{13} - z_1)\frac{z_3 - z_1}{z_1 - z} = (z_{13} - z_1) + (z_1 - z_3) \\
\Leftrightarrow\quad & (z_{13} - z_1)\left(1 + \frac{\alpha_1}{\alpha_3}\frac{z_3 - z}{z - z_1}\right) = z_3 - z_1 = z_3 - z + z - z_1.
\end{aligned}$$

Thus $(z_{13} - z_1)$ is a Möbius transformation of $(z_3 - z)$:

$$z_{13} - z_1 = L(z_1, z, \alpha_1, \alpha_3)[z_3 - z],$$

where

$$L(z_1, z, \alpha_1, \alpha_3) = \begin{pmatrix} 1 & z - z_1 \\ \frac{\alpha_1}{\alpha_3(z-z_1)} & 1 \end{pmatrix}.$$

is its matrix representation and

$$\widetilde{z} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z :\Leftrightarrow \widetilde{z} = \frac{az+b}{cz+d}.$$

Going arround the cube once, we have

$$\begin{aligned} z_{123} - z_{12} &= L(z_{12}, z_1, \alpha_2, \alpha_3)[z_{13} - z_1] \\ z_{123} - z_{12} &= L(z_{12}, z_2, \alpha_1, \alpha_3)[z_{23} - z_2]. \end{aligned}$$

This equality of these two values of z_{123} follows from the stronger claim

$$L(z_{12}, z_1, \alpha_2, \alpha_3)L(z_1, z, \alpha_1, \alpha_3) = L(z_{12}, z_2, \alpha_1, \alpha_3)L(z_2, z, \alpha_2, \alpha_3).$$

which shall be checked as an exercise.
The last equation (on the top face) then follows from symmetry. □

Now we will show that the Darboux butterflies can be put to all the faces of a combinatorial cube consistently.

Corollary 3.10 (Consistency of the 2D Darboux transformation)**.** *Given a point z and its three neighbours $z_1, z_2, z_3 \in \mathbb{C}$, let z_{ij} be the Darboux transforms associated to the faces of a cube. Then there exists a unique $z_{123} \in \mathbb{C}$ such that the faces $(z_1, z_{13}, z_{123}, z_{12})$, $(z_2, z_{23}, z_{123}, z_{12})$, and $(z_3, z_{13}, z_{123}, z_{23})$ are Darboux butterflies.*

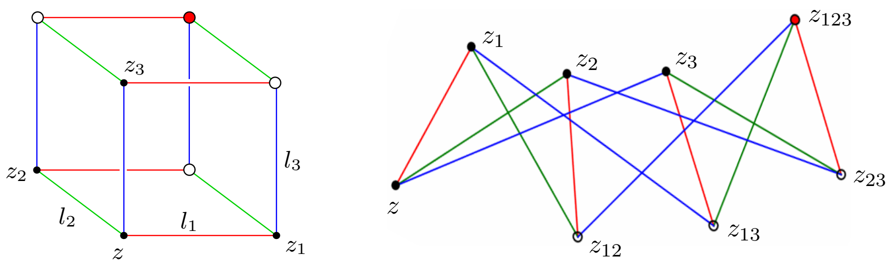


Figure 13: The combinatorial Darboux cube for the given data and the two dimensional embeding of the corresponding Darboux butterflies together with the unique consistent completion.

Proof. This follows from the previous Theorem 3.9 if one sets $\alpha_i = l_i^2$, where $l_i = |z - z_i|$. □

The idea of consistency (or compatibility) is in the core of the theory of integrable systems. One is faced with it already in the very beginning when defining *complete integrability* of a Hamiltonian flow in the Liouville-Arnold sence, which means exactly that the flow may be included into a complete family of commuting (compatible) Hamiltonian flows. The 3D-consistency is an example of this phenomenon in the discrete setup. Moreover, the consistency phenomenon has developed into one of the fundamental principles of discrete differential geometry, the *consistency principle* already mentioned in Section 1.

Next we give a simple example to show how this principle implies some facts from the corresponding smooth theory.

3.2 Darboux transformation and tangent flow

The tangent flow can be seen as an infinitesimal Darboux transformation. This observation is based upon a simple

Lemma 3.11 (Tangent flow as infinitesimal Darboux transformation)**.** *Let $\gamma_{k-1}, \gamma_k, \gamma_{k+1}$ be three consecutive vertices of an arclength parametrized curve. A Darboux transform of this curve is determined by choosing a vertex η_k corresponding to the vertex γ_k infinitesimally close to γ_{k-1}:*

$$\eta_k = \gamma_{k-1} + \varepsilon w + o(\varepsilon), \quad \varepsilon \to 0$$

with some $w \in \mathbb{C}$. Then the next vertex of the Darboux transform is given by

$$\eta_{k+1} = \gamma_k + \varepsilon v_k \langle w, T_{k-1} \rangle + o(\varepsilon), \tag{1}$$

where v_k is the tangent flow at γ_k:

$$v_k = \frac{T_k + T_{k-1}}{1 + \langle T_k, T_{k-1} \rangle}.$$

In particular, if $w = v_{k-1}$ then also $\eta_{k+1} = \gamma_k + \varepsilon v_k + o(\varepsilon)$.

Proof. The distance between γ and its Darboux transform η is given by

$$l^2 = \|\gamma - \eta\|^2 = \|T_{k-1} - \varepsilon w + o(\varepsilon)\|^2 = 1 - 2\varepsilon \langle w, T_{k-1} \rangle + o(\varepsilon).$$

For the cross-ratio this implies

$$q := \mathrm{cr}(\gamma_{k-1}, \gamma_k, \eta_k, \eta_{k-1}) = \frac{1}{l^2} = 1 + 2\varepsilon \langle w, T_{k-1} \rangle + o(\varepsilon).$$

Resolving the cross-ratio formula $q = \operatorname{cr}(\gamma_k, \gamma_{k+1}, \eta_{k+1}, \eta_k)$ for η_{k+1} we obtain

$$\eta_{k+1} - \gamma_k = \frac{(1-q)(\gamma_k - \gamma_{k+1})(\eta_k - \gamma_k)}{\gamma_k - \gamma_{k+1} + q(\eta_k - \gamma_k)}.$$

In the limit $\varepsilon \to 0$ this yields

$$\eta_{k+1} - \gamma_k = \varepsilon\langle w, T_{k-1}\rangle \frac{T_k \cdot T_{k-1}}{T_k + T_{k-1}} + o(\varepsilon),$$

which is equivalent to (1). □

If we apply such a Darboux transformation to the left end vertex of γ_0 of a discrete curve $\gamma : \{0, .., N\} \to \mathbb{R}^2$ we obtain an infinitesimal tangent flow of all vertices (except maybe γ_0).

Theorem 3.12. *The Darboux tranformation of discrete arclength parametrized curves is compatible with its tangent flow. This means:*
Given two discrete arclength parametrized curves $\gamma, \widetilde{\gamma} : I \to \mathbb{R}^2$ evolving under the tangential flow $t \mapsto \gamma(t, \cdot)$, $t \mapsto \widetilde{\gamma}(t, \cdot)$. If the curves are in Darboux correspondence at some $t = t_0$, then they are Darboux transforms of each other for all t.

Proof. This fact can be derived from the permutability of the Darboux transformations. The three directions of the compatibility cube for Darboux transformations get three different interpretations:

Let γ and $\widetilde{\gamma}$ be a Darboux pair. Let η be an infinitesimal Darboux transform of γ as in Lemma 3.11, i.e. with $\eta_k \to \gamma_{k-1}$. The Darboux condition determines vertices $\widetilde{\eta}_k$ and $\widetilde{\eta}_{k+1}$, as in the picture, uniquely (from Corollary 3.10 we know that everything is consistent).

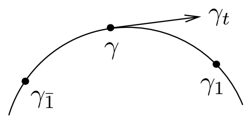
From the cross-ratio $\operatorname{cr}(\gamma_k, \eta_k, \widetilde{\eta}_k, \widetilde{\gamma}_k)$ it is easy to see that the vertex $\widetilde{\eta}_k$ of the Darboux cube also satisfies $\widetilde{\eta}_k \to \widetilde{\gamma}_{k-1}$ and therefore, due to Lemma 3.11, the curve $\widetilde{\eta}$ is given by the tangent flow of $\widetilde{\gamma}$.

Passing to the limit $\varepsilon \to 0$ we obtain the claim of the theorem.

□

Remark. (Möbius geometry tangent flow).
For the Möbius Darboux transformation the corresponding construction leads to the flow

$$\gamma_t = \frac{\Delta\gamma \cdot \Delta\gamma_{\bar{1}}}{\Delta\gamma + \Delta\gamma_{\bar{1}}}$$



which is tangent to the circle through $\gamma_{\bar{1}}, \gamma$ and γ.

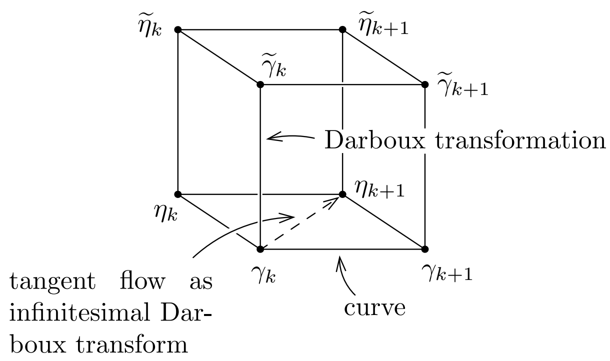


Figure 14: The Darboux compatibility cube with different interpretations of the three dimensions.

Remark. (Generalization to space curves).
There exists also a generalization of the Darboux transformation to a consistent system in an associative algebra $\mathcal{A}$. The case of quaternions, $\mathcal{A} = \mathbb{H}$, leads to a Darboux transformation for space curves (with twist).

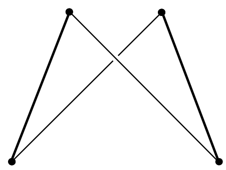


Figure 15: A Darboux butterfly for space curves is not planar.

4 Discrete elastica

Consider variations γ of a (space) curve of length L which preserve:

- the length of the curve
- the endpoints
- the tangent directions at the endpoints.

Parametrizing by arclength we get mappings $\gamma : [0, L] \to \mathbb{R}^3$ and $T = \gamma' : [0, L] \to S^2$ with constant

- $\gamma(0)$, $\gamma(L) \in \mathbb{R}^3 \Rightarrow \gamma(L) - \gamma(0) = \int_0^L T(s)ds$
- $T(0), T(L)$

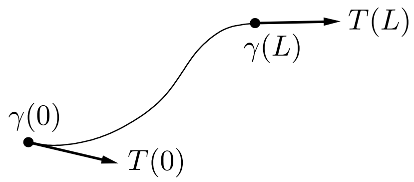


Figure 16: One admissible variation under given boundary conditions.

Definition 4.1 (Smooth Elastica, Bernoulli's elastica)**.** *Elastica* are extremals (critical points) of the bending energy

$$E = \int_0^L \kappa^2(s) ds.$$

Here $\kappa^2 = \|T'\|^2$ is the curvature of a curve.

Basic fact from the calculus of variations. The critical points (extremals) of the functional

$$\mathcal{S} = \int_0^L \mathcal{L}(q, q') ds$$

under the variations preserving the constraints

$$F_i := \int_0^L f_i(q, q') ds = c_i \in \mathbb{R} \quad i = 1, .., N$$

are also critical points of the unconstraint functional

$$\mathcal{S}_\lambda = \mathcal{S} + \sum_{i=1}^N \lambda_i F_i$$

with some constants $\lambda_i \in \mathbb{R}$. These constants are determined by the conditions $F_i = c_i \ \forall \ i$ and are called *Lagrange multipliers*.

Basic fact from Lagrangian mechanics. The trajectory $q(t)$ of a mechanical system with the potential energy P and the kinetic energy K is critical for the *action functional*

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L}(q, q') dt$$

with the Lagrangian $\mathcal{L} = K - P$.

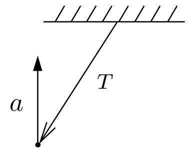
Theorem 4.2. *An arclength parametrized curve $\gamma : [0, L] \to \mathbb{R}^3$ is elastica if and only if its tangent vector $T = \gamma' : [0, L] \to S^2$ describes the evolution of the axis of a spherical pendulum.*

Proof. The extrema of the functional

$$\mathcal{S} = \int_0^L \left(\frac{1}{2}\langle T', T' \rangle - \langle a, T \rangle \right) ds, \quad a \in \mathbb{R}^3, \quad T : [0, L] \to S^2,$$

can be equivalently seen as:

1. extrema of the functional $\int\limits_0^L \langle T', T' \rangle ds$ with the constraints $\int\limits_0^L T ds = c \in \mathbb{R}^3$, i.e. elastica.
2. extrema of the Lagrangian of a spherical pendulum with the kinetic energy $\frac{1}{2}\langle T', T' \rangle$ and the gravitational energy $\langle a, T \rangle$ (cp. second basic fact).



The constant vector $a \in \mathbb{R}^3$ is interpreted as gravitation vector in the pendulum case and as Lagrange multipliers for the elastica variational problem. □

Remarks. • The spherical pendulum in S^n and elastica in $\mathbb{R}^n$ always lie in 3-dimensional euclidean spaces spanned by the initial resp. boundary conditions

$$T(0),\, T'(0),\, a \quad (\text{or } \gamma(L) - \gamma(0),\, T(0),\, T(L)).$$

• Planar elastica were classified by Euler (cp. Fig. 17)

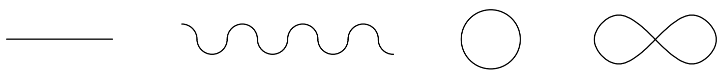


Figure 17: Examples of planar elastica, among them the two only closed ones: circle and Euler's elastic eight.

• These curves correspond to the classical mathematical pendulum in the plane.

Now what is a proper bending energy for a discrete arclength parametrized curve $\gamma : I \to \mathbb{R}^3,\ T : I \to S^2$?

As we will see later, there are deep mathematical arguments to define the bending energy of a discrete curve at vertex γ_i as

$$\mathcal{E}_i := \log\left(1 + \frac{\kappa_i^2}{4}\right).$$

Here φ is the bending angle and $\kappa = 2\tan\frac{\varphi}{2}$ is the curvature of the osculating circle touching two neighbouring edges at the midpoints (cp. Fig. 9). For now we just check that this definition agrees with physical intuition:

- In the smooth limit $\varphi_i \to 0$ we have $\kappa \to 0$ and $\mathcal{E}_i \to \frac{1}{4}\kappa_i^2$, which agrees with the definition in the smooth case.
- For singular curves $\varphi_i \to \pi$ the energy becomes infinite.
- One gets this bending energy assuming that the bending force is proportional to the curvature. For the bending energy this implies

$$\begin{aligned}\int_0^{\varphi} \kappa(\psi)d\psi &= \int_0^{\varphi} 2\tan\frac{\psi}{2}d\psi = -4\log\left(\cos\frac{\varphi}{2}\right) \\ &= -2\log\left(\cos^2\frac{\varphi}{2}\right) = 2\log\left(1+\tan^2\frac{\varphi}{2}\right).\end{aligned}$$

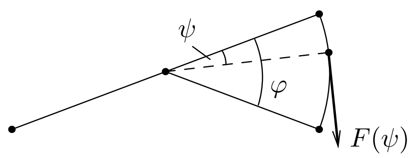


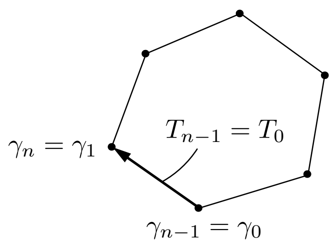
Figure 18: The bending energy as the integral of the bending force.

Definition 4.3 (Discrete Elastica)**.** A discrete arclength parametrized curve $\gamma : I \to \mathbb{R}^3$ with the tangent vectors $T : I \to S^2$ is called *discrete elastica* if it is critical for the functional

$$\mathcal{S} = \sum_i \log\left(1 + \frac{\kappa_i^2}{4}\right) \cong \sum_i \log(1 + \langle T_i, T_{i-1}\rangle) \cong \sum_i \log \|T_{i-1} + T_i\|.$$

Here $\kappa_i = \tan\frac{\varphi_i}{2}$ is the curvature and $\cong$ means that the functionals are equivalent (i.e. have the same critical points). The permissible variations should preserve the end points and the end edges γ_0, γ_n, T_0, T_{n-1}; $I = [0, n] \subset \mathbb{Z}$.

A special case of boundary conditions is the case of closed discrete elastica. The conditions $\gamma_{n-1} = \gamma_0$, $\gamma_n = \gamma_1$ imply $T_0 = T_{n-1}$. Without loss of generality one can assume that γ_0 and γ_1 are fixed.



Theorem 4.4 (Euler-Lagrange equations for discrete elastica)**.** *A discrete arclength parametrized curve* $\gamma : I \to \mathbb{R}^3$ *is a discrete elastica if and only if there exist vectors* $a, b \in \mathbb{R}^3$ *such that*

$$\frac{T_k \times T_{k-1}}{1 + \langle T_k, T_{k-1} \rangle} = a \times \gamma_k + b, \quad \textit{where } T_k = \gamma_{k+1} - \gamma_k. \tag{2}$$

Proof. Let us derive the equations for critical points of the functional

$$\mathcal{S} = \sum_k \log\left(1 + \langle T_{k-1}, T_k \rangle\right).$$

The constraints $\sum_{k=0}^{n-1} T_k = const$ and $\langle T_k, T_k \rangle = 1 \ \forall k$ can be taken into account by Lagrange multipliers. This leads to the functional

$$\mathcal{S}_\lambda = \sum_k \left(\log\left(1 + \langle T_{k-1}, T_k \rangle\right) - c_k \langle T_k, T_k \rangle - \langle a, T_k \rangle\right).$$

Here $c_k \in \mathbb{R}$ and $a \in \mathbb{R}^3$ are Lagrange multipliers.
The equation for critical points

$$\frac{\partial \mathcal{S}_\lambda}{\partial T_k} = 0 \quad \forall k$$

reads

$$\frac{T_{k-1}}{1 + \langle T_k, T_{k-1} \rangle} + \frac{T_{k+1}}{1 + \langle T_k, T_{k+1} \rangle} = a + 2c_k T_k.$$

The cross product with T_k implies

$$\frac{T_{k+1} \times T_k}{1 + \langle T_k, T_{k+1} \rangle} - \frac{T_k \times T_{k-1}}{1 + \langle T_k, T_{k-1} \rangle} = a \times T_k = a \times (\gamma_{k+1} - \gamma_k),$$

which is equivalent to (2). □

In the smooth limit $\langle T_k, T_{k+1} \rangle$ we get

$$T_k \times T_{k-1} = (T_k - T_{k-1}) \times T_{k-1} \to T' \times T,$$

and thus the equation (2) yields the Euler-Lagrange equation

$$\gamma'' \times \gamma = a \times \gamma + b \tag{3}$$

for smooth elastica.

Definition 4.5 (Discrete spherical pendulum)**.** A *discrete spherical pendulum* is a map $T : I \to S^2$ which is critical for the Lagrangian

$$\mathcal{L} = \log\left(1 + \langle T_k, T_{k-1}\rangle\right) - \langle a, T_k\rangle.$$

Here the terms $\log(1 + \langle T_k, T_{k-1}\rangle)$ and $\langle a, T_k\rangle$ are interpreted as the kinetic and the potential energies of the pendulum with the gravitation vector $a \in \mathbb{R}^3$.

Imediately one obtains a discrete version of Theorem 4.2:

Theorem 4.6. *A discrete arclength parametrized curve $\gamma : I \to \mathbb{R}^3$ is a discrete elastica if and only if its tangent vector $T_k = \gamma_{k+1} - \gamma_k$, $T : I \to S^2$, describes the evolution of a discrete spherical pendulum.*

Proof. Analogous to the proof of Theorem 4.2 □

4.1 Discrete Heisenberg flow and elastica

Definition 4.7 (Discrete Heisenberg flow)**.** The flow

$$\gamma_t = \frac{T_k \times T_{k-1}}{1 + \langle T_k, T_{k-1}\rangle} \tag{4}$$

on discrete arclength parametrized curves is called *Heisenberg flow.*

In the smooth case the Heisenberg flow on arclength parametrized curves is given by $\gamma_t = T' \times T$. Differentiation one obtains $T_t = T'' \times T$ which is an equation for the Heisenberg magnetic.

There is the following important relation between the discrete Heisenberg flow and the tangential flow we discussed earlier:

going to be adopted for general discrete curves

Lemma 4.8. *The Heisenberg flow is the only local flow in the binormal direction which commutes with the tangent flow on discrete arclength parametrized curves.*

Proof. to be completed (direct long computation). □

The discrete Heisenberg flow allows another characterization of discrete elastica as follows:

Theorem 4.9. *A discrete arclength parametrized curve is a discrete elastica if and only if the Heisenberg flow preserves its form, i.e. under the action of this flow the curve evolves by an Euclidean motion.*

For the proof of this theorem we need a convenient analytic description of rigid (Euclidean) motions. Therefore we will now give a *quaternionic description* of these, which will be also convenient for treatment of other problems in these lectures.

4.1.1 Quaternions

The *quaternion algebra* $\mathbb{H}$ is a real-4-dimensional generalization of complex numbers. Let $\{1, \mathfrak{i}, \mathfrak{j}, \mathfrak{k}\}$ be the standard basis of $\mathbb{H}$ for which holds

$$\mathfrak{i}\mathfrak{j} = \mathfrak{k},\ \mathfrak{j}\mathfrak{k} = \mathfrak{i},\ \mathfrak{k}\mathfrak{i} = \mathfrak{j} \quad \text{and} \quad \mathfrak{i}\mathfrak{i} = \mathfrak{j}\mathfrak{j} = \mathfrak{k}\mathfrak{k} = -1.$$

A quaternion

$$q = q_0 \cdot 1 + q_1 \cdot \mathfrak{i} + q_2 \cdot \mathfrak{j} + q_3 \cdot \mathfrak{k}$$

is decomposed in its *real* and *imaginary parts*

$$\mathrm{Re}(q) := q_0 \in \mathbb{R}, \quad \mathrm{Im}(q) := q_1\mathfrak{i} + q_2\mathfrak{j} + q_3\mathfrak{k} \in \mathrm{Im}\mathbb{H}.$$

The *conjugated quaternion* $\bar{q}$ and the *absolute value* $|q|$ are given by

$$\bar{q} := q_0 - q_1\mathfrak{i} - q_2\mathfrak{j} - q_3\mathfrak{k}, \quad |q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2.$$

Then for any $q, p \in \mathbb{H}$ one has

$$\overline{qp} = \bar{p} \cdot \bar{q}$$

and the *inverse quaternion* is given by

$$q^{-1} = \frac{\bar{q}}{|q|^2}.$$

We identify vectors in $\mathbb{R}^3$ with imaginary quaternions

$$v = (v_1, v_2, v_3) \in \mathbb{R}^3 \longleftrightarrow v = v_1\mathfrak{i} + v_2\mathfrak{j} + v_3\mathfrak{k} \in \mathrm{Im}\mathbb{H}.$$

For the quaternionic product of imaginary quaternions this implies

$$vw = -\langle v, w\rangle + v \times w \tag{5}$$

where $\langle v, w\rangle$ and $v \times w$ are the scalar and vector product in $\mathbb{R}^3$.
Unitary quaternions

$$\mathbb{H}_1 = \{q \in \mathbb{H} \mid |q| = 1\}$$

can be parametrized as $q = \cos\alpha + \sin\alpha w$, where $\alpha \in [0, \pi]$ and $w \in \mathrm{Im}\mathbb{H}$ is a unitary vector $|w| = 1$.

Proposition 4.10. *The map $R_q : \mathbb{R}^3 \to \mathbb{R}^3$ induced by the quaternionic mapping $v \mapsto qvq^{-1}$ with $q = \cos\alpha + \sin\alpha w \in \mathbb{H}_1$ ($w \in Im\mathbb{H}$, $|w| = 1$) is the rotation about the vector w by the angle 2α.*

Proof. Let $v \in \mathrm{Im}\mathbb{H}$. Decompose $v = v_{\|} + v_{\perp}$ with $v_{\|} \parallel w$, $v_{\perp} \perp w$. From (5) follows

$$v_{\|}\, w = -\langle v_{\|}, w\rangle = w\, v_{\|} \tag{6}$$

and

$$v_{\perp}\, w = v_{\perp} \times w = -w\, v_{\perp}. \tag{7}$$

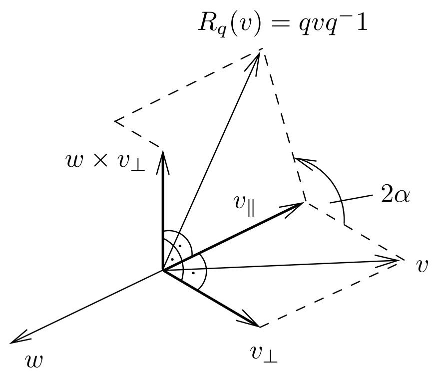
Since $|q| = 1$ we have $q^{-1} = \bar{q} = \cos\alpha - \sin\alpha w$ and therefore

$$\begin{aligned} q\, v_{\|}\, q^{-1} &= (\cos\alpha + \sin\alpha w)\, v_{\|}\, (\cos\alpha - \sin\alpha w) \\ &\overset{(6)}{=} (\cos\alpha + \sin\alpha w)(\cos\alpha - \sin\alpha w)\, v_{\|} \\ &= qq^{-1} v_{\|} = v_{\|} \end{aligned}$$

as well as

$$\begin{aligned} q\, v_{\perp}\, q^{-1} &= (\cos\alpha + \sin\alpha w)\, v_{\perp}\, (\cos\alpha - \sin\alpha w) \\ &\overset{(7)}{=} (\cos\alpha + \sin\alpha w)(\cos\alpha + \sin\alpha w)\, v_{\perp} \\ &= (\cos^2\alpha - \sin^2\alpha + 2\cos\alpha\sin\alpha w)\, v_{\perp} \\ &= \cos(2\alpha)\, v_{\perp} + \sin(2\alpha)\, w \times v_{\perp}. \end{aligned}$$

Hence R_q is a rotation about w of amount 2α since the parallel ratio of v is preserved and the perpendicular one is rotated by 2α in a plane perpendicular to w.



□

Remark. Since for any axis of rotation one finds a unit vector w, all rotations of $\mathbb{R}^3$ can be described in the above way. Moreover for each rotation R_q there are exactly two unitary quaternions which describe this rotation, namely q and $-q$, which means that unitary quaternions are a double covering of $SO(3)$, the rotation group of $\mathbb{R}^3$.

As mentioned before an *Euclidean motion* is a composition of a rotation and a translation and therefore can be described as

$$v \mapsto qvq^{-1} + p, \quad q \in \mathbb{H}_1, p \in \mathrm{Im}\mathbb{H}$$

in the quaternionic setup.
Let $t \mapsto (q(t), p(t))$ be a one parameter family of Euclidean motions (here t is interpreted as time). The derivative of this flow is a vector field called an *infinitesimal Euclidean motion.*

Corollary 4.11. *Infinitesimal Euclidean motions are vector fields of the form*

$$v_t = a \times v + b, \quad a, b \in Im\mathbb{H}. \tag{8}$$

Here $a, b \in \mathbb{R}^3$ are called angular velocity *and* (translation) velocity *of the motion.*

Proof. Differentiating $v(t) = q(t)v_0q^{-1}(t) + p(t)$ we get

$$v_t = [q'q^{-1}, v] + p'.$$

The quaternions $q'q^{-1}$ and p' are imaginary and are identified with vectors in $\mathbb{R}^3$. Indeed $q\bar{q} = 1$ implies $q'\bar{q} + q\bar{q}'$, which is the imaginarity condition for the quaternion $q'q^{-1} = q'\bar{q}$. The Euclidean motion flow corresponding to the vector field (8) is given by

$$q(t) = e^{at}, \quad p(t) = bt.$$

□

This leads to an easy

Proof of Theorem 4.9. The claim follows immediately from Theorem 4.4 and Corollary 4.11. □

what about generalizations and problems from lecture notes p.30? beeing adapted

5 Discrete surfaces

many different views of discrete surfaces, general definitions

6 Curvatures of polyhedral surfaces

In the smooth case the Gaussian curvature of a surface at a point p is defined as the quotient of oriented areas

$$K(p) = \lim_{\varepsilon \to 0} \frac{A\left(N\left(U_\varepsilon(p)\right)\right)}{A\left(U_\varepsilon(p)\right)},$$

where $U_\varepsilon(p)$ is an ε-neighborhood of p on the surface, and $N(U_\varepsilon(p)) \subset S^2$ is its image under the Gauss map.

For a polyhedral surface the curvature concentrates at vertices: the area $N(U_\varepsilon(p))$ vanishes for all internal points of faces and edges. For a vertex it is equal to the area of the corresponding geodesic spherical polygon, which is $2\pi - \sum_i \alpha_i$. Here α_i are the external angles of the spherical polygon, which conincide with the angles of the different faces of the polyhedral surface at vertex p (cp. Fig. 19). Thus $\sum \alpha_i$ is the total vertex angle at p. picture p. 36a

Figure 19: α_2 is the angle between n_2 and m_2. Rotating by $\frac{\pi}{2}$ about N_2 we get α_2 as the angle between $N_1 \times N_2$ and $N_3 \times N_2$.

Definition 6.1 (Discrete Gaussian curvature)**.** For a polyhedral surface S the angle defect

$$K(p) := 2\pi - \sum_i \alpha_i$$

at a vertex p is called the *Gaussian curvature* of S at p. The *total Gaussian curvature* is defined as the sum

$$K(S) := \sum_{p \in V} K(p).$$

The points with $K(p) > 0$, $K(p) = 0$ and $K(p) < 0$ are called *spherical*, *euclidean*, and *hyperbolic* respectively.

Remark. Since the discrete Gaussian curvature is defined intrinsically (the normals are not involved) it is preserved by isometries as in the smooth case (Gauss' Theorema Egregium).

Theorem 6.2 (Polyhedral Gauss-Bonnet)**.** *The total Gaussian curvature of a compact, closed polyhedral surface S is given by*

$$K(S) = 2\pi\chi(S).$$

Proof. We have

$$K(S) = \sum_{p \in V} K(p) = 2\pi|V| - \sum_{\text{all angles of } S} \alpha_i.$$

The angles $\pi - \alpha_i$ are the (oriented) external angles of a polygon. Their sum is

$$\sum_{\text{all angles of the polygon}} (\pi - \alpha_i) = 2\pi.$$

Therefore the sum over all faces gives

$$\sum_{\text{all angles of } S} (\pi - \alpha_i) = 2\pi|F|$$

which leads to

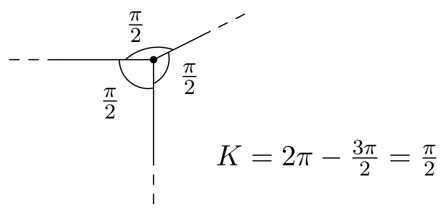
$$2\pi|E| - \sum_{\text{all angles of } S} \alpha_i = 2\pi|F|.$$

Here we used the following: In the sum each angle gives π and the number of angles is equal to $2|E|$ (each edge is associated with 4 attached angles but each angle comes with two edges). Together we have

$$K(S) = 2\pi(|V| - |E| + |F|) = 2\pi\chi(S).$$

□

Example 6.3 (Gaussian curvature of a cube)**.** The Gaussian curvature of every vertex of a cube is $\frac{\pi}{2}$.



6.1 Principal curvatures

Extrinsic curvatures of a smooth surface are defined as follows. Consider a one parameter family of spheres $S(\kappa)$ of signed curvature κ touching the surface at a point p (κ is positive if the sphere lies at the same side of the tangent plane as the normal and negative otherwise). Let M be the set of tangent spheres intersecting any neighborhood $U \subset p$ in more than just the point p. The values

$$\kappa_1 = \inf_{S \in M} \kappa(S), \quad \kappa_2 = \sup_{S \in M} \kappa(S)$$

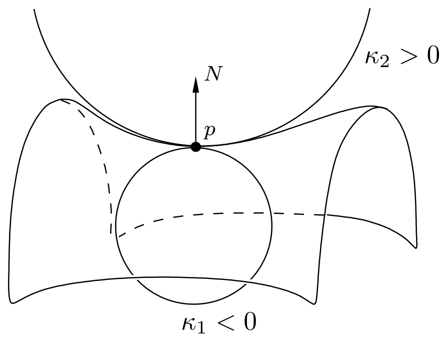


Figure 20: The curvature spheres touching the surface in p.

are called the *principal curvatures* of the surface at p (cp. Fig. 20).
The spheres $S(\kappa_1)$ and $S(\kappa_2)$ have a second order contact with the surface. The contact directions are orthogonal (curvature directions). The *mean curvature* is defined as

$$H = \frac{1}{2}(\kappa_1 + \kappa_2).$$

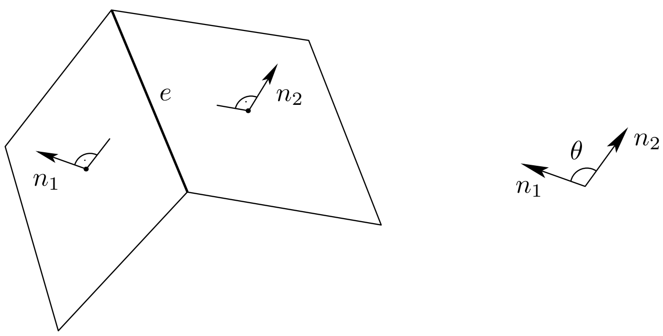
For the Gaussian curvature one has

$$K = \kappa_1 \kappa_2.$$

Definition 6.4 (Discrete mean curvature)**.** The *discrete mean curvature* of a polyhedral surface S is a function on edges $e \in E$ given by

$$H(e) := \frac{1}{2}\theta(e)l(e),$$

where $l(e)$ is the length of e, and $\theta(e)$ is the oriented angle between the normals of the adjacent faces sharing the edge e (the angle is considered positive in the convex case and negative otherwise).



The *total mean curvature* again is defined as the sum over all edges

$$H(S) := \sum_{e \in E} H(e) = \frac{1}{2} \sum_{e \in E} \theta(e)l(e).$$

This definition can be motivated by Steiner's formulas for polyheder and smooth surfaces:

Theorem 6.5 (Steiner's theorem for polyheder)**.** *Let $\mathcal{P}$ be a convex polyhedron and $\mathcal{P}_\rho$ its parallel body at the distance ρ*

$$\mathcal{P}_\rho := \left\{ p \in \mathbb{R}^3 \mid d(p, \mathcal{P}) \leq \rho \right\}.$$

The volume of the convex body $\mathcal{P}_\rho$ is a cubic polynomial in ρ given by

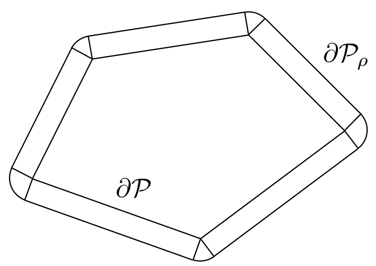
$$V(\mathcal{P}_\rho) = V(\mathcal{P}) + A(\partial\mathcal{P})\rho + H(\partial\mathcal{P})\rho^2 + \frac{4\pi}{3}\rho^3, \tag{9}$$

where $\partial\mathcal{P}$ is the boundary surface of $\mathcal{P}$ and $A(\partial\mathcal{P})$ is its area. The area of the boundary surface $\partial\mathcal{P}_\rho$ is given by

$$A(\partial\mathcal{P}_\rho) = A(\partial\mathcal{P}) + 2H(\partial\mathcal{P})\rho + 4\pi\rho^2. \tag{10}$$

Proof. The area of $\partial\mathcal{P}_\rho$ consists of three parts:

Plane pieces congruent to the faces of $\partial\mathcal{P}$, cylindrical pieces of radius ρ, angle θ and length $l(e)$ along the edges of $\partial\mathcal{P}$ and spherical pieces at the vertices of $\mathcal{P}$.



Merged together by parallel translation the spherical pieces comprise a round sphere of radius ρ. The formula for the volume follows by integration in ρ. □

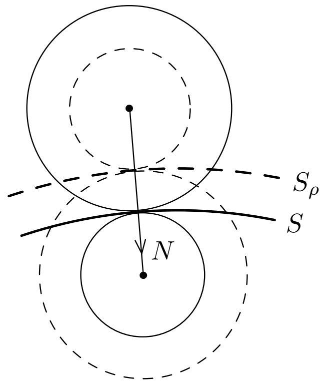
In the smooth case things look like follows:

The *normal shift* of a smooth surface S with the normal map N is defined as

$$S_\rho := S + \rho N.$$

For sufficiently small ρ the surface S_ρ is also smooth. Interpreting S as an enveloping surface of the principal sphere congruences one can show that the centers of the principal curavature spheres of S and S_ρ coincide. The radii change by ρ

$$\frac{1}{\kappa_{1\rho}} = \frac{1}{\kappa_1} + \rho, \quad \frac{1}{\kappa_{2\rho}} = \frac{1}{\kappa_2} + \rho.$$



Theorem 6.6. *Let S be a smooth surface and S_ρ its smooth normal shift for sufficiently small ρ. Then the area of S_ρ is a quadratic polynomial in ρ.*

$$A(S_\rho) = A(S) + 2H(S)\rho + K(S)\rho^2,$$

where $H(S) = \int_S H$ and $K(S) = \int_S K$ are the total mean and Gaussian curvatures of S.

Proof. Let ω_S and ω_{S_ρ} be the area forms of S and S_ρ. The normal shift preserves the Gauss map, therefore for these area forms one has

$$K\omega_S = K_\rho \omega_{S_\rho},$$

where K and K_ρ are the corresponding Gaussian curvatures. For the area this implies

$$\begin{aligned} A(S_\rho) &= \int_{S_\rho} \omega_{S_\rho} = \int_S \frac{K}{K_\rho}\omega_S = \int_S \kappa_1\kappa_2\left(\frac{1}{\kappa_1} + \rho\right)\left(\frac{1}{\kappa_2} + \rho\right)\omega_S \\ &= \int_S \left(1 + (\kappa_1 + \kappa_2)\rho + \kappa_1\kappa_2\rho^2\right)\omega_S = A(S) + 2H(S)\rho + K(S)\rho^2. \end{aligned}$$

Comparing the above equations with (10) justifies Definition 6.4. □

7 Curvatures of line congruence nets

Since line congruence nets come with a one parameter family of parallel polyhedral surfaces it is natural to use Steiner's formula (10) to define curvatures. Observe that the faces of parallel surfaces in this case are polygons with parallel edges, therfore we first deal with polygons with parallel edges and their properties.

Definition 7.1 (Oriented Area)**.** The *oriented area* of a triangle $\Delta = (p_0, p_1, p_2)$, is defined as

$$A(\Delta) = \frac{1}{2}\left([p_0, p_1] + [p_1, p_2] + [p_2, p_0]\right).$$

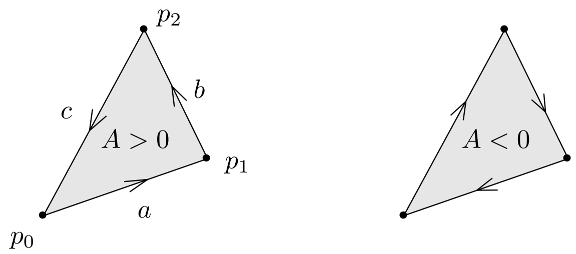
Here the vertices p_0, p_1, p_2 are given as vectors and $[\cdot, \cdot]$ denotes the *area form* of the plane

$$[x, y] = \langle x \times y, n\rangle = \det(x, y, n)$$

where n is the unit normal of the plane.

Lemma 7.2. *For a triangle $\Delta = (p_0, p_1, p_2)$ with oriented edges $a = p_1 - p_0$, $b = p_2 - p_1$, $c = p_0 - p_2$ holds*

$$A(\Delta) = \frac{1}{2}[a, b] = \frac{1}{2}[b, c] = \frac{1}{2}[c, a].$$



Proof.

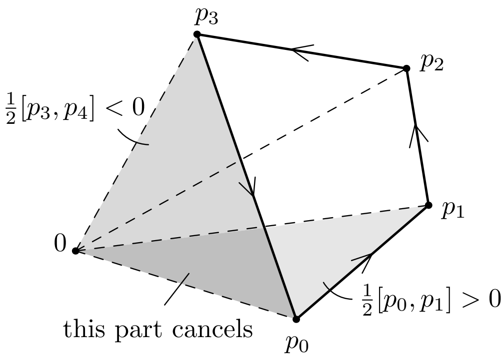
$$\begin{aligned}
2A(\Delta) &= [p_0, p_1] + [p_1, p_2] + [p_2, p_0] \\
&= [p_0, p_0 + a] + [p_0 + a, p_0 + a + b] + [p_0 - c, p_0] \\
&= [p_0 + a, a] + [p_0 + a, b] - [c, p_0] \\
&= [p_0 + a, a + b] + [p_0, c] \\
&= [p_0, a + b] + [a, a + b] + [p_0, c] \overset{a+b+c=0}{=} [a, b].
\end{aligned}$$

Since we chose p_0 as starting point arbitrary one has $[a, b] = [b, c] = [c, a]$. □

Lemma 7.3 (and Definition)**.** *The oriented area of a k-gon in a plane with vertices $p_0, .., p_k = p_0$ is equal*

$$A(P) = \frac{1}{2}\sum_{i=0}^{k-1}[p_i, p_{i+1}].$$

Proof. The generalization for k-gons follows from the decomposition like in the following figure.



□

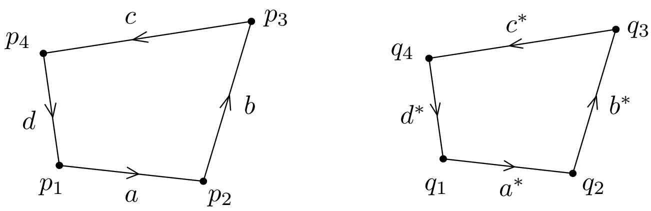
Lemma 7.4. *For a quadrilateral* $Q = (p_0, p_1, p_2, p_3)$ *with oriented edges* $a = p_1 - p_0$, $b = p_2 - p_1$, $c = p_3 - p_2$, $d = p_0 - p_3$ *holds*

$$A(Q) = \frac{1}{2}([a, b] + [c, d]).$$

Proof. Decompose Q in two triangles (p_0, p_1, p_2), (p_0, p_2, p_3) and use Lemma 7.2. □

7.1 Space of polygons with parallel edges and mixed areas

Polygons with parallel edges (not nescessarily convex nor embedded) build a vector space. This will be explained in little more detail in 7.1.1.



quadrilaterals with parallel edges: $a \parallel a^*$ etc.

Definition 7.5 (Mixed area)**.** Let P and Q be two polygons with the vertices $p_0, \ldots, p_k = p_0$ and $q_0, \ldots, q_k = q_0$ which have parallel edges, $(p_i, p_{i+1}) \parallel (q_i, q_{i+1})$. The sum

$$A(P, Q) := \frac{1}{4} \sum_{i=0}^{k-1} ([p_i, q_{i+1}] + [q_i, p_{i+1}]) \tag{11}$$

is called the *mixed area* of P and Q.

By a direct computation we get:

Proposition 7.6. *Let P and Q be two polygons with parallel edges. Then the oriented area of their linear combination is given by a quadratic polynomial*

$$A(P + tQ) = A(P) + 2tA(P, Q) + t^2 A(Q).$$

7.1.1 The spaces $\mathcal{P}(t)$ and $\widetilde{\mathcal{P}}(t)$

The sequence of tangent vectors (considered as lines)

$$t_1, \ldots, t_k \in \mathbb{R}P^1 = S^1/_{\{\pm I\}}$$

or equivalently of the normals

$$n_1, \ldots, n_k \in \mathbb{R}P^1 = S^1/_{\{\pm I\}}$$

uniquely specifies the space of polygons $p_0, \ldots, p_k = p_0$ with parallel edges $(p_i, p_{i+1} \parallel t_i)$.

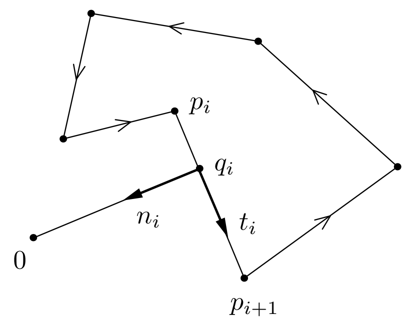


Figure 21: A polygon as a sequence of oriented edges parallel to the given set of tangent vectors and the corresponding normals.

Let us choose representatives $t_1, \ldots, t_k \in S^1$ and $n_1, \ldots, n_k \in S^1$ such that each corresponding pair (n_i, t_i) builds a positively oriented basis. The *oriented lengths* l_i are uniquely defined by

$$p_{i+1} - p_i = l_i t_i.$$

Let h_i be the *oriented hights* from the the origin to the edges of the polygon, i.e.

$$\vec{0q_i} = h_i n_i,$$

where q_i is the the vertical projection of 0 to the line containing the edge (cp. Fig. 21).

Denote $\mathcal{P}(t)$, $t = (t_1, \ldots, t_k)$ the vector space of polygons, for which hold that the i-th edge is parallel to the corresponding t_i. This is an k-dimensional vector space with the coordinates $h = (h_1, \ldots, h_k)$. If one factorizes this space by translations, one gets the (k-2)-dimensional vector space $\widetilde{\mathcal{P}}(t)$, for which one has natural coordinates $l = (l_1, \ldots, l_k)$ satisfying the condition

$$\sum_{i=1}^{k} l_i t_i = 0 \tag{12}$$

which means that each polygon has to be closed.

7.1.2 Properties of the mixed area

Proposition 7.7. *$A(P, Q)$ is a symmetric bilinear form on the spaces $\mathcal{P}(t)$ and $\widetilde{\mathcal{P}}(t)$.*

Proof. • Obviously $A(P, Q) = A(Q, P)$

- $A(P, Q)$ is linear in coordinates of the points and therefore also linear in h
- $A(P, Q)$ is invariant with respect to translations and therefore well defined on $\widetilde{\mathcal{P}}(t)$ and linear in l.

□

Corollary 7.8. *$A(P) = A(P, P)$ is a quadratic form on $\mathcal{P}(t)$ and $\widetilde{\mathcal{P}}(t)$.*

A useful geometric representation of the mixed area is given by

Proposition 7.9.

$$A(P, Q) = \frac{1}{2} \sum_{i=1}^{k} h_i(P) l_i(Q) = \frac{1}{2} \sum_{i=1}^{k} h_i(Q) l_i(P).$$

Proof. Since $A(P, Q)$ is symmetric one has

$$A(P, Q) = \sum_{i,j} a_{ij} h_i(P) h_j(Q)$$

with a symmetric matrix (a_{ij}). Algebra gives

$$\frac{\partial A(Q, Q)}{\partial h_i} = 2 \sum_{j} a_{ij} h_j(Q)$$

and from geometric consideration follows

$$\frac{\partial A(Q,Q)}{\partial h_i} = l_i(Q).$$

Combining both formulas one ends up with

$$A(P,Q) = \sum_i h_i(P) \sum_j a_{ij} h_j(Q) = \frac{1}{2} \sum_{i=1}^{k} h_i(P) l_i(Q).$$

□

Remark. Usually we take an advantage of the translational invariance and therefore will deal with the space $\widetilde{\mathcal{P}}(t)$.

7.1.3 Dual quadrilaterals

Definition 7.10 (Dual quadrilaterals)**.** Two quadrilaterals P, Q with parallel edges are called *dual* to each other if their mixed area vanishes,

$$A(P,Q) = 0.$$

Proposition 7.11. *Two quadrilaterals $P = (p_1, p_2, p_3, p_4)$ and $Q = (q_1, q_2, q_3, q_4)$ with parallel edges are dual if and only if their diagonals are* antiparallel*, i.e.*

$$(p_1, p_3) \parallel (q_2, q_4), \quad (p_2, p_4) \parallel (q_1, q_3).$$

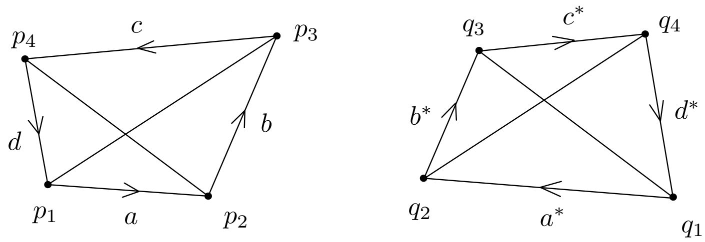


Figure 22: Dual quadrilaterals.

Proof. Denote the edges of the quadrilaterals P and Q as in Fig. 22. From Lemma 7.4 follows that the area of the quadrilateral $P + tQ$ is given by

$$A(P + tQ) = \frac{1}{2} \left([a + ta^*, b + tb^*] + [c + tc^*, d + td^*] \right)$$

and from Proposition 7.6 we know that twice the mixed area $A(P,Q)$ comes as coefficient of the linear term in the equation for the area $A(P+tQ)$. Hence comparison of coefficients leads to

$$\begin{aligned} A(P,Q)=0 \quad &\Leftrightarrow \quad [a,b^*]+[a^*,b]+[c,d^*]+[c^*,d]=0 \\ &\overset{a\|a^*\ldots}{\Leftrightarrow} \quad [a+b,b^*]+[a^*,a+b]+[c+d,d^*]+[c^*,c+d]=0 \\ &\overset{a+b=-(c+d)}{\Leftrightarrow} \quad [a+b,b^*-a^*-d^*+c^*]=0. \end{aligned}$$

Now the last of the above equations is equivalent to the anti-parallelity of the diagonals, $(a+b)\parallel(b^*+c^*)$. □

Lemma 7.12 (Existence and uniqueness of the dual quadrilateral)**.** *For every planar quadrilateral a dual one exists and is unique up to scaling and translation.*

Proof. First we show the **existence**:
Let (A,B,C,D) be the given quadrilateral and denote with e_1 and e_2 some vectors along the diagonals which gives

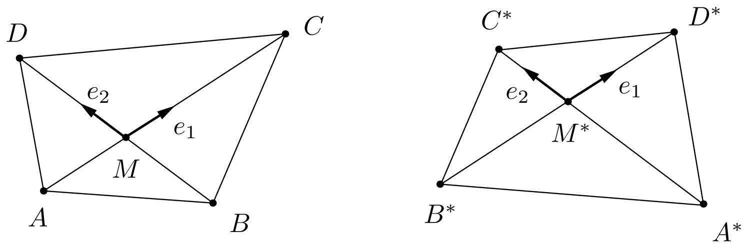
$$\vec{MA}=\alpha e_1,\ \vec{MB}=\beta e_2,\ \vec{MC}=\gamma e_1,\ \vec{MD}=\delta e_2.$$

Then for the edges holds

$$\vec{AB}=\beta e_2-\alpha e_1,\ \vec{BC}=\gamma e_1-\beta e_2,\ \vec{CD}=\delta e_2-\gamma e_1,\ \vec{DA}=\alpha e_1-\delta e_2.$$

Take an arbitrary M^* as intersection point of the diagonals of the quadrilateral (A^*,B^*,C^*,D^*) to be constructed as follows:

$$\vec{M^*A^*}=-\frac{e_2}{\alpha},\ \vec{M^*B^*}=-\frac{e_1}{\beta},\ \vec{M^*C^*}=-\frac{e_2}{\gamma},\ \vec{M^*D^*}=-\frac{e_1}{\delta}.$$



The diagonals of the quadrilaterals are obviously parallel. For the edges parallelity can easily be checked:

$$\begin{aligned} \vec{A^*B^*}&=\frac{e_2}{\alpha}-\frac{e_1}{\beta}=\frac{1}{\alpha\beta}\vec{AB} \\ \vec{B^*C^*}&=\frac{e_1}{\beta}-\frac{e_2}{\gamma}=\frac{1}{\beta\gamma}\vec{BC} \\ \vec{C^*D^*}&=\frac{e_2}{\gamma}-\frac{e_1}{\delta}=\frac{1}{\gamma\delta}\vec{CD} \\ \vec{D^*A^*}&=\frac{e_1}{\delta}-\frac{e_2}{\alpha}=\frac{1}{\delta\alpha}\vec{DA}. \end{aligned}$$

From Proposition 7.11 follws that the two quadrilaterals are dual.

Uniqueness of the dual quadrilateral follows from the geometric construction: choose two diagonals and one vertex. This determines a quadrilateral with parallel edges uniquely if it closes. Since resulting quadrilaterals would only differ by scaling, the closeness condition is independent from the choice of the initial vertex. Hence the existence of the above constructed closed quadrilateral implies that the construction always closes. □

Let P be a quadrilateral and denote $\widetilde{\mathcal{P}}(P)$ the space of quadrilaterals with parallel edges, factorized by translations. Since one has two degrees of freedom this is a 2-dimensional vector space.

Proposition 7.13. *Let P be a quadrilateral with non-vanishing oriented area, $A(P) \neq 0$. Then P and P^* build an orthogonal basis of the space $\widetilde{\mathcal{P}}(P)$ (with respect to $A(P,Q)$). Moreover $A(\lambda P + \mu P^*) = \lambda^2 A(P) + \mu^2 A(P^*)$.*

Proof. The space is 2-dimensional and P and P^* are linearily independent. The orthogonality follows from the definition. □

Corollary 7.14. *The quadratic form $A : \widetilde{\mathcal{P}}(P) \to \mathbb{R}$ is definite (indefinite) if $A(P)A(P^*) > 0$ ($A(P)A(P^*) < 0$ respectively).*

With this background we will now go back to line congruence nets.

7.2 Curvatures of line congruence nets

Consider a line congruence net $(S, N) : \mathbb{S} \to \mathbb{R}^{3+3}$. Here S is a discrete surface with planar faces comprized by the vertices of the line congruence net, and N is its normal map, normalized by the condition that $S + tN$ build a family of surfaces parallel to S (i.e. in general the normals are not of the same length). Note that for a given line congruence net N is fixed as soon as the length of the normal at one vertex is prescribed.

Theorem 7.15. *The surface area of the parallel surface $S^t = S + tN$ obeys the law*

$$A(S^t) = \sum_{f \in F} \left(1 - 2tH_f + t^2 K_f\right) A(f) \tag{13}$$

where

$$H_f = -\frac{A(f, N(f))}{A(f)}, \quad K_f = \frac{A(N(f))}{A(f)}. \tag{14}$$

Here f and $N(f)$ are the corresponding faces of S and N.

Proof. The corresponding faces of S and N are parallel, which implies the parallelity of the corresponding edges. Then the claim follows from Proposition 7.6. □

Definition 7.16 (Mean and Gaussian curvature for line congruence nets)**.** Let (S, N) be a line congruence net with the normal map $N : \mathbb{S} \to \mathbb{R}^3$ normalized as above. Its *mean* and *Gaussian curvatures* are defined by the formulas (14).

Remark. Although the values of the curvatures at faces are defined up to a multiplication by a common constant, discrete surfaces with constant curvature are well defined.

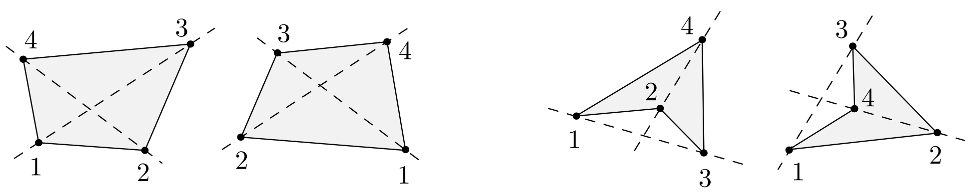
Now we can use what we have learned in 7.1 to define principal curvatures of a quadrilateral face of a line congruence net.
Consider a face f and its parallel face f^t. The desired formulas $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ and $K = \kappa_1\kappa_2$ imply the factorization

$$A(f^t) = \left(1 - 2tH + t^2K\right) A(f) = (1 - t\kappa_1)(1 - t\kappa_2)A(f).$$

Real valued $\kappa_1 \neq \kappa_2$ in this formula exist if and only if the form $A : \widetilde{\mathcal{P}}(f) \to \mathbb{R}$ is indefinite. If A is semidefinite, $\kappa_1 = \kappa_2$. If they are real, we call κ_1, $\kappa_2 \in \mathbb{R}$ the *principal curvatures* of the quadrilateral f.

Example 7.17. Wheather $A(f)$ is definite or indefinite depends on the structure of f:

convex f non-convex f



$A(f) > 0$ $A(f^*) < 0$ $A(f) > 0$ $A(f^*) > 0$

Exercise: Classify all quadrilaterals with indefinite A (consider also non-embedded quadrilaterals).
Note that this is a property of the set $\{t_1, t_2, t_3, t_4\} \subset \mathbb{R}P^1$.

8 Line congruence nets with constant curvatures. Elementary properties.

We consider quad-surfaces of line congruence nets. Let $\mathbb{S}$ be an abstract quad-graph and $(f, N) : \mathbb{S} \to \mathbb{R}^{3+3}$ a line congruence net. $f : \mathbb{S} \to \mathbb{R}^3$ and $N : \mathbb{S} \to \mathbb{R}^3$ are discrete surfaces with parallel planar quad faces. The discrete surfaces $f + tN$, $t \in \mathbb{R}$ are parallel surfaces of f.

Definition 8.1 (Dual quad-surfaces)**.** Two quad-surfaces $f, f^* : \mathbb{S} \to \mathbb{R}^3$ with planar faces are called *dual* to each other if any elementary quadrilateral of the net f^* is dual to the corresponding quadrilateral of the net f.

It is easy to characterize *minimal* line congruence nets, i.e. those with vanishing mean curvature.

Proposition 8.2 (Minimal line congruence nets)**.** *A line congruence net $(f, N) : \mathbb{S} \to \mathbb{R}^6$ has zero mean curvature, $H \equiv 0$, if and only if $N = f^*$.*

Proof. This is a statement about each face, so let Q be a face of $\mathbb{S}$. We have

$$H = 0 \Leftrightarrow A(f(Q), N(Q)) = 0 \Leftrightarrow N(Q) = f^*(Q).$$

□

Proposition 8.3 (Line congruence nets with constant mean curvature)**.** *A line congruence net $(f, N) : \mathbb{S} \to \mathbb{R}^6$ has constant mean curvature $H_0 \neq 0$ if and only if there exists a parallel surface $f + dN$ at constant distance $d \in R$, which is dual to f, i.e. $f^* = f + dN$.*
The line congruence net (f^*, N) has constant mean curvature $-H_0$ and the mid-surface $\frac{1}{2}(f + f^*)$ is a discrete surface with constant positive Gaussian curvature $4H_0^2$. For the distance holds $d = \frac{1}{H_0}$. picture?

Proof. The condition

$$A(f(Q), N(Q)) = -H_0 A(f(Q))$$

can be rewritten as

$$A(f(Q), f(Q) + \frac{1}{H_0} N(Q)) = 0$$

which is equivalent to

$$f^*(Q) = f(Q) + \frac{1}{H_0} N(Q).$$

Interpreting f as a parallel surface of $f + dN$, we get the statement about the mean curvature of $(f + dN, N)$ (since N has the same orientation for both nets, the shifts, and therefore the curvatures, need to have different signs). For the Gaussian curvature of the mid-surface $f + \frac{d}{2} N$ we get what about the $A(f, f)$-term?

$$\begin{aligned} K\left(f(Q) + \frac{d}{2} N(Q)\right) &= \frac{A(N(Q))}{A(f(Q) + \frac{d}{2} N(Q))} \\ &= \frac{A(N(Q))}{A(f(Q) + dN(Q), f(Q)) + \frac{d^2}{4} A(N(Q))} = 4H_0^2. \end{aligned}$$

□

Proposition 8.4. *Let (f, N) be a line congruence net with constant mean curvature. Then its parallel surfaces $(f + tN, N)$ are* linar Weingarten, *i.e. their mean and Gaussian curvatures stisfy a linear condition*

for each face?

$$\alpha H(t) + \beta K(t) = 1 \tag{15}$$

CMC ⇔ CGC ?

with constant coefficients $\alpha, \beta \in \mathbb{R}$.

Proof. Compute the curvatures $H(t)$ and $K(t)$ of the parallel surface $f+tN$ of a CMC line congruence net. Let H and K be the curvatures of the basic surface f. Then we have

$$\begin{aligned}\frac{A((f+(t+\delta)N)(Q))}{A((f+tN)(Q))} &\overset{(13)}{=} \frac{1-2H(t+\delta)+K(t+\delta)^2}{1-2Ht+Kt^2} \\ &= 1 - 2\delta\frac{H-Kt}{1-2Ht+Kt^2} + \delta^2\frac{K}{1-2Ht+Kt^2}.\end{aligned}$$

Finally, considering $f + (t + \delta)N$ as a parallel surface of $f + tN$, from (13) follows

$$H(t) = \frac{H-Kt}{1-2Ht+Kt^2}, \quad K(t) = \frac{K}{1-2Ht+Kt^2}.$$

Note that H is independent of the face, whereas K is varying. Therfore, with the above values for $H(t)$ and $K(t)$, the condition (15) is equivalent to

cp. last comment

$$\frac{\alpha H}{1-2Ht} = \frac{\beta - \alpha t}{t^2} = 1,$$

which implies

$$\alpha = \frac{1}{H} - 2t, \quad \beta = \frac{t}{H} - t^2.$$

□

8.1 Discrete Gauss maps

We see that any dualizable discrete surface can be extended to a minimal line congruence net (i.e. $H \equiv 0$), or a one with constant mean curvature, by an appropiate choice of the “normals”. Indeed,

$(f, N = f^*)$	is minimal, and
$(f, N = f^* - f)$	has constant mean curvature.

However, so defined $N : \mathbb{S} \to \mathbb{R}^3$ in such generality can hardly interpreted as a discrete Gauss map. One can (and should) put additional requirements on N which bring it closer to a Gauss map of a surface. In the smooth case N is a map to the unit sphere. It is natural to suggest the following three dicrete versions of this fact:

1. $N(\mathbb{S})$ has to be a polyhedral surface with all vertices on the unit sphere S^2. Obviously this implies that all faces of $N(S)$ are circular. This condition holds thus for any parallel surface. In particular this implies that all faces of $f(\mathbb{S})$ are also circular. We call such nets *circular*. pictures?

2. $N(S)$ has to be a polyhedral surface with all faces touching the unit sphere S^2. This implies that for any vertex p there is a cone of revolution with the tip at p touching all faces of $N(S)$ which are incident to p. Obviously this property holds true for any parallel surface, in particular for $f(\mathbb{S})$. We call such nets *conical*.

3. $N(S)$ has to be a polyhedral surface with all edges touching the unit sphere S^2 (such polyhedra are called *Koebe polyhedra*). For any vertex p all edges incident to p lie on a cone of revolution with the tip at p. Obviously this property holds true for any parallel surface, in particular for $f(\mathbb{S})$. Equivalently one can characterize them by the condition that at each vertex ther exists a sphere touching all edges incident to the vertex. We call such nets *of Koebe type*.

These additional requirements turn out to be compatible with the theory developed, and make it more complicated. can't read lecture notes... pictures?

9 Koenigs nets

In this section we forget about the additional structure of line congruence nets and simply characterize dualizible discrete surfaces $f : \mathbb{S} \to \mathbb{R}^3$, where $\mathbb{S}$ is a simply connected quad-graph.

Definition 9.1 (Discrete Koenigs net)**.** A quad-surface $f : \mathbb{S} \to \mathbb{R}^3$ with planar faces is called a *discrete Koenigs net* if it admits a dual net $f^* : \mathbb{S} \to \mathbb{R}^3$.

Since we assume $\mathbb{S}$ to be simply connected, it is bipartite. Therefore we can color its vertices black and white such that two vertices sharing an edge are colored differently. Diagonals of the quadrilaterals connect vertices of the same color, i.e. we have two sub nets (black and white) of diagonals. Denote $\vec{E}_d$ the set of all oriented diagonals and define a function $q : \vec{E}_d \to \mathbb{R}$ by

$$q(\vec{AC}) = \frac{\vec{MC}}{\vec{MA}}, \tag{16}$$

where M is the intersection point of the diagonals of the quad-face $ABCD$ of $f(\mathbb{S})$. Note that $q(\vec{CA}) = -\frac{1}{q(\vec{AC})}$, and for convex quadrilaterals one has $q < 0$.

Theorem 9.2 (Algebraic characterization of Koenigs nets). *A quad-surface $f : \mathbb{S} \to \mathbb{R}^3$ is a discrete Koenigs net if and only if for any cycle of directed diagonals (white or black) the product of all quantities q along the cycle is equal to 1.*

Proof. Let us try to dualize a quad-surface starting with an arbitrary quadrilateral. It is easy to see that obstructions may apperal when running along closed chains of elementary quadrilaterals in which any two subsequent quadrilaterals share an edge. Consider a vertex of value 4 and apply Lemma 7.12 to the four quadrilaterals incident to this vertex. Let the diagonals be divided by their intersection points in the relations $\gamma_k : \alpha_k$ and $\delta_k : \beta_k$ (as in Fig. *). The dual quadrilaterals are determined up to scaling factors λ_k $(k = 1, \dots, 4)$. picture & reference

Matching the edge shared by the dual quadrilaterals 1 and 2, we find for their scaling factors

$$\frac{\lambda_1}{\alpha_1\delta_1} = \frac{\lambda_2}{\alpha_2\beta_2} \Leftrightarrow \frac{\lambda_1}{\lambda_2} = \frac{\alpha_1\delta_1}{\alpha_2\beta_2}.$$

Similarly one finds

$$\frac{\lambda_2}{\lambda_3} = \frac{\alpha_2\delta_2}{\alpha_3\beta_3}, \quad \frac{\lambda_3}{\lambda_4} = \frac{\alpha_3\delta_3}{\alpha_4\beta_4}, \quad \frac{\lambda_4}{\lambda_1} = \frac{\alpha_4\delta_4}{\alpha_1\beta_1}.$$

The compatibility condition reads

$$\frac{\delta_1}{\beta_1}\frac{\delta_2}{\beta_2}\frac{\delta_3}{\beta_3}\frac{\delta_4}{\beta_4} = 1,$$

which is the statement of the theorem. The same proof holds for vertices of other valence. Finally any cycle on a simply connected discrete surface can be decomposed to a combination of elementary cycles arround vertices. □

Let us consider the special case $\mathbb{S} = \mathbb{Z}^2$.

Theorem 9.3 (Geometric characterization of discrete Koenigs nets). *A discrete surface $f : \mathbb{Z}^2 \to \mathbb{R}^N$ with planar faces and non-planar vertices is a discrete Koenigs net if and only if the intersection points of diagonals of any four quadrilaterals sharing a vertex are co-planar.* white & black subnets non-planar...

Proof. Let Q_1, Q_2, Q_3, Q_4 be four neighbouring quadrilaterals meeting at P and P_1, P_2, P_3, P_4 the intersection points of their diagonals. Denote $\mathcal{P}$ the plane through P_1, P_2, P_3. Now consider the 3-dimensional space W generated by the neighbours $A_1, \dots, A_4$ of P. Let h_i, $i = 1, \dots, 4$ be the distances of the correspondig vertices A_i to the plane $\mathcal{P}$. Obviously one has picture with P

$$q(A_i\vec{A}_{i+1}) = \frac{h_{i+1}}{h_i}, \quad i = 1, 2, 3.$$

This implies that $P_4 \in \mathcal{P}$ is equivalent to

$$q(\vec{A_1 A_2})q(\vec{A_2 A_3})q(\vec{A_3 A_4})q(\vec{A_4 A_1}) = 1.$$

□ picture

Theorem 9.4 (Geometric characterization of generic Koenigs nets in terms of vertices). *Let $f : \mathbb{Z}^2 \to \mathbb{R}^N$, $N \geq 4$ be a discrete surface with planar faces and vertices of full dimension (i.e. four edges meeting at a vertex are linearly independent). Then f is a discrete Koenigs net if and only if any vertex f and its next-neighbours $f_{\pm 1, \pm 2}$ lie in a 3-dimensional subspace $V \subset \mathbb{R}^N$, not containing any of the four points $f_{\pm 1}, f_{\pm 2}$ (either none or any point is contained in this space).*

Proof. Theorem 9.3 implies that if $f : \mathbb{Z}^2 \to \mathbb{R}^N$ is Koenigs, then the five points $f, f_{\pm 1, \pm 2}$ lie in a 3-dimensional subspace.
On the other hand, let V be the 3-dimensional space of $f, f_{\pm 1, \pm 2}$ and W the 3-dimensonal space of $f_{\pm 1}, f_{\pm 2}$. Both lie in a (by assumption) common 4-dimensional space of $f, f_{\pm 1}, f_{\pm 2}$. The intersection points of the diagonals lie in $V \cap W$ which is generically two dimensional. □

Remarks. • In the case of a vertex of valence 3 the Koenigs condition picture & label

$$\frac{\vec{P_1 A_2}}{\vec{A_1 P_1}} \cdot \frac{\vec{P_2 A_3}}{\vec{A_2 P_2}} \cdot \frac{\vec{P_3 A_1}}{\vec{A_3 P_3}} = -1 \tag{17}$$

is equivalent to the co-linearity of P_1, P_2, P_3, due to Menelaus' theorem.

- The Koenigs condition of Theorem 9.2 in the generic case can be interpreted geometrically using the generalized *Menelaus' theorem*:

  Consider a vertex of valence k and assume that its edges are linearly independent. The net is discrete Koenigs if and only if the intersection points of the diagonals of the k neighbouring quadrilaterals lie in an (k-1)-dimensional space.

- An embedded picture of three quadrilaterals like in Fig. * can also be dualized in the case of the multiratio equals 1. This is the case of the *Ceva's theorem*: reference

  P_1, P_2, P_3 are generated by three intersecting lines in a triangle. This net should be considered as a double cover, and the dual net consists of six neighbouring quadrilaterals. picture

- It is important that exactly in the same way one can define and consider multidimensional Koenigs nets $f : \mathbb{Z}^m \to \mathbb{R}^N$. It is non-trivial but true that multidimensional nets can be dualized consistently. Here we see the consistency discretization principle at work.

10 Laplace operators on graphs

11 Dirichlet energy of piecewise linear maps

12 Delaunay tesselations of polyhedral surfaces

13 Delaunay tesselation as a minimizer of the Dirichlet energy

14 Discrete Laplace-Beltrami operator

15 Discrete mean curvature and minimal surfaces

16 Cell decompositions of surfaces

17 Circle packings and circle patterns

18 Analytic description of circle patterns

19 Variational description, uniqueness and existence