Properties of the interface of the symbiotic branching model

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Abstract

The symbiotic branching model describes the evolution of two interacting populations and if started with complementary Heaviside functions, the interface where both populations are present remains compact. In this paper, we show tightness of the diffusively rescaled solutions and thus provide a first step towards a scaling limit for the interface. The crucial estimate involves a mixed fourth moment bound which we analyse using a particle system duality. As a corollary, we obtain an estimate on the moments of the width of an approximate interface.

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1 Introduction

1.1 The Symbiotic branching model and its interface

The 'symbiotic branching model' of Etheridge and Fleischmann [EF04] is a stochastic spatial model of two interacting populations, parametrized by $\varrho \in [-1, 1]$ governing the correlation between the two driving noises. More precisely, it is described by the initial value problem associated with the system of stochastic partial differential equations

$$\mathrm{SBM}(\varrho,\gamma)_{u_0,v_0}: \quad \begin{cases} du_t(x) &= \frac{\Delta}{2}u_t(x) + \sqrt{\gamma u_t(x)v_t(x)}\, dW_t^1(x), \\ dv_t(x) &= \frac{\Delta}{2}v_t(x) + \sqrt{\gamma u_t(x)v_t(x)}\, dW_t^2(x), \end{cases} \tag{1}$$

with positive suitable initial conditions $u_0(x) \geq 0, v_0(x) \geq 0, x \in \mathbb{R}$. Here, $\gamma > 0$ is the branching rate and $\mathbf{W} = (W^1, W^2)$ is a pair of correlated standard Gaussian white noises on $\mathbb{R}_+ \times \mathbb{R}$ with correlation $\varrho \in [-1, 1]$, i.e., for $t_1, t_2 \geq 0$,

$$\mathbb{E}\big[W_{t_1}^i(A_1)W_{t_2}^j(A_2)\big] = \begin{cases} (t_1 \wedge t_2)\ell(A_1 \cap A_2), & i = j, \\ \varrho(t_1 \wedge t_2)\ell(A_1 \cap A_2), & i \neq j, \end{cases} \tag{2}$$

where ℓ denotes the Lebesgue measure and A_1, A_2 are Borel sets. Solutions of this model have been considered rigorously in the framework of the corresponding martingale problem

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in Theorem 4 of [EF04], which states that, under natural conditions on the initial conditions $u_0(\cdot), v_0(\cdot)$, a solution exists for all $\varrho \in [-1,1]$. Further, the martingale problem is well-posed for all $\varrho \in [-1,1)$, which implies the strong Markov property except in the boundary case $\varrho = 1$. In [EF04] it has also been observed that for $\varrho = -1$, the system reduces to the *heat equation with Wright-Fisher noise* discussed e.g. by Tribe in [Tri95], and that for $\varrho = 0$, the system is the so-called *mutually catalytic model* of Dawson and Perkins [DP98].

An important tool for the analysis of the symbiotic branching model is the following uniform version of a result on the asymptotic behaviour of the moments of SBM obtained by Blath, Döring and Etheridge in [BDE11], Theorem 2.5. They define the so-called 'critical curve' of the symbiotic branching model $p : (-1,1) \to (1,\infty)$ by

$$p(\varrho) = \frac{\pi}{\arccos(-\varrho)}, \tag{3}$$

and denote its inverse by $\varrho(p) = -\cos(\frac{\pi}{p})$ (for $p > 1$). This curve separates the upper right quadrant in two ares: below the characteristic curve, where moments remain bounded, and above the characteristic curve, where moments increase to infinity as $t \to \infty$:

Theorem 1.1 ([BDE11])**.** *Suppose (u_t, v_t) is a solution to the symbiotic branching model with initial conditions $u_0 = v_0 \equiv 1$. Let $\varrho \in (-1,1)$ and $\gamma > 0$. Then, for every $x \in \mathbb{R}$,*

$$\varrho < \varrho(p) \quad \text{iff} \quad \mathbb{E}^{1,1}\big[u_t(x)^p\big] \text{ is bounded uniformly in all } t \geq 0.$$

In particular, if $\varrho < \varrho(p)$, there exists a constant $C(\gamma,\varrho)$ so that, uniformly for all $x \in \mathbb{R}$ and $t \geq 0$,

$$\mathbb{E}^{1,1}\big[u_t(x)^p\big] \leq C(\gamma,\varrho), \quad t \geq 0.$$

Remark 1.2. (i) Of course, due to symmetry, the same result holds for the v population. That there is a finite bound independent of x follows since the system is under the $(\mathbf{1},\mathbf{1})$ starting condition translation invariant.

(ii) Moreover, for any $x_1, \ldots, x_4$ we have by the generalized Hölder inequality that

$$\mathbb{E}^{1,1}[u_t(x_1)u_t(x_2)v_t(x_3)v_t(x_4)] \leq \max_{i=1,\ldots,4} \mathbb{E}[u_t(x_i)^4]^{\frac{1}{4}} \leq C(\gamma,\varrho),$$

and similarly if some of the v's are replaced by u (and vice versa).

Natural questions about such (systems of) SPDEs are related to their longterm behaviour, in particular the speed of propagation of waves and interfaces for suitable initial conditions, such as 'complementary Heaviside initial conditions', i.e.

$$u_0(x) = \mathbf{1}_{\mathbb{R}^-}(x) \quad \text{and} \quad v_0(x) = \mathbf{1}_{\mathbb{R}^+}(x), \quad x \in \mathbb{R}.$$

Definition 1.3. The interface at time t of a solution (u_t, v_t) of the symbiotic branching model $\mathrm{cSBM}(\varrho,\kappa)_{u_0,v_0}$ with $\varrho \in [-1,1]$ is defined as

$$\mathrm{Ifc} = \mathrm{cl}\big\{x : u_t(x)v_t(x) > 0\big\},$$

where $\mathrm{cl}(A)$ denotes the closure of the set A in $\mathbb{R}$.

The main question addressed in [EF04] is whether for the above initial conditions the so-called 'compact interface property' holds, that is, whether the interface is compact at each time almost surely. This is answered affirmatively in their Theorem 6, together with the assertion that the interface propagates with at most linear speed, i.e. there exists a constant $c = c(\gamma)$ such that for each $\varrho \in [-1, 1]$, there is a (almost-surely) finite random time T_0 such that, almost surely, for all $T \geq T_0$,

$$\bigcup_{t \leq T} \mathrm{Ifc}_t \subseteq \big[- cT, cT \big]. \tag{4}$$

However, due to the scaling property of the symbiotic branching model, see Lemma 8 of [EF04], which states that if (u_t, v_t) is a solution to $\mathrm{SBM}(\varrho, \gamma)_{\mathrm{u}_0, \mathrm{v}_0}$, then

$$\big(u_t^K(x), v_t^K(x)\big) := \big(u_{Kt}(\sqrt{K}x), v_{Kt}(\sqrt{K}x)\big), \quad x \in \mathbb{R}, K > 0,$$

is a solution to $\mathrm{cSBM}(\varrho, \sqrt{\mathrm{K}} \cdot \gamma)_{\mathrm{u}_0^{\mathrm{K}}, \mathrm{v}_0^{\mathrm{K}}}$ with suitably transformed initial states (u_0^K, v_0^K), one might expect that the fluctuations of the position of the interface should be of order $t^{1/2}$. Indeed, with the help of the moment estimates of Theorem 1.1, it is possible to strengthen (4) for a (rather small) parameter range, see [BDE11]:

Theorem 1.4 ([BDE11])**.** *Suppose (u_t, v_t) is a solution of $\mathrm{SBM}(\varrho, \gamma)_{1_{\mathbb{R}^-}, 1_{\mathbb{R}^+}}$ with $\varrho < \varrho(35)$. Then there is a constant $C(\gamma, \varrho) > 0$ and a finite random-time T_0 such that almost surely*

$$\bigcup_{t \leq T} \mathrm{Ifc}_t \subseteq \Big[- C\sqrt{T\log(T)}, C\sqrt{T\log(T)} \Big],$$

for all $T > T_0$.

The restriction to $\varrho < \varrho(35)$ seems artificial and comes from the technique of the proof. Though $\varrho(35) \approx -0.9958$ is rather close to -1 the result is still interesting since it shows that sub-linear speed of propagation is not restricted to situations in which solutions are uniformly bounded as for instance for $\varrho = -1$. The proof is based on the method of Tribe from [Tri95] for the heat equation with Wright-Fisher noise employed with improved bounds on the moments of the symbiotic branching model based on the critical curve, circumventing the lack of uniform boundedness of the population sizes.

In the light of the scaling property, one might hope that for a rather large parameter set, and possibly all $\varrho \leq 0$, a diffusive time-space rescaling could lead to a tight sequence of stochastic processes. Indeed, this programme has been carried out for the discrete space version of (1) the symbiotic branching model. For mutually catalytic model $\varrho = 0$, Klenke and Mytnik construct in a series of papers [KM10, KM11a, KM11b], a non-trivial limiting process for $\gamma \to \infty$ and study their long-term properties. This limit is called the "infinite rate mutually catalytic branching process". Moreover, Klenke and Oeler [KO10] give a Trotter type approximation. Regarding the interface, in Corollary 1.2 of [KO10] the authors *conjecture* that, under suitable assumptions, a non-trivial interface for the limiting process exists, which would in turn predict a square-root order for the fluctuations of the interface.

Recently, this programme has been extended by Döring and Mytnik to the case $\varrho \leq 0$ in [DM11a, DM11b].

These observations and the above conjecture are the starting point of our investigation.

1.2 Main results and open problems

We first need to introduce some suitable notation. For a pair of (continuous) functions (u, v), we define

$$R(u, v) := \sup\{x : u(x) > 0\}, \quad L(u, v) = \inf\{x : v(x) > 0\}. \tag{5}$$

For the solution (u_t, v_t) of the symbiotic branching model with complementary Heaviside initial conditions, we note that the interface is contained in the set $[L(u_t, v_t), R(u_t, v_t)]$, an interval whose width we call the *diameter of the interface* (this notion is well defined due to the compactness result of [EF04]). It is proved in [Tri95] for $\varrho = -1$ and for initial conditions $u_0 = 1 - v_0$ which satisfy $-\infty < L(u_0, v_0) \leq R(u_0, v_0) < \infty$ that under Brownian rescaling, the motion of the position of the right endpoint of the interface $t \mapsto R(u_{n^2t}, 1 - u_{n^2t})/n, t \geq 0$, converges to a Brownian motion as $n \to \infty$.

The first central idea in the proof in [Tri95] is to consider the following measure valued processes

$$\mu_t^n(dx) = u_{n^2t}(nx)dx \quad \text{and} \quad \nu_t^n(dx) = v_{n^2t}(nx)dx, \tag{6}$$

and show that the sequences of these processes are tight. In fact, since in the case of $\varrho = -1$, $v = 1 - u$, it suffices to consider only one of these processes. The second step is to identify the limit and it is shown that for $\varrho = -1$, $(\mu_t^n)_{t\geq 0}$ converges in law to the measure-valued process $(\mathbb{1}_{x\leq B_t})_{t\geq 0}$ for $(B_t)_{t\geq 0}$ a standard Brownian motion.

In this note, we take the first step in this programme and show tightness of the measure-valued process defined in (6). Here, the measure-valued processes are treated as elements of $\mathcal{C}((0, \infty), \mathcal{M}_{\text{tem}}^2)$ the space of continuous processes taking values in the space of (pairs of) tempered measures, see also the Appendix A.1.

Theorem 1.5. *Assume $\varrho < \varrho(4) = -\frac{1}{\sqrt{2}}$. Let (u_t, v_t) be a solution to the symbiotic branching initial value problem with complementary Heaviside initial conditions. Then, the processes $(\mu_t^n, \nu_t^n)_{t\geq 0}$ are tight in $\mathcal{C}((0, \infty), \mathcal{M}_{\text{tem}}^2)$.*

It would be interesting to see if the point $\varrho(4)$ is really significant or mereley due to technicalities. Therefore, one should check whether ϱ affects other, finer, properties of the interface.

The essential step proof of the tightness result 1.5, is a fourth moment estimate. In the case $\varrho = -1$, [MT97] exploit the corresponding result to get a estimate on the moments of the width of the interface $|R(u_t, v_t) - L_(u_t, v_t)|$. However, this moment estimate heavily relies on the fact that there are "no holes" in the system where both u and v are zero. In our case, we can imitate the reasoning to get an estimate for the approximate interface defined in the following way. For any $\varepsilon > 0$, define an approximate left end point of the interface as

$$L_t(\varepsilon) = \inf\Big\{x : \int_{-\infty}^{x} u_t(y)v_t(y)dy \geq \varepsilon\Big\} \wedge R(u_t, v_t).$$

and similarly, for the right end point

$$R_t(\varepsilon) = \sup\Big\{x : \int_{x}^{\infty} u_t(y)v_t(y)dy \geq \varepsilon\Big\} \vee L(u_t, v_t).$$

Since $|R(u_t,v_t)|, |L(u_t,v_t)|$ are almost surely finite, $R_t(\varepsilon), L_t(\varepsilon)$ are well-defined. Our next result states that this approximate width of the interface remains small uniformly in t in the following way.

Theorem 1.6. *Supose $(u_0,v_0) = (\mathbb{1}_{\mathbb{R}^-}, \mathbb{1}_{\mathbb{R}^+})$ and $\varepsilon > 0$. Then for any $\varrho < \varrho(4) = -\frac{1}{\sqrt{2}}$, for any $p \in (0,1)$, there exists a constant $C = C(p,\varepsilon,\gamma,\varrho)$ such that for all $t > 0$,*

$$\mathbb{E}^{\mathbf{1}_{\mathbb{R}^-},\mathbf{1}_{\mathbb{R}^+}}((R_t(\varepsilon) - L_t(\varepsilon))^+)^p \leq C.$$

Remark 1.7. *Open problems.* Ideally, one would like to show that the measure-valued process are not only tight, but also converge in a suitable topology. One can show that for n fixed the densities $(u^{(n)}, v^{(n)})$ satisfy a martingale problem which is the continuous analogue of the discrete space infinite rate symbiotic problem, see [DM11b, Prop. 3.3] (where one has to apply the operator Δ to the test function). Therefore, one way of showing convergence, is to show that the continuum problem has a unique solution (under some condition which says that the two measures have essentially disjoint support), which is however still open.

A second problem is to improve the result of Theorem 1.6 and replace the approximate left and right end points by the exact bounds on the interface $L(u_t,v_t)$ and $R(u_t,v_t)$ and carry out the remaining programme of [MT97].

1.3 A coloured particle moment dual

There are several dual processes for the symbiotic branching model. Here, we aim to describe the asymptotic behaviour of mixed moments of the form

$$\mathbb{E}^{u_0,v_0}\big[u_t(x_1)\cdots u_t(x_n)v_t(x_{n+1})\cdots v_t(x_{n+m})\big].$$

The dual works as follows for $\varrho \in (-1,1)$. Consider $n+m$ particles in $\mathbb{R}$ which can take on two colours, say colour 1 and 2. Each particle moves like a Brownian motion independently of all other particles. At time 0, we place n particles of colour 1 at positions $x_1,...,x_n$, respectively, and m particles of colour 2 at positions $x_{n+1},...,x_{n+m}$. As soon as two particles meet, they start collecting collision local time. If both particles are of the same colour, one of them changes colour when their collision local time exceeds an (independent) exponential time with parameter γ. Denote by $L_t^=$ the total collision local time collected by all pairs of the same colour up to time t, and let $L_t^{\neq}$ be the collected local time of all pairs of different colour up to time t. Finally, let $l_t := (l_t^1, l_t^2), t \geq 0$, be the corresponding particle process, that is, $l_t^1(x)$ denotes the number of particles of colour 1 at x at time t and $l_t^2(x)$ is defined accordingly for particles of colour 2. Our mixed moment duality function will then be given, up to an exponential correction involving both $L_t^=$ and $L_t^{\neq}$, by a moment duality function

$$(u,v)^{l_t} := \prod_{\substack{x\in\mathbb{R}:\\ l_t^1(x) \text{ or } l_t^2(x)\neq 0}} u(x)^{l_t^1(x)} v(x)^{l_t^2(x)}.$$

Note that since there are only $n+m$ particles the potentially uncountably infinite product is actually a finite product and hence well-defined. The following lemma is taken from Section 3 of [EF04].

Lemma 1.8. *Let (u_t, v_t) be a solution of* $\mathrm{dSBM}(\varrho,\gamma)_{u_0,v_0}$ *with $\varrho \in (-1,1)$. Then, for any $x \in \mathbb{R}$, $t \geq 0$,*

$$\mathbb{E}^{u_0,v_0}\Big[u_t(x_1)\cdots u_t(x_n)v_t(x_{n+1})\cdots v_t(x_{n+m})\Big] = \mathbb{E}\Big[(u_0,v_0)^{l_t}e^{\gamma(L_t^{=}+\varrho L_t^{\neq})}\Big], \tag{7}$$

where the dual process $\{l_t\}$ behaves as explained above, starting in $l_0 = (l_0^1, l_0^2)$ with particles of colour 1 located in $(x_1,\ldots,x_n)$ and particles of colour 2 respectively in $(x_{n+1},\ldots,x_{n+m})$.

Note that if $u_0 = v_0 \equiv 1$ the first factor in the expectation of the right-hand side equals 1.

2 Proofs

The proof of Theorem 1.5 splits into three main parts.

- First, we need to prove an analogue of [Tri95, Lemma 2.1] to obtain a bound on integrated mixed fourth moments. Although the result is similar to Tribe's, the proof is very different since we have to work with the coloured particle moment dual with exponential correction instead of the system of coalescing Brownian motion available in the case of the heat equation with Wright-Fisher noise. This will be done in Section 2.1
- Next, we prove tightness of the diffusively rescaled coordinate processes with the help of the fourth moment bound obtained above in Section 2.2.
- Then, we check tightness of the measure-valued processes on path-space. Here, we employ a variant of Jakubowski's criterion, which requires to check a compact containment condition. This is trivial in Tribe's case, but requires extra work in the case $\varrho \in (-1, -\frac{1}{\sqrt{2}})$. See Section 2.3
- Finally, in Section 2.4, we show the moment estimate on the width of the interface of Theorem 1.6.

Notation: We have collect some of the basic facts and notations about measure-valued process in Appendix A.1. Moreover, Appendix A.2 is a collection of estimates for Brownian motion and its local time. Throughout this paper, we will denote by c, C generic constants, whose value may change from line to line. If the dependence on parameters is essential we will indicate this correspondingly.

2.1 A bound on integrated fourth mixed moments

Lemma 2.1 (Mixed moments)**.** *Let (u_t, v_t) be a solution to the symbiotic branching initial value problem (1) with inital values $u_0 = \mathbb{1}_{\mathbb{R}^-}, v_0 = \mathbb{1}_{\mathbb{R}^+}$. Then, for $\varrho < \varrho(4) = -\frac{1}{\sqrt{2}}$,*

$$\mathbb{E}^{u_0,v_0}\Big[\iint u_t(x)u_t(y)v_t(x)v_t(y)\,dx\,dy\Big] \leq C(u_0,v_0;\gamma)$$

uniformly for all $t \geq 0$.

Note that by Fubini's theorem and a simple substitution, it is sufficient to prove that for $z > 0$,

$$\mathbb{E}^{u_0,v_0}\Big[\int u_t(x)u_t(x-z)v_t(x)v_t(x-z)\,dx\Big]\,,$$

is integrable in z. Our Ansatz is to use the moment duality from Lemma 1.8 and combine it with the moment bounds of Theorem 1.1. However, Theorem 1.1 requires constant initial conditions, which simplifies the moment duality considerably. In our case, we have to be much more careful. Then, the duality in (7) reads

$$\mathbb{E}^{1_{\mathbb{R}^-},1_{\mathbb{R}^+}}\big[u_t(x)u_t(x-z)v_t(x)v_t(x-z)\big] = \mathbb{E}_{l_0^1=(x,x-z),l_0^2=(x,x-z)}\Big[(u_0,v_0)^{l_t}e^{\gamma(L_t^{=}+\varrho L_t^{\neq})}\Big]$$

To describe the dynamics of $\{l_t\}$, we introduce a system of four independent Brownian motions $\{B_t^i,\ i=1,\dots,4\}$ with respective types (colours) $c_i(t)\in\{1,2\}$ at time t. A possible type change may occur when two particles of the same type collect a substantial amount of collision local time. Initially, we have locations $B_0^1=0, B_0^2=0, B_0^3=z, B_0^4=z$ and colours $c_1(0)=c_3(0)=1$, while $c_2(0)=c_4(0)=2$. Defining

$$f^1 := u_0 = \mathbb{1}_{\mathbb{R}^-},\quad f^2 := v_0 = \mathbb{1}_{\mathbb{R}^+},$$

we can write the duality, by translation invariance and symmetry as

$$\mathbb{E}^{u_0,v_0}[u_t(x)v_t(x)u_t(x-z)v_t(x-z)] = \mathbb{E}_{l_0^1=(0,z),l_0^2=(0,z)}\Big[\prod_{i=1}^4 f^{c_i(t)}(x-B_t^i)e^{\gamma(L_t^{=}+\varrho L_t^{\neq})}\Big].$$

We now integrate over x and estimate the integral. Note that the exponential term does not depend on x. Hence, we may restrict our attention to

$$\int\prod_{i=1}^4 f^{c_i(t)}(x-B_t^i)\,dx\,, \tag{8}$$

for different type configurations. First observe that

$$f^1(x-B_t) = \mathbb{1}\{x<B_t\}\quad\text{and}\quad f^2(x-B_t)=\mathbb{1}\{x>B_t\}, \tag{9}$$

so that one should think of the integral in (8) as an integral over a product of Heaviside functions centred at B_t^i, where the type determines the shape.

Now, if we denote by $r(t)$ the index of the left-most Brownian motion of type 1, i.e. $c^{r(t)}(t) = 1$ and

$$B_t^{r(t)} \le B_t^i \text{ for all } i \text{ such that } c^i(t)=1,$$

(where we choose the smaller index to resolve ties). Similarly by $\ell(t)$ the index of the right-most Brownian motion of type 2, i.e. $c^{\ell(t)}(t)=2$ and

$$B_t^{\ell(t)} \ge B_t^i \text{ for all } i \text{ such that } c^i(t)=2,$$

(with the smaller index to resolve ties).

Observe that, due to the definition of our dual particle system $\{l_t\}$, if we start with four particles and two colours, there will always be at least one particle of type 1 and at least particle of type 2 around at any time, no matter what the actual type changes were (type changes can only occur if two particles of the same colour meet). Moreover, with the above notation, the integral in (8) is 0 unless $B_t^{r(t)} > B_t^{\ell(t)}$ and since the product is either 0 or 1, we obtain

$$\int \prod_{i=1}^{4} f^{c_i(t)}(x - B_t^i)\, dx = (B_t^{r(t)} - B_t^{\ell(t)})^+.$$

Altogether, we arrive at

$$\mathbb{E}^{u_0,v_0} \int u_t(x)u_t(x-z)v_t(x)v_t(x-z)\, dx = \mathbb{E}_{(0,z),(0,z)}\Big[(B^{r(t)} - B^{\ell(t)})^+ e^{\gamma(L_t^= + \varrho L_t^{\neq})}\Big] \qquad (10)$$

and need to show that, for $z > 0$, this expression is integrable in z. We prepare this with a lemma which covers the important case where at least two particles in the middle are at the same location.

Lemma 2.2. *Let $-\infty < x < y < z < \infty$, for $\varrho < \varrho(4)$, and $\delta > 0$. Then, for any initial configuration $l_0 = \underline{x}$ that contains four particles in positions x, y, y, z and two of each colour, i.e.*

$$\underline{x} \in \Big\{(x, y), (y, z); (y, z), (x, y); (x, z), (y, y); (y, y), (x, z)\Big\},$$

we have

$$\mathbb{E}_{\underline{x}}\Big[(B_t^{r(t)} - B_t^{\ell(t)})^+ e^{\gamma(L_t^= + \varrho L_t^{\neq})}\Big] \leq C(\varrho, \gamma, \delta) \min\Big\{\frac{(z - y + 1)(y - x + 1)}{t^{\frac{1}{2} - \delta}}, 1 \vee t^{\delta}\Big\}.$$

Proof. Pick ϱ' so that $\varrho < \varrho' < \varrho(4)$ and let $\delta \in (0, 1)$. Using the (generalized) Hölder inequality twice for $p_1, p_2, p_3 \geq 1$ with $p_3 = (1-\delta)^{-1}$ and $p_1 = p_2$ such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, we obtain

$$\begin{aligned} &\mathbb{E}_{\underline{x}}\Big[(B^{r(t)} - B^{\ell(t)})^+ e^{\gamma(L_t^= + \varrho L_t^{\neq})}\Big] \\ &\quad\leq \mathbb{E}_{\underline{x}}\Big[((B^{r(t)} - B^{\ell(t)})^+)^{p_1}\Big]^{\frac{1}{p_1}}\ \mathbb{E}_{\underline{x}}\Big[e^{p_2\gamma(L_t^= + \varrho' L_t^{\neq})}\Big]^{\frac{1}{p_2}} \mathbb{E}_{\underline{x}}\Big[e^{-p_3\gamma(\varrho' - \varrho)L_t^{\neq}}\Big]^{\frac{1}{p_3}}. \end{aligned} \qquad (11)$$

The second expectation in (11) corresponds to the fourth mixed moment of a system with branching rate p_2 and correlation parameter ϱ'. Since $\varrho' < \varrho(4)$, this expression is bounded by a constant (depending only on p_2, γ) uniformly in $t \geq 0$, see also Remark 1.2.

For the first expectation on the right hand side in (11), we claim that

$$\mathbb{E}_{\underline{x}}\Big[((B_t^{r(t)} - B_t^{\ell(t)})^+)^{p_1}\Big]^{\frac{1}{p_1}} \leq C(p_1) t^{\frac{1}{2}} \qquad (12)$$

The claim follows if we can show that the expectation on the left hand side does not depend on the distances of the starting points $z - y, y - x$. We recall that the particles are labelled from left to right according to the initial positions. In particular $2, 3$ are the labels of the particles started in y. Also, we can always assume that $B_t^{\ell_t} < B_t^{r(t)}$ since this is the only scenario when when we observe a positive contribution to the expectation. *Case 1:* $\{r(t),\ell(t)\}=\{2,3\}$. Then

$$\mathbb{E}_{\underline{x}}\Big[((B_t^{r(t)}-B_t^{\ell(t)})^+)^{p_1}\mathbb{1}_{\{r(t),\ell(t)\}=\{2,3\}\}}\Big]^{\frac{1}{p_1}}\leq \mathbb{E}_{y,y}[|B_t^2-B_t^3|^{p_1}]^{1/p_1}\leq C(p_1)t^{\frac{1}{2}}, \tag{13}$$

by the scale invariance of Brownian motion. Next, we consider *Case 2*: $\{r(t),\ell(t)\}=\{1,4\}$. By definition of the labels $r(t),\ell(t)$, there are no particles in between $1,4$ at time t and therefore either one of the particles $2,3$ ends up to the right of $r(t)$ and the other to the left of $\ell(t)$ and we can bound $(B_t^{r(t)}-B_t^{\ell(t)})^+\leq|B_t^2-B_t^3|$ and then proceed as in (13). The other possibility is that both 2 and 3 end up to the left of $1,4$ or both to the right. Say both $2,3$ end up to the left of $1=\ell_t$ (the other possibilities work analogously). Then, one of the particles 2 or 3 must have collided before time t with particle 4, for otherwise the three particles on the left do not interact with 4 and therefore, cannot all have the same type, which contradicts the assumptions that $B_t^{\ell_t}<B_t^{r(t)}$. Hence, if $\tau_{i,j}$ is the first collision time of particles i,j, we can assume that $\tau_{2,4}\leq t$ and in particular we can estimate the left hand side of (12) using the strong Markov property by

$$\begin{aligned}\mathbb{E}_{\underline{x}}[\mathbb{1}_{\{\tau_{2,4}\leq t\}}|B_t^4-B_t^1|^{p_1}]^{\frac{1}{p_1}}&\leq \mathbb{E}_{\underline{x}}\Big[\mathbb{1}_{\{\tau_{2,4}\leq t\}}\mathbb{E}\big[\sup_{0\leq s\leq t-\tau}|B^2_{\tau+s}-B^4_{\tau+s}|^{p_1}|\mathcal{F}(\tau)\big]\Big]^{\frac{1}{p_1}}\\&\leq \mathbb{E}_{0,0}\Big[\sup_{0\leq s\leq t}|B_s^2-B_s^4|^{p_1}\Big]^{\frac{1}{p_1}}\leq C(p_1)t^{\frac{1}{2}}\end{aligned} \tag{14}$$

Finally, we consider *Case 3*, where the labels $\{r(t),\ell(t)\}$ correspond to one particle started at y and the other started at x or z, with loss of generality we assume that $\{r(t),\ell(t)\}=\{1,2\}$. If $\tau_{1,2}\leq t$, then we can argue as in (14) to get the right bound. Otherwise, if $\tau_{1,2}>t$, then necessarily $\ell(t)=1$, $r(t)=2$ and if particle 3 ends up to the left of 1, then we can estimate $(B_t^{\ell(t)}-B_t^{r(t)})^+\leq|B_t^2-B_t^3|$ and the argument in 13 gives the required bound, while if 1 does not meet 3, necessarily 4 has to meet 1 (otherwise the particles on the right cannot all have the same type) and the argument before (14) applies. These three cases combined yield the estimate (12).

Thus, we can conclude from (11) that

$$\mathbb{E}_{\underline{x}}\Big[(B^{r(t)}-B^{\ell(t)})^+e^{\gamma(L_t^=+\varrho L_t^{\neq})}\Big]\leq C(p_1,p_2,\gamma,\varrho)t^{\frac{1}{2}}\mathbb{E}_{\underline{x}}\Big[e^{-\gamma p_3(\varrho'-\varrho)L_t^{\neq})}\Big]^{\frac{1}{p_3}}.$$

Thus, recalling that $\frac{1}{p_3}=1-\delta$, we see that in order to complete the proof it suffices to show that for any $s>0$ there is a constant $C(s)$, such that for all $t\geq 0$,

$$\mathbb{E}_{\underline{x}}[e^{-sL_t^{\neq}}]\leq C(s)\min\Big\{\frac{(z-x+\log(t\wedge e))(y-x+\log(t\vee e))}{t},(\log(t\wedge e))t^{-\frac{1}{2}}\Big\}, \tag{15}$$

where we note that the $\log(t\vee e)$ term can be bounded by $t^\delta\vee 1$.

First, recall that for for the collision local time L_t up to time t of two independent Brownian motions, started in positions $x\leq y$, we have the classic bound that for all $t\geq 1$,

$$P_{x,y}\{L_t<\alpha\log t\}\leq(2\alpha\log t+y-x)t^{-\frac{1}{2}},\quad \alpha>0, \tag{16}$$

see for example Corollary A.5. Now, fix $s>0$ and let $c=\frac{2}{s}$. We distinguish the three cases:

(i) $L_t^{\neq} \geq c \log t$,

(ii) $L_t^{\neq} < c \log t$, but $L_t^{\text{tot}} := L_t^{=} + L_t^{\neq} \geq 2c \log t$,

(iii) $L_t^{\neq} < c \log t$ and $L_t^{\text{tot}} < 2c \log t$.

Regarding (i), we can estimate

$$\mathbb{E}_{\underline{x}}\Big[e^{-sL_t^{\neq}} \mathbb{1}_{\{L_t^{\neq} \geq c \log t\}}\Big] \leq t^{-sc}.$$

For (ii), we have in particular that $L_t^{=} \geq c \log t$. Now, from our classic fourth moment bounds (Theorem 1.1 and Remark 1.2 for the system with branching rate s/ϱ) we can deduce that

$$\begin{aligned}
\mathbb{E}_{\underline{x}}\Big[e^{-sL_t^{\neq}} \mathbb{1}_{\{L_t^{\neq} < c \log t, L^{\text{tot}} \geq 2c \log t\}}\Big] &\leq t^{-\frac{cs}{|\varrho|}} \mathbb{E}_{\underline{x}}\Big[e^{\frac{s}{|\varrho|}(L_t^{=} + \varrho L_t^{\neq})} \mathbb{1}_{\{L_t^{\neq} < c \log t, L^{\text{tot}} \geq 2c \log t\}}\Big] \\
&\leq t^{-\frac{cs}{|\varrho|}} \mathbb{E}_{\underline{x}}\Big[e^{\frac{s}{|\varrho|}(L_t^{=} + \varrho L_t^{\neq})}\Big] \\
&\leq C(s, \varrho) t^{-\frac{cs}{|\varrho|}} \leq C(s, \varrho) t^{-cs}.
\end{aligned}$$

Finally, consider case (iii). Here, note that if the total collision local time is small, then in particular the collision local time between the two Brownian motions started at y is small. That is, using (16),

$$\mathbb{E}_{\underline{x}}\Big[e^{-sL_t^{\neq}} \mathbb{1}_{L_t^{\neq} < c \log t, L_t^{\text{tot}} < 2c \log t}\Big] \leq P_{y,y}\{L_t < c \log t\} \leq 2c(\log t)\, t^{-\frac{1}{2}}.$$

A different bound can be reached by considering the collision local times between each pair of Brownian motions started in y, z and y, x respectively, leading to (again using (16))

$$\begin{aligned}
\mathbb{E}_{\underline{x}}\Big[e^{-sL_t^{\neq}} \mathbb{1}_{L_t^{\neq} < c \log t, L_t^{\text{tot}} < 2c \log t}\Big] &= P_{x,y}\{L_t \leq 2c \log t\} P_{y,z}\{L_t < 2c \log t\} \\
&\leq (4c \log t + y - x)(4c \log t + z - y) t^{-1}.
\end{aligned}$$

This completes the proof since we notice that by our choice of $c = \frac{2}{s}$, the dominating contribution is obtained by taking the minimum in the last scenario. □

Proof of Lemma 2.1. Fix $0 < \varepsilon < \frac{1}{2}$. By (10), it suffices to show that there exists a constant C such that for all $z > 0$,

$$\mathbb{E}_{(0,z),(0,z)}\Big[(B_t^{r(t)} - B_t^{\ell(t)})^+ e^{\gamma(L_t^{=} + \varrho L_t^{\neq})}\Big] \leq C(1 \wedge z^{-2(1-\varepsilon)}), \tag{17}$$

which is clearly integrable in z.

We condition on the time of the first collision of certain pairs of the four Brownian motions. Indeed, let $\tau_{i,j}$ denote the first hitting time of Brownian motions with index i and j, and consider the stopping time

$$\tau := \tau_{1,3} \wedge \tau_{1,4} \wedge \tau_{2,3} \wedge \tau_{2,4},$$

which is the first time that a motion started in 0 meets with a motion started in z.

Note that we can always assume that $\tau \leq t$, for otherwise the expectation in (17) is 0. Then, if $(\mathcal{F}(t))_{t \geq 0}$ denotes the filtration of the dual process, we can apply the strong

Markov property and use that up to time τ there are no particles of the same type that accumulate local time. In particular, none of the particles have switched type up to time τ, so the positions of B^i_τ at time τ and the type configuration at time τ satisfy the assumptions of Lemma 2.2, and we thus obtain that for $\delta = \frac{\varepsilon}{4}$, there exists a constant $C(\varrho, \gamma, \varepsilon)$ such that

$$\begin{aligned}
&\mathbb{E}_{(0,z),(0,z)}\Big[(B_t^{r(t)} - B_t^{\ell(t)})^+ e^{\gamma(L_t^= + \varrho L_t^{\neq})}\Big] \\
&= \mathbb{E}_{(0,z),(0,z)}\Big[\mathbb{E}\Big[(B_t^{r(t)} - B_t^{\ell(t)})^+ e^{\gamma(L_t^= - L_\tau^= + \varrho(L_t^{\neq} - L_\tau^{\neq}))} | \mathcal{F}(\tau)\Big] e^{\gamma(L_\tau^= + \varrho L_\tau^{\neq})}\Big] \qquad (18) \\
&\leq 4C(\varrho, \gamma, \varepsilon)\mathbb{E}_{(0,z),(0,z)}\Big[\mathbb{1}_{\{\tau = \tau_{2,3} \leq t\}} \min\big\{\tfrac{(B_\tau^4 - B_\tau^3 + 1)(B_\tau^2 - B_\tau^1 + 1)}{(t-\tau)^{\frac{1}{2}-\delta}}, (t-\tau)^\delta \vee 1\big\} e^{\varrho\gamma(L_\tau^{1,2} + L_\tau^{3,4})}\Big].
\end{aligned}$$

Here, we also used that the four possible cases $\tau = \tau_{1,3}, \tau_{1,4}, \tau_{2,3}, \tau_{2,4}$ are all equally likely and in all cases we obtain the same bound from Lemma 2.2. Moreover, in this scenario $L_\tau^{\neq} = L_\tau^{1,2} + L_\tau^{3,4}$.

In the analysis of the right hand side of (18), we distinguish four cases (where we always assume $\tau \leq t$):

(i) $\tau \leq z^{2-\varepsilon}$,

(ii) $\tau > z^{2-\varepsilon}$ and ($z^{2-\varepsilon} > t^{\frac{1}{4}}$ or $t \leq 2$)

(iii) $\tau > z^{2-\varepsilon}$, but $z^{2-\varepsilon} \leq t^{\frac{1}{4}}$ and $\tau \leq t^{1/2-\delta}$ for $\delta = \frac{\varepsilon}{4}$, $t \geq 2$.

(iv) $\tau > z^{2-\varepsilon}$, $z^{2-\varepsilon} \leq t^{\frac{1}{4}}$, but $\tau > t^{1/2-\delta}$ for $\delta = \frac{\varepsilon}{4}$, $t \geq 2$.

Case (i). On the event that $\tau \leq z^{2-\varepsilon} \wedge t$, we obtain

$$\begin{aligned}
\mathbb{E}_{(0,z),(0,z)}\Big[\mathbb{1}_{\{\tau=\tau_{2,3}\leq z^{2-\varepsilon}\wedge t\}} &\min\big\{\tfrac{(B_\tau^4 - B_\tau^3 + 1)(B_\tau^2 - B_\tau^1 + 1)}{(t-\tau)^{\frac{1}{2}-\delta}}, (t-\tau)^\delta \vee 1\big\} e^{\varrho\gamma L_\tau^{\neq}}\Big] \\
&\leq \mathbb{E}_{(0,z),(0,z)}\Big[\mathbb{1}_{\{\tau=\tau_{2,3}\leq z^{2-\varepsilon}\}}(B_\tau^4 - B_\tau^3 + 1)(B_\tau^2 - B_\tau^1 + 1)\Big] \\
&\leq \mathbb{E}_{(0,0)}\Big[\max_{s \leq z^{2-\varepsilon}} (B_s^2 - B_s^1 + 1)^2\Big] \mathbb{P}_{(0,z)}\{\tau_{1,2} \leq z^{2-\varepsilon}\}^{\frac{1}{2}} \\
&\leq C(1 \vee z^{2-\varepsilon-\frac{1}{4}\varepsilon}) e^{-\frac{1}{8}z^\varepsilon},
\end{aligned}$$

where we used in the penultimate step Cauchy-Schwarz and for the estimate of the first collision time that if $\tau(0)$ denotes the first hitting time of 0 for a single Brownian moton started at z, we have that

$$\begin{aligned}
\mathbb{P}_{(0,z)}\{\tau_{1,2} \leq z^{2-\varepsilon}\} &= \mathbb{P}_z\{\tau(0) \leq 2z^{2-\varepsilon}\} = \mathbb{P}_0\{\max_{s \leq 2z^{2-\varepsilon}} B_t \geq z\} \\
&= 2\mathbb{P}_0\{B_{2z^{2-\varepsilon}} \geq z\} \leq 2\frac{1}{\sqrt{\pi}} z^{-\frac{1}{2}\varepsilon} e^{-\frac{z^\varepsilon}{4}},
\end{aligned}$$

where we used a standard Gaussian estimate, see e.g. [MP10, Lemma 12.9], in the last step. This shows that in case (i) we obtain an upper bound on (17) that is integrable in z.

Case (ii). In this scenario, we can find an upper bound on the expectation on the right hand side in (18)

$$\begin{aligned}
&\mathbb{E}_{(0,z),(0,z)}\Big[\mathbb{1}_{\{z^{2-\varepsilon}<\tau=\tau_{2,3}\le t\}}\min\big\{\tfrac{(B^4_\tau-B^3_\tau+1)(B^2_\tau-B^1_\tau+1)}{(t-\tau)^{\frac12-\delta}},(t-\tau)^\delta\vee 1\big\}e^{\varrho\gamma(L^{1,2}_\tau+L^{3,4}_\tau)}\Big]\\
&\qquad\le \mathbb{E}_{(0,z),(0,z)}\Big[\mathbb{1}_{\{z^{2-\varepsilon}<\tau_{2,3}=\tau\le t\}}(1\vee t^\delta)e^{\gamma\varrho(L^{1,2}_\tau+L^{3,4}_\tau)}\Big]\\
&\qquad\le (1\vee t^\delta)\mathbb{E}_{0,0}\big[e^{\gamma\varrho L^{1,2}(z^{2-\varepsilon})}\big]^2\le C(1\vee t^\delta)(1\wedge z^{-2+\varepsilon}),
\end{aligned}$$

where we used the independence of the two pairs of Brownian motions and then Lévy's equivalence, see Lemma A.3, to calculate the asymptotics. However, if we assume that either $t\le 2$ or $z^{2-\varepsilon}>t^{\frac14}$, then this latter expression can be bounded by $C(1\wedge z^{-2+\varepsilon+4\delta})$, which by our choice of $\delta=\frac{\varepsilon}{4}$ is of the required form.

Case (iii). In this case, we can assume that $t\ge 2$ so that in particular we can estimate here that since $\tau\le t^{\frac12-\delta}$,

$$(t-\tau)^{-(\frac12-\delta)}\le(t-t^{\frac12-\delta})^{-(\frac12-\delta)}\le Ct^{-\frac12+\delta}.$$

Hence, we can use the first estimate in (18) and deduce that

$$\begin{aligned}
&\mathbb{E}_{(0,z),(0,z)}\Big[\mathbb{1}_{z^{2-\varepsilon}\le\tau\le t^{1/2-\delta}}\min\big\{\tfrac{(B^4_\tau-B^3_\tau+1)(B^2_\tau-B^1_\tau+1)}{(t-\tau)^{\frac12-\delta}},(t-\tau)^\delta\vee 1\big\}e^{\varrho\gamma(L^{1,2}_\tau+L^{3,4}_\tau)}\Big]\\
&\qquad\le C\mathbb{E}_{(0,z),(0,z)}\Big[\mathbb{1}_{\{z^{2-\varepsilon}\le\tau=\tau_{2,3}\le t^{\frac12-\delta}\}}\tfrac{(B^4_\tau-B^3_\tau+1)(B^2_\tau-B^1_\tau+1)}{(t-\tau)^{\frac12-\delta}}e^{\gamma\varrho(L^{1,2}_\tau+L^{3,4}_\tau)}\Big]\\
&\qquad\le Ct^{-\frac12+\delta}\mathbb{E}_{(0,z),(0,z)}\Big[\max_{s\le t^{\frac12-\delta}}(|B^4_s-B^3_s|+1)(|B^2_s-B^1_s|+1)e^{\gamma\varrho(L^{1,2}_{z^{2-\varepsilon}}+L^{3,4}_{z^{2-\varepsilon}})}\Big]
\end{aligned}$$

Now, applying Hölder's inequality, with $p=\frac{1}{1-\frac{\varepsilon}{2}}$ and q its conjugate, and then using the independence, we obtain an upper bound

$$\begin{aligned}
&Ct^{-\frac12+\delta}\mathbb{E}_{(0,z),(0,z)}\Big[\max_{s\le t^{\frac12-\delta}}(|B^4_s-B^3_s|+1)(|B^2_s-B^1_s|+1)e^{\gamma\varrho(L^{1,2}_{z^{2-\varepsilon}}+L^{3,4}_{z^{2-\varepsilon}})}\Big]\\
&\qquad\le Ct^{-\frac12+\delta}\mathbb{E}_{(0,0)}\Big[\max_{s\le t^{\frac12-\delta}}(|B^2_s-B^1_s|+1)^q\Big]^{\frac2q}\mathbb{E}_{0,0}\big[e^{\gamma\varrho pL^{1,2}_{z^{2-\varepsilon}}}\big]^{\frac2p}\\
&\qquad\le Ct^{-\frac12+\delta}\mathbb{E}_{(0,0)}\Big[\max_{s\le 1}(t^{\frac12(\frac12-\delta)}|B^2_s-B^1_s|+1)^q\Big]^{\frac2q}\mathbb{E}_{0,0}\big[e^{\gamma\varrho pL^{1,2}_{z^{2-\varepsilon}}}\big]^{\frac2p}\\
&\qquad\le C(p,q)(1\wedge z^{-(2-\varepsilon)\frac1p})
\end{aligned}$$

where we used Brownian scaling (and $t\ge 2$) to evaluate the first term and Lévy's equivalence, see Lemma A.3, for the second term. In particular, we obtain that the latter expression is bounded by $C(1\wedge z^{-(2-\varepsilon)\frac1p})\le(1\wedge z^{-2(1-\varepsilon)})$, by our choice of p.

Case (iv). For the remaining case (where by (ii) we can assume $t\ge 2$), we can use the second estimate in (18) and the independence of the Brownian motions to get an upper bound on (18)

$$\begin{aligned}
&\mathbb{E}_{(0,z),(0,z)}\Big[\mathbb{1}_{\{t^{\frac12-\delta}\le\tau=\tau_{2,3}\le t\}}\min\big\{\tfrac{(B^4_\tau-B^3_\tau+1)(B^2_\tau-B^1_\tau+1)}{(t-\tau)^{\frac12-\delta}},(t-\tau)^\delta\vee 1\big\}e^{\varrho\gamma(L^{1,2}_\tau+L^{3,4}_\tau)}\Big]\\
&\qquad\le C(1\vee t^\delta)\mathbb{E}_{(0,z),(0,z)}\Big[\exp\big\{\gamma\varrho(L^{1,2}_{t^{1/2-\delta}}+L^{3,4}_{t^{1/2-\delta}})\big\}\Big]\\
&\qquad\le C(1\vee t^{-\frac12+2\delta})\le C(1\vee z^{4(2-\varepsilon)(-\frac12+2\delta)}),
\end{aligned}$$

where we used Lévy's equivalence again and finally that $z^{2-\varepsilon} \leq t^{\frac{1}{4}}$. Hence, the resulting expression is of the form (17), since $2\delta = \frac{1}{2}\varepsilon < \frac{1}{4}$.

These cases exhaust all possibilities so that the Lemma is proved via (18). □

2.2 Tightness of the coordinate functions of the rescaled interface

Recall that for $n \in \mathbb{N}$ and $t > 0, x \in \mathbb{R}$ the rescaled solutions are $u_t^{(n)}(x) = u_{n^2t}(nx)$ and similarly $v^{(n)}(x) = v_{n^2t}(nx)$. We first establish tightness of $u^{(n)}$ and $v^{(n)}$ integrated against suitable test functions. For a discussion of the spaces involved, see Appendix A.1.

Lemma 2.3. *Suppose $\varrho < -\frac{1}{\sqrt{2}}$ and $(u_0, v_0) = (\mathbb{1}_{\mathbb{R}^-}, \mathbb{1}_{\mathbb{R}^+})$. If $\phi \in \mathcal{C}_{\text{rap}}$, then the coordinate processes $\{\langle \phi, u_t^{(n)} \rangle : t \geq 0\}_{n \in \mathbb{Z}_+}$ and $\{\langle \phi, v_t^{(n)} \rangle : t \geq 0\}_{n \in \mathbb{Z}_+}$ are tight in the space $D_{(0,\infty)}(\mathbb{R})$.*

Having established the fourth moment bound in Lemma 2.1, the proof of the tightness follows closely the proof of [Tri95, Lemma 4.1].

Proof. We denote by $(S_t)_{t \geq 0}$ the heat semigroup. By the scaling property of the model, see [EF04, Lemma 8], $(u^{(n)}, v^{(n)})$ is a solution of the original model when the branching rate γ is replaced by $n\gamma$. In particular, the Green's function representation for the symbiotic branching model, see Proposition A.2, yields for $\phi \in \mathcal{C}_{\text{rap}}$,

$$\begin{aligned} \langle \phi, u_t^{(n)} \rangle &= \langle S_t \phi, u_0^{(n)} \rangle + n^{1/2} \int_{[0,t] \times \mathbb{R}} \sqrt{u_s^{(n)}(x) v_s^{(n)}(x)} S_{t-s} \phi(x) d\overline{W}_s^1(x) \\ \langle \phi, v_t^{(n)} \rangle &= \langle S_t \phi, v_0^{(n)} \rangle + n^{1/2} \int_{[0,t] \times \mathbb{R}} \sqrt{u_s^{(n)}(x) v_s^{(n)}(x)} S_{t-s} \phi(x) d\overline{W}_s^2(x) \end{aligned} \tag{19}$$

for a pair of Gaussian white noises $(\overline{W}_s^1(x), \overline{W}_s^2(x))$ with correlation given by (2). Note that for our initial condition, the first term on the right hand in (19) side is equal to $\langle S_t \phi, 1_{(-\infty,0]} \rangle$. We check Kolmogorov tightness criterion for the stochastic integral in (19). For $0 < s < t$, and $i = 1, 2$,

$$\begin{aligned} n^{1/2} &\int_{[0,t] \times \mathbb{R}} \sqrt{u_r^{(n)}(x) v_r^{(n)}(x)} S_{t-r} \phi(x) d\overline{W}_r^i(x) - n^{1/2} \int_{[0,s] \times \mathbb{R}} \sqrt{u_r^{(n)}(x) v_r^{(n)}(x)} S_{s-r} \phi(x) d\overline{W}_r^i(x) \\ &= n^{1/2} \int_{[s,t] \times \mathbb{R}} \sqrt{u_r^{(n)}(x) v_r^{(n)}(x)} S_{t-r} \phi(x) d\overline{W}_r^i(x) \\ &\quad + n^{1/2} \int_{[0,s] \times \mathbb{R}} \sqrt{u_r^{(n)}(x) v_r^{(n)}(x)} (S_{t-r} \phi(x) - S_{s-r} \phi(x)) d\overline{W}_r^i(x) \end{aligned} \tag{20}$$

For the fourth moment of the first term on the right hand side in (20) we obtain using first the Burkholder-Davis-Gundy inequality, then Jensen's inequality and finally the fourth moment bound, Lemma 2.1 for $\varrho < -\frac{1}{\sqrt{2}}$,

$$\begin{aligned}\mathbb{E}\Big[\Big(n^{1/2}\int_{[s,t]\times\mathbb{R}}\sqrt{u_r^{(n)}(x)v_r^{(n)}(x)}S_{t-r}\phi(x)d\overline{W}^i_r(x)\Big)^4\Big]\\ &\leq C\|\phi\|^4_\infty(t-s)^2\mathbb{E}\Big[\Big(\frac{1}{t-s}\int_{[s,t]}\int_{\mathbb{R}}nu_r^{(n)}(x)v_r^{(n)}(x)drdx\Big)^2\Big]\\ &\leq C(\phi)(t-s)\mathbb{E}\Big[\int_s^t\Big(\int_{\mathbb{R}}u_{n^2r}(x)v_{n^2r}(x)dx\Big)^2dr\Big]\\ &\leq C(\phi,\gamma,\varrho)(t-s)^2\end{aligned}$$

Similarly, using the bound $\|S_t\phi - S_s\phi\|_\infty \leq \|\phi\|_\infty(|t-s|s^{-1}\wedge 1)$ and the Burkholder-Davis-Gundy inequality, we have that the expectation of the fourth power of the second term on the right hand side in (20 is bounded above by

$$\begin{aligned}\mathbb{E}\Big[\Big(n^{1/2}\int_{[0,s]\times\mathbb{R}}\sqrt{u_r^{(n)}(x)v_r^{(n)}(x)}(S_{t-r}\phi(x)-S_{s-r}\phi(x))d\overline{W}^i_r(x)\Big)^4\Big]\\ &\leq \|\phi\|^2_\infty\mathbb{E}\ \Big[\Big(\int_0^s\int_{\mathbb{R}}nu_{n^2r}(nx)v_{n^2r}(nx)dx\,(|t-s|^2(s-r)^{-2}\wedge 1)dr\Big)^2\Big]\end{aligned} \tag{21}$$

Now, note if $s\in[\frac{t}{2},t)$, that by an explicit calculation

$$t-s\leq\int_0^s(1\wedge|t-s|^2(s-r)^{-2})dr = 2(t-s)-\frac{(t-s)^2}{s}\leq 2(t-s). \tag{22}$$

In particular, if we define $f(r)=1\wedge|t-s|^2(s-r)^{-2}$, then we can rewrite the left hand side (21), and then apply Jensen and finally the fourth moment bound,

$$\begin{aligned}\|\phi\|^2_\infty\Big(\int_0^s f(r)dr\Big)^2\mathbb{E}\ \Big[\Big(\frac{1}{\int_0^s f(r)dr}\int_0^s\int_{\mathbb{R}}nu_{n^2r}(nx)v_{n^2r}(nx)dx\,f(r)dr\Big)^2\Big]\\ &\leq\|\phi\|^2_\infty\int_0^s f(r)dr\int_0^s\mathbb{E}\Big[\Big(\int_{\mathbb{R}}u_{n^2r}(x)v_{n^2r}(x)dx\Big)^2\Big]f(r)dr\\ &\leq C(\phi,\gamma,\varrho)\Big(\int_0^s f(r)dr\Big)^2\leq 4C(\phi,\gamma,\varrho)|t-s|^2,\end{aligned}$$

by the estimate (22). Moreover, if $s\in[0,\frac{1}{2}]$, so that in particular $t-s\geq s$, we find that $\int_0^s f(r)dr = s$ and the same argument shows that the latter expression is bounded by $C(\phi,\gamma,\varrho)s^2\leq C(\phi,\gamma,\varrho)(t-s)^2$.

Combining the fourth moment estimates of the two terms in (20), one can deduce that

$$\mathbb{E}\Big[\Big(n^{1/2}\int_{[s,t]\times\mathbb{R}}\sqrt{u_r^{(n)}(x)v_r^{(n)}(x)}S_{t-r}\phi(x)d\overline{W}^i_r(x)\Big)^4\Big]\leq C(\phi,\varrho,\gamma)(t-s)^2,$$

confirming that the stochastic integral satisfies Kolmogorov's tightness criterion, and thus completing the proof. □

2.3 Tightness of the measure-valued processes on path space

In this section we will prove tightness of the measure-valued processes $(\mu_t^n)_{t\geq 0}$ and $(\nu_t^n)_{t\geq 0}$ in the Skorohod path space on the space of tempered measures, see Appendix A.1 for a discussion of these spaces. A nice exposition of the general strategy in the same setting of tempered measures can be found in [DEF$^+$02, Section 4.1].

We start with a uniform bound on the first moments of $u^{(n)}$ integrated against a suitable test function.

Lemma 2.4. *For any $\varrho \leq 0$ and for each $T > 0$ and $\varphi \in \mathcal{C}_{\text{rap}}$,*

$$\sup_n \mathbb{E}\Big[\sup_{0\leq t\leq T} |\left\langle u_t^{(n)}, \varphi\right\rangle|\Big] < \infty,$$

and analgously for $u^{(n)}$ replaced by $v^{(n)}$.

Proof. We can assume that $\varphi \in C_\lambda(\mathbb{R})$ for some $\lambda > 0$, then it suffices to verify the statement for $\varphi_\lambda(x) = e^{-\lambda|x|}, x \in \mathbb{R}$ since $|\phi|_\lambda \leq |\phi|_\lambda \phi_\lambda$, see also the discussion in Appendix A.1. In fact, it even suffices to check the claim for ψ_λ defined via (29) as the mollified version of ϕ_λ (by inequality (30)). Recall, that the rescaled solution $u^{(n)}$ is a solution of the symbitic branching model, where the branching rate γ is replaced by $n\gamma$, see [EF04, Lemma 8]. In particular, $u^{(n)}$ satisfies a suitable martingale problem, see [EF04, Definition 3]), more precisely

$$M_t^{1,n}(\varphi) := \left\langle u_t^{(n)}, \varphi\right\rangle - \left\langle u_0^{(n)}, \varphi\right\rangle - \int_0^t \left\langle u_s^{(n)}, \frac{1}{2}\Delta\varphi\right\rangle ds,$$

is a continuous square-integrable martingale with a covariance structure given by

$$\langle\langle M^{1,n}\rangle\rangle_t = \gamma \int_0^t \int n\varphi^2(x) u_r^{(n)}(x) v_r^{(n)}(x)\, dx\, dr$$

Hence, we can estimate the first moment by

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}\Big|\left\langle u_t^{(n)}, \psi_\lambda\right\rangle\Big|\Big] \leq \mathbb{E}\Big[\sup_{0\leq t\leq T} |M_t^{1,n}(\psi_\lambda)|\Big] + \left\langle u_0^{(n)}, \psi_\lambda\right\rangle + \int_0^T \langle \mathbb{E}[u_s^{(n)}], \frac{1}{2}|\Delta\psi_\lambda|\rangle\, ds. \quad (23)$$

We deal with each of the summands on the right hand side separately. The second summand is bounded since the initial density is bounded by 1. The last summand is controlled, since first of all, by (30) there exists c_λ such that $|\Delta\psi_\lambda(x)| \leq c_\lambda e^{-\lambda|x|}$ for all $x \in \mathbb{R}$. Secondly, $\mathbb{E}[u_s^{(n)}(x)] = S_{n^2 s} u_0(nx) \leq 1$. Finally, we consider the first summand in (23). Using first Burkholder-Davis-Gundy and then Jensen, we obtain

$$\begin{aligned}
\mathbb{E}\Big[\sup_{0\leq t\leq T} |M_t^{1,n}(\psi_\lambda)|\Big] &\leq \mathbb{E}\Big[\big(\langle\langle M_t^{1,n}(\psi_\lambda)\rangle\rangle\big)^{1/2}\Big] \leq \Big(\mathbb{E}\langle\langle M_t^{1,n}(\psi_\lambda)\rangle\rangle\Big)^{1/2} \\
&= \Big(\mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} n\psi_\lambda^2(x) u_r^{(n)}(x) v_r^{(n)}(x) dx dr\Big]\Big)^{1/2}.
\end{aligned}$$

Now, using the particle duality we can write the latter expectation as

$$\begin{aligned}\mathbb{E}\Big[\int_0^t\int n\psi_\lambda^2(x)u_r^{(n)}(x)v_r^{(n)}(x)dxdr\Big]\\ &= \int_0^t\int_{\mathbb{R}} n\psi_\lambda^2(x)\mathbb{E}_{(0,0)}[1_{\mathbb{R}_-}(B^{(1)}_{n^2s}+nx)1_{\mathbb{R}_+}(B^{(2)}_{n^2s}+nx)\exp\{\gamma\varrho L^{1,2}_{n^2s}\}]dxds\\ &= \int_0^t\int_{\mathbb{R}} \psi_\lambda^2(x)\mathbb{E}_{(0,0)}[1_{\mathbb{R}_-}(B^{(1)}_{s}+x)1_{\mathbb{R}_+}(B^{(2)}_{s}+x)n\exp\{\gamma\varrho nL^{1,2}_{s}\}]dxds\,,\end{aligned}$$

where we used the Brownian scale invariance in the last step. We can continue to estimate using a simple application of Tanaka's formula, see Lemma A.6, to get an upper bound

$$\begin{aligned}\sup_x\{\psi_\lambda^2(x)\}\int_0^t\mathbb{E}_{(0,0)}\Big[\int_{\mathbb{R}}1_{\mathbb{R}_-}(B^{(1)}_s+x)1_{\mathbb{R}_+}(B^{(2)}_s+x)n\exp\{\gamma\varrho nL^{1,2}_s\}\Big]dxds\\ &\le \sup_x\{\psi_\lambda^2(x)\}\int_0^t\mathbb{E}_{(0,0)}\big[(B^{(2)}_s-B^{(1)}_s)^+n\exp\{\gamma\varrho nL^{1,2}_s\}\big]ds\\ &\le \sup_x\{\psi_\lambda^2(x)\}\int_0^t\mathbb{E}_{0}\big[(B_{2s})^+n\exp\{\gamma\varrho nL^{0}_{2s}\}\big]ds\\ &= \sup_x\{\psi_\lambda^2(x)\}\mathbb{E}_0\Big[\frac{1}{2\gamma|\varrho|}(1-e^{\varrho\gamma nL^0_{2t}})\Big],\end{aligned}$$

where $(L_s^0)_{s\ge0}$ is the local time of a single Brownian in zero. This expression is clearly bounded since $\varrho\le0$, which completes the proof. □

Now, we can combine the previous lemma with the tightness of the coordinate functions to show the tightness of the measure-valued processes.

Lemma 2.5. *The measure-valued processes $\{\mu_t^n,t\ge0\}_{n\in\mathbb{N}}$ and $\{\nu_t^n,t\ge0\}_{n\in\mathbb{N}}$ are tight on the Skorohod space $D_{[0,\infty)}(\mathcal{M}_{\text{tem}})$ on the space of tempered measures.*

Proof. By a standard argument, known as Jakubowski's criterion, see for example [Daw93, Thm. 3.6.4], tightness follows in the Skorohod space if we can show a a compact containment condition together with tightness of the coordinate functions.

To show the compact containment condition, we define the relative compact subset

$$K=K((c_m)_{m\ge1}):=\{\nu\in\mathcal{M}_{\text{tem}}:\langle\nu,\phi_{1/m}\rangle\le c_m,m\ge1\},$$

where $(c_m)_{m\ge1}$ is a sequence of positive numbers. Then, given $\varepsilon>0$ and any $m\in\mathbb{N}$, we can find by Lemma 2.4, a number $c_m>0$ such that for all $n\in\mathbb{N}$,

$$\mathbb{P}\Big\{\sup_{0\le t\le T}\langle u^{(n)},\phi_{1/m}\rangle\ge c_m\Big\}\le\frac{\varepsilon}{2^m},$$

In particular, it follows that for all $n\in\mathbb{N}$

$$\mathbb{P}\big\{u_t^{(n)}\in K((c_m)_{m\ge1})\text{ for all }t\in[0,T]\big\}\ge1-\varepsilon. \tag{24}$$

The same statement also holds for $v^{(n)}$.

Secondly, we need tightness of

$$\{\langle \phi, u_t^{(n)} \rangle : t \geq 0\}_{n \in \mathbb{Z}_+} \quad \text{and} \quad \{\langle \phi, v_t^{(n)} \rangle : t \geq 0\}_{n \in \mathbb{Z}_+} \tag{25}$$

for any test function $\phi \in \mathcal{C}_{\text{rap}}$, which we already showed in Lemma 2.3. Hence, the compact containment condition (24) combined with the tightness of the coordinate functions (25) yields tightness of the measure-valued processes $(\mu_t^n)_{t \geq 0}$ and $(\nu_t^n)_{t \geq 0}$ on the space $D((0,\infty), \mathcal{M}_{\text{tem}})$. Since all our processes are continuous, tightness also follows in the $\mathcal{C}$-space. □

2.4 Bounds on the width of the interface

In this section, we will prove the pth moment estimate on the approximate width of the interface $(R_t(\varepsilon) - L_t(\varepsilon))$ of Theorem 1.6 using the fourth moment estimates established in Lemma 2.1 gives a bound on the width of the interface. We recall that

$$L_t(\varepsilon) = \inf \Big\{ x : \int_{-\infty}^{x} u_t(y) v_t(y) dy \geq \varepsilon \Big\} \wedge R(t).$$

s and similarly, for the right end point

$$R_t(\varepsilon) = \sup \Big\{ x : \int_{x}^{\infty} u_t(y) v_t(y) dy \geq \varepsilon \Big\} \vee L(t).$$

Proof of Theorem 1.6. First, we recall from (17) in the proof of Lemma 2.1 that since $\varrho < -\frac{1}{\sqrt{2}}$, we have that for any $\tilde{\varepsilon} \in (0, \frac{1}{2})$, there exists a constant $C = C(\gamma, \varrho) > 0$ such that for all $z > 0$ and all $t \geq 0$,

$$\begin{aligned} &\mathbb{E}^{\mathbf{1}_{\mathbb{R}^-}, \mathbf{1}_{\mathbb{R}^+}} \Big[\int_{\mathbb{R}} u_t(x) v_t(x) u_t(x+z) v_t(x+z) \, dx \Big] \\ &\qquad = \mathbb{E}^{\mathbf{1}_{\mathbb{R}^-}, \mathbf{1}_{\mathbb{R}^+}} \Big[\int_{\mathbb{R}} u_t(x) v_t(x) u_t(x-z) v_t(x-z) \, dx \Big] \\ &\qquad \leq C(1 \vee z^{-2(1-\tilde{\varepsilon})}). \end{aligned} \tag{26}$$

If we define for $q \in (0,1)$,

$$I_q(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} |x-y|^q u_t(x) v_t(x) u_t(y) v_t(y) \, dx \, dy,$$

and choosing $\tilde{\varepsilon} = \frac{1}{4}(1-q)$, the estimate in (26) shows that

$$\begin{aligned} \mathbb{E}^{\mathbf{1}_{\mathbb{R}^-}, \mathbf{1}_{\mathbb{R}^+}} [I_q(t)] &= 2 \int_0^\infty |z|^q \, \mathbb{E}^{\mathbf{1}_{\mathbb{R}^-}, \mathbf{1}_{\mathbb{R}^+}} \Big[\int_{\mathbb{R}} u_t(x) v_t(x) u_t(x+z) v_t(x+z) \, dx \, dz \Big] \\ &\leq C \int_0^\infty z^q (1 \wedge z^{-2(1-\tilde{\varepsilon})}) \, dz \leq C(\gamma, \varrho) \Big(1 + \int_1^\infty z^{-2+2\tilde{\varepsilon}+q} \, dz \Big) < \infty \end{aligned}$$

for all $t \geq 0$, since by our choice of $\tilde{\varepsilon}$, we have that $2\tilde{\varepsilon} + 1 = \frac{1}{2} + \frac{1}{2}q < 1$. Fix $z > 0$, then on the event that $R_t(\varepsilon) - L_t(\varepsilon) > z$, we can estimate using the definition of $L_t(\varepsilon), R_t(\varepsilon)$ that

$$I_q(t) \geq z^q \int_{-\infty}^{L_t(\varepsilon)} u_t(x) v_t(x) dx \int_{R_t(\varepsilon)}^{\infty} u_t(y) v_t(y) dy \geq \varepsilon^2 z^q.$$

Hence, we can conclude that

$$\begin{aligned}\mathbb{P}^{\mathbf{1}_{\mathbb{R}^-},\mathbf{1}_{\mathbb{R}^+}}\{R_t(\varepsilon)-L_t(\varepsilon)>z\} &\le \varepsilon^{-2}z^{-q}\mathbb{E}^{\mathbf{1}_{\mathbb{R}^-},\mathbf{1}_{\mathbb{R}^+}}[I_q(t)\mathbb{1}_{\{R_t(\varepsilon)-L_t(\varepsilon)>z\}}] \\ &\le \varepsilon^{-2}z^{-q}\mathbb{E}^{\mathbf{1}_{\mathbb{R}^-},\mathbf{1}_{\mathbb{R}^+}}[I_q(t)] \le C(q,\gamma,\varrho)\varepsilon^{-2}z^{-q}.\end{aligned}$$

Thus, we have by Fubini that for any $0<p<q<1$,

$$\begin{aligned}\mathbb{E}^{\mathbf{1}_{\mathbb{R}^-},\mathbf{1}_{\mathbb{R}^+}}\left[((R_t(\varepsilon)-L_t(\varepsilon))^+)^p\right] &= p\int_0^\infty z^{p-1}\mathbb{P}\{R_t(\varepsilon)-L_t(\varepsilon)>z\}\,dz \\ &\le C(q)p\varepsilon^{-2}\int z^{p-q-1}dz,\end{aligned}$$

which shows that the p-th moment is finite. □

A Appendix

A.1 Martingale problems and Green function representations

The following two characterizations of solution to the symbiotic branching model can be found in [EF04] and will be important tools in our investigation. To state them properly, however, we first need to collect a considerable amount of notation.

For $\lambda\in\mathbb{R}$, let $\phi_\lambda(x):=e^{-\lambda|x|}, x\in\mathbb{R}$, and for $f:\mathbb{R}^d\to\mathbb{R}$ let $|f|_\lambda=||f/\phi_\lambda||_\infty$, where $||\cdot||_\infty$ is the supremum norm. Denote by $\mathcal{B}_\lambda$ the space of all measurable functions $f:\mathbb{R}^d\to\mathbb{R}$ with $|f|_\lambda<\infty$, and such that $f(x)/\phi_\lambda(x)$ has a finite limit as $|x|\to\infty$. Introduce the spaces

$$\mathcal{B}_{\text{rap}}=\mathcal{B}_{\text{rap}}(\mathbb{R}^d)=\bigcap_{\lambda>0}\mathcal{B}_\lambda \quad \text{and} \quad \mathcal{B}_{\text{tem}}=\mathcal{B}_{\text{tem}}(\mathbb{R}^d)=\bigcap_{\lambda>0}\mathcal{B}_{-\lambda}, \tag{27}$$

of *exponentially decreasing* and *tempered* measurable functions on $\mathbb{R}^d$ respectively. We write $\mathcal{C}_\lambda, \mathcal{C}_{rap}, \mathcal{C}_{\text{tem}}$ for the respective subspaces of continuous functions.

For each $\lambda\in\mathbb{R}$, the linear space $\mathcal{C}_\lambda$ equipped with the norm $|\cdot|_\lambda$ is a separable Banach space, and the space $\mathcal{C}_{\text{rap}}$ is topologized by the metric

$$d^{\mathcal{C}}_{\text{rap}}(f,g)=\sum_{n=1}^\infty 2^{-n}(|f-g|_{-n}\wedge 1), \quad f,g\in\mathcal{C}_{\text{rap}} \tag{28}$$

which turns it into a Polish space. Finally, $\mathcal{C}_{\text{tem}}$ is Polish if we topologize with the analogous metric with $f,g\in\mathcal{C}_{\text{tem}}$.

We also need to use the smoothed version of ϕ_λ, see e.g. Section 2.1 in [DEF$^+$02] for a discussion of the relevant facts. for this reason consider the mollifier

$$\varrho(x)=c_\varrho\mathbb{1}_{\{|x|\le 1\}}\exp\{-1(1-x^2)\}, \quad x\in\mathbb{R},$$

where c_ϱ is such that ϱ is a probability density. Then, the ψ_λ, the mollfied version of ϕ_λ is defined as

$$\psi_\lambda(x):=\int_{\mathbb{R}}\phi_\lambda(y)\varrho(y-x)\,dy. \tag{29}$$

We will also need the following estimate for the derivatives of ψ_λ: for any $\lambda > 0, n \in \mathbb{N}_0$, there exist constants $\underline{c}_{\lambda,n}, \overline{c}_{\lambda,n} > 0$ such that

$$\underline{c}_{\lambda,n}\phi_\lambda(x) \leq \left|\frac{\partial^n}{\partial x^n}\psi_\lambda(x)\right| \leq \overline{c}_{\lambda,n}\phi_\lambda(x) \quad \text{for all } x \in \mathbb{R}. \tag{30}$$

Let $\mathcal{M} = \mathcal{M}(\mathbb{R}^d)$ denote the set of non-negative Radon measures μ on $\mathbb{R}^d$ and let d_0 be a complete metric on $\mathcal{M}$ inducing the vague topology. We identify μ with its density if it exists, and use the notation $\langle \mu, f\rangle$ for the integral of the function f with respect to the measure μ. Denote by $\mathcal{M}_F(\mathbb{R}^d)$ the space of finite non-negative Radon measures μ on $\mathbb{R}^d$. We need the space $\mathcal{M}_{\text{tem}} = \mathcal{M}_{\text{tem}}(\mathbb{R}^d)$ of all measures μ in $\mathcal{M}$ such that $\langle \mu, \phi_\lambda\rangle < \infty$ for all $\lambda > 0$, and topologize this set of *tempered measures* by the metric

$$d^{\mathcal{M}}_{\text{tem}} = d_0(\mu,\nu) + \sum_{n=1}^{\infty} 2^{-n}(|\mu-\nu|_{-1/n} \wedge 1), \quad \mu,\nu \in \mathcal{M}_{\text{tem}} \tag{31}$$

where $|\mu-\nu|_\lambda = |\langle \mu, \phi_\lambda\rangle - \langle \nu, \phi_\lambda\rangle|$. Note that $(\mathcal{M}_{\text{tem}}, d^{\mathcal{M}}_{\text{tem}})$ is also Polish.

Write $\mathfrak{C} = \mathcal{C}((0,\infty),(\mathcal{C}^+_{\text{tem}})^2)$ for the set of all continuous paths $t \mapsto f_t$ in $(\mathcal{C}^+_{\text{tem}})^2$ where $((\mathcal{C}^+_{\text{tem}})^2, (d^{\mathcal{C}}_{\text{tem}})^2)$ is defined as the Cartesian product of $(\mathcal{C}^+_{\text{tem}}, d^{\mathcal{C}}_{\text{tem}})$. When endowed with the metric

$$d_{\mathfrak{C}}(f_\cdot, \widetilde{f}_\cdot) = \sum_{n=1}^{\infty} 2^{-n}\left(\sup_{1/n\leq t\leq n}(d^{\mathcal{C}}_{\text{tem}})^2\left((f_t,\widetilde{f}_t)\wedge 1\right)\right), \quad f_\cdot, \widetilde{f}_\cdot \in \mathfrak{C}, \tag{32}$$

$\mathfrak{C}$ is a Polish space. Let $\mathcal{M}_1(\mathfrak{C})$ denote the set of all probability measures on $\mathfrak{C}$. Equipped with the Prohorov metric $d_{\mathcal{M}_1(\mathfrak{C})}$, $\mathcal{M}_1(\mathfrak{C})$ is also a Polish space. Define $\mathcal{C}((0,\infty),(\mathfrak{C}^+_{\text{rap}})^2)$ analogously.

Similarly, given any Polish space $\mathcal{S}$, one can turn the space $D_{[0,\infty)}(\mathcal{S})$ of càdlàg paths on $\mathcal{S}$ into a Polish space using the usual Skorohod metric, see e.g. [EK86].

We define random objects over a sufficiently large stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_\cdot, \mathbb{P})$ satisfying the usual hypotheses. If $Y = \{Y_t : t \geq 0\}$ is a stochastic process, the law of Y is denoted $\mathbb{P}^Y$, and we use $\mathcal{F}^Y_t$ to denote the completion of the σ-field $\cap_{\epsilon>0}\sigma\{Y_s : s \leq t+\epsilon\}, t \geq 0$.

Let $p = p^\kappa$ denote the heat kernel in $\mathbb{R}$ related to $\frac{1}{2}\Delta$,

$$p_t(a) = \frac{1}{(2\pi t)^{1/2}}\exp\left\{-\frac{|a|^2}{2t}\right\}, \quad t > 0, a \in \mathbb{R}^d, \tag{33}$$

write $S = \{S_t : t \geq 0\}$ for the semigroup of the associated Brownian motion.

Definition A.1. The Symbiotic Branching model in $\mathbb{R}$ is characterized via the following martingale problem. Fix $\varrho \in [-1,1]$ and $(u_0,v_0) \in (\mathcal{B}^+_{\text{tem}})^2$ (resp. $(\mathcal{B}^+_{\text{rap}})^2$). A stochastic process $(u_t,v_t), t \geq 0$ with law $\mathbb{P}_{(u_0,v_0)}$ on the path space $\mathcal{C}((0,\infty),(\mathcal{C}^+_{\text{tem}})^2)$ (resp. $\mathcal{C}((0,\infty),(\mathcal{C}^+_{\text{rap}})^2)$) is a solution to the martingale problem for Symbiotic Branching if for each test function $\phi \in \mathcal{C}^{(2)}_{\text{rap}}$ (resp. $\mathcal{C}^{(2)}_{\text{tem}}$),

$$M_t^u(\phi) = \langle \phi, u_t \rangle - \langle \phi, u_0 \rangle - \int_0^t \langle \frac{\kappa^2}{2} \Delta\phi, u_s \rangle ds, \quad t \geq 0,$$

(analogously for v) is a pair $(M^u(\phi), M^v(\phi))$ of continuous square-integrable martingales null at zero with covariance structure

$$\langle\langle M^k(\phi), M^l(\phi) \rangle\rangle_t = \varrho_{kl}\gamma \int_0^t \int \phi(x)^2 L_{[u,v]}(ds, dx)$$

where

$$\varrho_{kl} = \begin{cases} 1 & k = l \text{ (i.e. } k = l = u \text{ or } k = l = v), \\ \varrho & k \neq l. \end{cases}$$

We proceed with the Green function representation, see [EF04, Corollary 19].

Proposition A.2. *For $\phi \in \mathcal{C}_{\text{rap}}$ (resp. $\mathcal{C}_{\text{tem}}$), $k = 1, 2$, and $t \geq 0$,*

$$\langle \phi, u_t \rangle = \langle S_t\phi, u_0 \rangle + \int_{[0,t]\times\mathbb{R}} M^u(d(s,a)) S_{t-s}\phi(a) \tag{34}$$

(similar for M^v) where $M^u(d(s,a)), M^v(d(s,a))$ is a pair of zero-mean martingale measures with covariance structure

$$\left\langle \int_{[0,\cdot]\times\mathbb{R}} M^k(d(s,a))) f_s^k(a), \int_{[0,\cdot]\times\mathbb{R}} M^l(d(s,a)) f_s^l(a) \right\rangle_t = \gamma\varrho_{kl} \int_0^t ds \langle u_s v_s, f_s^k f_s^l \rangle, \tag{35}$$

for $0 \leq t \leq T$ and $k, l \in \{u, v\}$ and f^u, f^v belong to the set of predictable functions f defined on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$ such that

$$\mathbb{E}_{\mathbf{x}} \left[\int_0^t ds \langle u_s v_s, (f_s)^2 \rangle \right] < \infty, \quad 0 \leq t \leq T. \tag{36}$$

A.2 Standard estimates for Brownian motion and its local time

In this section, we recall some of the standard facts (and its variations) on Brownian motion in a formulation adapted to our needs.

Lemma A.3. *If $(B_t)_{t\geq 0}$ is a Brownian motion started in $x \in \mathbb{R}$ with local time $(L_t^0)_{t\geq 0}$ in 0, then*

$$(L_t^0)_{t\geq 0} \stackrel{d}{=} ((M_t)^+)_{t\geq 0},$$

where $(M_t)_{t\geq 0}$ is the maximum process of a Brownian motion started in $-|x|$.

Proof. We adapt the proof of Theorem 7.38 in [MP10]. By Tanaka's formula [MP10, Thm. 7.33], we find that

$$|B(t)| - |x| = \int_0^t \text{sign}(B(s)) dB(s) + L^0(t).$$

It is clear that the stochastic integral is in distribution equal to a Brownian motion started in 0, so if we set

$$W(t) = -\big(|x| + \int_0^t \operatorname{sign}(B(s))dB(s)\big),$$

then W is a linear Brownian motion started at $-|x|$ and we have that

$$|B_t| = -W_t + L_t^0, \tag{37}$$

Let $(M_t)_{t\geq 0}$ denote the maximum process of $(W_t)_{t\geq 0}$. We want to show that for all $t \geq 0$, we have that $M_t = L_t^0$. It follows immediately from (37) that for any $s \leq t$, $W_s \leq L_s^0 \leq L_t^0$, so that by taking the maximum we obtain that $M_t \leq L_t^0$.

Now, suppose there exists a time t such that $M(t)^+ < L(t)$. Let $s = \inf\{r < t : L(r) = L(t)\}$. Since L^0 only increases on the set $\{s : |B(s)| = 0\}$, by continuity and since $L^0(t) > 0$, we have that $u > 0$ and so $|B(u)| = 0$. In particular, it follows $W(u) = L^0(u)$ and $u < t$. Thus, we can deduce that

$$M(u) \geq W(u) = L^0(u) = L^0(t) > M(s),$$

which yields a contradiction since $u < s$ and M is obviously increasing. Hence, $M^+ = L^0$ as claimed. □

Lemma A.4. *Let B_t be a Brownian motion started in $z \in \mathbb{R}$ and denote by L_t^0 its local time in 0. Then, for all $t > 0$*

$$\mathbb{P}_z\{L_t^0 \leq \alpha \log t\} \leq \sqrt{\frac{2}{\pi}}\,\frac{\alpha \log t + |z|}{t^{\frac{1}{2}}}.$$

Proof. Using Lemma A.3, we find that if M_t denotes the maximum process, then we can estimate

$$\begin{aligned}\mathbb{P}_z\{L_t^0 \leq \alpha \log t\} &= \mathbb{P}_{-|z|}\{M_t^+ \leq \alpha \log t\} = \mathbb{P}_0\{M_t \leq \alpha \log t + |z|\} \\ &= \mathbb{P}_0\{|B_t| \leq \alpha \log t + |z|\} \leq \sqrt{\frac{2}{\pi}}\,\frac{\alpha \log t + |z|}{t^{\frac{1}{2}}},\end{aligned}$$

where we used the reflection principle, see e.g. [MP10, Thm. 2.21], in the second to last step. □

Corollary A.5. *Suppose that $(B_t^1)_{t\geq 0}, (B_t^2)_{t\geq 0}$ are indepedent Brownian motions started in $x < y$ respectively and denote the collision local time as $(L_t^{1,2})_{t\geq 0}$. Then,*

$$\mathbb{P}_{x,y}\{L_t^{1,2} \leq \alpha \log t\} \leq \frac{1}{\sqrt{\pi}}\,\frac{2\alpha \log t + y - x}{t^{\frac{1}{2}}}.$$

Proof. This follows immediately from Lemma A.4. Note that $W_t := B_t^2 - B_t^1, t \geq 0$ is by definition a Brownian motion (with quadratic variation $2t$ and started in $y - x$) and thus $B_t = W_{t/2} - (y - x), t \geq 0$ is a standard Brownian motion. Moreover, $L_t^{1,2} = L_t^0(B^2 - B^1) = L_t^0(W)$. Now,

$$\begin{aligned}L_t^0(W) &= \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{\{|W_s|\leq\varepsilon\}} ds = \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{\{|B_{2s}+y-x|\leq\varepsilon\}} ds \\ &\overset{d}{=} \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{\{|\sqrt{2}B_s+y-x|\leq\varepsilon\}} ds = \frac{1}{\sqrt{2}} L_t^{\frac{x-y}{\sqrt{2}}}(B).\end{aligned}$$

Hence,

$$\mathbb{P}_{x,y}\{L_t^{1,2} \le \alpha \log t\} = \mathbb{P}_0\{L_t^{\frac{x-y}{\sqrt{2}}} \le \sqrt{2}\alpha \log t\} = \mathbb{P}_{\frac{y-x}{\sqrt{2}}}\{L_t^0 \le \sqrt{2}\alpha \log t\}$$
$$\le \sqrt{\frac{2}{\pi}} \frac{\sqrt{2}\alpha \log t + \frac{1}{\sqrt{2}}(y-x)}{t^{\frac{1}{2}}},$$

which proves the corollary. □

Lemma A.6. *For $(B_t)_{t\ge 0}$ a Brownian motion and L_t^0 its local time in* 0*, we have that for* $x > 0$*, ,*

$$\mathbb{E}_x[B_t^+ e^{-\beta L_t^0}] = \tfrac{1}{2\beta}\mathbb{E}_x(1 - e^{-\beta L_t})$$

Proof. First of all, integration by parts yields

$$B_t^+ e^{-\beta L_t^0} = x + \int_0^t e^{-\beta L_t^0}\, dB_s^+ - \beta \int_0^t B_s^+ e^{-\beta L_s^0}\, dL_s^0 .$$

Now, by Tanaka's formula we have that $B_t^+ + B_t^- = |B_t| = \int_0^t \operatorname{sign}(B_s)\, dB_s + L_t^0$. Combining it with the integration by parts and the fact $B_t^+ \overset{=}{d} B_t^-$, we obtain

$$\begin{aligned}\mathbb{E}_x[B_t^+ e^{-\beta L_t^0}] &= x + \tfrac{1}{2}\mathbb{E}_x \int_0^t e^{-\beta L_s^0} \operatorname{sign}(B_s)\, dB_s + \mathbb{E}_x \int_0^t e^{-\beta L_s^0}(\tfrac{1}{2} - \beta B_s^+)\, dL_s^0, \\ &= x + \tfrac{1}{2\beta}\mathbb{E}_x(1 - e^{-\beta L_t})\end{aligned}$$

where we used in the last step that all the other expressions are either zero or have zero expectation (the stochastic integral). □

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