

COEXISTENCE IN LOCALLY REGULATED COMPETING
POPULATIONS AND SURVIVAL OF BARW: FULL
TECHNICAL DETAILS AND ADDITIONAL REMARKS

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We propose two models of the evolution of a pair of competing populations. Both are lattice based. The first is a compromise between fully spatial models, which do not appear amenable to analytic results, and interacting particle system models, which don't, at present, incorporate all the competitive strategies that a population might adopt. The second is a simplification of the first in which competition is only supposed to act within lattice sites and the total population size within each lattice point is a constant. In a special case, this second model is dual to a branching-annihilating random walk. For each model, using a comparison with oriented percolation, we show that for certain parameter values both populations will coexist for all time with positive probability. As a corollary we deduce survival for all time of branching annihilating random walk for sufficiently large branching rates.

We also present a number of conjectures relating to the rôle of space in the survival probabilities for the two populations.

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1. Introduction. Natural populations interact with one another and with their environment in complex ways. No mathematical model can possibly incorporate all such interactions and yet remain analytically tractable. As a result, in order to understand the effects of a feature of a population's dynamics, it is often useful to study 'toy models'. In this paper we investigate two such toy models that aim to parody the evolution of two populations that are distributed in space and competing for the same resource. Both our models can be viewed as a compromise between fully spatial models which don't appear to be amenable to a rigorous mathematical analysis and interacting particle system models which don't, at present, incorporate all the competitive strategies that a population of, say, plants might adopt.

Although lattice based, our first model is highly reminiscent of the models in continuous space studied by Bolker and Pacala (1999) and Murrell and Law (2003), while admitting a rigorous mathematical analysis. It comprises a system of interacting diffusions, indexed by \mathbb{Z}^d , driven by independent Feller noises and coupled through a drift term that reflects migration and competition (both within and between species). Our second model is much simpler: first we suppose that the parameters governing migration of individuals within the two populations are the same and that competition between the populations acts only within individual lattice sites; second we suppose that the total population size within each lattice site is a fixed constant. When we further restrict to the *symmetric* case, in which the parameters governing the evolution of the two populations are the same, we shall exhibit a duality between this second process and a *branching annihilating random walk*. The latter is a process that has received considerable attention in the physics literature and we believe this duality to be of some interest in its own right.

A natural starting point for modelling two competing populations is the classical Lotka-Volterra model. This is a deterministic model for the evolution of the total sizes of the two populations, denoted $N_1(t), N_2(t)$. They are assumed to follow the following system of differential equations:

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1} - \alpha_{12} \frac{N_2}{K_1} \right), \quad (1)$$

$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - \frac{N_2}{K_2} - \alpha_{21} \frac{N_1}{K_2} \right), \quad (2)$$

where r_i, K_i are respectively the intrinsic growth rates and carrying capacities of the two species and the α_{ij} measure the interspecific competition. It is easy to check that longtime coexistence of the two populations is possible if $K_1 > \alpha_{12} K_2$ and $K_2 > \alpha_{21} K_1$. A number of models have been proposed

that extend this in two different ways. First they incorporate *spatial structure* into the populations and second they assume that the evolution of the populations is *stochastic*.

It is far from clear how spatial structure affects the chances of longterm coexistence for two competing populations. Traditionally ecologists have believed that the local nature of interactions between populations that are dispersed in space promotes coexistence. One reason is the so-called *competition-colonization tradeoff*: a weaker competitor that is good at colonization may be able to survive by exploiting ‘gaps’ between its competitors. It has also been claimed that because in spatial models the population tends to become segregated into clusters of a single type, the intraspecific competition will be more important than the interspecific competition. Pacala and Levin (1997) make an attempt to quantify this effect. On the other hand Neuhauser and Pacala (1999) propose and analyse a spatial stochastic model for competing species in which space actually makes coexistence harder. This suggests then that in their model it is actually the interactions at the cluster boundaries that dominate.

In order to obtain analytic results about spatial stochastic models, simplifying assumptions must be made. Murrell and Law (2003) point out that common assumptions are that the parameters of neighbourhoods over which individuals compete are the same, irrespective of species, or that dispersal and competition neighbourhoods are of the same size, but that dropping such symmetries can have profound consequences. They argue, using a simulation study and the method of moment closure for a specific stochastic model in two space dimensions, that spatial structure *can* promote coexistence, by showing that in the spatial setting two populations in which the overall strength of interspecific and intraspecific competition is the same can coexist, but only if the distance over which individuals sense their heterospecific neighbours (i.e. their competitors) is shorter than that over which they sense their conspecific neighbours. They coin the term *heteromyopia* for populations that are ‘shortsighted’ in this way. We explain this concept in a little more detail in the context of our first model in Section 2 below. Although this model admits such differences in neighbourhood size, our methods are not strong enough to confirm the numerical findings of Murrell and Law in this context. Indeed, even when the populations migrate in a symmetric way and intraspecific and interspecific competition neighbourhoods are of the same size, although we conjecture (in Section 2) that space does *not* make coexistence harder for our model, our methods are not strong enough to provide a rigorous proof of this claim.

Model I

Let us describe the first model that we have in mind. Following Bolker and Pacala (1999) we assume that the strategies for survival that individuals in our model can employ are: (i) to colonise relatively unpopulated areas quickly, (ii) to quickly exploit resources in those areas and (iii) to tolerate local competition. We take two different populations (species) and each can adopt a different combination of strategies for survival. In order to simplify the proofs of our results, we suppose our populations to be living on \mathbb{Z}^d (the biologically relevant case is $d = 2$). The dynamics of the model are entirely analogous to those considered by Bolker and Pacala (1999) and by Murrell and Law (2003). We write $\{X(t)\}_{t \geq 0} = \{X_i(t), i \in \mathbb{Z}^d\}_{t \geq 0}$ and $\{Y(t)\}_{t \geq 0} = \{Y_i(t), i \in \mathbb{Z}^d\}_{t \geq 0}$ for our two populations. We shall suppose that the pair of processes $\{X(t)\}_{t \geq 0}, \{Y(t)\}_{t \geq 0}$ solves the following system of stochastic differential equations:

$$\begin{aligned} dX_i(t) &= \sum_{j \in \mathbb{Z}^d} m_{ij} (X_j(t) - X_i(t)) dt \\ &\quad + \alpha \left(M - \sum_{j \in \mathbb{Z}^d} \lambda_{ij} X_j(t) - \sum_{j \in \mathbb{Z}^d} \gamma_{ij} Y_j(t) \right) X_i(t) dt + \sqrt{\sigma X_i(t)} dB_i(t), \end{aligned} \tag{3}$$

$$\begin{aligned} dY_i(t) &= \sum_{j \in \mathbb{Z}^d} m'_{ij} (Y_j(t) - Y_i(t)) dt \\ &\quad + \alpha' \left(M' - \sum_{j \in \mathbb{Z}^d} \lambda'_{ij} Y_j(t) - \sum_{j \in \mathbb{Z}^d} \gamma'_{ij} X_j(t) \right) Y_i(t) dt + \sqrt{\sigma Y_i(t)} dB'_i(t), \end{aligned} \tag{4}$$

where $\{\{B_i(t)\}_{t \geq 0}, \{B'_i(t)\}_{t \geq 0}, i \in \mathbb{Z}^d\}$ is a family of independent standard Brownian motions. The (bounded non-negative) parameters $m_{ij}, m'_{ij}, \lambda_{ij}, \lambda'_{ij}, \gamma_{ij}$, and γ'_{ij} are all supposed to be functions of $\|i - j\|$ alone and to vanish for $\|i - j\| > R$ for some $R < \infty$. In other words the range of both migration and interaction for the two populations will be taken to be finite.

Here, $\|\cdot\|$ can either denote the lattice distance (so that simple random walk is included) or the maximum norm on \mathbb{Z}^d , but it will be convenient to take the maximum norm. Moreover, notice that by a change of units there is no loss of generality in taking the same σ for both populations and indeed we may set $\sigma = 1$.

Assumption: for the rest of this work, $\sigma = 1$.

Remark (Existence and uniqueness for Model I). Note that existence of a solution for Model I under the assumptions above is not covered by the now standard results in Shiga and Shimizu (1980). However, an extension of their results using a comparison theorem of Geiß and Mantey (1994) yields, for each $p \geq 1$, (weak) existence of a continuous, positive solution $\{X_t\}_{t \geq 0}, \{Y_t\}_{t \geq 0}$, such that a.s. for all $t \geq 0$, X_t and Y_t live in the space

$$\ell_{\Gamma}^{4p} = \{x \in \mathbb{R}^{\mathbb{Z}^d} : \|x\|_{\Gamma, 4p} < \infty\}. \quad (5)$$

Here the weighted ℓ_{Γ}^{4p} -norm, $\|\cdot\|_{\Gamma, 4p}$, is defined by

$$\|x\|_{\Gamma, 4p} = \left(\sum_{i \in \mathbb{Z}^d} \Gamma_i |x_i|^{4p} \right)^{\frac{1}{4p}},$$

where the sequence $\{\Gamma_i\}_{i \in \mathbb{Z}^d}$ is such that $\Gamma_i > 0$ for $i \in \mathbb{Z}^d$, $\sum_i \Gamma_i < \infty$ and $\Gamma_i/\Gamma_j < f(\|i - j\|)$ for a continuous function $f : [0, \infty) \rightarrow [0, \infty)$. Note that, for example, $\Gamma_i = e^{-\|i\|}$ satisfies this condition. We prove existence of a solution to this system in the Appendix. Uniqueness remains open, after considerable efforts including those of several experts whom we have consulted. At first sight one expects to be able to prove uniqueness in a suitable weighted l^p space by an application of the (infinite-dimensional) Yamada-Watanabe Theorem. This works only in the special case when λ_{ij} and γ_{ij} both vanish for $i \neq j$. The non-local nature of the interaction destroys the vestiges of monotonicity available in this special case. \square

For the X -population, the first two strategies for survival listed above correspond to taking large m_{ij} and large αM , while the third corresponds to taking small λ_{ij} (conspecific competition) and γ_{ij} (interspecific competition). By varying M we can also model how efficiently the species uses the available resources: a species that can tolerate lower resource levels will have a higher value of M .

We now need to introduce some terminology to describe the long-term behaviour of the system.

Definition 1.1 (Notions of survival) *Let $p \in [0, 1)$. We shall say that the X population survives for all time with probability greater than p , if there exists $\kappa > 0$ such that*

$$\liminf_{t \rightarrow \infty} \mathbb{P}[X_0(t) > \kappa] > p.$$

We shall say that both populations persist for all time with probability greater than p , if there exists $\kappa > 0$ such that

$$\mathbb{P} \left[\forall t > 0, \exists i, j \in \mathbb{Z}^d : X_i(t), Y_j(t) > \kappa \right] > p.$$

Finally, we shall say that the populations exhibit longterm coexistence with probability greater than p , if there exists $\kappa > 0$ such that

$$\liminf_{t \rightarrow \infty} \mathbb{P} [X_0(t), Y_0(t) > \kappa] > p.$$

Observe that the third notion is much stronger than the second one. Notice also that if $\gamma_{ij} = \gamma'_{ij}$ is zero for all $i, j \in \mathbb{Z}^d$, then each population follows an independent copy of the so-called *stepping-stone version of the Bolker-Pacala model* introduced in Etheridge (2004). There it is proved that if the range of migration is at least as great as the range over which the population interacts with itself (here determined by the $\{\lambda_{ij}\}$), then provided that αM is sufficiently large the population will survive. A partial converse of this, proved there only in the context of a continuous space analogue of this model, suggests that this condition is actually necessary, a conclusion reached independently by Law, Murrell and Dieckmann (2003). We therefore assume from the outset that there is a constant $c > 0$ so that for all $i, j \in \mathbb{Z}^d$ we have that $m_{ij} > c\lambda_{ij}$ and $m'_{ij} > c\lambda'_{ij}$ whenever λ_{ij} resp. λ'_{ij} is strictly positive. Indeed, Theorem 1.5 in Etheridge (2004) then tells us that if $\alpha M > \sum_j m_{ij}$ is sufficiently large (depending on c), the single species model for X started from any non-trivial translation invariant initial condition survives with positive probability, i.e. there exists a $\kappa > 0$, such that

$$\liminf_{t \rightarrow \infty} \mathbb{P} [X_0(t) > \kappa] > 0.$$

Clearly, the corresponding result holds also for the Y -population in the absence of its competitor.

For the competing species model we will have to make similar and additional assumptions. In particular, we shall choose initial conditions for our processes in such a way that we can find a box where both populations are present, but not so prevalent that the competitive interaction between them is too large.

Notation and assumptions for Theorem 1.2

- The parameters m_{ij} , m'_{ij} , λ_{ij} , λ'_{ij} , γ_{ij} , and γ'_{ij} are non-negative functions of $\|i - j\|$ alone and vanish for $\|i - j\| > R$ for some $R < \infty$.

- $\{m_{ij}\}, \{m'_{ij}\}, \{\lambda_{ij}\}$ and $\{\lambda'_{ij}\}$ are fixed in such a way that there exists a constant $c > 0$ such that, for all $m_{ij}, m'_{ij} \neq 0$,

$$\frac{1}{c}\lambda_{ij} < m_{ij} < c\lambda_{ij}, \text{ and } \frac{1}{c}\lambda'_{ij} < m'_{ij} < c\lambda'_{ij}. \quad (6)$$

For all i, j such that $m_{ij} = 0$ resp. $m'_{ij} = 0$ we require $\lambda_{ij} = 0$ resp. $\lambda'_{ij} = 0$. Assume that $\{m_{ij}\}$ and $\{m'_{ij}\}$ are non-diagonal and of the same range and that $\lambda_{ii}, \lambda'_{ii} > 0$ for all $i \in \mathbb{Z}^d$.

- Let $L = \max\{\|j - i\| : m_{ij}, m'_{ij} \neq 0\} \leq R < \infty$.
- Assume that $\alpha M > \sum_{j \in \mathbb{Z}^d} m_{ij}$ and $\alpha' M' > \sum_{j \in \mathbb{Z}^d} m'_{ij}$.
- Let $b \in \mathbb{Z}$ such that $\max\{\|j - i\| : \gamma_{ij} \text{ or } \gamma'_{ij} \neq 0\} < (b - 1)L$. Note that this bounds the range over which the two populations interact competitively.
- For $m \in \mathbb{N} \cup \{\infty\}$ and $0 < \kappa_1 < \kappa_2 < \infty$, $0 < \kappa'_1 < \kappa'_2 < \infty$, we write

$$(X(0), Y(0)) \in H(\kappa_1, \kappa_2; \kappa'_1, \kappa'_2; m)$$

if $X(0), Y(0) \in \ell_{\Gamma}^{4p}$ and there exists a box $J = \{[-m, m]^d \cap \mathbb{Z}^d\} \subset \mathbb{Z}^d$, such that for all $i \in J$,

$$X_0(i) \in [\kappa_1, \kappa_2) \text{ and } Y_0(i) \in [\kappa'_1, \kappa'_2). \quad (7)$$

Remark. One can drop the assumption that the range of $\{m_{ij}\}$ and $\{m'_{ij}\}$ are the same, but this will make the proof much messier. \square

Theorem 1.2 *Under the above assumptions, there exist finite constants $M_0 > 0, M'_0 > 0$, such that*

1) *For each $M > M_0$ and $M' > 0$ there is a constant $\gamma = \gamma(M, M') > 0$ and constants $0 < \kappa_1 < \infty, 0 < \kappa'_2 < \infty$, such that if $\sum_j \gamma_{ij} < \gamma$ and*

$$(X(0), Y(0)) \in H(\kappa_1, \infty; 0, \kappa'_2; (b + 1/2)L),$$

the X -population survives for all time with probability greater than one half.

2) *Similarly, for each $M' > M'_0$ and $M > 0$ there is a constant $\gamma' = \gamma'(M, M') > 0$ and constants $0 < \kappa_2 < \infty, 0 < \kappa'_1 < \infty$, such that if $\sum_j \gamma'_{ij} < \gamma'$ and*

$$(X(0), Y(0)) \in H(0, \kappa_2; \kappa'_1, \infty; (b + 1/2)L),$$

the Y -population survives for all time with probability greater than one half.

Corollary 1.3 *Under the conditions of Theorem 1.2, for each pair (M, M') with $M > M_0$ and $M' > M'_0$, there is a pair (γ, γ') with $\gamma > 0, \gamma' > 0$ and constants*

$$0 < \kappa_1 < \kappa_2 < \infty, \quad 0 < \kappa'_1 < \kappa'_2 < \infty,$$

such that if $\sum_j \gamma_{ij} < \gamma, \sum_j \gamma'_{ij} < \gamma'$ and

$$(X(0), Y(0)) \in H(\kappa_1, \kappa_2; \kappa'_1, \kappa'_2; (b + 1/2)L),$$

then the X and Y populations both persist for all time with positive probability.

Moreover, for each such pair if $H(\kappa_1, \kappa_2; \kappa'_1, \kappa'_2; (b + 1/2)L)$ is replaced by $H(\kappa_1, \kappa_2; \kappa'_1, \kappa'_2; \infty)$, then there is longterm coexistence with positive probability, i.e. there exists $\kappa > 0$ such that

$$\liminf_{t \rightarrow \infty} \mathbb{P}[X_0(t), Y_0(t) > \kappa] > 0.$$

As we explain in Section 2, we would conjecture a very much stronger result than Theorem 1.2 (or Corollary 1.3). In particular, we provide evidence to support the claim that in the biologically relevant case of two dimensions, if we take the special case of our model in which $\alpha = \alpha', M = M'$ and $m_{ij} = m'_{ij}$, then provided that $\gamma'_{ij} \leq \lambda_{ij}$ and $\gamma_{ij} \leq \lambda'_{ij}$ with strict inequality whenever λ_{ij} resp. $\lambda'_{ij} \neq 0$, and the parameters are such that if γ_{ij} and γ'_{ij} were zero then the single species models would survive, then with positive probability the competing species model will coexist for all time. This would be precisely the prediction of the corresponding Lotka-Volterra model. If we drop the assumptions $\alpha = \alpha'$ and $M = M'$ then this conjecture must be modified to reflect competition-colonisation tradeoff. We formulate this and other conjectures more carefully in Section 2. In the process we are led to consider our second model of two competing species.

Model II

Suppose now that the neighbourhood over which each individual competes is just the site in which it lives so that the only interaction between different points in \mathbb{Z}^d is through migration. In addition we suppose that the migration mechanism for the two populations is the same and that the *total* population size in each site is constant (that is $X_i(t) + Y_i(t) \equiv N > 0$ for all $i \in \mathbb{Z}^d$ and all $t \geq 0$). Let us write $p_i(t) = X_i(t)/N$ for the proportion of the total population in i at time t that belongs to the X -population. Then, as we see

in Section 2, we arrive at the much simpler model

$$dp_i(t) = \sum_{j \in \mathbb{Z}^d} m_{ij} (p_j(t) - p_i(t)) dt + sp_i(t) (1 - p_i(t)) (1 - \mu p_i(t)) dt + \sqrt{\frac{1}{N} p_i(t) (1 - p_i(t))} dW_i(t), \quad (8)$$

where

$$s = \alpha M - \alpha' M' + (\alpha' \lambda'_{ii} - \alpha \gamma_{ii}) N,$$

and

$$\mu = \frac{(\alpha' \lambda'_{ii} - \alpha \gamma_{ii}) N + (\alpha \lambda_{ii} - \alpha' \gamma'_{ii}) N}{\alpha M - \alpha' M' + (\alpha' \lambda'_{ii} - \alpha \gamma_{ii}) N}$$

and finally $\{W_i(t), i \in \mathbb{Z}^d\}_{t \geq 0}$ is a family of independent Brownian motions. This model is a system of interacting Fisher-Wright diffusions for gene frequencies in a spatially structured population. From the results in Shiga and Shimizu (1980) it follows that if $p_i(0) \in [0, 1]$ for all $i \in \mathbb{Z}^d$, then this system has a continuous, pathwise unique $[0, 1]^{\mathbb{Z}^d}$ -valued strong solution for all times $t \geq 0$.

If $\mu < 1$, then in each site i there is selection in favour of either the X -type or the Y -type according to whether $s > 0$ or $s < 0$. If $\mu > 1$, in each site i we have selection in favour of *heterozygosity* if $s > 0$ and selection in favour of *homozygosity* if $s < 0$. In the ‘neutral’ case ($s = 0$), the process has a *moment dual*, the so-called *structured coalescent*, (see, for example, Shiga 1982) and it is easy to show that if $d \geq 3$ then with positive probability there will be longterm coexistence of our two populations, whereas if $d \leq 2$, with probability one, eventually only one population will be present.

Notice that we have selection in favour of heterozygosity precisely when

$$(\alpha \lambda_{ii} - \alpha' \gamma'_{ii}) N > \alpha M - \alpha' M', \quad \text{and} \quad (\alpha' \lambda'_{ii} - \alpha \gamma_{ii}) N > \alpha' M' - \alpha M.$$

We sketch a proof of the following result and present a more detailed analysis in a forthcoming work.

Theorem 1.4 *Let $\{p_i(t), i \in \mathbb{Z}^d\}_{t \geq 0}$ evolve according to Model II. Suppose that $\mu > 1$ and let $\varepsilon \in (0, 1/4]$. Then, if $p_i(0) \in (\varepsilon, 1 - \varepsilon)$ for some $i \in \mathbb{Z}^d$, there exists an $s_0 \in [0, \infty)$ such that for all $s > s_0$, we have*

$$\mathbb{P} \left[\forall t > 0, \exists i \in \mathbb{Z}^d : \varepsilon < p_i(t) < 1 - \varepsilon \right] > 0.$$

Moreover, if $p_i(0) \in (\varepsilon, 1 - \varepsilon)$ for all $i \in \mathbb{Z}^d$, then

$$\liminf_{t \rightarrow \infty} \mathbb{P} [\varepsilon < p_0(t) < 1 - \varepsilon] > 0.$$

In the case when the two populations evolve symmetrically, i.e. $\mu = 2$, Model II reduces to

$$dp_i(t) = \sum_j m_{ij} (p_j(t) - p_i(t)) dt + sp_i(t) (1 - p_i(t)) (1 - 2p_i(t)) dt + \sqrt{\frac{1}{N} p_i(t) (1 - p_i(t))} dW_i(t). \quad (9)$$

For general s there is no convenient moment dual, but in Lemma 2.1 we find an alternative duality with a system of *branching annihilating random walks*.

Definition 1.5 (Branching annihilating random walk) *The Markov process $\{n_i(t), i \in \mathbb{Z}^d\}_{t \geq 0}$ with values $n_i(t) \in \mathbb{Z}_+$ and dynamics described by*

$$\begin{aligned} \begin{cases} n_i \mapsto n_i - 1, \\ n_j \mapsto n_j + 1 \end{cases} & \text{at rate } n_i m_{ij} & \text{(migration)} \\ n_i \mapsto n_i + m & \text{at rate } s n_i & \text{(branching)} \\ n_i \mapsto n_i - 2 & \text{at rate } \frac{1}{2} n_i (n_i - 1) & \text{(annihilation)} \end{aligned}$$

is called a branching annihilating random walk with offspring number m and branching rate s .

Corollary 1.6 *There exists $s_0 \geq 0$ such that if $s > s_0$ the branching annihilating random walk with offspring number two, started from an even number of particles at time zero, will survive for all time with positive probability.*

Remark. Notice that in a branching annihilating random walk with offspring number two, a birth event results in one individual splitting into *three*, a net increase of two, whereas an annihilation event results in the loss of two particles. As a result, we have parity preservation: if we start from an odd number of particles in the system, then there will always be an odd number of particles in the system (so, in particular, at least one). This is why we restrict the initial number of particles in Corollary 1.6 to be even. \square

Branching annihilating random walk has received considerable attention from physicists (see Täuber (2003) for a review). Much of the work relates to the analogous process with instant annihilation of any two particles in the same site (and offspring therefore thrown onto neighbouring sites) which is easier to simulate numerically but, for example, Cardy and Täuber (1996), (1998) consider precisely the process described above. Our conjecture for Model II, stated in Section 2, is based on their results, which in turn are

based on perturbation theory and renormalisation group calculations.

We remark that Neuhauser and Pacala (1999) exploit a so-called annihilating duality between their interacting particle system and BARW with *instant* annihilation.

The rest of the paper is laid out as follows. In Section 2, we explain the relationship between our two models and the duality between the symmetric form of Model II and branching annihilating random walk. We also make some conjectures about the longtime behaviour of our two models that are then placed in context by discussing in more detail the relationship with results and conjectures for other toy models. The proof of our main result will rely upon a comparison with oriented $2N$ -dependent percolation and so in Section 3, we recall the definition of $2N$ -dependent percolation and state a suitable comparison result. The proofs of Theorem 1.2 and Corollary 1.3 are in Section 4 and a sketch of the proof of Theorem 1.4 is in Section 4.2.3. Corollary 1.6 will then be immediate from the duality of Model II and branching annihilating random walk.

2. Heuristics and comparison to existing models. In this section we provide heuristic arguments that explain the connection between our two models, justify our results and also suggest that much stronger statements should be true. En route we exhibit the duality between the symmetric version of Model II and branching annihilating random walk. We then briefly review results and conjectures for some of the related models in the literature.

2.1. Relationship between the two models. Suppose that the evolution of our population follows Model I, i.e. is determined by equations (3) and (4). We now derive the system of equations governing the proportion of the total population at time t at site i that belongs to the X -subpopulation. We need some notation. Write

$$N_i(t) = X_i(t) + Y_i(t), \quad \text{and} \quad p_i(t) = \frac{X_i(t)}{N_i(t)}.$$

Then an application of Itô's formula (and some rearrangement) gives

$$\begin{aligned}
dp_i(t) &= \sum_{j \in \mathbb{Z}^d} m_{ij} \frac{N_j(t)}{N_i(t)} (p_j(t) - p_i(t)) dt \\
&+ \sum_{j \in \mathbb{Z}^d} (m_{ij} - m'_{ij}) \frac{N_j(t)}{N_i(t)} p_i(t) (1 - p_j(t)) dt \\
&- \sum_{j \in \mathbb{Z}^d} (m_{ij} - m'_{ij}) p_i(t) (1 - p_i(t)) dt \\
&+ \left[\alpha M - \alpha' M' + \sum_{j \in \mathbb{Z}^d} (\alpha' \lambda'_{ij} - \alpha \gamma_{ij}) N_j(t) \right. \\
&\quad \left. + \sum_{j \in \mathbb{Z}^d} (\alpha \gamma_{ij} + \alpha' \gamma'_{ij} - \alpha' \lambda'_{ij} - \alpha \lambda_{ij}) N_j(t) p_j(t) \right] p_i(t) (1 - p_i(t)) dt \\
&+ \sqrt{\frac{1}{N_i(t)} p_i(t) (1 - p_i(t))} dW_i(t), \tag{10}
\end{aligned}$$

where $\{W_i(t), i \in \mathbb{Z}^d\}_{t \geq 0}$ is a family of independent Brownian motions.

We concentrate on the case when $m_{ij} = m'_{ij}$. Notice that if we also assume that $\lambda_{ij}, \lambda'_{ij}$ and $\gamma_{ij}, \gamma'_{ij}$ are zero for $i \neq j$ and that the population sizes $N_i(t)$ are in fact a fixed constant then we arrive at Model II:

$$\begin{aligned}
dp_i(t) &= \sum_j m_{ij} (p_j(t) - p_i(t)) dt + s p_i(t) (1 - p_i(t)) (1 - \mu p_i(t)) dt \\
&\quad + \sqrt{\frac{1}{N} p_i(t) (1 - p_i(t))} dW_i(t),
\end{aligned}$$

where

$$s = \alpha M - \alpha' M' + N (\alpha' \lambda'_{ii} - \alpha \gamma_{ii})$$

and

$$\mu = \frac{(\alpha' \lambda'_{ii} - \alpha \gamma_{ii}) N + (\alpha \lambda_{ii} - \alpha' \gamma'_{ii}) N}{\alpha M - \alpha' M' + (\alpha' \lambda'_{ii} - \alpha \gamma_{ii}) N}.$$

2.2. Conjectures for Model II. Our conjectures for Model II are based on the symmetric case, implying $\mu = 2$. The model then reduces to the system (9). In this case we are able to find a convenient dual process. First we transform the equations. Let $x_i(t) = 1 - 2p_i(t)$. Then

$$dx_i(t) = \sum_j m_{ij} (x_j(t) - x_i(t)) dt + \frac{1}{2} s (x_i^3(t) - x_i(t)) dt - \sqrt{(1 - x_i^2(t))} dW_i(t). \tag{11}$$

Lemma 2.1 *The system (11) is dual to branching annihilating random walk with branching rate $s/2$ and offspring number two, denoted $\{n_i(t), i \in \mathbb{Z}^d\}_{t \geq 0}$, through the duality relationship*

$$\mathbb{E} \left[\underline{x}(t)^{\underline{n}(0)} \right] = \mathbb{E} \left[\underline{x}(0)^{\underline{n}(t)} \right],$$

where

$$\underline{x}^{\underline{n}} \equiv \prod_{i \in \mathbb{Z}^d} x_i^{n_i}.$$

The proof is completely standard, see, e.g. Shiga (1980), and is omitted.

Cardy and Täuber (1996), (1998) consider the branching annihilating random walk model of Definition 1.5. In particular, their results suggest that when the offspring number is two, although in one dimension the optimal value for s_0 in Corollary 1.6 is strictly positive, in two dimensions one can take $s_0 = 0$. Based on this we make the following conjecture with some confidence.

Conjecture 2.2 *For Model II with $\mu = 2$ and $d = 1$, there is a critical value $s_0 > 0$ such that the populations described by system (9) will both persist for all time with positive probability if and only if $s > s_0$. In $d = 2$, there is positive probability that both populations will persist for all time if and only if $s > 0$. For $d \geq 3$ this probability is positive if and only if $s \geq 0$.*

Roughly speaking, for $d \geq 2$, if there is a homozygous advantage, then the population will initially form homogenic clusters, but ultimately it will be the interactions at the cluster boundaries that dominate and one type will go extinct. In the heterozygous advantage case, there will be long term coexistence of species. In one dimension, the heterozygous advantage must be ‘sufficiently strong’ if we are to see coexistence.

In fact we would go further. In view of the genetic interpretation of Model II, it would be odd if the case $\mu = 2$ were pathological and so we expect that in $d \geq 2$ we will have positive probability of coexistence for any $s > 0$, $\mu > 1$. With slightly less confidence we therefore extend our conjecture.

Conjecture 2.3 *Conjecture 2.2 holds true for any $\mu > 1$, where in one dimension s_0 will now also depend on μ .*

If this conjecture is true, then in dimensions greater than one, for

$$(\alpha' \lambda'_{ii} - \alpha \gamma_{ii}) N > \alpha' M' - \alpha M \quad \text{and} \quad (\alpha \lambda_{ii} - \alpha' \gamma'_{ii}) N > \alpha M - \alpha' M'$$

we have positive probability that both populations survive. Comparing the quantities $\alpha' \lambda'_{ii} - \alpha \gamma_{ii}$ and $\alpha \lambda_{ii} - \alpha' \gamma'_{ii}$ tells us about the relative effectiveness

of the X and Y populations as competitors. If the first is smaller, then the X -population is a less effective competitor. However, provided that $\alpha M > \alpha' M'$, we can even allow it to be negative and have positive probability of survival for the X -population. This reflects a competition-colonisation tradeoff.

2.3. Conjectures for Model I. We now turn to Model I. We assume that the migration mechanisms governing the two populations are the same. Suppose first that $\alpha = \alpha'$, $M = M'$, $\lambda_{ij} = \gamma'_{ij}$ and $\lambda'_{ij} = \gamma_{ij}$. We then see that the system of equations (10) looks like a selectively neutral stepping stone model with variable population sizes in each lattice site. If we condition on the trajectories of those population sizes, then this process will have a dual process: a system of coalescing random walks in a space-and-time varying environment. Showing that there is no long term coexistence of types amounts to showing that two independent random walks evolving in this environment will, with probability one, eventually meet and coalesce. If the environment is sufficiently well-behaved then one might expect this to be true. Problems will arise if the environment develops large ‘holes’, so that the walkers never meet, or very dense clumps so that when the walkers do meet, they do so in such a heavily populated site that they don’t coalesce before moving apart again. Much of our proof of Theorem 1.2 is devoted to showing that the environment does not clump and a special case of that result says that, provided that initially both populations are present in sufficient numbers in all sites, the probability that any given site is in a ‘hole’ at time t is uniformly bounded below. We therefore conjecture that in the neutral case, Model I will behave qualitatively in the same way as Model II. In the biologically relevant case of two spatial dimensions we have been unable to produce a proof.

More generally we believe, still assuming that $m_{ij} = m'_{ij}$ and $\alpha = \alpha'$, $M = M'$, provided that at least one population persists, the question of longtime coexistence of the populations described by Model I will not be changed by assuming that competition only acts within individual lattice sites and moreover in that case the question of coexistence will be the same as for the populations described by Model II. Namely we make the following conjecture.

Conjecture 2.4 *Let $m_{ij} = m'_{ij}$, $\alpha = \alpha'$, $M = M'$ be fixed. Suppose that both X and Y populations start from non-trivial translation invariant initial conditions and that the parameters are such that each population has positive chance of survival in the absence of the other. Further let $\lambda_{ij} = \lambda'_{ij}$, $\gamma_{ij} = \gamma'_{ij}$.*

1. *If $\lambda_{ij} < \gamma_{ij}$ for all j , then eventually only one population will be*

present.

2. If $\lambda_{ij} > \gamma_{ij}$ for all j , then if $d \geq 2$, with positive probability the populations will exhibit longterm coexistence.

In one dimension the same result will hold true provided that $\lambda_{ij} - \gamma_{ij}$ is sufficiently large.

3. If $\lambda_{ij} = \gamma_{ij}$ and $d \geq 3$ then with positive probability both populations will exhibit longterm coexistence.

If $d \leq 2$ then with probability one, eventually one of the populations will die out.

When $\alpha M \neq \alpha' M'$, we would expect once again to see a competition-colonisation tradeoff, but we do not have a precise formulation of a conjecture in this context.

2.4. Heteromyopia. In view of equation (10), it is easy to see that Murrell and Law's heteromyopia might lead to coexistence. They work in a continuous space with the strength of competition between individuals decaying with their distance apart according to a Gaussian kernel. The analogue of their model in our setting is the symmetric version of Model I with $\lambda_{ij} = \lambda(\|i - j\|)$, $\gamma_{ij} = \gamma(\|i - j\|)$, where the functions λ and γ are monotone decreasing and $\sum_j \lambda_{ij} = \sum_j \gamma_{ij}$, but the range of λ_{ij} is greater than that of γ_{ij} . We can think of the effect of this as follows. Over small scales we have homozygous advantage, over larger scales heterozygous advantage. Again we expect to see the population forming homogenic clusters, but now the cluster boundaries will be maintained because the heterogeneity there confers an advantage to individuals within the clusters which counteracts the disadvantage to the individuals actually on the boundary. Reversing the sign to give populations with 'heterohyperopia' gives the opposite effect. This is not stable as an individual's own offspring, which are necessarily born at the same location as their parent and are of the parental type, will act to destroy clusters. Murrell and Law observe founder control in this case, which means that the outcome of the competition is entirely determined by the initial conditions. We make the following conjecture.

Conjecture 2.5 *Suppose that the parameters of Model I are chosen to be symmetric and are such that in the absence of the other type, each population would survive for all time with positive probability and that the initial condition for each population is non-trivial and translation invariant.*

In dimensions $d \geq 2$, if the populations are heteromyopic, then with positive probability we will see longterm coexistence, whereas in $d = 1$, the populations must be strongly heteromyopic for there to be positive probability of longtime coexistence of types.

2.5. Relation to existing models. The Murrell-Law model

Our conjectures for Model I are entirely in agreement with the numerical results of Murrell and Law (2003). They analyse a stochastic version of a spatial Lotka-Volterra system, very similar to ours. Their model differs from ours in that the populations are distributed in a continuous two-dimensional space, not on a lattice. The evolution is characterised in terms of moment equations. These moment equations were derived from a stochastic individual based model by Dieckmann and Law (2000). Although the assumption of a spatially continuous environment is clearly desirable, the price that they pay is that there are very few analytic tools available for the study of the resulting population models and so they use moment closure, assuming in this case a ‘power-1’ closure. In particular they ignore dynamics of all *spatial* moments beyond order two. In view of the clustering behaviour that is characteristic of populations evolving according to spatial branching models in two dimensions, this method has potential pitfalls. In fact the control of the clumping of the populations that forms an essential part of our proof of Theorem 1.2 also adds considerable credibility to the moment closure technique for these models and hence to the numerical predictions of Murrell and Law.

The Neuhauser-Pacala model

In order to place these results and conjectures in a slightly broader context, let us now describe the model of Neuhauser and Pacala (1999). They too consider an explicitly spatial stochastic version of the Lotka-Volterra model. Their model is lattice based but in contrast to ours allows only a single individual to live at each lattice site. Moreover, there is instant recolonisation so that there will always be exactly one individual at each site in \mathbb{Z}^d . This fixed population size makes it more analogous to Model II than to Model I.

Definition 2.6 (Neuhauser-Pacala model) *The Markov process $\{\eta_i(t), i \in \mathbb{Z}^d\}_{t \geq 0}$ in which $\eta_i(t) \in \{1, 2\}$ and with dynamics*

1. If $\eta_i(t) = 1$, it becomes 2 at rate

$$\frac{\lambda f_2}{\lambda f_2 + f_1} (f_1 + \alpha_{12} f_2),$$

2. If $\eta_i(t) = 2$, it becomes 1 at rate

$$\frac{f_1}{\lambda f_2 + f_1} (f_2 + \alpha_{21} f_1),$$

where

$$f_k(i) = \frac{|\{j : \eta_j(t) = k : j \in \mathcal{N}_i\}|}{|\mathcal{N}_i|}$$

and $\mathcal{N}_i = i + \{j : 0 < \|j\| \leq R\}$ will be said to follow the Neuhauser-Pacala (stochastic spatial Lotka-Volterra) model.

The idea is that an individual of type k will die at a rate determined by the proportion of its neighbours that are conspecific plus some constant multiple of the proportion of heterospecific neighbours. Thus, for example, if in Model I we took λ_{ij} and γ_{ij} to have the same range and to be constant on that range, then a small value of α_{12} would correspond to the ratio γ_{ij}/λ_{ij} being small. The dead individual is immediately replaced by an offspring of one of its neighbours chosen according to a weight that reflects the relative fecundity of the two types. Thus, for example, $\lambda > 1$ would reflect type 2 being more fecund than type 1. In Model I this would be modelled by taking $\alpha' M' > \alpha M$.

Let us recall some results for this model.

Theorem 2.7 (Neuhauser and Pacala (1999), Theorem 1) *Suppose that $\lambda = 1$, $d = 1$ or 2 and $\alpha_{12} = \alpha_{21} = \alpha$.*

1. *When $\alpha = 0$, then, except for the one-dimensional nearest neighbour case, product measure with density $1/2$ is the limiting distribution starting from any nontrivial initial distribution.*
2. *If α is sufficiently small (depending on R), then coexistence is possible except for the one-dimensional nearest neighbour case.*

For Model I, a result entirely analogous to part (2) is a special case of Theorem 1.2. For Model II, the analogue is Theorem 1.4. If we believe Conjecture 2.2 then although in $d = 1$ we require the condition ‘ α sufficiently small’, in $d = 2$ the corresponding result is true for all $\alpha < 1$. This corresponds to Conjecture 1 of Neuhauser and Pacala (1999).

Theorem 2.8 (Neuhauser and Pacala (1999), Corollary 1)

Suppose that $\lambda = 1$. Write $n = |\mathcal{N}|$ for the number of lattice sites in a neighbourhood. Species 1 competitively excludes species 2 if

$$\alpha_{12} < \begin{cases} n\alpha_{21} - n + 1, & \text{for } \alpha_{21} \in \left(1 - \frac{1}{n}, 1\right], \\ \frac{1}{n}\alpha_{21} + 1 - \frac{1}{n}, & \text{for } \alpha_{21} > 1. \end{cases}$$

Species 2 competitively excludes species 1 if

$$\alpha_{12} > \begin{cases} \frac{1}{n}\alpha_{21} + 1 - \frac{1}{n}, & \text{for } \alpha_{21} \in (0, 1], \\ n\alpha_{21} - n + 1, & \text{for } \alpha_{21} > 1. \end{cases}$$

In particular, this result shows that the values of $(\alpha_{12}, \alpha_{21})$ for which both populations persist for all time are contained in the shaded region in Figure

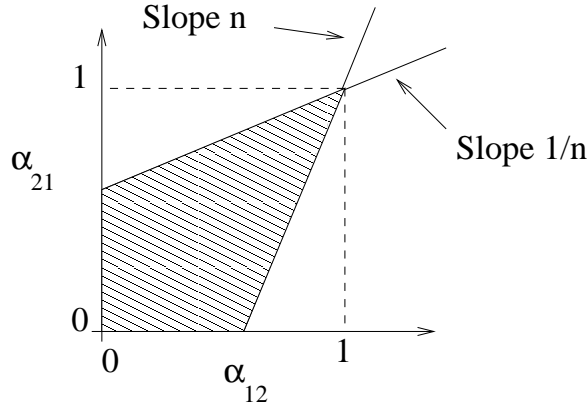


FIG 1. The region of the $(\alpha_{12}, \alpha_{21})$ -plane for which both populations can persist for all time in the Neuhauser-Pacala model is contained in the shaded region.

1. This is a reduction from the range of values predicted by the mean field model. The case $\lambda = 1$ corresponds in our setting to taking $m_{ij} = m'_{ij}$, $\alpha = \alpha'$ and $M = M'$, so in view of Conjecture 2.3, we expect that the coexistence region for Model II in two dimensions corresponds to the whole region $[0, 1) \times [0, 1)$ in the $(\alpha_{12}, \alpha_{21})$ -plane, i.e. the region predicted by the mean field model.

Cox and Perkins (2005) show that a sequence of processes following the Neuhauser-Pacala model, when suitably rescaled in space and time, converges to a superBrownian motion with a non-trivial drift. In low dimensions they restrict to long-range models whereas in dimensions $d \geq 3$ they can also consider the nearest neighbour case. SuperBrownian motion has emerged as a universal limit of critical spatial systems above the critical dimension and these results can be seen as special cases of a general convergence theorem for perturbations of the voter model. In a recent preprint (Cox and Perkins 2006) they show that in dimensions $d \geq 3$ the drift in the superBrownian motion is connected to questions of coexistence in the Neuhauser-Pacala model. Using this connection, they obtain additional information about the parameter regions in which survival of one type resp. coexistence holds. The biologically relevant case $d = 2$ is a topic of their current research.

3. $2N$ -dependent oriented percolation. We now turn to proving our results. Since our proofs will rely upon comparison with $2N$ -dependent oriented percolation we first briefly recall some well known facts which can

be found, for example, in Durrett (1995). The insistence on $2N$ - instead of N -dependent percolation will be explained in the remark below Theorem 3.5.

Oriented percolation will be defined on the lattice

$$\mathcal{L} := \{(x, n) \in \mathbb{Z}^2 : x + n \text{ is even}, n \geq 0\}.$$

This set is made into a graph by inserting edges from (x, n) to $(x + 1, n + 1)$ and to $(x - 1, n + 1)$. It is convenient to think of n as time. We introduce a family of $\{0, 1\}$ -valued random variables $\omega(x, n)$ at sites $(x, n) \in \mathcal{L}$. A site (x, n) is called *open* if $\omega(x, n) = 1$ and *closed* if $\omega(x, n) = 0$. Given such a family of random variables and integers $0 \leq m < n$, we say that $(y, n) \in \mathcal{L}$ can be reached from (x, m) if there is a sequence of points $x = x_m, x_{m+1}, \dots, x_n = y$ so that $|x_k - x_{k-1}| = 1$ and $\omega(x_k, k) = 1$ for $m \leq k \leq n$. We write this as $(x, m) \rightarrow (y, n)$. Finally, given an initial condition $\mathcal{W}_0 \subseteq 2\mathbb{Z} = \{x : (x, 0) \in \mathcal{L}\}$, we may define a percolation process $\{\mathcal{W}_n\}_{n \geq 0}$ by setting, for each $n > 0$, $\mathcal{W}_n = \{y : (x, 0) \rightarrow (y, n) \text{ for some } x \in \mathcal{W}_0\}$.

Definition 3.1 *Let $\theta \in (0, 1)$ and $N \in \mathbb{N}$. We say that an oriented percolation process $\{\mathcal{W}_n\}_{n \geq 0}$, determined by $\{\omega(x, n)\}_{(x, n) \in \mathcal{L}}$, is $2N$ -dependent with density at least $1 - \theta$, if, for any finite set of indices I such that $\|(x_k, n_k) - (x_l, n_l)\| > 2N$ for all $k \neq l \in I$, we have*

$$\mathbb{P}[\omega(x_k, n_k) = 0, k \in I] \leq \theta^{|I|}.$$

Define $\mathcal{C}_0 = \{(y, n) \in \mathcal{L} : (0, 0) \rightarrow (y, n)\}$ as the open cluster containing the origin. We say that *percolation occurs*, if $|\mathcal{C}_0| = \infty$. We first cite a result which gives us a lower bound for the probability of percolation depending on θ and N . A proof can be found in Durrett (1995).

Theorem 3.2 *If $\theta \leq 6^{-4(4N+1)^2}$, then*

$$\mathbb{P}[|\mathcal{C}_0| < \infty] \leq 55\theta^{1/(4N+1)^2} \leq \frac{1}{20}.$$

For particle system models of evolving populations, a standard strategy for showing survival of the population for all times with positive probability is to construct a suitable coupling with oriented percolation. Our approach amounts to a discretisation of our process and a modification of this strategy to cope with the interactions between the two populations and so we now describe the relevant comparison theorems. Once again we are citing Durrett (1995), but we also present a slightly modified version of the results which are more adequate for our purposes.

We first consider the classical setting. Adopting the slightly unconventional terminology of Durrett (1995), let $\{\xi_i(n), i \in \mathbb{Z}^d\}_{n \geq 0}$ denote a translation invariant time-homogeneous finite range flip process (spin system) with state space $\Omega = \{0, 1\}^{\mathbb{Z}^d}$, constructed from the usual graphical representation. Let $L \in \mathbb{N}$ be fixed. We write

$$H = \left\{ \xi \in \{0, 1\}^{\mathbb{Z}^d} : \xi_i = 1 \quad \forall i \in [-L/2, L/2]^d \cap \mathbb{Z}^d \right\}, \quad (12)$$

and $m \cdot H$ for the translation of H by some integer m with respect to the first component, i.e.

$$m \cdot H = \left\{ \xi \in \{0, 1\}^{\mathbb{Z}^d} : \xi_i = 1 \quad \forall i \in mL\mathbf{e}_1 + [-L/2, L/2]^d \cap \mathbb{Z}^d \right\}, \quad (13)$$

where \mathbf{e}_1 is the unit vector in the direction of the first component.

Definition 3.3 *Fix $N \in \mathbb{N}$ and $\theta \in (0, 1)$. We shall say that the process $\{\xi(n)\}_{n \geq 0}$ fulfils the classical comparison assumptions for N and θ , if for each configuration $\xi \in H$ there exists a “good event” G_ξ which is measurable with respect to the graphical representation of the flip process inside $[-NL, NL]^d \times [0, 1]$ and with $\mathbb{P}[G_\xi] > 1 - \theta$, so that if $\xi(0) = \xi$, then on G_ξ ,*

$$\xi(1) \in (+1) \cdot H \cap (-1) \cdot H.$$

This framework turns out to be slightly too narrow for our purposes, as the flip processes we are going to consider in the next section are functionals of more general underlying stochastic processes driven by independent Brownian motions and hence cannot be obtained from a graphical representation. However, a similar comparison result holds, if the flip process fulfils our *modified* comparison assumptions based on the behaviour of the Brownian increments.

More explicitly, we are going to construct our flip process in terms of the system of stochastic differential equations X, Y from Model I. Define $\{\bar{\xi}_i(n), i \in \mathbb{Z}^d\}_{n \in \mathbb{N}}$ by

$$\bar{\xi}_i(n) = \begin{cases} 1, & \text{if } X_i(2n) > c_1 \text{ and } Y_j(2n) < c_2 \quad \forall j \in i + [-bL, bL]^d \cap \mathbb{Z}^d, \\ 0, & \text{otherwise,} \end{cases} \quad (14)$$

where c_1, c_2 are positive finite numbers and b and L satisfy the assumption of Theorem 1.2. Note the different time scales for $\bar{\xi}$ and X, Y (the usefulness of this time-change will become clear in the next section), and observe that X, Y are time-homogeneous and the underlying system of driving Brownian motions is translation invariant.

Definition 3.4 *Assume $N := (b + 2)$ and $\theta \in (0, 1)$. Define the events H and $m \cdot H$ for some integer $m \in \mathbb{Z}$ in terms of the process $\{\bar{\xi}(n)\}_{n \geq 0}$ in the same way as in (12) and (13). Consider the σ -algebra*

$$\mathcal{F}^*(NL, [0, 2]) := \sigma \left\{ B_i(s), B'_j(t) : 0 \leq s, t \leq 2; i, j \in [-NL, NL]^d \cap \mathbb{Z}^d \right\}, \tag{15}$$

where $\{\{B_i(s)\}_{s \geq 0}, \{B'_j(t)\}_{t \geq 0}, i, j \in \mathbb{Z}^d\}$ is the family of independent standard Brownian motions as in Model I, see (3) and (4).

We shall say that the process $\{\bar{\xi}(n)\}_{n \geq 0}$ fulfils the modified comparison assumptions for NL and θ , if for each configuration $\bar{\xi} \in H$, there exists a “good event”

$$G_{\bar{\xi}} \in \mathcal{F}^*(NL, [0, 2]),$$

with $\mathbb{P}[G_{\bar{\xi}}] > 1 - \theta$, such that if $\bar{\xi}(0) = \bar{\xi}$, then on $G_{\bar{\xi}}$,

$$\bar{\xi}(1) \in (+1) \cdot H \cap (-1) \cdot H. \tag{16}$$

In other words, if $\bar{\xi}$ has all 1’s in the box of side length L about the origin at time 0, then at time 1 (measured in time units for the $\bar{\xi}$ process), with probability at least $1 - \theta$, it has successfully ‘invaded’ the boxes of side length L translated by $-Le_1$ and Le_1 in a way that is measurable with respect to the Brownian increments inside the box around the origin of side $2NL$ and up to time $[0, 2]$ (measured in time units for X, Y).

The following classical theorem, which can, e.g., be found in Durrett (1995), applies to both of the above settings.

Theorem 3.5 *If the classical resp. modified comparison assumptions hold for ξ resp. $\bar{\xi}$ for some $N \in \mathbb{N}$ and $\theta \in (0, 1)$, we may define random variables $\omega(x, n)$, such that $\mathcal{X}_n := \{(m, n) \in \mathcal{L} : \xi(n) \in m \cdot H\}$ resp. $\bar{\mathcal{X}}_n := \{(m, n) \in \mathcal{L} : \bar{\xi}(n) \in m \cdot H\}$ dominates a $2N$ -dependent oriented percolation process $\{\mathcal{W}_n\}$, defined on \mathcal{L} , with initial configuration $\mathcal{W}_0 = \mathcal{X}_0$ resp. $\bar{\mathcal{X}}_0$, and density parameter at least $1 - \theta$, i.e.*

$$\mathcal{W}_n \subseteq \mathcal{X}_n \text{ resp. } \bar{\mathcal{X}}_n \quad \text{for all } n \in \mathbb{N},$$

where $\mathcal{W}_n = \{y : (x, 0) \rightarrow (y, n) \text{ for some } x \in \mathcal{W}_0\}$.

Remarks.

1. Notice that irrespective of the dimension d of the lattice \mathbb{Z}^d on which our flip process lives, we are establishing a comparison with a 1 + 1-dimensional oriented percolation.

2. Why ‘ $2N$ -dependent’? For $(m, n) \in \mathcal{L}$ define (in time units measured for the $\bar{\xi}$ process),

$$\mathcal{R}_{m,n} = (mL\mathbf{e}_1, n) + \{[-NL, NL]^d \times [0, 1]\}.$$

If $\bar{\xi}_n \in m \cdot H$, then, under the modified comparison assumptions for $\{\bar{\xi}_n\}_{n \geq 0}$, the probability of invasion of $(m+1) \cdot H \cap (m-1) \cdot H$ by time $n+1$ will be bounded below by $1 - \theta$, irrespective of the behaviour outside $\mathcal{R}_{m,n}$ and before time $n+1$. If we take points (m, n) and (m', n') in \mathcal{L} with $\|(m, n) - (m', n')\| > 2N$, then the space-time regions $\mathcal{R}_{m,n}$ and $\mathcal{R}_{m',n'}$ are disjoint. \square

The following result is Theorem 4.2 in Durrett (1995).

Theorem 3.6 *Suppose that $\{\mathcal{W}_n\}_{n \geq 0}$ is a $2N$ -dependent oriented percolation process, started from the trivial initial state $\mathcal{W}_0(x) = 1$ for all x . If $\theta \leq 6^{-4(4N+1)^2}$, then*

$$\liminf_{n \rightarrow \infty} \mathbb{P}[0 \in \mathcal{W}_{2n}] \geq \frac{19}{20}.$$

Eventually, this theorem will be the key in proving the coexistence (with positive probability) of (X, Y) in Model I and Model II.

4. Proofs. We are now in a position to prove our main results. The key to the proof of Theorem 1.2 is to consider the Y -population as providing a random environment in which the X -population evolves. Of course the environment itself depends on the X -population, but we obtain some control of the behaviour of the environment that is *independent* of the evolution of the X -population. This ‘decoupling’ (and a symmetric argument for Y) then reduces the coexistence problem to that of survival of a single population: if the X and Y populations can each be shown to survive for all times with probability greater than one half, then longterm coexistence (with positive probability) will follow. We attack the question of survival of the X -population (resp. Y -population) by comparison with an oriented $2N$ -dependent percolation process using the results of the last section. To this end, we establish the existence of the corresponding ‘good events’ as required in Definition 3.4.

We prepare the necessary notation and technical estimates for this in the following subsection. Our results will rely heavily on comparisons to finite lattices of certain one-dimensional diffusions.

4.1. *A spin system and estimation of related flip probabilities.* The main step is to construct two spin systems, one for each of the X and Y

populations, that play the role of $\{\bar{\xi}_i(n), i \in \mathbb{Z}^d\}_{n \geq 0}$ of the last section for some suitable constants c_1, c_2 .

Indeed, we consider the spin system $\{\zeta_i(n), \eta_i(n), i \in \mathbb{Z}^d\}_{n \geq 0}$, where the process $\{\zeta_i(n), i \in \mathbb{Z}^d\}_{n \geq 0}$ is defined by

$$\zeta_i(n) = \begin{cases} 1, & \text{if } X_i(2n) > \frac{M}{K} \text{ and } Y_j(2n) < a'M' \quad \forall j \in i + [-bL, bL]^d \cap \mathbb{Z}^d, \\ 0, & \text{otherwise,} \end{cases} \quad (17)$$

and symmetrically,

$$\eta_i(n) = \begin{cases} 1, & \text{if } Y_i(2n) > \frac{M'}{K'} \text{ and } X_j(2n) < aM \quad \forall j \in i + [-bL, bL]^d \cap \mathbb{Z}^d, \\ 0, & \text{otherwise,} \end{cases} \quad (18)$$

where $K := 2\alpha Mc + 1$, $K' := 2\alpha' M'c + 1$ and a, a' are positive finite constants to be determined later (see (59) in the proof of Lemma 4.7 resp. the symmetric result for a). Recall that L denotes the range of the intra-species interaction and b denotes the smallest positive integer such that the range of $\{\gamma_{ij}\}$ resp. $\{\gamma'_{ij}\}$ is less than $(b-1) \cdot L$.

With these definitions, one expects that if the system $\{\zeta(n), \eta(n)\}_{n \geq 0}$ exhibits longterm coexistence in discrete time, then the system $\{X(t), Y(t)\}_{t \geq 0}$ exhibits coexistence in continuous time, and in fact this will follow from our proof.

The convenience of the time change $n \mapsto 2n$ in (17) and (18) will become clear when carrying out the comparison arguments in Section 4.2.1.

Outline of this subsection. In the rather technical Paragraph 4.1.1 we introduce the notation necessary to define special events that are based only on the behaviour of certain one-dimensional diffusions on finite lattices. This will later help to construct the suitably measurable “good events” G_ζ, G_η . Moreover, we prove two technical lemmas which provide estimates on the growth behaviour of these one-dimensional diffusions. These will eventually ensure that the good events can be made sufficiently likely.

Note that the technical results in this paragraph are obtained in an entirely standard way. The next two paragraphs contain the key arguments in the proof of the main coexistence results.

Indeed, Paragraph 4.1.2 provides “flip probabilities” related to the spin system ζ via comparisons in terms of the above-mentioned behaviour of the one-dimensional diffusions, under the additional condition that the system X evolves in a “safe environment”, i.e. given some bounds on the local Y -population.

Finally, in Paragraph 4.1.3, we will find conditions so that the “safe environment” assumption holds for the Y -population in a way that is independent of the evolution of the X -population, again by making use of comparisons to one-dimensional diffusions.

4.1.1. *Some notation and technical results for lattices of one-dimensional diffusions.*

Definition 4.1 *Let $\{\{B_i(s)\}_{s \geq 0}, \{B'_j(t)\}_{t \geq 0}, i, j \in \mathbb{Z}^d\}$ be the family of independent standard Brownian motions driving Model I. Fix $i \in \mathbb{Z}^d$. For $n \in \mathbb{N}, u > 0$ define the σ -algebras generated by the local Brownian increments*

$$\mathcal{F}(i, n, u) := \sigma\{B_i(n+s) - B_i(n) : s \in [0, u]\},$$

and

$$\mathcal{F}'(i, n, u) := \sigma\{B'_i(n+s) - B'_i(n) : s \in [0, u]\}.$$

Moreover, let

$$\mathcal{F}(i, NL, n, u) := \sigma\{B_j(n+s) - B_j(n) : s \in [0, u], j \in i + [-NL, NL]^d \cap \mathbb{Z}^d\},$$

and, similarly,

$$\mathcal{F}'(i, NL, n, u) := \sigma\{B'_j(n+s) - B'_j(n) : s \in [0, u], j \in i + [-NL, NL]^d \cap \mathbb{Z}^d\}.$$

These σ -algebras will be used to construct a suitably measurable events $G_{\bar{c}}, G_{\bar{\eta}}$. Recall from Definition 3.4, that

$$\mathcal{F}^*(NL, [0, 2]) = \mathcal{F}(0, NL, 0, 2) \vee \mathcal{F}'(0, NL, 0, 2).$$

The next objects of interest are certain finite systems of one-dimensional diffusions, which, via suitable comparisons, eventually lead to the estimates required for the comparison theorem.

Definition 4.2 *Let $i \in \mathbb{Z}^d$ and assume that the constants $\alpha', \bar{M}, \lambda, \bar{U} > 0$ are chosen such that $\lambda \bar{U} > 2\bar{M}$. Moreover, let $D_1, D_2 > 0$. Then, for each $j \in i + [-NL, NL]^d \cap \mathbb{Z}^d$, define the one-dimensional diffusions $\{Z_j(t)\}_{t \geq 0}, \{\bar{Z}_j(t)\}_{t \geq 0}, \{\hat{Z}_j(t)\}_{t \geq 0}$ and $\{\tilde{Z}_j(t)\}_{t \geq 0}$, driven by independent standard Brownian motions $\{W_j(t)\}_{t \geq 0}$, by*

$$dZ_j(t) = \alpha' (\bar{M} - \lambda Z_j(t)) Z_j(t) dt + \sqrt{Z_j(t)} dW_j(t) \quad (19)$$

(logistic Feller diffusion),

$$d\tilde{Z}_j(t) = D_1 \tilde{Z}_j(t) dt + \sqrt{\tilde{Z}_j(t)} dW_j(t) \quad (20)$$

(supercritical Feller diffusion),

$$d\hat{Z}_j(t) = D_2 dt + D_1 \hat{Z}_j(t) dt + \sqrt{\hat{Z}_j(t)} dW_j(t) \quad (21)$$

(supercritical Feller diffusion with constant positive immigration),

$$d\bar{Z}_j(t) = \alpha' (\bar{M} - \lambda \bar{U}) \bar{Z}_j(t) dt + \sqrt{\bar{Z}_j(t)} dW_j(t) \quad (22)$$

(subcritical Feller diffusion).

Since each of the four diffusions admits a (continuous) unique strong solution, we may assume them to be driven by some given family of independent Brownian motions, in particular those obtained either from (3) or from (4).

The next two technical lemmas collect some useful properties of the diffusions $\{Z(t)\}_{t \geq 0}$, $\{\tilde{Z}(t)\}_{t \geq 0}$ and $\{\bar{Z}(t)\}_{t \geq 0}$, derived by entirely standard techniques.

Lemma 4.3 *Let $\alpha', \lambda, \bar{M} > 0$. Consider the logistic Feller diffusion*

$$dZ(t) = \alpha' (\bar{M} - \lambda Z(t)) Z(t) dt + \sqrt{Z(t)} dW(t), \quad (23)$$

where $\{W(t)\}_{t \geq 0}$ is a Brownian motion. Let, for $w > 0$ and $Z(0) \geq w$,

$$\tau_w^Z := \inf \{t > 0 : Z(t) \leq w\}.$$

Then, provided that $w \geq 2\bar{M}/\lambda$ large enough, we have

$$\mathbb{P} \left[\tau_w^Z > \frac{1}{2} \mid Z(0) \in [w, \infty) \right] < \frac{2}{\alpha' \lambda (w - \bar{M}/\lambda)}. \quad (24)$$

Remarks.

1. The key fact in the lemma is that given w , the bound that we obtain is *uniform* in the value of $Z(0) \in [w, \infty)$. This is not surprising, given that the downward drift for large values of $Z(t)$ is quadratic.
2. Observe that by choosing w large we can make the right hand side of equation (24) arbitrarily small. \square

Proof. We estimate the expected value of τ_w^Z and apply Markov's inequality. First, we determine the *scale function*, $s(x)$, for the diffusion $Z(t)$.

$$\begin{aligned} s(x) &= \int_{\beta}^x \exp \left(-2 \int_{\beta}^z \alpha' (\bar{M} - \lambda \xi) d\xi \right) dz \\ &= \int_{\beta}^x \Gamma(\beta) \exp \left(-2\alpha' \bar{M} z + \alpha' \lambda z^2 \right) dz \end{aligned}$$

for some constants $\beta, \Gamma(\beta) \in [0, \infty)$. The *speed measure* is given by

$$m(dy) = \frac{2}{ys'(y)} dy$$

The expected value of the exit time from the interval (w, A) , denoted by $\tau_{(w,A)}^Z$, when being started in $x \in (w, A)$, is then

$$\begin{aligned} \mathbb{E}[\tau_{(w,A)}^Z] &= \frac{s(x) - s(w)}{s(A) - s(w)} \int_x^A \frac{s(A) - s(y)}{ys'(y)} dy \\ &\quad + \frac{s(A) - s(x)}{s(A) - s(w)} \int_w^x \frac{s(y) - s(w)}{ys'(y)} dy, \end{aligned}$$

see, for example, Revuz and Yor (1999) Chapter VII, Corollary 3.8. Substituting for $s(x)$ we have

$$\begin{aligned} \mathbb{E}[\tau_{(w,A)}^Z] &= \frac{\int_w^x \exp(-2\alpha' \bar{M}z + \alpha' \lambda z^2) dz}{\int_w^A \exp(-2\alpha' \bar{M}z + \alpha' \lambda z^2) dz} \int_x^A \frac{\int_y^A \exp(-2\alpha' \bar{M}z + \alpha' \lambda z^2) dz}{y \exp(-2\alpha' \bar{M}y + \alpha' \lambda y^2)} dy \\ &\quad + \frac{\int_x^A \exp(-2\alpha' \bar{M}z + \alpha' \lambda z^2) dz}{\int_w^A \exp(-2\alpha' \bar{M}z + \alpha' \lambda z^2) dz} \int_w^x \frac{\int_w^y \exp(-2\alpha' \bar{M}z + \alpha' \lambda z^2) dz}{y \exp(-2\alpha' \bar{M}y + \alpha' \lambda y^2)} dy \\ &\leq \int_w^x \exp(-2\alpha' \bar{M}z + \alpha' \lambda z^2) dz \int_x^A \frac{1}{y} \exp(2\alpha' \bar{M}y - \alpha' \lambda y^2) dy \\ &\quad + \int_w^x \frac{\int_w^y \exp(-2\alpha' \bar{M}z + \alpha' \lambda z^2) dz}{y \exp(-2\alpha' \bar{M}y + \alpha' \lambda y^2)} dy \\ &= \int_w^x \exp(\alpha' \lambda (z - \bar{M}/\lambda)^2) dz \int_x^A \frac{1}{y} \exp(-\alpha' \lambda (y - \bar{M}/\lambda)^2) dy \\ &\quad + \int_w^x \frac{\int_w^y \exp(\alpha' \lambda (z - \bar{M}/\lambda)^2) dz}{y \exp((\alpha' \lambda (y - \bar{M}/\lambda)^2)} dy \end{aligned}$$

Integrating by parts we have that

$$\begin{aligned} &\int_w^x \exp(\alpha' \lambda (z - \bar{M}/\lambda)^2) dz \\ &\leq \left[\frac{1}{2\alpha' \lambda (z - \bar{M}/\lambda)} \exp(\alpha' \lambda (z - \bar{M}/\lambda)^2) \right]_w^x \\ &\quad + \int_w^x \frac{1}{2\alpha' \lambda (w - \bar{M}/\lambda)^2} \exp(\alpha' \lambda (z - \bar{M}/\lambda)^2) dz, \end{aligned}$$

from which it follows that

$$\begin{aligned} & \int_w^x \exp(\alpha' \lambda (z - \bar{M}/\lambda)^2) dz \\ & \leq \frac{1}{1 - \frac{1}{2\alpha' \lambda (w - \bar{M}/\lambda)^2}} \frac{1}{2\alpha' \lambda (w - \bar{M}/\lambda)} \exp(\alpha' \lambda (x - \bar{M}/\lambda)^2). \end{aligned} \quad (25)$$

We may treat the second factor in a similar way:

$$\begin{aligned} & \int_x^A \frac{1}{y} \exp(-\alpha' \lambda (y - \bar{M}/\lambda)^2) dy \\ & = \left[-\frac{1}{2\alpha' \lambda y (y - \bar{M}/\lambda)} \exp(-\alpha' \lambda (y - \bar{M}/\lambda)^2) \right]_x^A \\ & \quad - \int_x^A \left(\frac{1}{2\alpha' \lambda (y - \bar{M}/\lambda)^2 y} + \frac{1}{2\alpha' \lambda (y - \bar{M}/\lambda) y^2} \right) \exp(-\alpha' \lambda (y - \bar{M}/\lambda)^2) dy, \end{aligned}$$

from which it follows that, since $w < x$,

$$\begin{aligned} & \int_x^A \frac{1}{y} \exp(-\alpha' \lambda (y - \bar{M}/\lambda)^2) dy \\ & \leq \frac{1}{1 + \left(\frac{1}{2\alpha' \lambda (w - \bar{M}/\lambda)(w)^2} + \frac{1}{2\alpha' \lambda (w - \bar{M}/\lambda)^2 w} \right)} \\ & \quad \times \frac{1}{2\alpha' \lambda (w - \bar{M}/\lambda)} \exp(-\alpha' \lambda (x - \bar{M}/\lambda)^2). \end{aligned}$$

The final integral can also be approximated via an integration by parts. First observe that from equation (25)

$$\begin{aligned} & \int_w^y \exp(\alpha' \lambda (z - \bar{M}/\lambda)^2) dz \\ & \leq \frac{1}{1 - \frac{1}{2\alpha' \lambda (w - \bar{M}/\lambda)^2}} \frac{1}{2\alpha' \lambda (y - \bar{M}/\lambda)} \exp(\alpha' \lambda (y - \bar{M}/\lambda)^2), \end{aligned}$$

and substitute to obtain

$$\begin{aligned} & \int_w^x \frac{\int_w^y \exp(\alpha' \lambda (z - \bar{M}/\lambda)^2) dz}{y \exp((\alpha' \lambda (y - \bar{M}/\lambda)^2)} dy \\ & \leq \int_w^x \frac{1}{1 - \frac{1}{2\alpha' \lambda (w - \bar{M}/\lambda)^2}} \frac{1}{2\alpha' \lambda (y - \bar{M}/\lambda)} \frac{1}{y} dy \\ & \leq \frac{1}{1 - \frac{1}{2\alpha' \lambda (w - \bar{M}/\lambda)^2}} \frac{1}{2\alpha' \lambda} \int_{w - \bar{M}/\lambda}^{\infty} \frac{1}{y^2} dy \\ & = \frac{1}{1 - \frac{1}{2\alpha' \lambda (w - \bar{M}/\lambda)^2}} \frac{1}{2\alpha' \lambda} \frac{1}{w - \bar{M}/\lambda} \end{aligned}$$

Combining the estimates above, we obtain

$$\begin{aligned} \mathbb{E}[\tau_{(w,A)}^Z] &\leq \frac{1}{1+K(w)} \frac{1}{1-2\alpha'\lambda L^2(w)} L^2(w) + \frac{L(w)}{1-2\alpha'\lambda L^2(w)} \\ &= \frac{1}{2\alpha'\lambda(w-\bar{M}/\lambda)} \left[\frac{1}{1+K(w)} \frac{L(w)}{1-2\alpha'\lambda L^2(w)} + \frac{1}{1-2\alpha'\lambda L^2(w)} \right], \end{aligned} \quad (26)$$

where

$$\begin{aligned} L(w) &= \frac{1}{2\alpha'\lambda(w-\bar{M}/\lambda)} \rightarrow 0, \\ K(w) &= \frac{1}{2\alpha'\lambda(w-\bar{M}/\lambda)w^2} + \frac{1}{2\alpha'\lambda(w-\bar{M}/\lambda)^2w} \rightarrow 0, \end{aligned}$$

as $w \rightarrow \infty$. The result now follows. \square

The second technical lemma is in the same spirit, but simpler.

Lemma 4.4 *Let $\alpha', \lambda, \bar{M}, D_1 > 0$ and assume $\bar{U} \geq 2\bar{M}/\lambda$. Consider the subcritical Feller diffusion*

$$d\bar{Z}(t) = \alpha'(\bar{M} - \lambda\bar{U})\bar{Z}(t)dt + \sqrt{\bar{Z}(t)}dW(t), \quad (27)$$

and the supercritical Feller branching diffusion

$$d\tilde{Z}(t) = D_1\tilde{Z}(t)dt + \sqrt{\tilde{Z}(t)}dW(t), \quad (28)$$

where $\{W(t)\}_{t \geq 0}$ denotes a standard Brownian motion. Suppose that

$$\bar{Z}(0) = \frac{3}{2}\bar{U}, \quad \tilde{Z}(0) = \frac{3}{2}\tilde{U}.$$

Define the first exit time of \bar{Z} from the interval $(\bar{U}, 2\bar{U})$ by

$$\tau_{(\bar{U}, 2\bar{U})}^{\bar{Z}} := \inf \left\{ t > 0 : \bar{Z}(t) \leq \bar{U} \text{ or } \bar{Z}(t) \geq 2\bar{U} \right\},$$

and denote by $\tau_{(\tilde{U}, 2\tilde{U})}^{\tilde{Z}}$ the corresponding exit time for \tilde{Z} from the interval $(\tilde{U}, 2\tilde{U})$. Let $\tau_{\bar{U}}^{\bar{Z}}$ be the first hitting time of level \bar{U} ,

$$\tau_{\bar{U}}^{\bar{Z}} := \inf \left\{ t > 0 : \bar{Z}(t) = \bar{U} \right\},$$

and similarly define $\tau_{\bar{U}}^{\bar{Z}}, \tau_{2\bar{U}}^{\bar{Z}}$ and $\tau_{2\bar{U}}^{\bar{Z}}$. Then, for the probability that \bar{Z} exits from the interval $(\bar{U}, 2\bar{U})$ through the upper bound, we have

$$\mathbb{P}\left[\tau_{2\bar{U}}^{\bar{Z}} < \tau_{\bar{U}}^{\bar{Z}}\right] \leq \exp(-\bar{\alpha}'M\bar{U}), \quad (29)$$

and similarly, that \tilde{Z} exits from the interval $(\tilde{U}, 2\tilde{U})$ through the lower bound

$$\mathbb{P}\left[\tau_{\tilde{U}}^{\tilde{Z}} < \tau_{2\tilde{U}}^{\tilde{Z}}\right] \leq \exp(-D_1\tilde{U}), \quad (30)$$

for all D_1 sufficiently large. Moreover, for the expected time to leave the interval $(\bar{U}, 2\bar{U})$ resp. $(\tilde{U}, 2\tilde{U})$, we have the bounds

$$\mathbb{E}[\tau_{(\bar{U}, 2\bar{U})}^{\bar{Z}}] > \frac{1}{24\alpha'\lambda\bar{U}}, \quad \mathbb{E}[\tau_{(\tilde{U}, 2\tilde{U})}^{\tilde{Z}}] > \frac{1}{32D_1}, \quad (31)$$

for all D_1 sufficiently large.

Proof. We first derive the results for the process $\{\bar{Z}(t)\}_{t \geq 0}$. Let $\bar{c} := -\alpha'(\bar{M} - \lambda\bar{U}) > 0$. Working with the same tools as in the proof of the preceding lemma, first note that the *scale function* s for the diffusion (27) is given by

$$\begin{aligned} s(x) &= \int_{\bar{U}}^x \exp\left(-2 \int_{\bar{U}}^y \frac{-\bar{c}r}{r} dr\right) dy \\ &= \int_{\bar{U}}^x \exp(2\bar{c}y - 2\bar{c}\bar{U}) dy \\ &= \frac{1}{2\bar{c}} [\exp(2\bar{c}(x - \bar{U})) - 1]. \end{aligned}$$

Hence, recalling $\bar{Z}(0) = \frac{3}{2}\bar{U}$ and so setting $x = \frac{3}{2}\bar{U}$, we have

$$\begin{aligned} \mathbb{P}[\tau_{2\bar{U}}^{\bar{Z}} < \tau_{\bar{U}}^{\bar{Z}}] &= \frac{s(\frac{3}{2}\bar{U}) - s(\bar{U})}{s(2\bar{U}) - s(\bar{U})} \\ &= \frac{\exp(3\bar{c}\bar{U}) - \exp(2\bar{c}\bar{U})}{\exp(4\bar{c}\bar{U}) - \exp(2\bar{c}\bar{U})} \\ &= \exp(-\bar{c}\bar{U}) \frac{1 - \exp(-\bar{c}\bar{U})}{1 - \exp(-2\bar{c}\bar{U})} \\ &\leq \exp(\alpha'M\bar{U}), \end{aligned}$$

which is bound (29). For the expected exit time from the interval $(\bar{U}, 2\bar{U})$,

we have

$$\begin{aligned}
\mathbb{E}[\tau_{(\bar{U}, 2\bar{U})}] &\geq \frac{s(2\bar{U}) - s(\frac{3}{2}\bar{U})}{s(2\bar{U}) - s(\bar{U})} \int_{\bar{U}}^{\frac{3}{2}\bar{U}} \frac{s(y) - s(\bar{U})}{ys'(y)} dy \\
&= \frac{\exp(4\bar{c}\bar{U}) - \exp(3\bar{c}\bar{U})}{\exp(4\bar{c}\bar{U}) - \exp(2\bar{c}\bar{U})} \int_{\bar{U}}^{\frac{3}{2}\bar{U}} \frac{1}{2\bar{c}y} (1 - \exp(2\bar{c}\bar{U} - 2\bar{c}y)) dy \\
&\geq \frac{1 - \exp(-\bar{c}\bar{U})}{1 - \exp(-2\bar{c}\bar{U})} \frac{1}{3\bar{c}\bar{U}} \int_{\frac{3}{4}\bar{U}}^{\frac{3}{2}\bar{U}} (1 - \exp(-\bar{c}\bar{U}/2)) dy \\
&\geq \frac{1}{12\bar{c}} \left[\frac{1 - \exp(-\bar{c}\bar{U})}{1 - \exp(-2\bar{c}\bar{U})} (1 - \exp(-\bar{c}\bar{U}/2)) \right] \\
&\geq \frac{1}{12\alpha'\lambda\bar{U}} \left[\frac{1 - \exp(-\bar{c}\bar{U})}{1 - \exp(-2\bar{c}\bar{U})} (1 - \exp(-\bar{c}\bar{U}/2)) \right],
\end{aligned}$$

by the definition of \bar{c} and the fact that $\bar{U} \geq 2\bar{M}/\lambda$. The result follows observing that the expression inside the brackets converges to 1.

For $\{\tilde{Z}(t)\}_{t \geq 0}$, note that the scale function \tilde{s} is given by

$$\begin{aligned}
\tilde{s}(x) &= \int_{\tilde{U}}^x \exp\left(-2 \int_{\tilde{U}}^y D_1 dr\right) dy \\
&= \int_{\tilde{U}}^x \exp(2D_1(\tilde{U} - y)) dy \\
&= \frac{1}{-2D_1} [\exp(2D_1(\tilde{U} - x)) - 1].
\end{aligned}$$

Hence, recalling $\tilde{Z}(0) = \frac{3}{2}\tilde{U}$ and so setting $x = \frac{3}{2}\tilde{U}$, we have

$$\begin{aligned}
\mathbb{P}[\tau_{\tilde{U}} < \tau_{2\tilde{U}}] &= \frac{\tilde{s}(2\tilde{U}) - \tilde{s}(\frac{3}{2}\tilde{U})}{\tilde{s}(2\tilde{U}) - \tilde{s}(\tilde{U})} \\
&= \frac{\exp(-2D_1\tilde{U}) - \exp(-D_1\tilde{U})}{\exp(-2D_1\tilde{U}) - 1} \\
&\leq \exp(-D_1\tilde{U}).
\end{aligned} \tag{32}$$

This is (30). For the expected exit time from the interval $(\tilde{U}, 2\tilde{U})$, we have

$$\begin{aligned} \mathbb{E}[\tau_{(\tilde{U}, 2\tilde{U})}^{\tilde{Z}}] &\geq \frac{\tilde{s}((3/2)\tilde{U}) - \tilde{s}(\tilde{U})}{\tilde{s}(2\tilde{U}) - \tilde{s}(\tilde{U})} \int_{(3/2)\tilde{U}}^{2\tilde{U}} \frac{\tilde{s}(2\tilde{U}) - \tilde{s}(y)}{y\tilde{s}'(y)} dy \\ &= \frac{\tilde{s}((3/2)\tilde{U}) - \tilde{s}(\tilde{U})}{\tilde{s}(2\tilde{U}) - \tilde{s}(\tilde{U})} \int_{(3/2)\tilde{U}}^{2\tilde{U}} \frac{\exp(-2D_1\tilde{U}) - \exp(2D_1(\tilde{U} - y))}{-2D_1y \exp(2D_1(\tilde{U} - y))} dy \\ &\geq \frac{\tilde{s}((3/2)\tilde{U}) - \tilde{s}(\tilde{U})}{\tilde{s}(2\tilde{U}) - \tilde{s}(\tilde{U})} \int_{(3/2)\tilde{U}}^{(7/4)\tilde{U}} \frac{1}{-4D_1\tilde{U}} (\exp(-2D_1(2\tilde{U} - y)) - 1) dy \\ &\geq \frac{1}{16D_1} \frac{\tilde{s}((3/2)\tilde{U}) - \tilde{s}(\tilde{U})}{\tilde{s}(2\tilde{U}) - \tilde{s}(\tilde{U})} (1 - \exp(-D_1\tilde{U}/2)). \end{aligned}$$

Since the second term (by 32) and the last term on the right-hand side converge to 1 as $D_1 \rightarrow \infty$, the result follows. \square

4.1.2. *Infection and recovery probabilities for the X-population.* Suppose that we are interested in the behaviour of the X-population within the time interval $[n, n+1]$ at site i and that we already know (recall (17)) that

$$\max_{t \in [n, n+1]} Y_j(t) < 2a'M' \quad \forall j \in i + [-bL, bL]^d \cap \mathbb{Z}^d. \quad (33)$$

Assume that the inter-species competition $\{\gamma_{ij}\}$ is chosen in such a way that (33) implies

$$\max_{t \in [n, n+1]} \sum_{l \in \mathbb{Z}^d} \gamma_{jl} Y_l(t) < 1 \quad \forall j \in i + [-L/2, L/2]^d \cap \mathbb{Z}^d. \quad (34)$$

This is possible since the range of $\{\gamma_{jl}\}$ is by assumption less than $(b-1)L$ (choose, e.g. $\sum_j \gamma_{ij} < (2a'M')^{-1}$). We will later (in Paragraph 4.1.3) construct events that are measurable with respect to either $\mathcal{F}'(i, NL, n, 2)$ or $\mathcal{F}'(i, NL, n-1, 2)$, which imply (33) and are of sufficiently high probability, cf. (54) and (55). For the moment, to aid intuition and to simplify notation, we will say that, in either case, a suitably measurable “safe environment condition” $G'_{sec}(i, n)$ holds at site i and time n , which implies that (33) (and for γ_{ij} small enough also (34)) holds, and which will be explicitly determined later.

We now consider ‘flip probabilities’ for the X-population that are closely linked to the flip probabilities of the ζ -population, introduced in (17) and (18), under the “safe environment condition” $G'_{sec}(i, n)$ at site i and at time $n \in \mathbb{N}$.

Lemma 4.5 (Infection and non-recovery) *Let $n \in \mathbb{N}$ and $i \in \mathbb{Z}^d$. Let $\alpha, \{m_{ij}\}, \{\lambda_{ij}\}$ be fixed. Given the parameters for the Y -population and some $a' > 0$, choose $\{\gamma_{ij}\}$ such that (33) implies (34). Then, for any $\varepsilon \in (0, 1)$, there exists a finite constant $M_0 > 0$, such that if $M > M_0$ and $K := 2\alpha M c + 1$, for each $j \in i + [-L/2, L/2]^d \cap \mathbb{Z}^d$, there exist events*

$$G_{non-rec}(i, n) \in \mathcal{F}(i, n, 1) \quad \text{and} \quad G_{infec}(i, j, n) \in \mathcal{F}(i, L/2, n, 1), \quad (35)$$

both measurable w.r.t. $\mathcal{F}(i, NL, n, 1)$, such that the following holds:

i) We have

$$\left\{ \{X_i(n) > M/K\} \cap G'_{sec}(i, n) \cap G_{non-rec}(i, n) \right\} \subset \{X_i(n+1) > M/K\}$$

(“non-recovery”). For the ‘non-recovery probability’ $p_{non-rec}(i, n)$, we have the bound

$$p_{non-rec}(i, n) := \mathbb{P}[G_{non-rec}(i, n)] > 1 - \varepsilon. \quad (36)$$

ii) Moreover,

$$\begin{aligned} \left\{ \{X_i(n) \leq M/K\} \cap \{\exists j : m_{ij} > 0, X_j(n) > M/K\} \cap G'_{sec}(i, n) \right\} \\ \subset \{X_i(n+1) > M/K\} \end{aligned}$$

(“infection by an occupied neighbour”). For the ‘infection probability’ $p_{infec}(i, j, n)$, we have the bound

$$p_{infec}(i, j, n) := \mathbb{P}[G_{infec}(i, j, n)] > 1 - \varepsilon. \quad (37)$$

Proof. i) We distinguish the two cases $X_i(n) \in (M/K, (3/2)M/K)$ and $X_i(n) \geq (3/2)M/K$.

Case 1. Suppose that $X_i(n) \geq (3/2)M/K$ and introduce the first hitting time of level M/K from above after time n :

$$\tau_{M/K}^{X_i}(n) := \inf \left\{ t > n : X_i(t) = \frac{M}{K} \right\}. \quad (38)$$

Our goal is to establish the existence of a suitably measurable event $G_{non-rec}(i, n) \in \mathcal{F}(i, n, 1)$, so that $G_{non-rec}(i, n)$ implies, under the above conditions, that $\tau_{M/K}^{X_i}(n) > 1$. To this end, we set up a suitable comparison to a one-dimensional diffusion.

Indeed, rearranging the drift in equation (3), as long as $Y_j(t) < a'M'$ for all $j \in i + [-bL, bL]^d \cap \mathbb{Z}^d$, and hence

$$\sum_l \gamma_{jl} Y_l(t) < 1 \quad \text{for all } j \in i + [-L/2, L/2]^d \cap \mathbb{Z}^d,$$

holds, and as long as $X_i \leq 2M/K$, we have

$$\begin{aligned} dX_i(t) &\geq \sum_{j \in \mathbb{Z}^d} \left(m_{ij} - \alpha \frac{2M}{K} \lambda_{ij} \right) X_j(t) dt \\ &\quad + \left(\alpha M - \sum_{j \in \mathbb{Z}^d} m_{ij} - \alpha \right) X_i(t) dt + \sqrt{X_i(t)} dB_i(t). \end{aligned} \quad (39)$$

Now we check that the first component of the drift on the right-hand side is positive. Indeed, from the assumption (6), we obtain

$$m_{ij} - \alpha \frac{2M}{K} \lambda_{ij} > m_{ij} - \alpha \frac{2M}{K} c m_{ij}, \quad (40)$$

which is positive by our choice of $K = 2\alpha M c + 1$, for all $j \in \mathbb{Z}^d$. Moreover, we have, for each $M > 1$,

$$\frac{M}{K} = \frac{M}{2\alpha M c + 1} \in \left(\frac{1}{2\alpha c + 1}, \frac{1}{2\alpha c} \right).$$

Under these conditions (39) implies

$$dX_i(t) \geq \left(\alpha M - \sum_{j \in \mathbb{Z}^d} m_{ij} - \alpha \right) X_i(t) dt + \sqrt{X_i(t)} dB_i(t) \quad (41)$$

and so while $X_i \in [0, 2M/K]$, using Corollary 5.3 to the Ikeda-Watanabe Comparison Theorem 5.2, both to be found in the Appendix, we may compare X_i to a dominated supercritical Feller diffusion \tilde{Z}_i defined in (20), with initial value $\tilde{Z}_i(n) := (3/2)M/K$ and

$$D_1 = D_1(M) = \left(\alpha M - \sum_{j \in \mathbb{Z}^d} m_{ij} - \alpha \right),$$

and driven by the same Brownian motion, i.e. $\{W_i(t)\}_{t \geq 0} := \{B_i(t)\}_{t \geq 0}$. It is an important observation in Corollary 5.3 that actually more is true: our domination argument does not only hold up to the time when X_i leaves the interval $[0, 2M/K]$ for the first time, but in fact as long as \tilde{Z}_i takes values inside this interval, i.e. up to the first exit time $\tau_{2M/K}^{\tilde{Z}_i}$ (defined as in 38).

Note that, for $M > \sum_j m_{ij}/\alpha$, the ‘supercriticality’ (i.e. positive drift) $D_1 = (\alpha M - \sum m_{ij} - \alpha)$ in (41) tends to ∞ as $M \rightarrow \infty$, while maintaining the condition $X_i(n) \geq 1/(2\alpha c)$.

Now we make use of the comparison. Indeed, for $t \geq n$, as long as $\tilde{Z}_i(t)$ stays inside the interval $[0, 2M/K]$ and given that initially $X_i(n) \geq \tilde{Z}_i(n) :=$

$(3/2)M/K$, we have that X_i dominates \tilde{Z}_i . To obtain a comparison that is valid throughout the whole time interval $[n, n+1]$, we go one step further and modify \tilde{Z}_i so that whenever \tilde{Z}_i hits level $2M/K$ (and thus is about to leave the area in which the comparison holds true), we restart the process \tilde{Z}_i at level $(3/2)M/K$ and repeat this procedure as often as necessary, so that the comparison holds for all times $t \in [n, n+1]$. More precisely, we define a sequence of stopping times, beginning with

$$\nu_{2M/K}^{\tilde{Z}_i}(n, 1) := \tau_{2M/K}^{\tilde{Z}_i}(n),$$

restart the \tilde{Z}_i process at this time, setting $\tilde{Z}_i(\nu_{2M/K}^{\tilde{Z}_i}(n, 1)) := (3/2)M/K$, and then iterate this procedure, considering, for $m \in \mathbb{N}$,

$$\nu_{2M/K}^{\tilde{Z}_i}(n, m+1) := \inf \left\{ t > \nu_{2M/K}^{\tilde{Z}_i}(n, m) : \tilde{Z}_i(t) = \frac{2M}{K} \right\}, \quad (42)$$

and again restarting the \tilde{Z}_i process accordingly, i.e. setting

$$\tilde{Z}_i(\nu_{2M/K}^{\tilde{Z}_i}(n, m+1)) = \frac{3}{2} \frac{M}{K}.$$

Note that $\nu_{2M/K}^{\tilde{Z}_i}(n, m) \uparrow \infty$ a.s. as $m \rightarrow \infty$. For definiteness, set $\nu_{2M/K}^{\tilde{Z}_i}(n, 0) := n$. We define the i.i.d. positive lengths of the corresponding upcrossing intervals from $(3/2)M/K$ to $2M/K$ for $m \geq 1$ by

$$\tilde{T}_m := \nu_{2M/K}^{\tilde{Z}_i}(n, m) - \nu_{2M/K}^{\tilde{Z}_i}(n, m-1). \quad (43)$$

Now observe that there is an event $G_{non-rec}^1(i, n)$, defined only in terms of the Brownian increments $\{B_i(n+s) - B_i(n) : s \in [0, 1]\}$, and hence being an element of $\mathcal{F}(i, n, 1)$, so that if we start our modified diffusion \tilde{Z}_i in $\tilde{Z}_i(n) = (3/2)M/K$, this event $G_{non-rec}^1(i, n)$ actually equals $\{\tau_{M/K}^{\tilde{Z}_i}(n) > n+1\}$. (The set $G_{non-rec}^1(i, n)$ contains all such ω , such that the corresponding Brownian increments lead to the desired behaviour if they drive the modified diffusion \tilde{Z}_i started at time n in $(3/2)M/K$). Moreover, by our comparison, the event $\{\tau_{M/K}^{\tilde{Z}_i}(n) > n+1\}$ implies that $\{\tau_{M/K}^X(n) > n+1\}$ which in turn implies that $X_i(n+1) > M/K$.

It remains to show that the event $G_{non-rec}^1(i, n)$ has sufficiently high probability. To this end, note that the number of upcrossings of the modified and suitably restarted process \tilde{Z}_i from level $(3/2)M/K$ to level $2M/K$ before the first downcrossing from $(3/2)M/K$ to M/K is a geometric random variable with positive parameter

$$\tilde{q}_{M,K} := \mathbb{P}[\tau_{M/K}^{\tilde{Z}_i}(n) < \tau_{2M/K}^{\tilde{Z}_i}(n)].$$

Note that by (30), applied with $\tilde{U} := M/K$, for D_1 and hence M sufficiently large,

$$\tilde{q}_{M,K} \leq \exp(-D_1\tilde{U}), \quad (44)$$

and this approaches 0 exponentially fast (recall that M/K remains bounded). Moreover, by (31), the expected time for \tilde{Z}_i to exit from the interval $(M/K, 2M/K)$ when being started in $(3/2)M/K$, is bounded below by $\frac{1}{32D_1}$ for D_1 large enough. Hence, for such D_1 ,

$$\mathbb{E}[T_1] \geq \mathbb{E}\left[\tau_{(M/K, 2M/K)}^{\tilde{Z}_i}\right] \geq \frac{1}{32D_1}. \quad (45)$$

Now let $\tilde{\mathcal{D}}$ denote the number of upcrossings before the first “success”, i.e. a downcrossing from $(3/2)M/K$ to M/K . For each $\tilde{N} \in \mathbb{N}$, we may then write

$$\begin{aligned} \mathbb{P}[\tau_{M/K}^{\tilde{Z}_i}(n) \leq n+1] &= \mathbb{P}[\tau_{M/K}^{\tilde{Z}_i}(n) \leq n+1; \tilde{\mathcal{D}} < \tilde{N}] + \mathbb{P}[\tau_{M/K}^{\tilde{Z}_i}(n) \leq n+1; \tilde{\mathcal{D}} \geq \tilde{N}] \\ &\leq \mathbb{P}[\tilde{\mathcal{D}} < \tilde{N}] + \mathbb{P}\left[\sum_{i=1}^{\tilde{\mathcal{D}}} \tilde{T}_i < 1; \tilde{\mathcal{D}} \geq \tilde{N}\right] \\ &\leq 1 - (1 - \tilde{q}_{M,K})^{\tilde{N}} + \mathbb{P}\left[\sum_{i=1}^{\tilde{N}} \tilde{T}_i < 1\right] \\ &\leq \tilde{N}\tilde{e}\tilde{x}\tilde{p}(-\tilde{D}_1\tilde{U}) + \mathbb{P}\left[\sum_{i=1}^{\tilde{N}} T_i < 1\right] \end{aligned}$$

where, in the last step, we applied (44) and Bernoulli’s inequality. Since by (45), for large D_1 , the expectation of the length of the of the i.i.d. upcrossing intervals $\{\tilde{T}_i\}$ of the modified and suitably restarted process is bounded below by $\frac{1}{32D_1}$, the number of such upcrossing intervals up to time 1 is at most of order D_1 . Hence by the Law of Large Numbers we can find a constant \tilde{d} , so that for $\tilde{N} := \tilde{d} \cdot D_1$ and all D_1 large enough, the last term on the right-hand side is bounded by $\varepsilon/4$. Since the first term on the right-hand side still decreases exponentially in D_1 once \tilde{d} is fixed (the linearly increasing pre-factor being squashed), for D_1 and hence M large enough, this bound holds simultaneously for the first and the last term, and we arrive at the desired result: Under the above conditions, with $\tilde{Z}_i(n) = (3/2)M/K$,

$$\mathbb{P}[\tau_{M/K}^{\tilde{Z}_i}(n) \leq n+1] \leq \frac{\varepsilon}{2}$$

for D_1 and hence M large enough, which in turn implies

$$\mathbb{P}[G_{non-rec}^1(i, n)] > 1 - \frac{\varepsilon}{2},$$

so that *Case 1* of Part i) follows.

Case 2. Now suppose that $M/K < X_i(n) < (3/2)M/K$. In this case, we cannot find a uniform lower bound on the probability of the previously considered event $\{\tau_{M/K}^{\tilde{Z}_i}(n) > n + 1\}$, and hence of the probability of $\{\tau_{M/K}^{X_i}(n) > n + 1\}$, that is sufficiently large.

However, we may still use the same comparison as above to a dominated supercritical Feller diffusion \tilde{Z}_i , so that the comparison works as long as \tilde{Z}_i stays below $2M/K$. This time, set $\tilde{Z}_i(n) = M/K < X_i(n)$ and observe that there is a constant $M_0^2 > 0$, so that for all $M > M_0^2$, the deterministic drift in the supercritical Feller diffusion \tilde{Z}_i will achieve two goals with sufficiently high probability: firstly, make \tilde{Z}_i hit level $(3/2)M/K$ within the time-interval $[n, n+1/2]$ with sufficiently high probability, and secondly, after hitting level $(3/2)M/K$, arguing just as in the first part of the lemma, ensure that there will be no further downcrossing from $(3/2)M/K$ to M/K up to time $[n+1]$. Thus, once again we can find a measurable event $G_{non-rec}^2(i, n) \in \mathcal{F}(i, n, 1)$, depending only on the corresponding Brownian increments, so that given $\tilde{Z}_i(n) = M/K$, by comparison, $G_{non-rec}^2(i, n)$ implies $X_i(n+1) > M/K$, and moreover

$$\mathbb{P}[G_{non-rec}^2(i, n)] > 1 - \frac{\varepsilon}{2}.$$

Hence the result holds also in *Case 2*. Finally, in the view of both cases, choose

$$G_{non-rec}(i, n) := G_{non-rec}^1(i, n) \cap G_{non-rec}^2(i, n) \in \mathcal{F}(i, n, 1) \subset \mathcal{F}(i, NL, n, 1)$$

and Part i) follows.

To prove Part ii), we begin with some preliminary considerations. Note that, by using the same comparison and similar arguments as before, again considering suitable up- and downcrossings, this time from M/K down to $M/(2K)$, we can actually go one step further and find a finite constant M_0^3 , so that if $M > M_0^3$, and for $j \in \mathbb{Z}^d$ so that $m_{ij} > 0$, there exists an event

$$G_{per-occ}(i, j, n) \in \mathcal{F}(i, NL, n, 1),$$

so that given $X_j(n) > \frac{M}{K}$ and $G'_{sec}(i, n)$, the event $G_{per-occ}(i, j, n)$ implies

$$X_j(n+1) > \frac{M}{K} \quad \text{and} \quad \tau_{M/(2K)}^{X_j}(n) > n + 1,$$

and moreover, we have

$$\mathbb{P}[G_{per-occ}(i, j, n)] > 1 - \frac{\varepsilon}{2}. \quad (46)$$

Note that once again we use the assumption that the range of the $\{\gamma_{ij}\}$ is less than $(b-1)L$ so that the “safe environment condition”, in particular (34), allows comparisons of the above type also for site j .

We are now prepared to consider the *infection probability* at a site i in the presence of at least one occupied neighbour, say, at j^* . Again, assuming (6), we use a comparison based on the Corollary 5.3 to the Ikeda-Watanabe Theorem from the Appendix. This time we rearrange the drift so as to highlight the rôle of immigration of mass to an unoccupied site from occupied neighbours. Once immigrated we can then compare the evolution of the mass to a supercritical continuous state branching process, as before. Indeed, considering the drift in equation (3), observe that as long as $X_i(t) \leq 2M/K$ and given the existence of at least one neighbour at some site $j^* \in \mathbb{Z}^d$ with $m_{ij^*} > 0$ and $X_{j^*}(n) > M/K$ (noting that m_{ij^*} is bounded below by some $\delta > 0$ since the family $\{m_{ij}\}$ is of finite range), then we have that, as long as t satisfies

$$n \leq t < \tau_{M/(2K)}^{X_{j^*}}(n), \quad (47)$$

by our choice $K = 2\alpha Mc + 1$,

$$\begin{aligned} dX_i(t) &\geq \sum_{j \in \mathbb{Z}^d} \left(m_{ij} - 2\alpha \frac{M}{K} \lambda_{ij} \right) X_j(t) dt \\ &\quad + \left(\alpha M - \sum_{j \in \mathbb{Z}^d} m_{ij} - \alpha \right) X_i(t) dt + \sqrt{X_i(t)} dB_i(t) \\ &\geq \left(m_{ij^*} - 2\alpha \frac{M}{K} \lambda_{ij^*} \right) \frac{M}{2K} dt \\ &\quad + \left(\alpha M - \sum_{j \in \mathbb{Z}^d} m_{ij} - \alpha \right) X_i(t) dt + \sqrt{X_i(t)} dB_i(t) \\ &\geq \left(m_{ij^*} - \frac{1}{c} \lambda_{ij^*} \right) \frac{1}{4\alpha c + 2} dt \\ &\quad + \left(\alpha M - \sum_{j \in \mathbb{Z}^d} m_{ij} - \alpha \right) X_i(t) dt + \sqrt{X_i(t)} dB_i(t), \quad (48) \end{aligned}$$

assuming $M > \max\{1, \sum_j m_{ij}/\alpha\}$. Thus at the uninfected site i , after time n , as long as (47) holds, we may compare the evolution of the process X_i to a dominated supercritical branching process \hat{Z}_i with immigration, as defined in (21), and driven by the same Brownian motion,

i.e. $\{W_i(t)\}_{t \geq 0} := \{B_i(t)\}_{t \geq 0}$, with constant strictly positive immigration, where

$$D_2 = \left(m_{ij^*} - \frac{1}{c}\lambda_{ij^*}\right) \frac{1}{4\alpha c + 2} > 0, \quad D_1 = \left(\alpha M - \sum_{j \in \mathbb{Z}^d} m_{ij} - \alpha\right) > 0.$$

Note that here we use the fact that we assumed the strict inequality $\frac{1}{c}\lambda_{ij^*} < m_{ij^*}$ from (6) to obtain a strictly positive immigration rate. Note also that the rate of immigration is bounded below independently of M . Again, the ‘supercriticality’ $(\alpha M - \sum_j m_{ij} - \alpha)$ tends to ∞ as M tends to ∞ . Hence, arguing as before in Part i), this time starting the dominated process in $\hat{Z}_i(n) = 0 \leq X_i(n)$, stopping \hat{Z}_i once it reaches level $2M/K$ and then restarting at $(3/2)M/K$ if necessary, we may find a constant $M_0^4 > 0$, so that if $M > M_0^4$, under the above conditions, the event

$$\hat{Z}_i(n+1) > \frac{M}{K} \quad \text{implies} \quad X_i(n+1) > \frac{M}{K}$$

and has probability greater than $\varepsilon/2$. As before, there is an event $G_{infec}^*(i, j^*, n) \in \mathcal{F}(i, n, 1)$, depending solely on the Brownian increments at i within the time-interval $[n, n+1]$, so that if we start our modified diffusion at time n in $\hat{Z}_i(n) = 0$, driven by the corresponding Brownian increments, we have $G_{infec}^*(i, j^*, n) = \{\hat{Z}_i(n+1) > M/K\}$. Moreover,

$$\mathbb{P}[G_{infec}^*(i, j^*, n)] > 1 - \frac{\varepsilon}{2}.$$

Now observe that, due to our preparations, (47) is actually being guaranteed by $G_{per-occ}(i, j^*, n)$ up to time $n+1$. Combining both events (which are actually independent), we define

$$G_{infec}(i, j^*, n) := G_{infec}^*(i, j^*, n) \cap G_{per-occ}(i, j^*, n) \in \mathcal{F}(i, L/2, n, 1)$$

and finally see that

$$\mathbb{P}[G_{infec}(i, j^*, n)] > 1 - \varepsilon.$$

Together with the fact that given $X_{j^*}(n) > M/K$, our event $G_{infec}(i, j, n)$ implies $X_i(n+1) > M/K$, defining

$$M_0 = \max\{M_0^1, M_0^2, M_0^3, M_0^4\}$$

finishes the proof of Lemma 4.5. \square

4.1.3. *Control of the environment.* We now find estimates for suitably measurable events, a combination of which will later provide the “safe environment condition” $G'_{sec}(i, n)$ at some site $i \in \mathbb{Z}^d$ and time $n \in \mathbb{N}$, which will in turn imply (33). Again, this is done via suitable comparisons to one-dimensional diffusions.

Lemma 4.6 (Control of the environment) *Let $n \in \mathbb{N}$ and $i \in \mathbb{Z}^d$. Let $\alpha', \{m'_{ij}\}, \{\lambda'_{ij}\}$ be fixed. Then, for any $\varepsilon \in (0, 1)$, there is a finite constant $v'_0 > 0$, such that for all $v' > v'_0$ there exists an event $E'(i, n)[v'] \in \mathcal{F}'(i, n, 2)$, so that for all $\alpha'M' > \sum_j m'_{ij}$,*

$$\mathbb{P}[E'(i, n)[v']] > 1 - \varepsilon.$$

Moreover,

$$E'(i, n)[v'] \subset \left\{ \sup_{0 \leq s \leq 1} Y_i(n+1+s) < 2v'M' \right\}, \quad (49)$$

and

$$\left\{ E'(i, n)[v'] \cap \{Y_i(n) \in [0, v'M']\} \right\} \subset \left\{ \sup_{0 \leq s \leq 2} Y_i(n+s) < 2v'M' \right\}. \quad (50)$$

These results hold true for any choice of $\{\gamma'_{ij}\}$ and any parameter values for the X -population in Model I.

Remarks.

1. In the proof of this lemma we use the assumption that there is a constant $c > 0$ such that $m'_{ij} < c\lambda'_{ij}$. We do not believe that this is necessary.
2. That the result should be true is again due to the fact that the downward drift resulting from overcrowding in a site is quadratic whereas the upward drift due to reproduction in the population is only linear. Moreover, for sufficiently crowded sites, immigration from neighbouring sites is being compensated by intra-species competition.
3. The independence of the X -parameters stems from the fact that competition by X actually facilitates a controlled environment. \square

Proof. We begin by setting up a suitable comparison, this time in a way such that Y_i is being *dominated* by a one-dimensional diffusion. Indeed, notice that, for $t \geq n$, as long as

$$Y_i(t) > m'_{ij}/(\alpha'\lambda'_{ij}) \quad \forall j \neq i,$$

an informal calculation shows

$$\begin{aligned}
dY_i(t) &= \sum_{j \in \mathbb{Z}^d} m'_{ij} \left(Y_j(t) - Y_i(t) \right) dt \\
&\quad + \alpha' \left(M' - \sum_{j \in \mathbb{Z}^d} \lambda'_{ij} Y_j(t) - \sum_{j \in \mathbb{Z}^d} \gamma'_{ij} X_j(t) \right) Y_i(t) dt + \sqrt{Y_i(t)} dB'_i(t) \\
&\leq \sum_{j \neq i, j \in \mathbb{Z}^d} \left[\left(m'_{ij} - \alpha' \lambda'_{ij} Y_i(t) \right) Y_j(t) \right] dt \\
&\quad + \left(\alpha' M' - \sum_{j \neq i, j \in \mathbb{Z}^d} m'_{ij} \right) Y_i(t) dt - \alpha' \lambda'_{ii} Y_i^2(t) dt + \sqrt{Y_i(t)} dB'_i(t) \\
&\leq \alpha' \left(M' - \frac{1}{\alpha'} \sum_{j \neq i, j \in \mathbb{Z}^d} m'_{ij} - \lambda'_{ii} Y_i(t) \right) Y_i(t) dt + \sqrt{Y_i(t)} dB'_i(t). \quad (51)
\end{aligned}$$

Hence the immigration of mass from site j is compensated for by the downward drift due to crowding at site j , and we may compare the evolution of Y_i (again applying the Corollary 5.3 to the Ikeda-Watanabe Comparison Theorem) to that of the solution Z_i of the logistic Feller diffusion (19), this time with driving Brownian motion given by $\{W_i(t)\}_{t \geq 0} := \{B'_i(t)\}_{t \geq 0}$. More precisely, we choose a constant $v' > 0$ so that

$$v' M' > \frac{m'_{ij}}{\alpha' \lambda'_{ij}} \quad \text{for all } j,$$

and then set

$$\lambda = \lambda'_{ii} > 0, \quad \text{and} \quad \bar{M} = M' - \sum_{j \neq i} m'_{ij} / \alpha' > 0 \quad (52)$$

in the comparison. Recall that \bar{M} is positive follows from our initial assumptions on Model I just after (6).

As before in the proof Lemma 4.5, a useful observation is the following. The comparison between Y_i and Z_i holds not only as long as Y_i and Z_i are *both* inside $[m'_{ij}/(\alpha' \lambda'_{ij}), \infty)$, but in fact as long as Z_i is above $m'_{ij}/(\alpha' \lambda'_{ij})$.

We now make use of the comparison. Let $Z_i(n) := \max\{Y_i(n), (3/2)v' M'\}$ and consider the stopping time

$$\tau_{v'M'}^{Z_i}(n) := \inf\{t > n : Z_i(t) = v' M'\}.$$

By the domination of Y_i by Z_i , at least up to time $\tau_{v'M'}^{Z_i}(n)$, due to our comparison, the event

$$\{\tau_{v'M'}^{Z_i}(n) < n + 1/2\} \in \mathcal{F}'(i, n, 1/2)$$

implies

$$\{\tau_{v'M'}^{Y_i}(n) < n + 1/2\}.$$

By Lemma 4.3, Equation (24), we may find a finite constant $v_0 > 0$ so that for all $v' > v_0$

$$\mathbb{P}[\tau_{v'M'}^{Z_i}(n) \leq 1/2] > 1 - \varepsilon/4 \quad (53)$$

thanks to our quadratic downward drift.

We now argue in a similar fashion as in the proof of Part i) in Lemma 4.5 and suitably restart Z_i once it hits level $v'M'$ at time $\tau_{v'M'}^{Z_i}(n)$ by then setting $Z_i(\tau_{v'M'}^{Z_i}(n)) := (3/2)v'M'$. As before in (42), we may define inductively stopping times $\nu_{v'M'}^{Z_i}(n, m)$ for each $m \in \mathbb{N}$ and corresponding interval-lengths $\{T_i\}$ as in (43). This modified and suitably restarted process remains inside $[v'M', \infty)$ for all times and hence the comparison to Y_i holds during the whole time interval $[n, n + 2]$.

To prove the result, we now need to make sure that given $\{\tau_{v'M'}^{Z_i}(n) < n + 1/2\}$, both processes do not grow from level $v'M'$ too quickly, so that they do not hit level $2v'M'$ within $[\tau_{v'M'}^{Z_i}(n), n + 2]$ with sufficiently high probability. To simplify calculations, after time $\tau_{v'M'}^{Z_i}(n)$, we apply the even cruder comparison of Z_i to the subcritical Feller diffusion \bar{Z}_i as defined in (22), driven by the same Brownian motion $\{B'_i(t)\}_{t \geq 0}$. Indeed, note that (51) also shows that

$$dY_i(t) \leq \alpha'(\bar{M} - \lambda\bar{U})Y_i(t) dt + \sqrt{Y_i(t)}dB'_i(t),$$

where we choose λ, \bar{M} as in (52) and let $\bar{U} = v'M'$. Then, the comparison applies for $v' > 2/\lambda$, noting that, by (52), we uniformly have $v' > 2\bar{M}/(\lambda M')$ for all $M' > \sum_j m'_{ij}/\alpha'$, and we may use the results obtained in Lemma 4.4.

Following our by now standard procedure, we stop \bar{Z}_i at level $v'M'$ and restart the diffusion at level $(3/2)v'M'$ in order to make the comparison work throughout the whole time-interval $[n, n + 2]$. Thus, if the suitably restarted process \bar{Z}_i does not rebound to level $2v'M'$ within the time-interval $[\tau_{v'M'}^{Z_i}(n), n + 2]$, then Y_i does not, either.

We finally need to ensure that the desired behaviour becomes sufficiently likely. We proceed in a way that is analogous to arguments in the proof of Part i) of Lemma 4.5. Indeed, note that, by Lemma 4.4, for the probability of \bar{Z}_i to leave the interval $(v'M', 2v'M')$ through its upper boundary, when

being started in $(3/2)v'M'$, we have, for v' large enough,

$$\bar{q}_{2v'M'} := \mathbb{P}[\tau_{2v'M'}^{\bar{Z}_i} < \tau_{v'M'}^{\bar{Z}_i}] \leq \exp(-\bar{\alpha}'\bar{M}v'M'),$$

which rapidly approaches 0 as $v' \rightarrow \infty$. The number of downcrossings of the modified process \bar{Z} from $(3/2)v'M'$ to $v'M'$ before the first time of hitting $2v'M'$ is geometric with parameter $\bar{q}_{2v'M'}$.

Again, as in (45), the expected time of the interval-lengths $\{\bar{T}_i\}$ can be bounded from below by $1/(24\alpha'\lambda v'M')$ using (31), once the product $v'M'$ is greater than $2\bar{M}/\lambda$ and since $M' > \sum_j m'_{ij}/\alpha'$ this holds once v' is sufficiently large.

Finally, let \bar{D} denote the number of such downcrossings before the first “success”, i.e. an upcrossing from $(3/2)v'M'$ to $2v'M'$. Then, for each $\bar{N} \in \mathbb{N}$, we may write, using the strong Markov property and continuity for \bar{Z}_i ,

$$\begin{aligned} & \mathbb{P}\left[\tau_{2v'M'}^{\bar{Z}_i}(\tau_{v'M'}^{\bar{Z}_i}(n)) \leq n + 2 \mid \tau_{v'M'}^{\bar{Z}_i}(n) \leq n + \frac{1}{2}\right] \\ & \leq \mathbb{P}\left[\tau_{2v'M'}^{\bar{Z}_i}(0) \leq 2; \bar{D} < \bar{N} \mid \bar{Z}_i(0) = \frac{3}{2}v'M'\right] \\ & \quad + \mathbb{P}\left[\tau_{2v'M'}^{\bar{Z}_i}(0) \leq 2; \bar{D} \geq \bar{N} \mid \bar{Z}_i(0) = \frac{3}{2}v'M'\right] \\ & \leq \mathbb{P}\left[\bar{D} < \bar{N} \mid \bar{Z}_i(n) = \frac{3}{2}v'M'\right] \\ & \quad + \mathbb{P}\left[\sum_{i=1}^{\bar{D}} \bar{T}_i < 2; \bar{D} \geq \bar{N} \mid \bar{Z}_i(0) = \frac{3}{2}v'M'\right] \\ & \leq 1 - (1 - q_{2v'M'})^{\bar{N}} + \mathbb{P}\left[\sum_{i=1}^{\bar{N}} \bar{T}_i < 2\right] \\ & \leq \bar{N} \exp(-\alpha'\bar{M}v'M') + \mathbb{P}\left[\sum_{i=1}^{\bar{N}} \bar{T}_i < 2\right], \end{aligned}$$

by Bernoulli’s inequality. Since for large v' , the expectation of \bar{T}_1 is bounded below by $1/(24\alpha'\lambda v'M')$, the number of downcrossing intervals up to time 2 is a most of order $v'M'$. Hence by the Law of Large Numbers we can find a constant \bar{d} , so that for $\bar{N} := \bar{d} \cdot v'M'$, the last term on the right-hand side is bounded by $\varepsilon/4$. Since the first term on the right-hand side still decreases exponentially in $v'M'$ once \bar{d} is fixed (the linearly increasing pre-factor being squashed), for v' large enough, this bound holds simultaneously for the first and the last term, and we arrive at the desired result.

Altogether, in a similar fashion as before, we may find a set $E'_1(i, n)[v'] \in \mathcal{F}'(i, n, 2)$ which is determined by precisely those Brownian increments $\{B'_i(n+s) - B'_i(n) : s \in [0, 2]\}$, so that for *all* initial values $Z_i(n) \in [(3/2)M/K, \infty)$, if the diffusion Z_i is being driven by the above increments, then

$$\left\{ \tau_{v'M'}^Z(n) \leq n + 1/2; \tau_{2v'M'}^Z(\tau_{v'M'}^Z(n)) > n + 2 \right\}$$

holds. Note that, by the above calculation and (53),

$$\begin{aligned} \mathbb{P}[E'_1(i, n)[v']] &= \mathbb{P}\left[\tau_{2v'M'}^{\bar{Z}}(\tau_{v'M'}^{\bar{Z}}(n)) \leq n + 2 \mid \tau_{v'M'}^{\bar{Z}}(n) \leq n + \frac{1}{2} \right] \\ &\quad \times \mathbb{P}\left[\tau_{v'M'}^Z(n) \leq 1/2 \right] \\ &> 1 - \frac{\varepsilon}{2}. \end{aligned}$$

Finally, thanks to our comparison,

$$E'_1(i, n)[v'] \subset \left\{ \sup_{0 \leq s \leq 1} Y_i(n+1+s) < 2v'M' \right\}.$$

For the second implication, in a similar and actually simpler way, construct an event $E'_2(i, n)[v'] \in \mathcal{F}'(i, n, 2)$, so that it has the required properties, using the same comparisons as above and setting $\bar{Z}_i(n) := (3/2)v'M'$. Finally consider the event

$$E'(i, n)[v'] := E'_1(i, n)[v'] \cap E'_2(i, n)[v'] \in \mathcal{F}'(i, n, 2),$$

so that

$$\mathbb{P}[E'(i, n)[v']] > 1 - \varepsilon,$$

and (49) and (50) hold. \square

Remark.

Note that for $a' > v'_0$ in Lemma 4.6 (and of course also for the stronger condition $a'/2 > v'_0$), at time $n \in \mathbb{N}$, recalling $N = b + 2$,

$$\bigcap_{j \in i + [-bL, bL]^d \cap \mathbb{Z}^d} E'(j, n-1)[a'] \in \mathcal{F}'(i, bL, n-1, 2) \subset \mathcal{F}'(i, NL, n-1, 2) \quad (54)$$

implies (33). Moreover, together with the additional condition $Y_j(n) \leq a'M'$ for all $j \in i + [-bL, bL]^d \cap \mathbb{Z}^d$, both

$$\bigcap_{j \in i + [-bL, bL]^d \cap \mathbb{Z}^d} E'(j, n-1)[a'] \in \mathcal{F}'(i, bL, n-1, 2) \subset \mathcal{F}'(i, NL, n-1, 2) \quad (55)$$

and

$$\bigcap_{j \in i + [-bL, bL]^d \cap \mathbb{Z}^d} E'(j, n)[a'] \in \mathcal{F}'(i, bL, n, 2) \subset \mathcal{F}'(i, NL, n, 2) \quad (56)$$

imply (33), too. Thus, all three events can be considered as instances of the “safe environment condition” $G'_{sec}(i, n)$ in the sense of Lemma 4.5. \square

4.2. Comparison arguments.

4.2.1. *Comparison of ζ resp. η to $2N = 2(b+2)$ -dependent oriented percolation.* We first focus on the X resp. ζ population and construct a suitable “good event” G_ζ . Recall that, by our technical assumption, L denotes the maximum range of both migration matrices $\{m_{ij}\}, \{m'_{ij}\}$, that is

$$L = \max \{ \|j - i\| : m_{ij}, m'_{ij} \neq 0 \},$$

and b is the smallest positive integer such that the range of $\{\gamma_{ij}\}$ and $\{\gamma'_{ij}\}$ is less than $(b-1) \cdot L$. Recall also from (17) the definition

$$\zeta_i(n) = \begin{cases} 1, & \text{if } X_i(2n) > \frac{M}{K} \text{ and } Y_j(2n) < a'M' \quad \forall j \in i + [-bL, bL]^d \cap \mathbb{Z}^d, \\ 0, & \text{otherwise,} \end{cases} \quad (57)$$

and from the comparison assumptions (12) the notation

$$\zeta(n) \in H \quad \text{if} \quad \zeta_i(n) = 1 \text{ for all } i \in [-L/2, L/2]^d \cap \mathbb{Z}^d,$$

and finally recall from (13) the notion of translation of H by mL , for some $m \in \mathbb{Z}$, denoted by $m \cdot H$.

Lemma 4.7 (Comparison) *Let $\theta \in (0, 1)$. Suppose we are given fixed parameters $\alpha, \alpha', \{m_{ij}\}, \{m'_{ij}\}, \{\lambda_{ij}\}, \{\lambda'_{ij}\}$.*

Then, under the assumptions on Model I for Theorem 1.2, there exist finite constants $M_0 > 0$ and $a' > 0$, such that if $M > M_0$, $K := 2\alpha Mc + 1$, $M' > \sum_j m'_{ij}/\alpha'$, then there is a finite $\gamma = \gamma(a'M') > 0$, so that if $\sum_j \gamma_{ij} < \gamma$, and for all $\{\gamma'_{ij}\}$, the process $\{\zeta(n)\}_{n \geq 0}$ fulfils the modified comparison assumptions (16) for NL and θ . In particular, for each configuration $\zeta \in H$, there exists a “good event”

$$G_\zeta \in \mathcal{F}^*(NL, [0, 2]),$$

where $N = b + 2$, with $\mathbb{P}[G_\zeta] > 1 - \theta$, such that if $\zeta(0) = \zeta$, then on G_ζ ,

$$\zeta(1) \in (+1)H \cap (-1)H.$$

Consequently, using Theorem 3.5, the process

$$\mathcal{X}_n := \{(m, n) \in \mathcal{L} : \zeta(n) \in m \cdot H\}$$

dominates a $2N$ -dependent oriented percolation process $\{\mathcal{W}_n\}_{n \geq 0}$ on \mathcal{L} with density at least $1 - \theta$ and initial condition $\mathcal{W}_0 = \mathcal{X}_0$.

Remarks.

1. A similar result is true for the Y - resp. η -population, which, given $\theta > 0$, produces a similar threshold M'_0 and parameters M', a, γ' , which allow a comparison to a $2N = 2(b + 2)$ -dependent oriented percolation process of density at least $1 - \theta$ via a similar “good event” G_η .
2. The available degree of freedom in the choice of $\{\gamma'_{ij}\}$ in this result is crucial for the simultaneous comparison of $\{\zeta(n)\}_{n \geq 0}$ and $\{\eta(n)\}_{n \geq 0}$ which we will need to consider later. It is due to the fact the our results for the “control of the environment” in Lemma 4.6 are entirely independent of these $\{\gamma'_{ij}\}$, since competition by X only facilitates the “good environment condition” determined in terms of Y . \square

Proof. Fix $\theta > 0$ and let

$$\varepsilon = \frac{1}{2} \frac{\theta}{(4(b+2)L)^d}. \quad (58)$$

We begin with the specification of consistent parameter values for our model that will lead to the required comparison. First note that all of the constants $\alpha, \alpha', \{m_{ij}\}, \{m'_{ij}\}, \{\lambda_{ij}\}, \{\lambda'_{ij}\}, b, N, L$ will remain fixed throughout what follows. The only values we need to adjust suitably in order to produce the proof are $M, M', a', \{\gamma_{ij}\}$. The proof is entirely independent of the choice of $\{\gamma'_{ij}\}$ (as long as all parameter values remain compatible with the assumptions of Theorem 1.2).

First, we choose a' large enough, so that for any $M' > \sum_j m'_{ij}/\alpha'$,

$$a' > \frac{1}{2\alpha' M' c + 1}, \quad (59)$$

and moreover such that $a'/2 > v'_0$ in Lemma 4.6 with the above ε . Then, for each i and n , we have the bound

$$\mathbb{P}[E'(i, n)[a'/2]] > 1 - \varepsilon. \quad (60)$$

Note that this bound does not depend on $\{\gamma'_{ij}\}$ (and obviously not on $\{\gamma_{ij}\}$). From now on, a' remains fixed. Define, for any $M' > \sum_j m'_{ij}/\alpha'$ the constant

$\gamma = \gamma(a', M') := (2a'M')^{-1}$, so that, for each i ,

$$\sum_{j \in \mathbb{Z}^d} \gamma_{ij} 2a'M' < 1.$$

Finally, we can find $M_0 > 0$, such that for all $M > M_0$, the bounds of Lemma 4.5 for the “infection” and “non-recovery probabilities” hold with our choice of ε .

We now check that with these parameter values for M_0, a', γ , and for all $M > M_0$, assuming $\zeta(0) \in H$, there is a “good event” $G_\zeta \in \mathcal{F}^*(NL, [0, 2])$, which implies

$$\zeta(1) \in (+1)H \cap (-1)H, \quad (61)$$

and has probability at least $1 - \theta$. Recall that $\zeta(0) \in H$ means

- $X_i(0) > M/K$, where $K = 2\alpha M c + 1$ as in Section 4.1, for all $i \in [-L/2, L/2]^d \cap \mathbb{Z}^d$,
- $Y_j(0) < a'M'$, for all $j \in i + [-bL, bL]^d \cap \mathbb{Z}^d, i \in [-L/2, L/2]^d \cap \mathbb{Z}^d$.

To construct G_ζ , recall first that one time step for ζ corresponds to two time units for X and Y . We consider the corresponding time interval $[0, 2]$ in two parts $[0, 1]$ and $[1, 2]$.

In view of Lemma 4.6, applied with $v' = a' > v'_0$, we see that

$$\begin{aligned} \mathbb{P} \left[\bigcap_i E'(i, 0)[a'] : i \in [-(b+1/2)L, (b+1/2)L]^d \cap \mathbb{Z}^d \right] \\ > 1 - (2(b+1/2)L)^d \varepsilon, \end{aligned}$$

and recall that this event, denoted for short by

$$E'(0, (b+1/2)L, 0, 2)[a'] \in \mathcal{F}'(0, (b+1/2)L, 0, 2),$$

implies, since $\zeta(0) \in H$, by Lemma 4.6,

$$\sup_{0 \leq s \leq 2} \{Y_i(s) : i \in [-(b+1/2)L, (b+1/2)L]^d \cap \mathbb{Z}^d\} < 2a'M'.$$

Next, Lemma 4.5 tells us that (recalling that, for $n = 0$, each $G_{non-rec}(i, 0) \in \mathcal{F}(i, 0, 1)$),

$$\mathbb{P} \left[\bigcap_i G_{non-rec}(i, 0) : i \in [-L/2, L/2]^d \cap \mathbb{Z}^d \right] > 1 - L^d \varepsilon,$$

and note that this event, denoted by

$$G_{non-rec}(0, L/2, 0, 1) \in \mathcal{F}(0, L/2, 0, 1),$$

implies that

$$X_i(1) > \frac{M}{2\alpha Mc + 1} \text{ for all } i \in [-L/2, L/2]^d \cap \mathbb{Z}^d.$$

Now, applying Lemma 4.6 once again, this time with $v' = a'/2$, we see that

$$\mathbb{P}\left[\bigcap_i E'(i, 0)[a'/2] : i \in [-NL, NL]^d \cap \mathbb{Z}^d\right] > 1 - \varepsilon(2NL)^d,$$

and this event, again denoted shorthand by

$$E'(0, NL, 0, 2)[a'/2] \in \mathcal{F}'(0, NL, 0, 2),$$

ensures that

$$\sup_{s \in [1, 2]} \{Y_i(s) : i \in [-NL, NL]^d \cap \mathbb{Z}^d\} < a'M'.$$

Combining all these events, so far then we have guaranteed that with probability at least

$$1 - (2NL)^d \varepsilon - L^d \varepsilon - (2(b + 1/2)L)^d \varepsilon,$$

we have that

$$X_i(1) > \frac{M}{K} = \frac{M}{2\alpha Mc + 1} \text{ for all } i \in [-L/2, L/2]^d \cap \mathbb{Z}^d \quad (62)$$

and

$$\sup_{1 \leq t \leq 2} \{Y_j(t) : j \in [-NL, NL]^d \cap \mathbb{Z}^d\} < a'M', \quad (63)$$

where $N = b + 2$. Thus, throughout the time interval $[1, 2]$, a “safe environment condition” $G'_{sec}(i, 1) \in \mathcal{F}'(i, NL, 1, 1)$ holds at time $n = 1$ for all $i \in [-(3/2)L, (3/2)L]^d \cap \mathbb{Z}^d$, and hence the local Y -population is not ‘too big’, that is, for all $j \in [-(3/2)L, (3/2)L]^d \cap \mathbb{Z}^d$,

$$\max_{t \in [1, 2]} \sum_{l \in \mathbb{Z}^d} \gamma_{jl} Y_l(t) < 1.$$

Thus, the ζ -process can safely invade the neighbouring boxes, i.e., conditional on (62) and given the above instance of the “safe environment condition”, for each site

$$i \in \left\{ \{L\mathbf{e}_1 + [-L/2, L/2]^d\} \cup \{-L\mathbf{e}_1 + [-L/2, L/2]^d\} \right\} \cap \mathbb{Z}^d, \quad (64)$$

where \mathbf{e}_1 denotes the first unit vector in \mathbb{Z}^d , the ‘infection event’ $G_{infec}(i, j, 1)$ at i has probability greater than $1 - \varepsilon$, by Lemma 4.5 (noting that by our choice of L , and the fact that m_{ij} is a function of $\|i - j\|$ alone, each such site i has at least one occupied neighbour $j \in [-L/2, L/2]$). Hence, after all these prerequisites, the probability that simultaneously for all such sites i taken from the set in (64), at time 1, the event

$$G_{infec}(i, j, 1) \in \mathcal{F}(i, NL, 1, 1) \quad \text{holds for some } j \in [-L/2, L/2]^d \cap \mathbb{Z}^d,$$

implying, under the above conditions, that

$$X_i(2) > \frac{M}{K} = \frac{M}{2\alpha M c + 1},$$

is at least $1 - 2L^d \varepsilon$. We denote this simultaneous event by

$$G_{infec}(L \uparrow L/2, 1, 1) \in \mathcal{F}(0, NL, 0, 2).$$

Thus, we may define the $\mathcal{F}^*(NL, [0, 2])$ -measurable ‘good event’

$$\begin{aligned} G_\zeta := & E'(0, (b + 1/2)L, 0, 2)[a'] \cap G_{non-rec}(i, L/2, 0, 1) \\ & \cap E'(i, NL, 0, 2)[a'/2] \cap G_{infec}(i, L \uparrow L/2, 1, 1), \end{aligned} \quad (65)$$

which implies, given $\zeta = \zeta(0) \in H$,

$$\zeta(1) \in (+1)H \cap (-1)H,$$

and gladly observe

$$\begin{aligned} \mathbb{P}[G_\zeta] &> 1 - (2(b + 1/2)L)^d \varepsilon - L^d \varepsilon - (2NL)^d \varepsilon - 2L^d \varepsilon \\ &> 1 - 4(2NL)^d \varepsilon \\ &= 1 - \theta, \end{aligned}$$

which completes the comparison.

4.2.2. Simultaneous comparison and Proof of Theorem 1.2 and Corollary 1.3. Assume that

$$\theta \leq 6^{-4(b+2)+1}.$$

Then we may choose a, a' , so that for all $M > \sum_j m_{ij}/\alpha$ and $M' > \sum_j m'_{ij}/\alpha'$, (60) holds for both populations X and Y with $\varepsilon = \frac{1}{4} \frac{\theta}{(2NL)^d}$. The point is that this can be done simultaneously, since the bounds for the

control of the environment do not depend on the behaviour of the competitor.

Then, we may pick M, M' and simultaneously γ, γ' such that Lemma 4.5 holds with $\varepsilon = \frac{1}{4} \frac{\theta}{(2NL)^d}$ for both the X - and the Y -population. Condition (60) is unaffected by this, since the bounds on the environment do not depend on $\{\gamma_{ij}\}$ and $\{\gamma'_{ij}\}$ and hold for all $M > \sum_j m_{ij}/\alpha$ and $M' > \sum_j m'_{ij}/\alpha'$.

Assuming then that $\zeta(0)$ and $\eta(0) \in H$, observing that condition (59) on a' ensures that

$$\kappa'_1 := \frac{M'}{2\alpha'M'c + 1} < a'M' =: \kappa'_2$$

(with a similar inequality for κ_1, κ_2), leads to the initial condition

$$(X(0), Y(0)) \in H(\kappa_1, \kappa_2; \kappa'_1, \kappa'_2; (b + 1/2)L)$$

specified in Theorem 1.2 with

$$\kappa_1 = \frac{M}{2\alpha Mc + 1}, \quad \kappa_2 = aM, \quad \kappa'_1 = \frac{M'}{2\alpha'M'c + 1}, \quad \kappa'_2 = a'M'.$$

Hence we can simultaneously construct the corresponding good events G_ζ and G_η and infer from Theorem 3.2 that both the X and Y population survive each with probability greater than $\frac{19}{20}$, which yields *persistence* of $\{X, Y\}$ with positive probability.

Moreover, if we make the stronger assumption that the initial configurations of the X and Y populations satisfy

$$(X(0), Y(0)) \in H(\kappa_1, \kappa_2; \kappa'_1, \kappa'_2; \infty),$$

hence assuming

$$\zeta_i(0) = 1, \quad \eta_i(0) = 1, \quad \text{for all } i \in \mathbb{Z}^d,$$

then, according to Theorem 3.6,

$$\liminf_{n \rightarrow \infty} \mathbb{P}[\zeta_{2n}(0) = 1] \geq \frac{19}{20}.$$

The same result holds for η . Thus,

$$\liminf_{n \rightarrow \infty} \mathbb{P}[\zeta_{2n}(0) = 1, \eta_{2n}(0) = 1] \geq \frac{9}{10}.$$

By the definition of $\{\zeta_n, \eta_n\}_{n \geq 0}$ and our bounds from the last section, applied in a similar fashion, this implies that there is a uniform positive lower bound on $\mathbb{P}[X_0(t) \geq M/(2K), Y_0(t) \geq M/(2K) \forall t \in [0, 4] \text{ given } \zeta_{2n}(0) = 1, \eta_{2n}(0) = 1]$, and the proof is finished. \square

4.2.3. *Sketch of the comparison proof of Theorem 1.4.* Model II will be treated more thoroughly in future work. Here we simply sketch the proof of Theorem 1.4. Once again we proceed by comparison to oriented percolation.

This time we make two (symmetric) comparisons. In the first case, a site i is deemed to be *infected* at time n if $p_i(n) > \varepsilon$ for some fixed $\varepsilon \in (0, 1/4]$ and *recovered* otherwise. For simplicity, we assume $N \equiv 1$ and $\mu = 2$ ('symmetric case'). Bounds on infection and non-recovery rates in terms of s for an associated discrete time flip process based on suitably measurable events that will later constitute the necessary "good event" are entirely analogous to those in the proof of Theorem 1.2 (although now things are much simpler as we do not have to worry about a potentially big random environment and the number of parameters involved is much smaller).

We proceed exactly as in Section 4.1.2 noting that while $0 \leq p_i(t) \leq 2\varepsilon$, we can use Corollary 5.3 to compare $p_i(t)$ to a suitable one-dimensional diffusion. This time, in place of the Feller branching diffusion, we have a Fisher-Wright diffusion. Indeed, as long as $0 \leq p_i(t) \leq 2\varepsilon$ we have

$$\begin{aligned} dp_i(t) &= \sum_{j \in \mathbb{Z}^d} m_{ij} (p_j(t) - p_i(t)) dt \\ &\quad + sp_i(t) (1 - p_i(t)) (1 - 2p_i(t)) dt + \sqrt{p_i(t) (1 - p_i(t))} dW_i(t), \\ &\geq \left[s(1 - 2\varepsilon)(1 - 4\varepsilon) - \sum_{j \in \mathbb{Z}^d} m_{ij} \right] p_i(t) dt + \sqrt{p_i(t)(1 - p_i(t))} dW_i(t). \end{aligned} \tag{66}$$

For a site j with $p_j(n) > \varepsilon$ (we call such a site '*occupied*'), we introduce the stopping time $\mathcal{S}_n^j(\varepsilon) := \inf\{t \geq n : p_j(t) < \varepsilon/2\}$. By (66) we see that by increasing the value of s , we can arrange that the probability that $\mathcal{S}_n^j(\varepsilon) > n + 1$ for such a site j is sufficiently close to 1.

Again following the arguments in Section 4.1.2, for a site $i \in \mathbb{Z}^d$, given the existence of an *occupied neighbour* (that means there is a $j \in \mathbb{Z}^d$ with $m_{ij} > 0$ and $p_j(t) > \varepsilon$), as long as $t < \mathcal{S}_n^j(\varepsilon)$ and $p_i(t) < \varepsilon$, we can compare $p_i(t)$ to a Fisher-Wright diffusion with immigration and positive drift using the observation

$$\begin{aligned} dp_i(t) &\geq \frac{\varepsilon}{2} m_{ij} dt + sp_i(t)(1 - p_i(t))(1 - 2p_i(t)) dt \\ &\quad - \sum_{j \in \mathbb{Z}^d} m_{ij} p_i(t) dt + \sqrt{p_i(t)(1 - p_i(t))} dW_i(t). \end{aligned}$$

Now it is easy to provide bounds on the infection probability (in the presence of an occupied neighbour) and the recovery probability in a similar fashion

to Lemma 4.5. Such bounds then establish the required lower bound on the probability of the corresponding “good event”, and hence allow a comparison to oriented percolation whenever there exists at least one $i \in \mathbb{Z}^d$ with $p_i(0) > \varepsilon$. Moreover, if $p_i(0) > \varepsilon$ for all $i \in \mathbb{Z}^d$, using Theorem 3.6, we have, for s large enough,

$$\liminf_{n \rightarrow \infty} \mathbb{P} [p_0(n) > \varepsilon] > \frac{19}{20}.$$

Since $1 - p_i(t)$ satisfies the same equations as $p_i(t)$ the result follows. \square

5. Appendix.

5.1. *Existence for Model I.* Although existence and uniqueness of solutions to Model II is classical (see Shiga and Shimizu 1980), existence of solutions to Model I is not covered by standard results and so we outline it here.

For the state space of Model I, recall the notation from (5), i.e., for $p \geq 1, p \in \mathbb{N}$, let

$$\ell_\Gamma^p = \{x \in \mathbb{R}^{\mathbb{Z}^d} : \|x\|_{\Gamma,p} < \infty\}, \quad (67)$$

where the weighted ℓ^p -norm $\|\cdot\|_{\Gamma,p}$ is defined, for $\Gamma = \{\Gamma_i\}_{i \in \mathbb{Z}^d} \in l^1((0, \infty)^{\mathbb{Z}^d})$ by

$$\|x\|_{\Gamma,p} = \left(\sum_{i \in \mathbb{Z}^d} \Gamma_i |x_i|^p \right)^{\frac{1}{p}}.$$

We assume $\Gamma_i/\Gamma_j < f(\|i - j\|)$ for some continuous function $f : [0, \infty) \rightarrow [0, \infty)$, where $\|\cdot\|$ either denotes the lattice distance or the maximum norm on \mathbb{Z}^d . For example, we can take $\Gamma_i = e^{-\|i\|}$ for $i \in \mathbb{Z}^d$.

Theorem 5.1 (Existence for Model I) *Fix $p \geq 1, p \in \mathbb{N}$. Suppose $X(0) \in \ell_\Gamma^{4p}$ non-negative. Let $\alpha, M \in (0, \infty)$ and assume that λ_{ij}, m_{ij} are positive, bounded and of finite range for all $i, j \in \mathbb{Z}^d$. Let $\{B_i(t), i \in \mathbb{Z}^d\}_{t \geq 0}$ be a family of independent Brownian motions. Then, almost surely, there exists a weak non-negative solution to*

$$\begin{aligned} dX_i(t) = & \sum_{j \in \mathbb{Z}^d} m_{ij} (X_j(t) - X_i(t)) dt \\ & + \alpha \left(M - \sum_{j \in \mathbb{Z}^d} \lambda_{ij} X_j(t) \right) X_i(t) dt + \sqrt{X_i(t)} dB_i(t), \end{aligned} \quad (68)$$

taking values in ℓ_Γ^{4p} , which is continuous in each component.

Remark. The above theorem also covers the competing species case by re-indexing and an appropriate relabelling of the parameters. Moreover, the initial condition in (7) guarantees that $X(0) \in \ell_\Gamma^{4p}$ for all $p \geq 1, p \in \mathbb{N}$ \square

Proof. We construct a sequence of approximating (finite dimensional) systems of stochastic differential equations converging weakly to a component-wise continuous limit in ℓ_Γ^{4p} , which solves (68). For $n \in \mathbb{N}$, let

$$I_n = [-n, n]^d \cap \mathbb{Z}^d \quad (69)$$

with $I_n \uparrow \mathbb{Z}^d$ as $n \rightarrow \infty$. Define X^n to be the solution of the system of interacting stochastic differential equations given by

$$\begin{aligned} dX_i^n(t) &= \sum_{j \in \mathbb{Z}^d} m_{ij} (X_j^n(t) - X_i^n(t)) dt \\ &\quad + \alpha \left(M - \sum_{j \in \mathbb{Z}^d} \lambda_{ij} X_j^n(t) \right) X_i^n(t) ds + \sqrt{X_i^n(t)} dB_i(t) \end{aligned} \quad (70)$$

and $X_i^n(0) = X_i(0)$ if $i \in I_n$, and $X_i^n(t) = 0$ for all $t \geq 0$ if $i \notin I_n$, assuming the initial condition $X^n(0) \in \ell_\Gamma^{4p}$. Note that the above system is essentially finite-dimensional with locally Lipschitz coefficients for each n and hence we can apply standard results to obtain existence and uniqueness up to a random explosion time which we denote by ϑ_{X^n} . Now, for $\varepsilon > 0$, consider the *dominating system*

$$\begin{aligned} d\tilde{X}_i^n(t) &= \sum_{j \in \mathbb{Z}^d} m_{ij} (\tilde{X}_j^n(t) - \tilde{X}_i^n(t)) dt \\ &\quad + \alpha M \tilde{X}_i^n(t) dt + \varepsilon dt + \sqrt{\tilde{X}_i^n(t)} dB_i(t) \end{aligned} \quad (71)$$

with $\tilde{X}_i^n(0) = X_i^n(0)$ for $i \in I_n$ and $\tilde{X}_i^n(t) = 0$ for $i \notin I_n$. Denote its explosion time by $\vartheta_{\tilde{X}^n}$. Applying the comparison theorem of Geiß and Manthey (1994) for finite-dimensional stochastic differential equations, we have that

$$\mathbb{P} \left[X_i^n(t) \leq \tilde{X}_i^n(t), t \in [0, \vartheta_{X^n} \wedge \vartheta_{\tilde{X}^n}] \right] = 1. \quad (72)$$

As the drift in the system (71) fulfils a linear growth condition and is Lipschitz, using Gronwall's inequality (arguing exactly as in the proof of Theorem 5.2.1 in Sturm (2002)), we have that if $X^n(0) = \tilde{X}^n(0) \in \ell_\Gamma^{4p}$, then

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|X^n(t)\|_{\Gamma, 4p}^{4p} \right] \leq \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\tilde{X}^n(t)\|_{\Gamma, 4p}^{4p} \right] < \tilde{C}(4p, T) < \infty, \quad (73)$$

where the bound is independent of n , showing that both explosion times are a.s. infinite. Moreover, note that $X^n(t)$ is also in ℓ_Γ^{2p} for all t a.s.

A crucial step in the existence proof will be to establish a suitable growth condition on the drift coefficients of the infinite dimensional system. Usually, one needs to assume a linear growth condition, but the fact that we have a non-exploding dominating system allows us to work with a more general quadratic growth condition instead. As a consequence we must assume that our initial conditions lie in ℓ_Γ^{4p} instead of ℓ_Γ^{2p} (which is the situation of Theorem 2.3 in Sturm (2003), see also the proof of Theorem 5.2.1 in Sturm (2002) for more details). To establish the quadratic growth condition, we compute

$$\begin{aligned} & \left\| \alpha \left(M - \sum_{j \in \mathbb{Z}^d} \lambda_{ij} X_j^n(t) \right) X_i^n(t) \right\|_{\Gamma, 2p}^{2p} \\ &= \sum_{i \in I_n} \Gamma_i \left(\alpha \left(M - \sum_{j \in \mathbb{Z}^d} \lambda_{ij} X_j^n(t) \right) X_i^n(t) \right)^{2p} \\ &\leq c' \sum_{i \in I_n} \Gamma_i (\alpha M X_i^n(t))^{2p} + c' \sum_{i \in I_n} \Gamma_i \left(\alpha X_i^n(t) \sum_{j \in \mathbb{Z}^d} \lambda_{ij} X_j^n(t) \right)^{2p} \end{aligned}$$

for some finite $c' > 0$, where in the first equality we have used the fact that $X_i^n(t) = 0$ for all $t > 0$ if $i \notin I_n$. So, modifying the value of c' from line to line as necessary, we conclude that

$$\begin{aligned} & \left\| \alpha \left(M - \sum_{j \in \mathbb{Z}^d} \lambda_{ij} X_j^n(t) \right) X_i^n(t) \right\|_{\Gamma, 2p}^{2p} \\ &\leq c' \sum_{i \in I_n} \Gamma_i (X_i^n(t))^{2p} + c' \sum_{i \in I_n} \Gamma_i \sum_{j \in \mathbb{Z}^d} (\lambda_{ij} X_i^n(t) X_j^n(t))^{2p} \\ &\leq c' \sum_{i \in I_n} \Gamma_i (X_i^n(t))^{2p} + c' \sum_{i \in I_n} \Gamma_i (\lambda_{ii} X_i^n(t))^{4p} \\ &\quad + c' \sum_{i \in I_n} \Gamma_i \sum_{i \neq j \in \mathbb{Z}^d} (\lambda_{ij} X_i^n(t) X_j^n(t))^{2p}. \end{aligned}$$

Observe that since λ_{ij} is of finite range, under our assumptions on Γ , Γ_i/Γ_j is bounded for all i, j within this range, and so, replacing the mixed terms

by the square of the diagonal terms ensures that

$$\begin{aligned}
& \left\| \alpha \left(M - \sum_{j \in \mathbb{Z}^d} \lambda_{ij} X_j^n(t) \right) X_i^n(t) \right\|_{\Gamma, 2p}^{2p} \\
& \leq c' \sum_{i \in I_n} \Gamma_i(X_i^n(t))^{2p} + c' \sum_{i \in I_n} \Gamma_i(X_i^n(t))^{4p} \\
& = c' \left(\|X^n(t)\|_{\Gamma, 2p}^{2p} + \|X^n(t)\|_{\Gamma, 4p}^{4p} \right).
\end{aligned}$$

Note that although we have not considered the migration term in the above computations, it is easy to see that it satisfies a linear growth condition. Hence, using (73) we have for $0 \leq t \leq T$

$$\begin{aligned}
\mathbb{E} \left[\left\| \sum_{j \in \mathbb{Z}^d} m_{ij} (X_j^n(t) - X_i^n(t)) + \alpha \left(M - \sum_{j \in \mathbb{Z}^d} \lambda_{ij} X_j^n(t) \right) X_i^n(t) \right\|_{\Gamma, 2p}^{2p} \right] \\
\leq C + \tilde{C}(4p, T) < \infty \quad (74)
\end{aligned}$$

After these preparations, we now prove tightness of the approximating sys-

tem. For $0 \leq s \leq t \leq T$ with $|t - s| \leq \delta < 1$, consider

$$\begin{aligned}
& \mathbb{E} \left[\|X^n(t) - X^n(s)\|_{\Gamma, 2p}^{2p} \right] \\
&= \mathbb{E} \left[\sum_{i \in I_n} \Gamma_i \left(\int_s^t \left(\sum_{j \in \mathbb{Z}^d} m_{ij} (X_j^n(u) - X_i^n(u)) \right. \right. \right. \\
&\quad \left. \left. \left. + \alpha [M - \sum_{j \in \mathbb{Z}^d} \lambda_{ij} X_j^n(u)] X_i^n(u) \right) du + \int_s^t \sqrt{X_i^n(u)} dB_i(u) \right)^{2p} \right] \\
&\leq C_p \mathbb{E} \left[\sum_{i \in I_n} \Gamma_i \left(\int_s^t \sum_{j \in \mathbb{Z}^d} m_{ij} (X_j^n(u) - X_i^n(u)) \right. \right. \\
&\quad \left. \left. + \sum_{i \in I_n} \alpha [M - \sum_{j \in \mathbb{Z}^d} \lambda_{ij} X_j^n(u)] X_i^n(u) du \right)^{2p} \right] \\
&\quad + C_p \mathbb{E} \left[\sum_{i \in I_n} \left(\int_s^t \sqrt{X_i^n(u)} dB_i(u) \right)^{2p} \right] \\
&\leq C_p \delta^{2p-1} \mathbb{E} \left[\int_s^t \left\| \sum_{j \in \mathbb{Z}^d} m_{ij} (X_j^n(u) - X_i^n(u)) \right. \right. \\
&\quad \left. \left. + \alpha [M - \sum_{j \in \mathbb{Z}^d} \lambda_{ij} X_j^n(u)] X_i^n(u) \right\|_{\Gamma, 2p}^{2p} du \right] \\
&\quad + C_p \delta^{p-1} \sum_{i \in I_n} \Gamma_i \int_s^t \mathbb{E}[(X_i^n(u))^p] du \\
&\leq C_{p, \delta, T} < \infty,
\end{aligned}$$

where we applied Jensen's Inequality to the drift terms, Burkholder's Inequality followed by Jensen's Inequality to the diffusion term and finally used the quadratic growth condition (74). Since $\Gamma_i > 0$ for each $i \in I$ we have by Markov's inequality for $a_i \geq 0$, using (73) again,

$$\sup_n \mathbb{P} \left[\sup_{0 \leq t \leq T} |X_i^n(t)| > a_i \right] < \frac{\tilde{C}(4p, T)}{a_i^{4p} \Gamma_i} \quad (75)$$

and

$$\sup_n \sup_{|t-s| \leq \delta} \sup_{0 \leq t \leq T} \mathbb{P} \left[|X_i^n(t) - X_i^n(s)| > a_i \right] < \frac{C_{p, \delta, T}}{a_i^{2p} \Gamma_i} \quad (76)$$

and so combining (75) and (76) with Theorem 3.8.6 of Ethier and Kurtz (1986), we have that X_i^n are tight in the space of càdlàg paths $D([0, \infty), \mathbb{R})$ equipped with the usual Skorohod topology. Moreover, since $C([0, \infty), \mathbb{R})$ is

a closed subspace with respect to the induced topology we also have that X_i^n are tight in the space $C([0, \infty), \mathbb{R})$ for each $i \in \mathbb{Z}^d$. By Cantor diagonalisation we can find a weakly convergent subsequence in $C([0, \infty), \mathbb{R}^{\mathbb{Z}^d})$ denoted by $X^{n'}$, converging to a limit which is continuous in each component.

It is easy to show that every limit indeed solves (68). Using arguments from Shiga and Shimizu (1980), one can also show positivity of the limit.

Finally, we need to show that the limit lives in ℓ_Γ^{4p} . Since we have that $\tilde{X}_i^n(t) = 0$ for all $t \geq 0$ for all $i \notin I_n$, the dominating system, representing a supercritical branching process, is monotone as $I_n \uparrow \mathbb{Z}^d$ and so by the Monotone Convergence Theorem we have that

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|\tilde{X}^n(t)\|_{\Gamma, 4p}^{4p} \right] \uparrow \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\tilde{X}(t)\|_{\Gamma, 4p}^{4p} \right] < \infty,$$

and we note that the same arguments as above give a limit for the \tilde{X}^n processes.

Then using the comparison theorem from Geiß and Manthey (1994) and Dominated Convergence we have that $\mathbb{E}[\sup_{0 \leq t \leq T} \|X^n\|_{\Gamma, 4p}^{4p}]$ remains bounded as $n \rightarrow \infty$, showing that the limiting system lives in ℓ_Γ^{4p} for all $t \geq 0$ almost surely.

5.2. *A comparison theorem.* The following Ikeda-Watanabe (1981) Comparison Theorem for one-dimensional diffusions, and in particular its subsequent simple corollary, are tailored for our purposes in Section 4.

Theorem 5.2 (Ikeda-Watanabe, 1981) *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a filtered probability space and let $x_1(t, \omega), x_2(t, \omega)$ be two real $\{\mathcal{F}_t\}$ -adapted processes. Let $B(t, \omega)$ be a one-dimensional $\{\mathcal{F}_t\}$ -Brownian motion such that $B(0) = 0$ a.s. and let $\beta_1(t, \omega), \beta_2(t, \omega)$ be two real $\{\mathcal{F}_t\}$ -adapted previsible drifts. Assume that with probability one,*

$$x_i(t) - x_i(0) = \int_0^t \sqrt{x_i(s)} dB(s) + \int_0^t \beta_i(s) ds, \quad i = 1, 2,$$

and that pathwise uniqueness of solutions holds for at least one of the equations. Moreover, assume that with probability one,

- $x_1(0) \leq x_2(0)$,
- $\beta_1(t) \leq b_1(t, x_1)$ for every $t \geq 0$,
- $\beta_2(t) \geq b_2(t, x_2)$ for every $t \geq 0$,

for two real continuous functions $b_1(t, x), b_2(t, x)$ defined on $[0, \infty) \times \mathbb{R}$ such that

$$b_1(t, x) \leq b_2(t, x) \quad (77)$$

for all $t \geq 0$ and $x \in \mathbb{R}$. Then,

$$x_1(t) \leq x_2(t) \quad \text{for every } t \geq 0.$$

The proof is standard and can be found in Ikeda and Watanbe (1981).

Note that, for example, the theorem holds if one of the processes is a positive one-dimensional non-exploding diffusion with locally Lipschitz drift, so that a unique strong solution exists. The following corollaries allow a comparison even if (77) holds for intervals only.

Corollary 5.3 *In the framework of Theorem 5.2, assume that x_1 and x_2 are positive and non-exploding. Let $\delta > 0$.*

- a) *Suppose that assumption (77) on the functions b_1, b_2 is required only for all $x \in [\delta, \infty)$. Assume $x_1(0) \leq x_2(0)$. Define*

$$\tau_\delta^{x_2} := \inf\{t \geq 0 : x_2(t) \leq \delta\}.$$

Then, with probability one,

$$x_1(t \wedge \tau_\delta^{x_2}) \leq x_2(t \wedge \tau_\delta^{x_2}) \quad \text{for all } t \geq 0.$$

- b) *Suppose that assumption (77) on the functions b_1, b_2 is required only for all $x \in [0, \delta]$. Assume $x_1(0) \leq x_2(0)$. Define*

$$\tau_\delta^{x_1} := \inf\{t \geq 0 : x_1(t) \geq \delta\}.$$

Then, with probability one,

$$x_1(t \wedge \tau_\delta^{x_1}) \leq x_2(t \wedge \tau_\delta^{x_1}) \quad \text{for all } t \geq 0.$$

Proof. This follows directly from the arguments (due to Le Gall) in the proof in Rogers and Williams (1987), 43.1. By the usual localisation, we may assume that b_1, b_2 are Lipschitz with some finite Lipschitz constant $\mathcal{K} > 0$. For a), observe that, abbreviating $\tau = \tau_\delta^{x_2}$,

$$\begin{aligned} & (x_1(t \wedge \tau) - x_2(t \wedge \tau))^+ \\ &= \int_0^{t \wedge \tau} 1_{\{x_1(s) - x_2(s) > 0\}} \left(\sqrt{x_1(s)} - \sqrt{x_2(s)} \right) dB_i(s) \\ &+ \int_0^{t \wedge \tau} 1_{\{x_1(s) - x_2(s) > 0\}} \left(\beta_1(s) - \beta_2(s) \right) ds. \end{aligned}$$

The diffusion coefficients $\sqrt{\cdot}$ satisfy the Yamada-Watanabe condition (see Rogers and Williams (1987), V 40.1), and therefore the local time of $x_1(t) - x_2(t)$ in level 0 is identically equal to 0. Hence

$$\begin{aligned}
0 &\leq \mathbb{E} \left[(x_1(t) - x_2(t))^+; t \leq \tau \right] \\
&= \mathbb{E} \left[(x_1(t \wedge \tau) - x_2(t \wedge \tau))^+ \right] \\
&\leq \mathbb{E} \int_0^{t \wedge \tau} 1_{\{x_1(s) - x_2(s) > 0\}} \left(b_1(x_1(s)) - b_2(x_2(s)) \right) ds \\
&\leq \mathbb{E} \int_0^{t \wedge \tau} 1_{\{x_1(s) - x_2(s) > 0\}} \left(b_1(x_1(s)) - b_1(x_2(s)) \right) ds \\
&\leq \mathcal{K} \mathbb{E} \int_0^{t \wedge \tau} (x_1(s) - x_2(s))^+ ds, \\
&\leq \mathcal{K} \int_0^t \mathbb{E} \left[(x_1(s) - x_2(s))^+; s \leq \tau \right] ds,
\end{aligned}$$

from which we infer that

$$\mathbb{P}[\{t \leq \tau : x_2(t) > x_1(t)\}] = 0,$$

by Gronwall's inequality. The second statement b) follows in a similar fashion. \square

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