An $L^2$ model for selfadjoint elliptic differential operators with constant coefficients on bounded domains

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The selfadjoint realization of a second order elliptic differential expression with Dirichlet boundary conditions is shown to be unitarily equivalent to the maximal multiplication operator with the independent variable in an explicit $L^2$ model space.

1 Introduction

It is well known that every selfadjoint operator in a Hilbert space is unitarily equivalent to a multiplication operator in an abstract $L^2$ space. For the case of a selfadjoint Sturm–Liouville differential operator on $(0, \infty)$, where, e.g., $\infty$ is in the limit point case and 0 is a regular endpoint, the integral representation of the classical Titchmarsh–Weyl $m$-function gives rise to a multiplication operator model in a more explicit $L^2$ space; cf. [4, 10, 13, 14]. The main objective of the present note is to construct an $L^2$ model space in a similar way for the Dirichlet realization $A$ of a second order elliptic differential expression with constant coefficients on a bounded domain $\Omega \subset \mathbb{R}^n$, $n > 1$. It will be shown that the maximal multiplication operator in this model space is unitarily equivalent to $A$. $L^2$ models for other selfadjoint realizations can be constructed analogously.

2 An $L^2$ model for a selfadjoint elliptic operator with Dirichlet boundary conditions

Let $\Omega \subset \mathbb{R}^n$, $n > 1$, be a bounded domain with a smooth boundary $\partial \Omega$ and denote by $H^s(\Omega)$ and $H^s(\partial \Omega)$, $s \in \mathbb{R}$, the Sobolev spaces of order $s$ on $\Omega$ and $\partial \Omega$, respectively. The trace of $u \in H^s(\Omega)$, $s > 1/2$, on $\partial \Omega$ is denoted by $u|_{\partial \Omega}$ and belongs to the space $H^{s-1/2}(\partial \Omega)$. The inner product $(\cdot, \cdot)$ on $L^2(\partial \Omega)$ can be extended by continuity to $H^{1/2}(\partial \Omega) \times H^{-1/2}(\partial \Omega)$. Let $\iota \pm$ be isomorphisms from $H^{1/2}(\partial \Omega)$ onto $L^2(\partial \Omega)$ with $(x, y)_{1/2} = (\iota_{-} x, \iota_{+} y)$ for all $x \in H^{1/2}(\partial \Omega)$ and $y \in H^{-1/2}(\partial \Omega)$. If $\mathcal{H}, \mathcal{K}$ are Hilbert spaces, the space of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ is denoted by $\mathcal{L}(\mathcal{H}, \mathcal{K})$, or $\mathcal{L}(\mathcal{H})$ if $\mathcal{K} = \mathcal{H}$.

Let $a_{jk} \in \mathbb{C}$, $j, k = 1, \ldots, n$, suppose that the $n \times n$-matrix $(a_{jk})_{j,k=1}^n$ is positive and let $c > 0$. In the following we consider the elliptic differential expression $A = -\sum_{j,k=1}^n a_{jk} \partial_j \partial_k + c$. It is well known that the operator

$$Au = \Lambda u - \sum_{j,k=1}^n a_{jk} \partial_j \partial_k u + cu,$$

$$\text{dom } A = \{ u \in H^2(\Omega) : u|_{\partial \Omega} = 0 \},$$

(1)

is a positive selfadjoint operator in $L^2(\Omega)$ with compact resolvent, see, e.g., [6]. Besides the selfadjoint operator $A$ we shall make use of the so-called minimal operator $A_{\text{min}} = \Lambda u$, dom $A_{\text{min}} = \{ u \in H^2(\Omega) : u|_{\partial \Omega} = 0 \}$, where $\partial^{\nu}_{\nu} u|_{\partial \Omega}$ denotes the normal vector pointing outwards. Clearly, the minimal operator is a restriction of $A$ and hence symmetric. Furthermore, dom $A_{\text{min}}$ is dense in $L^2(\Omega)$ and $A_{\text{min}}$ is a closed operator with infinite deficiency indices. The adjoint $A_{\text{min}}^*$ is the maximal operator $A_{\text{max}}$ associated to $A$ which is defined on dom $A_{\text{max}} = \{ u \in L^2(\Omega) : \Lambda u \in L^2(\Omega) \}$. According to [9, Theorem 2.1] the trace map $u \mapsto u|_{\partial \Omega}$, $u \in H^s(\Omega)$, $s > 1/2$, can be extended by continuity to a surjective mapping from dom $A_{\text{max}}$ onto $H^{1/2}(\partial \Omega)$, where dom $A_{\text{max}}$ is equipped with the graph norm. As $A$ is positive and dom $A_{\text{max}} = \text{dom } A + \text{ker } A_{\text{max}}$ holds, it follows that for $y \in L^2(\partial \Omega)$ there is a unique function $u_0(y) \in \text{ker } A_{\text{max}}$ such that $y = \iota_{-} u_0(y)|_{\partial \Omega}$.

Theorem 2.1 For $\lambda$ from the resolvent set $\rho(A)$ of $A$ and $y \in L^2(\partial \Omega)$ we define

$$M(\lambda) y := -\lambda \iota_{+} (\partial^{\nu}_{\nu} (A - \lambda)^{-1} u_0(y))|_{\partial \Omega}.$$

Then $M(\lambda)$ is a bounded operator in $L^2(\partial \Omega)$, and the function $M : \rho(A) \to \mathcal{L}(L^2(\partial \Omega))$, $\lambda \mapsto M(\lambda)$ is an operator-valued Nevanlinna function, which admits an integral representation

$$M(\lambda) = \alpha + \int_{\mathbb{R}} \left( \frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right) d\Sigma(t),$$

(2)

where $\alpha \in \mathcal{L}(L^2(\partial \Omega))$ is a selfadjoint operator and $\Sigma : \mathbb{R} \to \mathcal{L}(L^2(\partial \Omega))$ is a nondecreasing operator function which satisfies $\int_{\mathbb{R}} (1 + t^2)^{-1} d\Sigma(t) \in \mathcal{L}(L^2(\partial \Omega))$.

The proof of Theorem 2.1 will be published elsewhere. It makes use of the notion of boundary triplets and Weyl functions.

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associated to symmetric operators from [5, 7], see also [1, 3, 8] for the elliptic case.

Let $\Sigma : \mathbb{R} \to L(L^2(\partial \Omega))$ be the nondecreasing operator function from the integral representation (2). The space $L^2_\Sigma(L^2(\partial \Omega))$ is defined as in [2, 7, 12]. Very roughly speaking it consists of $L^2(\partial \Omega)$-valued functions on $\mathbb{R}$ which are square-integrable with respect to the measure $d\Sigma$. The next theorem is the main result in this note.

**Theorem 2.2** The Dirichlet operator $A$ in (1) is unitarily equivalent to the maximal multiplication operator with the independent variable in $L^2_\Sigma(L^2(\partial \Omega))$.

**Proof.** The proof of Theorem 2.2 consists of two steps. In the first step it will be shown that the span of the defect spaces of the minimal operator $A_{\min}$ is dense in $L^2(\Omega)$. In the second step a unitary operator $U : L^2(\Omega) \to L^2_\Sigma(L^2(\partial \Omega))$ will be constructed, which fulfills $A = U^* A \Sigma U$, where $A \Sigma$ is the maximal multiplication operator with the independent variable in the model space $L^2_\Sigma(L^2(\partial \Omega))$.

**Step 1.** We claim that $A_{\min}$ has no eigenvalues. In fact, assume that $u \in \text{dom } A_{\min}$ is a solution of $A_{\min}u = \lambda u$ for some $\lambda \in \mathbb{R}$ and define the function $\tilde{u}$ to be the extension of $u$ by $0$ on $\mathbb{R}^n \setminus \Omega$. Then $u|_{\partial \Omega} = \partial L \lambda u|_{\partial \Omega} = 0$ and the equivalence of the graph norm induced by $A_{\min}$ to the $H^2$ norm imply $\tilde{u} \in H^2(\mathbb{R}^n)$. It follows that $\tilde{u}$ satisfies the equation $A \tilde{u} = \lambda \tilde{u}$ on $\mathbb{R}^n$. Hence $\tilde{u}$ is an eigenfunction of the selfadjoint operator $\tilde{A}$ associated to $\Lambda$ in $L^2(\mathbb{R}^n)$ defined on $\text{dom } \tilde{A} = H^2(\mathbb{R}^n)$. But $\tilde{A}$ has no eigenvalues (this can be seen, for example, with the help of the Fourier transform), and therefore $\tilde{u} = 0$. This implies $u = 0$ and hence $A_{\min}$ has no eigenvalues.

Since the spectrum of the selfadjoint operator $A$ in (1) consists only of eigenvalues it follows that $A_{\min}$ does not contain a nontrivial selfadjoint part, i.e., there is no nontrivial subspace $H \subset L^2(\Omega)$ which is invariant for the operator $A_{\min}$ such that the restriction $A_{\min} \upharpoonright \{H \cap \text{dom } H\}$ is selfadjoint in $H$. It is well known (see, e.g., [11]) that this is equivalent to

$$L^2(\Omega) = \text{span}\{\ker(A_{\min} - \lambda) : \lambda \in \mathbb{C} \setminus \mathbb{R}\} = \text{span}\{\ker(A - \lambda) : \lambda \in \mathbb{C} \setminus \mathbb{R}\}. \quad (3)\]$$

**Step 2.** Let $A_S$ be the maximal multiplication operator with the independent variable in $L^2_\Sigma(L^2(\partial \Omega))$ and denote the restriction of $A_S$ onto the dense subspace $\{f \in \text{dom } A_S : \int f d\Sigma = 0\}$ by $S_S$. For further details and the precise definition of $\text{dom } S_S$ we refer to [12, §7]. For $\lambda \in \mathbb{C} \setminus \mathbb{R}$ we define $\gamma(\lambda) \in L(L^2(\partial \Omega), L^2(\Omega))$ and $\tilde{\gamma}(\lambda) \in L(L^2(\partial \Omega), L^2_\Sigma(L^2(\partial \Omega)))$ by

$$\gamma(\lambda)y = (I + \lambda(A - \lambda)^{-1}u_0(y))y \quad \text{and} \quad \tilde{\gamma}(\lambda)y = (I - \lambda)^{-1}y, \quad y \in L^2(\partial \Omega),$$

where $u_0(y)$ is the unique solution in $\ker A_{\max}$ such that $\iota \cdot u_0(y)|_{\partial \Omega} = y$. Then we have $\text{ran } \gamma(\lambda) = \ker(A_{\max} - \lambda)$ and $\text{ran } \tilde{\gamma}(\lambda) = \ker(S_S - \lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Moreover, the equation

$$\gamma(\mu)^* \gamma(\lambda) = \frac{M(\lambda) - M(\mu)^*}{\lambda - \mu}, \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}, \quad (4)$$

holds, and $\gamma(\lambda) = (I + (\lambda - i)(A - \lambda)^{-1})\gamma(i)$ and $\tilde{\gamma}(\lambda) = (I + (\lambda - i)(A_S - \lambda)^{-1})\tilde{\gamma}(i)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. It follows from (3) and (4) that

$$V \left(\sum_{j=0}^{l} \gamma(\lambda_j) y_j \right)_{j=0}^{l} = \sum_{j=0}^{l} \gamma(\lambda_j) y_j, \quad \text{dom } V = \left\{ \sum_{j=0}^{l} \gamma(\lambda_j) y_j : \lambda_j \in \mathbb{C} \setminus \mathbb{R}, y_j \in L^2(\partial \Omega), j = 0, \ldots, l, l \in \mathbb{N} \right\},$$

is a well-defined isometric operator with dense domain in $L^2(\Omega)$. As a consequence of [12, Proposition 7.9 (ii)] $\text{ran } V$ is dense in $L^2_\Sigma(L^2(\partial \Omega))$ and hence $V$ admits a unique unitary extension $U : L^2(\Omega) \to L^2_\Sigma(L^2(\partial \Omega))$. For $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the equation

$$U(A - \lambda)^{-1} \gamma(i) = U \frac{1}{\lambda - i} (\gamma(\lambda) - \gamma(i)) = \frac{1}{\lambda - i} \left(\tilde{\gamma}(\lambda) - \tilde{\gamma}(i)\right) = (A_S - \lambda)^{-1} \tilde{\gamma}(i) = (A_S - \lambda)^{-1} U \gamma(i).$$

This implies $A_S U u = U A u$ for all $u \in \text{dom } A$, that is, $A$ and $A_S$ are unitarily equivalent. $\square$

**References**