Linearizations of a class of elliptic boundary value problems

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We construct linearizations for a class of second order elliptic eigenvalue dependent boundary value problems on smooth bounded domains with rational operator-valued Nevanlinna functions in the boundary condition.

1 Introduction

Various types of boundary value problems with eigenparameter dependent boundary conditions appear in many physical applications and have extensively been studied in a more or less abstract framework in the last decades. A lot of attention has been drawn to λ-dependent boundary value problems for ordinary differential operators and for large classes of functions in the boundary condition the theory is well understood. In this note we consider a second order elliptic differential expression \( L \) (which is regarded as an unbounded operator in \( L^2(\Omega) \)) defined on the so-called Beals space \( D_1(\Omega) \) subject to a \( λ \)-dependent boundary condition involving a rational \( \Sigma L^2(\partial\Omega) \)-valued Nevanlinna function \( τ \) and the traces and conormal derivatives of the functions in \( D_1(\Omega) \), see (1) below and [3] for a more abstract treatment. Here \( \Sigma L^2(\partial\Omega) \) denotes the space of bounded everywhere defined linear operators in \( L^2(\partial\Omega) \). In Theorem 3.1 we construct a self-adjoint operator \( A \) in the product Hilbert space \( L^2(\Omega) \oplus L^2(\partial\Omega) \oplus \cdots \oplus L^2(\partial\Omega) \) and we show that its compressed resolvent \( P_{L^2(\Omega)}(A-λ)^{-1} |_{L^2(\Omega)} \) onto \( L^2(\Omega) \) yields a solution of the \( λ \)-dependent boundary value problem (1). For the special case of \( λ \)-linear boundary condition we retrieve some results from [4, 5] in Corollary 3.2.

2 Elliptic differential operators defined on the Beals space

Let \( Ω \) be a bounded domain in \( \mathbb{R}^m \) with \( C^\infty \) boundary \( ∂Ω \) and closure \( \overline{Ω} \). We consider the differential expression

\[
(\mathcal{L}f)(x) := - \sum_{j,k=1}^{m} (D_j a_{jk} D_k f)(x) + \sum_{j=1}^{m} (a_j D_j f - D_j \pi_j f)(x) + a(x)f(x), \quad x \in Ω,
\]

with coefficients \( a_{jk}, a_j, a \in C^\infty(\overline{Ω}) \). We assume \( a_{jk}(x) = \overline{a_{jk}(x)} \) for all \( x \in \overline{Ω} \) and \( j, k = 1, \ldots, m \), and that \( a \) is real valued. Moreover, we assume that \( \sum_{j,k=1}^{m} a_{jk}(x) \xi_j \xi_k \geq C \sum_{k=1}^{m} \xi_k^2 \) holds for some constant \( C > 0 \) and all \( x \in Ω \), \((\xi_1, \ldots, \xi_m)^T \in \mathbb{R}^m \), i.e., \( L \) is a uniformly elliptic differential expression which is symmetric. Denote by \( n(x) \) the outward normal vector at \( x \in ∂Ω \). We say that \( f \in H^2_{loc}(Ω) \) has \( L^2 \) boundary value on \( ∂Ω \) if the limit \( f|_{∂Ω} := \lim_{x \to ∂Ω, f(x) - εn(x)} \) exists in \( L^2(∂Ω) \). The differential expression \( L \) is then regarded as an operator in \( L^2(Ω) \) which is defined on the so-called Beals space \( D_1(Ω) := \{ f \in L^2(Ω) | f \in L^2(Ω), f, \frac{∂f}{∂ν}, \ldots, \frac{∂^rf}{∂ν^r} \text{ have } L^2 \text{ boundary values on } ∂Ω \} \), cf. [2] and e.g. [1]. For the general spectral theory of operators associated with \( L \) we refer the reader to [7, 8, 9] and to e.g. [3, 6] for a more abstract extension theory. We recall that the mapping \( \frac{∂f}{∂ν}|_{∂Ω} := \sum_{j,k=1}^{m} a_{jk} n_j \frac{∂f}{∂ν}|_{∂Ω} + \sum_{j=1}^{m} \pi_j f|_{∂Ω}, \quad f \in D_1(Ω), \) is surjective onto \( L^2(∂Ω) \) and Green’s identity (\( \mathcal{L}f, g|_{Ω} - (f, \mathcal{L}g)|_{Ω} = (f|_{∂Ω}, \frac{∂g}{∂ν}|_{∂Ω})_{∂Ω} - (\frac{∂f}{∂ν}|_{∂Ω}, g|_{∂Ω})_{∂Ω} \) holds for all \( f, g \in D_1(Ω) \).

3 An eigenvalue dependent elliptic boundary value problem

Let \( A_1, B_i \in \Sigma L^2(∂Ω), \quad i = 1, \ldots, n, \) be bounded self-adjoint operators in \( L^2(∂Ω) \) and assume that the \( B_i \) are uniformly positive, that is, \( σ(B_i) \subset (0, \infty), \quad i = 1, \ldots, n \). Then the function

\[
C \setminus \mathbb{R} \ni λ \mapsto τ(λ) := A_1 + λB_1 + \sum_{j=2}^{n} B_j^{1/2}(A_j - λ)^{-1} B_j^{1/2} \in \Sigma L^2(∂Ω)
\]

is an \( \Sigma L^2(∂Ω) \)-valued Nevanlinna function, i.e., \( τ \) is holomorphic on \( C \setminus \mathbb{R} \), \( τ(λ) = τ(λ)^* \), \( λ \in C \setminus \mathbb{R} \), and \( \text{Im} \ τ(λ) \) is a nonnegative operator for all \( λ \in C^+ \). Note that by the assumption \( 0 \notin σ(B_i), \quad B_i \geq 0, \) here the operator \( \text{Im} \ τ(λ) \) is even uniformly positive (uniformly negative) for \( λ \in C^+ \) (\( λ \in C^- \) respectively). Moreover \( τ \) can be analytically continued to all real \( λ \) which belong to \( σ(A_2) \cap \cdots \cap σ(A_n) \).

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We consider the following elliptic boundary value problem: For a given \( g \in L^2(\Omega) \) and \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) find a function \( f \in \mathcal{D}_1(\Omega) \) such that

\[
(\mathcal{L} - \lambda)f = g \quad \text{and} \quad \tau(\lambda) \frac{\partial f}{\partial \nu}|_{\partial \Omega} + f|_{\partial \Omega} = 0
\]

holds. In the next theorem we show how this problem can be solved with the help of the compressed resolvent of a self-adjoint operator \( \tilde{A} \) in \( L^2(\Omega) \oplus (L^2(\partial \Omega))^n \).

**Theorem 3.1** The operator \( \tilde{A} = \{f, h_1, \ldots, h_n\} \) defined on

\[
\text{dom } \tilde{A} = \left\{ \left\{ f, h_1, \ldots, h_n \right\} : \frac{\partial f}{\partial \nu}|_{\partial \Omega} = B_1^{-1/2}h_1 = B_j^{-1/2}(h_j' - A_jh_j), \quad j = 2, \ldots, n, \right\}
\]

is self-adjoint in the Hilbert space \( L^2(\Omega) \oplus (L^2(\partial \Omega))^n \). For all \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) the unique solution \( f \in \mathcal{D}_1(\Omega) \) of the boundary value problem \( (1) \) is given by \( f = P_{L^2(\Omega)}(\tilde{A} - \lambda)^{-1}|_{L^2(\Omega)}g \).

**Proof.** Let us first verify that \( \tilde{A} \) is well defined as an operator. In fact, if \( f = 0 \) and \( h_1 = \cdots = h_n = 0 \) then obviously \( \mathcal{L}f = 0 \) and the boundary conditions reduce to \( 0 = \frac{\partial f}{\partial \nu}|_{\partial \Omega} = B_2^{-1/2}h_2^i = \cdots = B_n^{-1/2}h_n^i \) and \( f|_{\partial \Omega} = -B_1^{-1/2}h_1^i \). Since \( 0 \in \rho(B_j), \quad j = 1, \ldots, n \) we obtain \( h_1^i = \cdots = h_n^i = 0 \), i.e., \( \tilde{A} \) is an operator. Next we check that \( \tilde{A} \) is symmetric in \( L^2(\Omega) \oplus (L^2(\partial \Omega))^n \). For this, let \( f = \{f, h_1, \ldots, h_n\} \), \( g = \{g, k_1, \ldots, k_n\} \) \( \in \text{dom } \tilde{A} \) and \( \tilde{A}f = \{\mathcal{L}f, h_1', \ldots, h_n'\} \), \( \tilde{A}g = \{\mathcal{L}g, k_1', \ldots, k_n'\} \). Making use of Green’s identity we obtain

\[
(\tilde{A}f, g) - (f, \tilde{A}g) = (f|_{\partial \Omega}, \frac{\partial f}{\partial \nu}|_{\partial \Omega} - \frac{\partial g}{\partial \nu}|_{\partial \Omega}) + \sum_{i=1}^{n}((h_i', k_i)|_{\partial \Omega} - (h_i, k_i)|_{\partial \Omega}) \tag{2}
\]

and a straightforward calculation using the boundary conditions satisfied by \( f, g \in \text{dom } \tilde{A} \) shows that \( (2) \) is zero and hence \( \tilde{A} \) is symmetric. For the self-adjointness of \( \tilde{A} \) it is now sufficient to prove \( \text{ran } (\tilde{A} - \lambda_+)^{-1} = \text{ran } (\tilde{A} - \lambda_-)^{-1} \) for some \( \lambda_+ \in \mathbb{C}^+ \) and \( \lambda_- \in \mathbb{C}^- \). This follows from a perturbation argument as in [3, Theorem 5.1].

Let now \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) and set \( f := P_{L^2(\Omega)}(\tilde{A} - \lambda)^{-1}|_{L^2(\Omega)}g \). If we denote the element \( P_{L^2(\Omega)}(\tilde{A} - \lambda)^{-1}|_{L^2(\Omega)}g \) by \( \{h_1, \ldots, h_n\} \), \( h_i \in L^2(\partial \Omega) \), then \( \{f, h_1, \ldots, h_n\} \) belongs to \( \text{dom } \tilde{A} \) and \( \tilde{A}\{f, h_1, \ldots, h_n\} = \{g + \lambda f, \lambda h_1, \ldots, \lambda h_n\} \) holds. Hence we have \( \tilde{A}f = g + \lambda f \) and it remains to show that the boundary condition in \( (1) \) is satisfied. In fact, since \( \phi = \{f, h_1, \ldots, h_n\} \in \text{dom } \tilde{A} \) and \( h_i' = \lambda_i h_i, \quad i = 1, \ldots, n \), we obtain

\[
\tau(\lambda) \frac{\partial f}{\partial \nu}|_{\partial \Omega} = (A_1 + \lambda B_1 + \sum_{j=2}^{n} B_j^{1/2}(A_j - \lambda)^{-1}B_j^{1/2}) \frac{\partial f}{\partial \nu}|_{\partial \Omega} = \sum_{j=2}^{n} B_j^{1/2}(A_j - \lambda)^{-1}B_j^{1/2} \frac{\partial f}{\partial \nu}|_{\partial \Omega} = 0
\]

and hence \( f = P_{L^2(\Omega)}(\tilde{A} - \lambda)^{-1}|_{L^2(\Omega)}g \) solves the boundary value problem \( (1) \). The uniqueness follows as in [3].

**Corollary 3.2.** Let \( A \) and \( B \) be bounded self-adjoint operators in \( L^2(\partial \Omega) \) and assume that \( B \) is uniformly positive. Then \( \tilde{A} = \{f, B^{1/2} \frac{\partial f}{\partial \nu}|_{\partial \Omega} \} \) is a self-adjoint operator in \( L^2(\Omega) \oplus L^2(\partial \Omega) \) and the unique solution \( f \in \mathcal{D}_1(\Omega) \) of the \( \lambda \)-linear boundary value problem

\[
(\mathcal{L} - \lambda)f = g, \quad (A + \lambda B)\frac{\partial f}{\partial \nu}|_{\partial \Omega} + f|_{\partial \Omega} = 0,
\]

is given by \( f = P_{L^2(\Omega)}(\tilde{A} - \lambda)^{-1}|_{L^2(\Omega)}g \).

**References**