Cubature for infinite dimensional SDEs

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Introduction to SPDEs

Stochastic differential equations in infinite dimensional spaces are increasingly popular as models, e. g. for the stochastic evolution of interest rate curves.

- Let H be a separable real Hilbert space and consider the SDE

$$dX_t^x = (AX_t^x + \alpha(X_t^x))dt + \sum_{i=1}^d \beta_i(X_t^x)dB_t^i, \quad X_0^x = x \in H \tag{1}$$

where $A : \mathcal{D}(A) \subset H \to H$ is the (possibly unbounded) generator of a C_0-semigroup $(T_t)_{t\geq 0}$ on H, $\alpha, \beta_1, \dots, \beta_d : H \to H$ denote C^∞-bounded vector fields and $(B_t)_{t\geq 0}$ denotes a d-dimensional Brownian motion. For details see [1].

- X_t^x is called mild solution of (1) if it satisfies the *variation of constants* formula

$$X_t^x = T_t x + \int_0^t T_{t-s}\alpha(X_s^x)ds + \sum_{i=1}^d \int_0^t T_{t-s}\beta_i(X_s^x)dB_s^i. \tag{2}$$

We are interested in *weak approximations* of the solution, i. e. we want to approximate $u(t,x) = E(f(X_t^x))$ for suitable functionals $f : H \to \mathbf{R}$. In the case of an interest rate model, f might represent the payoff of an interest rate option and u its price.

Usually, this problem is numerically solved by finite-element or finite-difference schemes, see [2]. We propose an alternative method which closely fits to the concept of mild solutions.

Introduction to cubature on Wiener space

Cubature on Wiener space is a method for numerical solution of finite-dimensional SDEs, see [3]. Let $\lambda_j > 0$ and $\omega_j : [0,T] \to \mathbf{R}^d$ be of bounded variation, $j = 1, \dots, n$. If

$$E\left(\int_{0<t_1<\cdots<t_k<T} \circ dB_{t_1}^{i_1} \cdots \circ dB_{t_k}^{i_k}\right) = \sum_{j=1}^n \lambda_j \int_{0<t_1<\cdots<t_k<T} d\omega_j^{i_1}(t_1)\cdots d\omega_j^{i_k}(t_k)$$

for all $(i_1,\dots,i_k) \in \{0,\dots,d\}^k$, $k \geq 0$ up to degree m, then the weights λ_j and the paths ω_j form a *cubature formula on Wiener space*.

Given a cubature formula, the numerical method looks as follows:

1. Fix a (uniform) partition of $[0,T]$ with size $l+1$ and denote by $\omega_{j_1,\dots,j_l}$ the concatenation of the cubature paths $\omega_{j_r} : [0, T/l] \to \mathbf{R}^d$, $j_r \in \{1,\dots,n\}$, $r \in \{1,\dots,l\}$.
2. For the concatenated paths solve the ODE given by formally replacing all occurrences of "dB" in the SDE with "$d\omega_{j_1,\dots,j_l}$" and denote the result by $X_T^x(\omega_{j_1,\dots,j_l})$.
3. Approximate the quantity of interest $E(f(X_T^x))$ by a weighted average of the $f(X_T^x(\omega_{j_1,\dots,j_l}))$. For f smooth (or, with slight modifications, f Lipschitz), the methods converge (deterministically) with order $\frac{m-1}{2}$, i. e.
$$\sup_x \left| E(f(X_T^x)) - \sum_{(j_1,\dots,j_l)\in\{1,\dots,n\}^l} \lambda_{j_1}\cdots\lambda_{j_l} f(X_T^x(\omega_{j_1,\dots,j_l}))\right| \leq CT\left(\frac{T}{l}\right)^{(m-1)/2}.$$
4. Instead of calculating $X_T^x(\omega_{j_1,\dots,j_l})$ for all multi-indices $(j_1,\dots,j_l)$, one can approximate the weighted average by doing Monte-Carlo simulation on the "cubature tree".

Theoretical remarks and results

- Euler schemes are not easily transferred to infinite dimensional spaces because they conceptually rely on strong solutions, which are often not given due to unboundedness of the differential operator.
- Cubature on Wiener space, however, relies on mild solutions and can be immediately transferred to Hilbert spaces. Indeed, for a path of bounded variation $\omega : [0,T] \to \mathbf{R}^d$ define $X_t^x(\omega)$ as solution of
$$X_t^x(\omega) = T_t x + \int_0^t T_{t-s}\alpha(X_s^x(\omega))ds + \sum_{i=1}^d \int_0^t T_{t-s}\beta_i(X_s^x(\omega))d\omega^i(s),$$
i. e. as the mild solution of the corresponding ODE in H, and proceed as in the finite dimensional case.
- In general, we can prove (weak) convergence of the method, but without an order of convergence. The proof relies on the Yosida approximation of A and requires some additional smoothness assumptions on the vector fields.
- The finite dimensional result can be fully recovered, including the order, if the SDE (1) can be effectively restricted to the Fréchet space $\mathcal{D}(A^\infty)$, the projective limit of the Hilbert spaces $\mathcal{D}(A^k)$ endowed with their graph norms
$$\|x\|^2_{\mathcal{D}(A^k)} = \|x\|_H^2 + \sum_{i=1}^k \|A^i x\|_H^2, \quad x \in \mathcal{D}(A^k).$$

Theorem. *Assume that $\alpha, \beta_1, \dots, \beta_d$ are smooth vector fields $H \to \mathcal{D}(A^\infty)$ and that their restrictions to $\mathcal{D}(A^k)$ are C^∞-bounded as maps $\mathcal{D}(A^k) \to \mathcal{D}(A^k)$ for each $k \in \mathbf{N}$. Given $f \in C^\infty(\mathcal{D}(A^\infty); \mathbf{R})$ such that $\sup_{y\in\mathcal{D}(A^\infty)} |\beta_{i_1}\cdots\beta_{i_k} f(y)| < \infty$ for all multi-indices $m < \deg(i_1,\dots,i_k) \leq m+2$, $k \in \mathbf{N}$, we have*

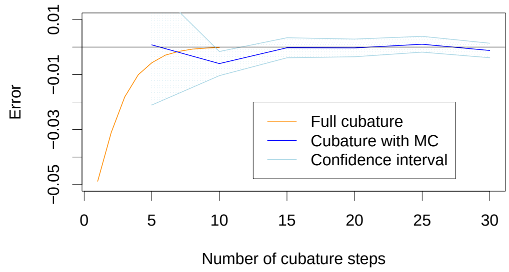
$$\sup_{x\in\mathcal{D}(A^\infty)} \left| E(f(X_T^x)) - \sum_{(j_1,\dots,j_l)\in\{1,\dots,n\}^l} \lambda_{j_1}\cdots\lambda_{j_l} f(X_T^x(\omega_{j_1,\dots,j_l}))\right| \leq CT\left(\frac{T}{l}\right)^{(m-1)/2}.$$

This means that the method converges with order $\frac{m-1}{2}$ and has deterministic a-priori error bounds.

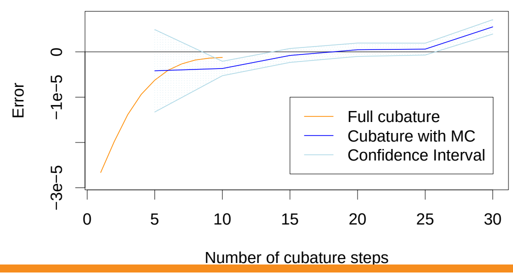
Numerical results

The examples are based on $H = L^2(]0,1[)$. We consider the Dirichlet-Laplacian Δ and SDEs driven by one-dimensional Brownian motion and approximate $E(\int_0^1 X_1^x(u)du)$.

- The Ornstein-Uhlenbeck SDE $dX_t^x = \Delta X_t^x dt + \phi dB_t$, $\phi(u) = \sin(\pi u)$, $u \in]0,1[$, can be solved explicitly and we see fast convergence of our method.



- As a more realistic example we use a stochastic perturbation by a Nemicky operator: $dX_t^x = \Delta X_t^x dt + \sin \circ X_t^x dB_t$, initial value $x(u) = \sqrt{(1 - 2|u - \frac{1}{2}|)/\sqrt{|u - \frac{1}{2}|}}$. Even though we are now outside the scope of the theorem, the results look promising.



Outlook

1. Even deterministic methods are possible since we can apply recombination schemes as presented in [4].
2. If the differential operator exhibits smoothing properties similar to the Laplace operator, then we expect the full order of convergence to hold.
3. The method can be extended to the case of infinitely many Brownian motions by truncation. Maybe it is even possible to find infinite dimensional cubature paths by an extension of Chow's Theorem of differential geometry to Hilbert manifolds.

References

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