Optimal Order Scheduling for Deterministic Liquidity Patterns

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Abstract

We consider a broker who has to place a large order which consumes a sizable part of average daily trading volume. By contrast to the previous literature, see, e.g., Obizhaeva and Wang [7], Predoiu et al. [8], we allow the liquidity parameters of market depth and resilience to vary deterministically over the course of the trading period. The resulting singular optimal control problem is shown to be tractable by methods from convex analysis and, under minimal assumptions, we construct an explicit solution to the scheduling problem in terms of some concave envelope of the resilience adjusted market depth.

Keywords: Order scheduling, liquidity, convexification, singular control, convex analysis, envelopes, optimal order execution

1 Introduction

It is well-known that market liquidity exhibits deterministic intraday patterns; see, e.g., Chordia et al. [3] or Kempf and Mayston [6] for some empirical investigations. The academic literature on optimal order scheduling,
however, only considers time-invariant specifications of market depth and resilience; cf. Obizhaeva and Wang [7], Alfonsi et al. [2], Predoiu et al. [8]. It thus becomes an issue how to account for time-varying specifications of these liquidity parameters when minimizing the execution costs of a trading schedule.

Using dynamic programming techniques and calculus of variation, this problem was addressed by Fruth et al. [5]. These authors show that under certain additional assumptions on these patterns there is a time-dependent level for the ratio of the number of orders still to be scheduled and the current market impact which signals when additional orders should be placed. The thesis [4] discusses conditions under which this structure persists in case of stochastically varying liquidity parameters. In discrete time, Acevedo and Alfonsi [1] use backward induction arguments in discrete time and then pass to continuous time to compute optimal policies for nonlinear specifications of market impacts which are scaled by a time-dependent factor satisfying some strong regularity conditions.

By contrast to these approaches, we show how to reduce our optimization problem to a convex one. Hence, we can use convex analytic first-order characterizations of optimality which we show are intimately related to the construction of generalized concave envelopes of a resilience-adjusted form of market depth. Under minimal assumptions, this allows us to characterize when optimal schedules exist and, if so, to construct them explicitly in terms of these envelopes. We illustrate our findings by recovering the analytic solution of Obizhaeva and Wang [7] and we show how optimal schedules depend on fluctuations in market depth and the level of resilience.

2 Setup

We consider a broker who has to acquire a total number of $x > 0$ shares of some stock. The broker knows that, due to limited liquidity of the stock, these orders will be executed at a mark-up over some reference stock price. This mark-up will depend on the broker’s past and present trades. For our specification of the mark-up we adopt the model proposed by Obizhaeva and Wang [7], see also Alfonsi et al. [2] and Predoiu et al. [8] for further motivation of this approach. By contrast to these papers, but in line with Fruth et al. [5] and Acevedo and Alfonsi [1], we will allow for the market’s liquidity characteristics of depth and resilience to be changing over time.
according to a deterministic pattern.

Specifically, given the broker’s cumulative purchases \( X = (X_t)_{t \geq 0} \), a right-continuous increasing process with \( X_0^- \triangleq 0 \), the resulting mark up evolves according to the dynamics

\[
\begin{align*}
\eta_0^X & \triangleq \eta_0 \geq 0, \\
d\eta_t^X &= \frac{dX_t}{\delta_t} - r_t \eta_t^X \, dt
\end{align*}
\]

where \( \delta_t \) describes the market’s depth at time \( t \geq 0 \) and where \( r_t \) measures its current resilience. Thus (1) has the right-continuous solution

\[
\eta_t^X = \left( \eta_0 + \int_{[0,t]} \frac{\rho_s}{\delta_s} \, dX_s \right) / \rho_t \quad \text{with} \quad \rho_t \triangleq \exp \left( \int_0^t r_s \, ds \right), \quad t \geq 0,
\]

under

**Assumption 2.1.** The resilience pattern is given by a strictly positive locally Lebesgue-integrable function \( r : [0, \infty) \to (0, \infty) \).

In the sequel we shall require furthermore

**Assumption 2.2.** The pattern of market depth \( \delta : [0, \infty) \to [0, \infty) \) is nonnegative, not identically zero, bounded and upper-semicontinuous with \( \limsup_{t \to \infty} \delta_t / \rho_t = 0 \).

The broker’s aim is to minimize the cumulative mark-up costs:

\[
(3) \quad \text{Minimize } C(X) \triangleq \int_{[0,\infty)} \left( \eta_t^X + \frac{\Delta_t^X}{2\delta_t} \right) \, dX_t \text{ subject to } X \in \mathcal{X}^c
\]

where \( \Delta_t^X \triangleq X_{t^+} - X_{t^-} \) and

\[
\mathcal{X}^c \triangleq \{ (X_t)_{t \geq 0} \text{ right-cont., incr. } : X_{0^-} = 0, X_\infty = x, C(X) < \infty \}.
\]

**Remark 2.3.** 1. Note that the \( \frac{\Delta_t^X}{2\delta_t} \)-term in (3) accounts for the costs a non-infinitesimal order will incur due to its own mark-up effect; cf., e.g., Alfonsi et al. [2] or Predoiu et al. [8]. Note also that, since we let \( X_{0^-} \triangleq 0 \), a value of \( X_0 > 0 \) corresponds to an initial jump of size \( \Delta_0 X = X_0 \) in the order schedule.
2. To impose liquidation over a finite time horizon $T \geq 0$, one merely has to let the market depth $\delta_t = 0$ for $t > T$. Indeed, following the convention that $1/0 = \infty$ in the integration (2), $\eta^X$ and thus the costs $C(X)$ will then be infinite for any order schedule $X$ which increases after $T$.

3. Strict positivity of $r$ comes without loss of generality since if resilience $r = 0$ vanishes almost everywhere on an interval $[t_0, t_1]$ there is no need to trade it off against market depth there and it is optimal to trade whatever amount is to be traded at the moment(s) when market depth $\delta$ attains its maximum over this period; cf. Proposition 4.1. Hence, $\delta$ could be assumed to take this maximum value at $t_0$ and the interval $(t_0, t_1]$ then be removed from consideration.

4. The assumption of upper-semicontinuous market depth $\delta$ is necessary to rule out obvious counterexamples for existence of optimal schedules. For unbounded $\delta$ one can easily show that $\inf_X C = x\eta_0/\rho_\infty$ whence no optimal schedule exists. The lim sup-condition is needed to rule out the optimality of deferring part of the order indefinitely.

5. Including a discount factor with locally Lebesgue-integrable discount rate $\bar{r} = (\bar{r}_t) \geq 0$ in our mark-up costs is equivalent to considering $\tilde{\delta}_t \triangleq \delta_t \exp(\int_0^t \bar{r}_s \, ds)$ and $\tilde{r}_t \triangleq r_t + \bar{r}_t$, $t \geq 0$ instead of $\delta$ and $r$ above.

3 Main result and sketch of its proof

The main result of this paper is the solution to problem (3):

**Theorem 3.1.** Suppose Assumptions 2.1 and 2.2 hold, let $\lambda_t \triangleq \delta_t/\rho_t$, $\tilde{\lambda}_t \triangleq \sup_{u \geq t} \lambda_u$ and define

\[
L_t^* = \inf_{u > t} \frac{\tilde{\lambda}_u - \tilde{\lambda}_t}{\lambda_u/\rho_u - \tilde{\lambda}_t/\rho_t}, \quad t \geq 0,
\]

where we follow the convention that $0/0 \triangleq 0$.

Then the optimal order schedule strategy is to buy shares at any time $t \geq 0$ if and while the resulting mark-up is no larger than $y^* L_t^* / \rho_t$, i.e.,

\[
X_t^* = \lambda_0(y^* L_0^* - \eta_0)^+ + \int_{(0,t]} \lambda_s \, \sup_{0 \leq s \leq s} \{ (y^* L_s^*) \lor \eta_0 \}, \quad t \geq 0,
\]
provided the constant $y^* > 0$ in (5) can be chosen such that $X^*_\infty = x$. This is the case if and only if the right hand side of (5) with $y^* \triangleq 1$ remains bounded as $t \uparrow \infty$. If this is not the case, we have $\inf_{X \in \mathcal{X}} C(X) = 0$ and the problem does not have a solution.

The following results outline the proof of this theorem and may be of independent interest. Our first auxiliary result provides a mathematically more convenient formulation of problem (3):

**Proposition 3.2.** Suppose Assumptions 2.1 and 2.2 hold, let $\lambda \triangleq \delta/\rho$, $\kappa \triangleq \lambda/\rho = \delta/\rho^2$ and define, for increasing and right-continuous $Y = (Y_t)_{t \geq 0}$,

$$K(Y) \triangleq \frac{1}{2} \int_{[0, \infty)} \kappa_t \, d(Y_t^2).$$

Then

$$Y_t = \eta_0 + \int_{[0, t]} \frac{dX_s}{\lambda_s}, \quad Y_0^- \triangleq \eta_0, \quad \text{and} \quad X_t = \int_{[0, t]} \lambda_s \, dY_s, \quad X_0^- \triangleq 0, \quad t \geq 0,$$

define mappings from $\mathcal{X}$ to

$$\mathcal{Y} \triangleq \left\{ (Y_t)_{t \geq 0} \text{ right-cont., incr.} : Y_0^- \triangleq \eta_0, \int_{[0, \infty)} \lambda_t \, dY_t = x, K(Y) < \infty \right\}$$

and vice versa such that

$$C(X) = K(Y).$$

As a result, with these choices of $\kappa$ and $\lambda$, optimization problem (3) is equivalent to the following problem:

$$\text{(7) Minimize } K(Y) \triangleq \frac{1}{2} \int_{[0, \infty)} \kappa_t \, d(Y_t^2) \text{ subject to } Y \in \mathcal{Y}.$$ 

Neither problem (3) nor problem (7) is convex in general:

**Proposition 3.3.** For upper-semicontinuous $\kappa$, the functional $K = K(Y)$ of (7) is (strictly) convex for increasing $Y$ with $Y_0^- = \eta_0$ if and only if $\kappa$ is (strictly) positive and (strictly) decreasing.

Convexity can always be arranged for, though, in the following sense:
Theorem 3.4. Let \( \lambda, \kappa \) be as in Proposition 3.2. Then optimization problem (7) has the same value as the convex optimization problem

\[
\text{(8) } \quad \underset{\tilde{Y}}{\text{Minimize}} \; \tilde{K}(\tilde{Y}) \triangleq \frac{1}{2} \int_{[0,\infty)} \tilde{\kappa}_t d(\tilde{Y}_t^2) \text{ subject to } \tilde{Y} \in \tilde{\mathcal{Y}}
\]

where \( \tilde{\kappa}_t \triangleq \tilde{\lambda}_t/\rho_t, \; \tilde{\lambda}_t \triangleq \sup_{u \geq t} \lambda_u, \; t \geq 0, \) and 

\( \tilde{\mathcal{Y}} \triangleq \left\{ (\tilde{Y}_t)_{t \geq 0} \text{ right-cont., incr. } : \tilde{Y}_{0-} \triangleq \eta_0, \int_{[0,\infty)} \tilde{\lambda}_t d\tilde{Y}_t = x, \tilde{K}(\tilde{Y}) < \infty \right\} \).

Moreover, any solution \( \tilde{Y}^* \) to (8) with \( \{d\tilde{Y}^* > 0\} \subset \{\tilde{\lambda} = \lambda\} \) will also be a solution to (7).

Remark 3.5. For an increasing process \( Y = (Y_t)_{t \geq 0} \) we say that \( t \) is a point of increase towards the right and write \( dY_t > 0 \) if \( Y_{t-} < Y_u \) for any \( u > t \). A similar convention applies to decreasing processes and points of decrease towards the right.

The (necessary and sufficient) first-order conditions for optimality in problem (8) are described in Proposition 3.6.

Proposition 3.6. For \( \tilde{\kappa}, \tilde{\lambda} \geq 0 \) as in Theorem 3.4, \( \tilde{Y}^* \in \tilde{\mathcal{Y}} \) solves (8) if and only if there is a constant \( y > 0 \) such that

\[
\text{(9) } \quad - \int_{[t,\infty)} \tilde{Y}_u^* d\tilde{\kappa}_u \geq y\tilde{\lambda}_t \text{ for } t \geq 0 \text{ with } \geq \text{ whenever } d\tilde{Y}_t^* > 0.
\]

Constructing right-continuous increasing \( \tilde{Y}^*_\geq 0 \) satisfying the first order conditions of (9) can be done by using a time-change and concave envelopes; see also Figure 3 below:

Theorem 3.7. Under Assumptions 2.1 and 2.2, consider the level passage times \( \tau_k \triangleq \inf \{ t \geq 0 : \tilde{\kappa}_t \leq k \} \) and let \( \tilde{\Lambda}_k \triangleq k\rho_{\tau_k}, \; k \in (0,\tilde{\kappa}_0] \) and \( \tilde{\Lambda}_0 \triangleq 0 \).

Then \( \tilde{\Lambda} \) is a continuous increasing map on \( [0,\tilde{\kappa}_0] \). Its concave envelope \( \hat{\tilde{\Lambda}} \) is absolutely continuous with a left-continuous, decreasing density \( \partial \hat{\tilde{\Lambda}} = (\partial \hat{\tilde{\Lambda}})_0^{k \leq \tilde{\kappa}_0} \geq 0 \). Moreover, letting \( \partial \hat{\tilde{\Lambda}}_0 \triangleq \partial \hat{\tilde{\Lambda}}_{\tilde{\kappa}_0+} \), we have that for any \( y > 0 \) and \( \eta_0 \geq 0 \), \( \tilde{Y}^*_\geq = (y\partial \hat{\tilde{\Lambda}}_{\tilde{\kappa}_0}) \vee \eta_0, \; t \geq 0, \) with \( \tilde{Y}^*_\geq \triangleq \eta_0 \) yields a right-continuous increasing process satisfying (9).
Combining the previous results, we shall obtain the following solution to our original problem (3) which also provides a characterization different from that outlined in Theorem 3.1; see also Figure 2 below:

**Corollary 3.8.** Under the assumptions of Theorem 3.7 and using its notation we have the following dichotomy:

In case $|\partial \hat{\Lambda}|_{L^2} \triangleq \int_{k=0}^{\infty} (\partial \hat{\Lambda}_k)^2 \, dk^{1/2} < \infty$ we can choose $y^* > 0$ uniquely such that

\[
X_t^* \triangleq \lambda_0 (y^* \partial \hat{\Lambda}_{\kappa_0} - \eta_0)^+ + \int_{[0,t]} \lambda_s \, d\left\{ (y^* \partial \hat{\Lambda}_{\kappa_s}) \vee \eta_0 \right\}, \quad t \geq 0,
\]

increases from $X_{0-}^* \triangleq 0$ to $X_\infty^* = x$; this $X^* \in \mathcal{X}$ is an optimal order schedule for problem (3). In the special case where $\eta_0 = 0$, $y^* = x/|\partial \hat{\Lambda}|_{L^2}$ and the minimal costs are given by $C(X^*) = x^2/(2|\partial \hat{\Lambda}|_{L^2}^2)$.

If, by contrast, $|\partial \hat{\Lambda}|_{L^2} = \infty$ then we have $\inf_{X \in \mathcal{X}} C(X) = 0$ and problem (3) does not have a solution.

4 Illustrations

4.1 Constant market depth and resilience

Let us first illustrate how to recover the solution of Obizhaeva and Wang [7] who consider a time horizon $T > 0$ and constant market depth $\delta_t \equiv \delta_0 1_{[0,T]}$ and constant market resilience $r_t \equiv r_0 > 0$. In this case we have

$$
\lambda_t = \tilde{\lambda}_t = \delta_0 e^{-r_0 t} 1_{[0,T]}(t) \quad \text{and} \quad \kappa_t = \tilde{\kappa}_t = \delta_0 e^{-2r_0 t} 1_{[0,T]}(t).
$$

Hence,

$$
\rho_{\kappa_k} = \sqrt{\delta_0 / (k \vee \kappa_T)} \quad \text{and} \quad \tilde{\Lambda}_k = \sqrt{\delta_0} k \wedge \sqrt{\delta_0 / \kappa_T k}, \quad 0 \leq k \leq \delta_0.
$$

Thus, $\tilde{\Lambda}$ is its own concave envelope, i.e., $\tilde{\Lambda} = \tilde{\Lambda}$, and its left-continuous density is

$$
\partial \tilde{\Lambda}_k = \begin{cases}
\frac{1}{2} \sqrt{\delta_0 / k}, & k > \kappa_T, \\
\frac{\sqrt{\delta_0} / \kappa_T}{e^{r_0 T}}, & k \leq \kappa_T.
\end{cases}
$$
Obviously $\partial \hat{\Lambda}$ is square integrable (and hence the problem is well-posed) if and only if $T < \infty$. In that case, we compute

$$\hat{Y}_t \triangleq \partial \hat{\Lambda}_t = \begin{cases} \frac{1}{2} \sqrt{\frac{\delta_0}{\kappa_t}} = \frac{1}{2} e^{r_0 t}, & t < T, \\ \sqrt{\frac{\delta_0}{\kappa_T}} = e^{r_0 T}, & t \geq T, \end{cases}$$

and each of the order schedules from (10),

$$X^y_t \triangleq \delta_0 \left( \frac{1}{2} y - \eta_0 \right)^+ + \frac{1}{2} y \delta_0 r_0 (t \wedge T - \tau^y)^+ + \frac{1}{2} \delta_0 y 1_{[T, \infty)}(t), \quad \text{quadt} \geq 0,$$

with

$$\tau^y \triangleq \begin{cases} \left( \frac{1}{r_0} \log \frac{2 \eta_0}{y} \right)^+ \wedge T, & y \geq 2 \eta_0 e^{-r_0 T}, \\ T, & \eta_0 e^{-r_0 T} \leq y \leq 2 \eta_0 e^{-r_0 T}, \\ \infty, & y < \eta_0 e^{-r_0 T}, \end{cases}$$

is optimal for the total volume it trades. In particular, if $\eta_0 = 0$, we find that

$$X^y_t = \frac{y \delta_0}{2} \left( 1 + r_0 t \wedge T + 1_{[T, \infty)}(t) \right), \quad t \geq 0,$$

and so choosing $y^* \triangleq x / (\delta_0 (1 + r_0 T / 2))$ yields $X^* = X^{y^*}$ with $X^\infty = x$. We therefore recover that result of Obizhaeva and Wang [7]: if $\eta_0 = 0$, i.e., if there have been no previous orders, it is optimal to place orders of size $y^* \delta_0 / 2$ at both $t = 0$ and $t = T$, and to place orders at the constant rate $y^* \delta_0 r_0 / 2$ in between; cf. Figure 1.

### 4.2 Time-varying market depth

We next illustrate that the above order placement strategy of [7] is indeed strongly dependent on constant market depth and resilience. Figure 2 below exhibits how a fluctuating market depth affects the timing of the optimal order flow as provided by Corollary 3.8. Note that we include a shut-down period for the market over the time period $(t_0, t_1)$ when market depth vanishes. The corresponding concepts introduced by Theorem 3.7 are illustrated in Figure 3 below.

If we decrease the resilience parameter to $r_0 = 0$ the focus on peaks of market depth sharpens, to the extent that eventually only one huge order is placed when market depth reaches its global maximum; see Figure 4.
Figure 1: Optimal order schedule $X^*$ (black) for constant market depth $\delta$ (blue), its resilience adjustment $\lambda = \tilde{\lambda}$ (red), $\tilde{\kappa}$ (green) over a finite horizon $T$.

Figure 2: A specification of market depth $\delta$ (blue) with finite horizon $T$, its resilience adjustment $\lambda$ (purple), the corresponding decreasing envelope $\tilde{\lambda}$ (red) and $\tilde{\kappa}$ (green) along with an optimal order schedule $X^*$ (black).
Figure 3: The decreasing envelope of resilience adjusted market depth \( \tilde{\Lambda} \) (red), its concave envelope \( \hat{\Lambda} \) (orange) and the density \( \partial \hat{\Lambda} \) (black).

Figure 4: Optimal order schedule \( X^* \) (black) without market resilience and time-varying market depth \( \delta \) (blue).
Proposition 4.1. If \( r \equiv 0 \) and \( \delta \) satisfies Assumption 2.2, the solutions to optimization problem (3) are precisely those order schedules \( X^* \in \mathcal{X} \) with \( \{dX^* > 0\} \subset \arg \max \delta \).

**Proof.** When \( r \equiv 0, \rho \equiv 1 \) and so \( \eta_t^X = \int_{[0,t]} \frac{dX_s}{\delta_s} \geq \frac{X_t}{\max \delta}, t \geq 0. \) Thus,

\[
C(X) \geq \frac{x^2}{2 \max \delta}, \quad X \in \mathcal{X},
\]

with equality for all \( X^* \in \mathcal{X} \) with \( \{dX^* > 0\} \subset \arg \max \delta. \)

Conversely, with high resilience, orders tend to be spread out more around local maxima of market depth as illustrated by Figure 5.

![Figure 5: Optimal order schedule \( X^* \) (black) with strong market resilience for time-varying market depth \( \delta \) (blue).](image)

5 Proofs

We first prove that the original problem (3) can indeed be reformulated as (7) by giving the
Proof of Proposition 3.2 We first observe that for $X \in \mathcal{X}$ the mapping in (6) defines an increasing right-continuous $Y$ with $Y = \rho \eta^X$. Because $C(X) < \infty$, $\eta^X$ is $dX$-integrable and thus finite on $\{X < x\}$. Hence, $Y$ is finite on this set as well and we conclude $dX = \lambda dY$. It follows by elementary calculus that $K(Y) = C(X)$ and, thus, $Y \in \mathcal{Y}$ as desired.

Conversely, for $Y \in \mathcal{Y}$, $\kappa = \lambda/\rho$ is $dY$-integrable. Since $\rho > 0$ is continuous this implies that $\lambda$ is locally $dY$-integrable and so $X$ given by (6) is right-continuous and increasing with $dX = \lambda dY$. By the same reasoning as above this implies $C(X) = K(Y)$ as well as $X \in \mathcal{X}$.

We next characterize when problem (7) is convex:

Proof of Proposition 3.3 If $\kappa$ is upper semi-continuous and decreasing, it is also left-continuous and we can use Fubini’s theorem to write

$$K(Y) = \frac{1}{2} \left( \kappa_\infty (Y^2 - \eta_0^2) - \int_{[0,\infty)} (Y_t^2 - \eta_0^2) \, d\kappa_t \right)$$

for any right-continuous increasing $Y$ with $Y_0 = \eta_0$. Hence, $K = K(Y)$ is obviously convex in such $Y$ with strict convexity holding true on its domain for strictly decreasing $\kappa$.

Conversely, consider for $0 \leq s < t \leq T$ the function $Y = \eta_0 + a 1_{[s,T]} + b 1_{[t,T]}$. Then

$$K(Y) = \frac{1}{2} \left( \kappa_s ((a + \eta_0)^2 - \eta_0^2) + \kappa_t ((a + b + \eta_0)^2 - (a + \eta_0^2)) \right)$$

$$= \frac{1}{2} \kappa_s a^2 + \kappa_t ab + \frac{1}{2} \kappa_t b^2 + \eta_0 (a \kappa_s + b \kappa_t)$$

is convex in $a, b > 0$ if and only if $\kappa_s \geq \kappa_t \geq 0$, with strict inequalities corresponding to strict convexity. 

In order to prepare the proof of Theorem 3.4 let us recall that for any increasing $Z : [0, \infty) \to \mathbb{R}$ we let

$$\{dZ > 0\} \triangleq \{t \geq 0 : Z_{t-} < Z_u \text{ for all } u > t\}$$

denote the collection of all points of increase towards the right. For a decreasing $Z$ we let $\{dZ < 0\} \triangleq \{d(-Z) > 0\}$. In either case we let $\text{supp} \, dZ$ denote the support of the measure $dZ$, i.e., the smallest closed set whose complement has vanishing $dZ$-measure.
Lemma 5.1. For upper-semicontinuous, bounded $\lambda : [0, \infty) \to \mathbb{R}$, we have that $\tilde{\lambda}_t \triangleq \sup_{u \geq t} \lambda_u$ is left-continuous and decreasing with

\begin{equation}
\{d\tilde{\lambda} < 0\} \subset \{\tilde{\lambda} = \lambda\}.
\end{equation}

Moreover, we have the partition

\begin{equation}
\mathbb{R} = \{d\tilde{\lambda} < 0\} \cup \bigcup_{n \in N_1} [l_n, r_n) \cup \bigcup_{n \in N_2} (l_n, r_n)
\end{equation}

where $(l_n, r_n), n \in N$, are the disjoint open intervals forming $\mathbb{R} \setminus \text{supp } d\tilde{\lambda}$ and where $N_1 = \{n \in N : l_n \geq 0, \Delta_{l_n} \tilde{\lambda} = 0\}$ and $N_2 = N \setminus N_1$.

Proof. Left-continuity of $\tilde{\lambda}$ and relation (11) are immediate. Note next that $\{d\tilde{\lambda} < 0\} \subset \text{supp } d\tilde{\lambda}$ whence $\mathbb{R} \setminus \{d\tilde{\lambda} < 0\} \supset \bigcup_{n \in N}(l_n, r_n)$. Hence, to deduce partition (12) it suffices to observe that for $n \in N_1$ we have $t \notin \{d\tilde{\lambda} < 0\}$ and that for $t \geq 0$ such that $\tilde{\lambda}_t = \tilde{\lambda}_u$ for some $u > t$ we have $(t, u) \subset (l_n, r_n)$ for some $n \in N$, whence $t \in (l_n, r_n)$ or $t = l_n$ with $\Delta_{l_n} \tilde{\lambda} = 0$.

The main tool in the proof of Theorem 3.4 is the following

Lemma 5.2. Under the conditions of Theorem 3.4, we can find for any increasing, right-continuous $Y \geq \eta_0$ an increasing, right-continuous $\tilde{Y} \geq \eta_0$ such that $\tilde{Y} \geq Y$ and

\begin{enumerate}
\item[(i)] $\int_{[0, \infty)} \lambda_t dY_t = \int_{[0, \infty)} \lambda_t d\tilde{Y}_t$,
\item[(ii)] $\{d\tilde{Y} > 0\} \subset \{d\tilde{\lambda} < 0\}$,
\item[(iii)] $K(Y) \geq K(\tilde{Y}) = \tilde{K}(\tilde{Y})$.
\end{enumerate}

Proof. We let $I_n, n \in N$, denote the disjoint intervals of Lemma 5.1 forming the complement of $\{d\tilde{\lambda} < 0\}$ and we will use $l_n, r_n$ to denote their respective boundaries. For the one interval where before $l_n = \infty$ we now redefine, for simplicity of notation, $l_n \triangleq 0$ provided that $r_n > 0$; if, by contrast, this $I_n$ is just the negative half line we can and shall remove it from consideration in the sequel. Similarly, if $r_n = \infty$ for some $n \in N$, it follows from Assumption 2.2 that $\delta_t = \lambda_t = \kappa_t \equiv 0$ on $I_n$ which thus can be disregarded as well.
Observe then that

\[
\sup_{I_n} \lambda = \lambda_{r_n},
\]

by upper semi-continuity of \(\lambda\) and our choice when to include \(l_n\) in \(I_n\) and when not.

Let, for \(t \geq 0\),

\[
\tilde{Y}_t = \eta_0 + \int_{[0,t]} 1_{\{d\tilde{\lambda} < 0\}}(s) dY_s + \sum_{n \in N, r_n \leq t} \int_{I_n} \frac{\lambda_s}{\lambda_{r_n}} dY_s.
\]

We first note that \(\tilde{Y} \leq Y\). Indeed:

\[
Y_t - \tilde{Y}_t = \int_{[0,t]} 1_{R \setminus \{d\tilde{\lambda} < 0\}}(s) (dY_s - d\tilde{Y}_s)
= \sum_{n \in N, l_n \leq t} \left( \int_{I_n \cap [0,t]} dY_s - 1_{[r_n, \infty)}(t) \int_{I_n} \frac{\lambda_s}{\lambda_{r_n}} dY_s \right)
\]

which is nonnegative because of (13).

Assertion (i) is readily checked using the partition given by (12). For assertion (ii) it suffices to observe that all \(r_n, n \in N\), are contained in \(\{d\tilde{\lambda} < 0\}\).

In order to prove assertion (iii), we first note that \(K(\tilde{Y}) = \tilde{K}(\tilde{Y})\) is an immediate consequence of (ii) and (11). To establish \(K(Y) - K(\tilde{Y}) \geq 0\) we decompose this difference into its contributions from the different parts in the partition given by (12), each of which will be shown to be nonnegative.

From \(\{d\tilde{\lambda} < 0\} \setminus \{r_n : n \in N\}\) we collect

\[
\frac{1}{2} \int_{[0,T] \cap \{d\tilde{\lambda} < 0\} \setminus \{r_n : n \in N\}} \kappa_{t} \left[ d(Y_t^2) - d(\tilde{Y}_t^2) \right]
= \int_{[0,T] \cap \{d\tilde{\lambda} < 0\} \setminus \{r_n : n \in N\}} \kappa_{t} \left( Y_{t-} + \frac{1}{2} \Delta_t Y \right) dY_t - \left( \tilde{Y}_{t-} + \frac{1}{2} \Delta_t \tilde{Y} \right) d\tilde{Y}_t
\]

which is nonnegative because \(Y \geq \tilde{Y}\) and because \(dY_t = d\tilde{Y}_t\) for \(t \in \{d\tilde{\lambda} < 0\} \setminus \{r_n : n \in N\}\) by construction.

From \(I_n \cup \{r_n\}, n \in N\), we get the contribution

\[
\frac{1}{2} \left\{ \int_{I_n \cup \{r_n\}} \kappa_{t} d(Y_t^2) - \kappa_{r_n} \left[ (\tilde{Y}_{r_n-} + \int_{I_n \cup \{r_n\}} \frac{\lambda_s}{\lambda_{r_n}} dY_s)^2 - \tilde{Y}_{r_n-}^2 \right] \right\}
\]
for which we note that its \([\ldots]\)-part can be written as
\[
\frac{1}{2} \left[ \left( \tilde{Y}_{r_n^-} + \int_{I_n \cup \{r_n\}} \frac{\lambda_s}{\lambda_{r_n}} dY_s \right)^2 - \tilde{Y}_{r_n^-}^2 \right]
\]
\[
= \frac{1}{2} \left( \int_{I_n \cup \{r_n\}} \frac{\lambda_s}{\lambda_{r_n}} dY_s \right)^2 + \tilde{Y}_{r_n^-} - \int_{I_n \cup \{r_n\}} Y_{r_n} + \frac{1}{2} \sum_{\Delta_t Y \neq 0, t \in I_n \cup \{r_n\}} \left( \frac{\lambda_t}{\lambda_{r_n}} \right)^2 (\Delta_t Y)^2
\]
\[
+ \tilde{Y}_{r_n^-} - \int_{I_n \cup \{r_n\}} \frac{\lambda_s}{\lambda_{r_n}} dY_s.
\]

Hence, using (13) again, we obtain with \(y_n \triangleq Y_{l_n^-}\) if \(l_n \in I_n\) and \(y_n \triangleq Y_{l_n}\) otherwise that
\[
\frac{1}{2} \ldots \leq \int_{I_n \cup \{r_n\}} (Y_{l_n} - y_n) \frac{\lambda_t}{\lambda_{r_n}} dY_t + \frac{1}{2} \sum_{\Delta_t Y \neq 0, t \in I_n \cup \{r_n\}} \frac{\lambda_t}{\lambda_{r_n}} (\Delta_t Y)^2
\]
\[
+ \tilde{Y}_{r_n^-} - \int_{I_n \cup \{r_n\}} \frac{\lambda_s}{\lambda_{r_n}} dY_s
\]
\[
\leq \int_{I_n \cup \{r_n\}} (Y_{l_n} - y_n) \frac{\lambda_t}{\lambda_{r_n}} dY_t + \frac{1}{2} \sum_{\Delta_t Y \neq 0, t \in I_n \cup \{r_n\}} \frac{\lambda_t}{\lambda_{r_n}} (\Delta_t Y)^2
\]
\[
+ y_n \int_{I_n \cup \{r_n\}} \frac{\lambda_s}{\lambda_{r_n}} dY_s
\]
\[
= \frac{1}{2} \int_{I_n \cup \{r_n\}} \frac{\lambda_t}{\lambda_{r_n}} d(Y_t^2)
\]

where the second estimate holds since \(\tilde{Y}_{r_n^-} = \tilde{Y}_{l_n^-} \leq y_n\) because of (ii). Since \(\rho = \lambda/\kappa\) is increasing by assumption, we have
\[
\frac{\lambda_t}{\lambda_{r_n}} = \frac{\rho_t}{\rho_{r_n}} \frac{\kappa_t}{\kappa_{r_n}} \leq \frac{\kappa_t}{\kappa_{r_n}}
\]

and thus
\[
\frac{1}{2} \kappa_{r_n} \ldots \leq \frac{1}{2} \int_{I_n \cup \{r_n\}} \kappa_t d(Y_t^2)
\]
as remained to be shown. \(\square\)

With the preceding policy improvement lemma it is now easy to give the
Proof of Theorem 3.4  By Lemma 5.2 and using its notation, we can find for any $Y \in \mathcal{Y}$ a $\widetilde{Y} \in \mathcal{Y} \cap \mathcal{Y}$ such that
\[
\tilde{K}(Y) \geq K(Y) \geq K(\widetilde{Y}) = \tilde{K}(\widetilde{Y}).
\]
As a result, $\inf_{\mathcal{Y}} K = \inf_{\mathcal{Y}} \tilde{K}$. Moreover, if $\tilde{Y}^* \in \mathcal{Y}$ attains the latter infimum we can apply Lemma 5.2 to $\tilde{\lambda}$ and $\tilde{K}$ instead of $\lambda$ and $K$ to obtain another optimal $\tilde{Y}^{**} \in \mathcal{Y}$ which satisfies in addition $d\tilde{Y}^{**} > 0 \subset \{d\tilde{\lambda} < 0\}$. By Lemma 5.1, the latter set is contained in $\{\lambda = \tilde{\lambda}\} = \{\kappa = \tilde{\kappa}\}$ and thus this $\tilde{Y}^{**}$ is also contained in $\mathcal{Y}$ and optimal for (7) as well. \hfill \Box

Let us next derive the first-order conditions of the convexified problem (8) in the

Proof of Proposition 3.6  Recalling that $\tilde{\kappa}_\infty = 0$, we obtain by Fubini’s theorem
\[
(14) \quad \tilde{K}(Y) = -\frac{1}{2} \int_{[0,\infty)} (Y_t^2 - \eta_0^2) \, d\tilde{\kappa}_t.
\]
For necessity, we observe that for any $Y \in \mathcal{Y}$ and $0 < \varepsilon \leq 1$ we have
\[
0 \leq \tilde{K}(\varepsilon Y + (1 - \varepsilon) Y^*) - \tilde{K}(Y^*)
\]
\[
= -\varepsilon \int_{[0,\infty)} (Y_i - Y_i^*) Y_i^* \, d\tilde{\kappa}_i - \frac{\varepsilon^2}{2} \int_{[0,\infty)} (Y_i - Y_i^*)^2 \, d\tilde{\kappa}_i
\]
which, upon division by $\varepsilon > 0$ and letting $\varepsilon \searrow 0$, yields that $Y^*$ also solves the linear problem
\[
(15) \quad \text{Minimize } -\int_{[0,\infty)} Y_i^* Y_i \, d\tilde{\kappa}_i \text{ subject to } Y \in \mathcal{Y}.
\]
Equivalently, due to Fubini’s theorem, $Y^*$ is a solution to the problem:
\[
(16) \quad \text{Minimize } \int_{[0,\infty)} \left( -\int_{[t,\infty)} Y_u^* \, d\tilde{\kappa}_u \right) \, dY_t \text{ subject to } Y \in \mathcal{Y}.
\]
As a consequence, $Y^*$ can solve (15) only if $dY^*_t > 0$ exclusively at those times $t \geq 0$ when $-\int_{[t,\infty)} Y_u^* \, d\tilde{\kappa}_u/\tilde{\lambda}_t$ attains its infimum over $\{\tilde{\lambda} > 0\}$. Hence, this infimum is actually a minimum and is thus strictly positive. Denoting it by $y > 0$ shows the necessity of (9).
For sufficiency we use (14) again to deduce that for \( Y \in \tilde{\mathcal{Y}} \):
\[
\tilde{K}(Y) - \tilde{K}(Y^*) = -\frac{1}{2} \int_{[0,\infty)} ((Y_t)^2 - (Y_t^*)^2) \, d\tilde{\kappa}_t \geq -\int_{[0,\infty)} Y_t^* (Y_t - Y_t^*) \, d\tilde{\kappa}_t.
\]
The last term is nonnegative if \( Y^* \) solves (15), which due to the equivalence of (15) and (16) amounts to our first-order condition (9).

The construction of solutions to the first order conditions given in Theorem 3.7 can now be established:

**Proof of Theorem 3.7** \( \tilde{\Lambda} \) is continuous on \([0, \tilde{\kappa}_0]\) since so is \( k \mapsto \tau_k \) because of the strict monotonicity of \( \rho \) and, thus, \( \tilde{\kappa} \) on \( \{\tilde{\kappa} > 0\} \). Absolute continuity of the concave envelope \( \hat{\Lambda} \) follows from the continuity of \( \tilde{\Lambda} \).

The monotonicity of \( \tilde{Y}^* \) is obvious from the monotonicity of \( \tilde{\kappa} \) and \( \partial \hat{\Lambda} \).

For its right-continuity note that \( \lim_{t \downarrow t_0} \tilde{Y}_t^* = (y \partial \hat{\Lambda}_{\tilde{\kappa}_t} + \eta_t) \) by left-continuity of \( \partial \hat{\Lambda} \) and its definition at 0. Hence, our assertion amounts to \( \partial \hat{\Lambda}_{k_0} = \partial \hat{\Lambda}_{k_1} \) where \( k_0 \triangleq \tilde{\kappa}_{t_0^+} \) and \( k_1 \triangleq \tilde{\kappa}_{t_0^+} \geq k_0 \). If \( k_0 = k_1 \) there is nothing to show. In case \( k_0 < k_1 \), \( \tau_k = \tau_{k_1} \) for \( k \in [k_0, k_1) \) and, thus, \( \hat{\Lambda} \) is linear with slope \( \rho_{rk_0} \) on this interval. As a consequence, \( \hat{\Lambda} \) is linear there as well and, thus, \( \partial \hat{\Lambda}_{k_1} = \partial \hat{\Lambda}_{k_0^+} \) by left-continuity of \( \partial \hat{\Lambda} \). Hence, it suffices to show that there is no downward jump in \( \partial \hat{\Lambda} \) at \( k_0 \). If there was such a jump then, by the properties of concave envelopes, necessarily \( \hat{\Lambda}_{k_0} = \tilde{\Lambda}_{k_0} \) and \( \partial \hat{\Lambda}_{k_0^+} \geq \rho_{rk_0} \).

Hence, for \( k \leq k_0 \) we would have
\[
k_0 \leq \hat{\Lambda}_k \leq \hat{\Lambda}_{k_0} + \partial \hat{\Lambda}_{k_0^+} (k - k_0) \leq k_0 \rho_{rk_0},
\]
where the first estimate is due to the monotonicity of \( \rho \), the second is the envelope property of \( \hat{\Lambda} \), the third follows from its concavity and the last is a consequence of the just derived properties of \( \hat{\Lambda} \) and \( \partial \hat{\Lambda} \) at \( k_0 \). We would thus have equality everywhere in the above estimates and in particular \( \partial \hat{\Lambda}_{k_0} = \rho_{rk_0} \leq \partial \hat{\Lambda}_{k_0^+} \). This is a contradiction to the presumed downward jump of \( \partial \hat{\Lambda} \) at \( k_0 \).

To verify that \( \tilde{Y}^* \) satisfies the first order condition (9), let us first argue that
\[
-\int_{[t,\infty)} \tilde{Y}_u^* \, d\tilde{\kappa}_u \geq -y \int_{[t,\infty)} \partial \hat{\Lambda}_{\tilde{\kappa}_u} \, d\tilde{\kappa}_u = y \int_0^{\tilde{\kappa}_t} \partial \hat{\Lambda}_k \, dk = y \hat{\Lambda}_{\tilde{\kappa}_t}
\]
\[
\geq y \tilde{\Lambda}_{\tilde{\kappa}_t} = y \tilde{\kappa}_t \rho_{rk_t} = y \xi_t.
\]
Indeed, the first estimate is immediate from the definition of $\tilde{Y}^\ast$. The first identity follows by the change-of-time formula for Lebesgue-Stieltjes-integrals: just observe that $\tilde{\kappa}_\infty = 0$ by Assumption 2.2 and that $\partial \hat{\Lambda}$ is constant on those intervals contained in $[0, \tilde{\kappa}_0]$ which $\tilde{\kappa}$ jumps across because $\tilde{\Lambda}$ is linear on such intervals. The second identity follows from the absolute continuity of $\hat{\Lambda}$ and because $\hat{\Lambda}_0 = \tilde{\Lambda}_0 = 0$, again by Assumption 2.2. The second estimate holds because $\hat{\Lambda} \geq \tilde{\Lambda}$ by definition of concave envelopes and for the last identity we note that $\tau_l = t$ if $\tilde{\kappa}_t > 0$ and $\tilde{\kappa}_t = \hat{\Lambda}_t = 0$ otherwise. Finally, we observe that $d\tilde{Y}^\ast_t > 0$ can only happen when $y \partial \hat{\Lambda}_t$ has increased above $\eta_0$ which ensures equality in the first of the above estimates. Equality in the second holds for such $t$ as well because if $\partial \hat{\Lambda}_t$ increases at time $t$, $\partial \tilde{\Lambda}_t$ must decrease at $\tilde{\kappa}_t$, whence $\tilde{\Lambda}$ coincides with its concave envelope $\hat{\Lambda}$ at this point. $\square$

We are now in a position to wrap up and give the

**Proof of Corollary 3.8** Let $\tilde{Y}_t \triangleq \partial \tilde{\Lambda}_{\tilde{\kappa}_t}$, $\tilde{Y}_{0}^{-} \triangleq 0$ and define $Y^\ast_t \triangleq (y\tilde{Y}_t) \vee \eta_0$, $t \geq 0$, $Y^\ast_0 = \eta_0$.

We first show that $\int_{0+} (\partial \hat{\Lambda}_t)^2 dk < \infty$ if and only if $\lambda$ is $d\tilde{Y}$-integrable.

To see this we argue that with $\partial \hat{\Lambda}_{\tilde{\kappa}_0} \triangleq 0$ we have

$$\int_{[0,\infty)} \lambda_t \, d\tilde{Y}_t = \int_{[0,\infty)} \tilde{\Lambda}_{\tilde{\kappa}_t} \, d(\partial \hat{\Lambda}_{\tilde{\kappa}_t}) = \int_{[0,\infty)} \hat{\Lambda}_{\tilde{\kappa}_t} \, d(\partial \hat{\Lambda}_{\tilde{\kappa}_t}) = \int_{0}^{\tilde{\kappa}_0} \partial \hat{\Lambda}_t \partial \hat{\Lambda}_{\tilde{\kappa}_t} \, dl = \int_{0}^{\tilde{\kappa}_0} (\partial \hat{\Lambda})^2 \, dl .$$

Indeed, the first identity is just the definition of $\hat{Y}$ and $\tilde{\Lambda}$ and the second holds because $\hat{\Lambda} = \tilde{\Lambda}$ at points where $\partial \Lambda$ changes; the third identity follows from an application of Fubini’s theorem after writing $\hat{\Lambda}_{\tilde{\kappa}_t} = \int_{0}^{\tilde{\kappa}_t} \partial \hat{\Lambda}_t \, dl$ and the last equality holds since $\partial \hat{\Lambda}$ is left-continuous and constant over intervals that $\tilde{\kappa}$ jumps across.

So if $\int_{0+} (\partial \hat{\Lambda}_t)^2 dk < \infty$, then $X_y^\ast \triangleq \int_{[0,t]} \lambda \, dY^\ast$, $t \geq 0$, is real-valued, right-continuous and increasing in $t$. Moreover, $X_\infty^\ast$ is increasing in $y \geq 0$ with $X_\infty^0 = 0$ and $X_\infty^\ast \geq X_0^\ast \rightarrow \infty$ as $y \uparrow \infty$. In fact, $X_\infty^\ast = y\int_{[0,\infty)} \lambda \, d\tilde{Y}$ for $y \geq \eta_0/\tilde{Y}_0$ (where $0/0 \triangleq \infty$) and, for $y \in [\eta_0/\tilde{Y}_0, \eta_0/\tilde{Y}_0]$, $X_\infty^y = y\int_{[y,\infty)} \lambda \, d\tilde{Y} = \int_{[y,\infty)} \lambda \, d\tilde{Y}$ where $\tau_y \triangleq \inf \{ t \geq 0 : y > \eta_0/\tilde{Y}_t \}$. Hence $X_\infty^y$ is in fact continuously and strictly increasing from 0 to $\infty$ in $y \geq \eta_0/\tilde{Y}_\infty$ and we thus
obtain existence and uniqueness of $y^* > 0$ with $X^y_\infty = x$. Hence, we can conclude that $X^* \triangleq X^{y^*}$ is contained in $\mathcal{Y}$ (and that thus the corresponding $Y^* = Y^{y^*}$ of (6) is contained in $\mathcal{Y}$) once we have established that $K(Y^*) < \infty$. For this it suffices to observe that $K(Y^*) \leq (y^*)^2K(\tilde{Y})$ and that by the same arguments as in our previous calculation of $\int_{[0,\infty]} \lambda d\tilde{Y}$ we have

$$K(\tilde{Y}) = \tilde{K}(\tilde{Y}) = \frac{1}{2} \int_{[0,\infty]} \tilde{\kappa}_t d \left( (\partial \tilde{\Lambda}_{\tilde{\kappa}_t})^2 \right) = \frac{1}{2} \int_0^{\tilde{\kappa}_0} (\partial \tilde{\Lambda}_t)^2 dl < \infty.$$ 

We next show that $X^*$ and $Y^*$ are optimal, respectively, for problem (3) and problems (7) and (8). In fact, due to Theorem 3.7, $Y^* = (y^* \partial \tilde{\Lambda}_{\tilde{\kappa}}) \lor \eta_0$ satisfies the first order condition (9) and, by Proposition 3.6, is thus optimal for the convexified problem (8) provided that $Y^*$ is also contained in $\mathcal{Y}$. To see that indeed even $\int_{[0,\infty]} \tilde{\lambda} dY^* = x$ and to deduce the optimality of $Y^*$ also for problem (7) (and thus, by Proposition 3.2, optimality of $X^*$ for the original problem (3)) it suffices by Theorem 3.4 to check that

$$dY^*_t > 0 \text{ only at times } t_0 \geq 0 \text{ when } \tilde{\lambda}_{t_0} = \lambda_{t_0}.$$ 

We will show that even $d\tilde{\lambda}_{t_0} < 0$ for such $t_0$. Otherwise there is $t_1 > t_0$ such that $\tilde{\lambda}_{t_1} = \tilde{\lambda}_{t_0}$ for $t \in [t_0, t_1]$. Hence, $\tilde{\lambda}$ is constant on the interval $[\tilde{\kappa}_{t_1}, \tilde{\kappa}_{t_0}]$. Because $dY^*_t > 0$, the density $\partial \tilde{\lambda}$ must decrease at $k_0$ and so the envelope $\tilde{\lambda}$ coincides with $\tilde{\Lambda}$ at this point. Concavity and monotonicity of $\tilde{\Lambda}$ then imply, however, that $\partial \tilde{\lambda} = 0$ around $k_0$, a contradiction to its decrease there.

The formula for the minimal costs when $\eta_0 = 0$ is an immediate consequence of our above computations for $\tilde{Y}$. It thus remains to show that our optimization problems do not have a solution when $|\partial \tilde{\lambda}|_{L^2} = \infty$. To see this note that for any sufficiently large 0 $\leq S < T < \infty$ there is a schedule $X^{S,T} \in \mathcal{X}$ which is optimal for $\delta^{S,T} \triangleq \delta_{[S,T]}$ instead of $\delta$ when $\eta_0 = 0$. This follows from our earlier results once we note that any model with eventually vanishing market depth has a bounded density in the corresponding concave envelope $\tilde{\Lambda}^{S,T}$ and thus a solution if its market depth does not vanish identically. The latter property does hold for our $\delta^{S,T}$ when $T$ is chosen sufficiently large for otherwise $\delta \equiv 0$ after some time $S$ which would rule out the presumed explosion of $\partial \tilde{\lambda}$ at 0. Note that we can choose $S, T \uparrow \infty$ such that $\tilde{\lambda}$ coincides with $\tilde{\Lambda}_{\tilde{\kappa}}$ at these points. This ensures that $\tilde{\lambda} = \tilde{\Lambda}^{S,T}$ on $[\tilde{\kappa}_T, \tilde{\kappa}_S]$ and hence $|\partial \tilde{\Lambda}^{S,T}|_{L^2} \rightarrow \int_0^{\tilde{\kappa}_0} (\partial \tilde{\Lambda}_{k_{\tilde{\kappa}}, \tilde{\kappa}_S})^2 dk = \infty$ as $T \uparrow \infty$. 

19
Now because
\[ C(X^{S,T}) = \int_{[0,\infty)} \frac{\eta_0}{\rho_t} dX_t^{S,T} + C^0(X^{S,T}) \leq \frac{\eta_0 x}{\rho_S} + C^0(X^{S,T}) \]
where \( C^0(X) \) denotes the cost of any \( X \in \mathcal{X} \) when \( \eta_0 = 0 \), we obtain
\[ \inf_X C \leq \frac{\eta_0 x}{\rho_S} + \frac{x^2}{2|\partial \Lambda^{S,T}|_{L^2}^2} \]
where we used our formula for the optimal costs \( C^0(X^{S,T}) \). Because of our special choice of \( S, T \), the second term vanishes for any fixed \( S \) as \( T \uparrow \infty \).

The first term vanishes for \( S \uparrow \infty \) because \( \rho \) has to be unbounded for \( \partial \Lambda_k \) to increase to \( \infty \) as \( k \downarrow 0 \). Indeed: \( \partial \Lambda_{0+} = \sup_{k>0} \tilde{\Lambda}_k \rho_t = \sup_{k>0} \tilde{\Lambda}_k \rho_t \).

Finally let us show how Theorem 3.1 follows from Corollary 3.8:

**Proof of Theorem 3.1** In view of Corollary 3.8 it suffices to show that \( \sup_{0 \leq t \leq s} L_t^* = \partial \Lambda_{\tilde{k}_s}, \ s \geq 0 \). Now, from the properties of concave envelopes and because of the left-continuity of \( \partial \Lambda \) we have for any \( 0 < k \leq \tilde{k}_0 \) that
\[ \partial \Lambda_k = \sup_{l \in [k, \tilde{k}_0]} \inf_{m \in [0, l]} \frac{\Lambda_m - \Lambda_l}{m - l} . \]

With the changes of variables \( k = \tilde{k}_s, \ l = \tilde{k}_t, \ m = \tilde{k}_u \) the preceding ratio turns into the one occurring in (4), accomplishing our proof.

**References**


