Mathematics:
A Key Technology in Finance

Peter Bank
Institut für Mathematik
Humboldt–Universität zu Berlin
Unter den Linden 6
D–10099 Berlin, Germany
pbank@mathematik.hu-berlin.de

Introduction

For centuries the use of mathematics in banks was limited more or less to compound interest computation. As important and astonishing as these calculations may be from an economic point of view, the involved concepts of interest rate theory typically are not what a mathematician would call particularly interesting. During the last two decades, however, the picture has changed dramatically. Nowadays advanced mathematical techniques and ideas are used both in banks and insurance companies. Their effect is not only felt on a computational level: Mathematical concepts are also shaping new approaches to risk management in the financial industry. The aim of this note is to explain this mathematical revolution in the financial industry and to illustrate how the finance projects carried out in our new research center will contribute to this ongoing process.

1 Mathematics of financial products: an example

How mathematical 'high tech' has found its way deep into the financial industry can nicely be exemplified by a so called 'guarantee fund'. This is a new financial product which seeks to combine the advantages of two more traditional products: bonds and investment funds. It offers an investor the possibility to participate in the evolution of some stock market index, the DAX say, while limiting the downside risk by a previously fixed minimum payoff, the guarantee. Thus, if stock markets perform well, the proceeds from an investment in a guarantee fund will be considerably higher than the revenues from an investment in riskless bonds. On the other hand, in case of an unfavorable stock market evolution the investor can be sure to receive at least the guarantee; compare Figure 1. So a guarantee fund provides a new investment opportunity which may be

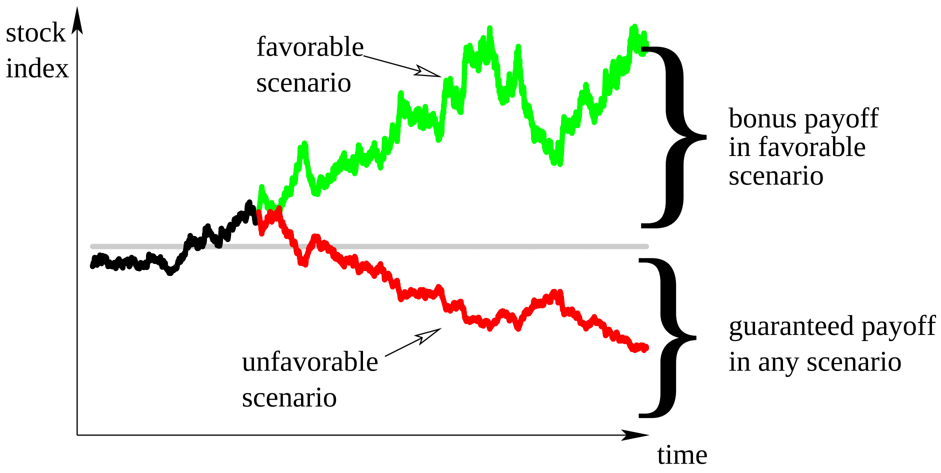

Figure 1: Payoff of a guarantee fund in two different stock market scenarios.

viewed as an attractive supplement to a private investor's retirement provisions.

There is an almost innumerable variety of other, considerably more complex financial products providing tailor made investment opportunities not only for private, but also for institutional investors. The complexity of all these products calls for new techniques and concepts of risk management. For example, a bank which has sold a large number of shares in guarantee funds faces the risk that a rising stock market will lead to a substantial increase in its over all payment obligations. On the other hand, when stock markets are falling the bank still needs to pay the guarantee. Now, the obvious question is how the bank can secure itself against *both* scenarios at the same time? The economic answer to this question is *hedging*: the bank should invest part of the collected fees in the stocks underlying the guarantee funds — as an 'insurance' against the risk of rising stock markets — and it should invest the remaining fees in riskless bonds thus ensuring that the guaranteed minimum payoff can be paid. This reduces the bank's problem to finding the optimal balance between riskless investments in bonds and risky investments in stocks. It is in determining this balance where advanced mathematical methods are being used. The introduction of these methods goes back to F. Black, M. Scholes, and

R. Merton[1] and it was awarded the Nobel Prize for Economics in 1997[2].

The starting point for the Black-Scholes-Merton approach is a mathematical model for the price fluctuations on a financial market, the so called geometric Brownian motion. Within this model it is possible to describe a dynamic trading strategy which allows the bank to exactly match its obligations from the contract, regardless of the concrete scenario which is going to unfold. The computation of the dynamic hedging strategy depicted in Figure 2 is based on a probabilistic extension of Leibniz' and Newton's infinitesimal calculus which is called *stochastic calculus*. This calculus was introduced by K. Itô in the 1940s and since then was further developed by many researchers in Stochastic Analysis. When it later was applied to Finance, it turned out to be almost tailor made for dealing with dynamic price fluctuations occurring in financial market models and for developing a successful theory of pricing and hedging financial derivatives.

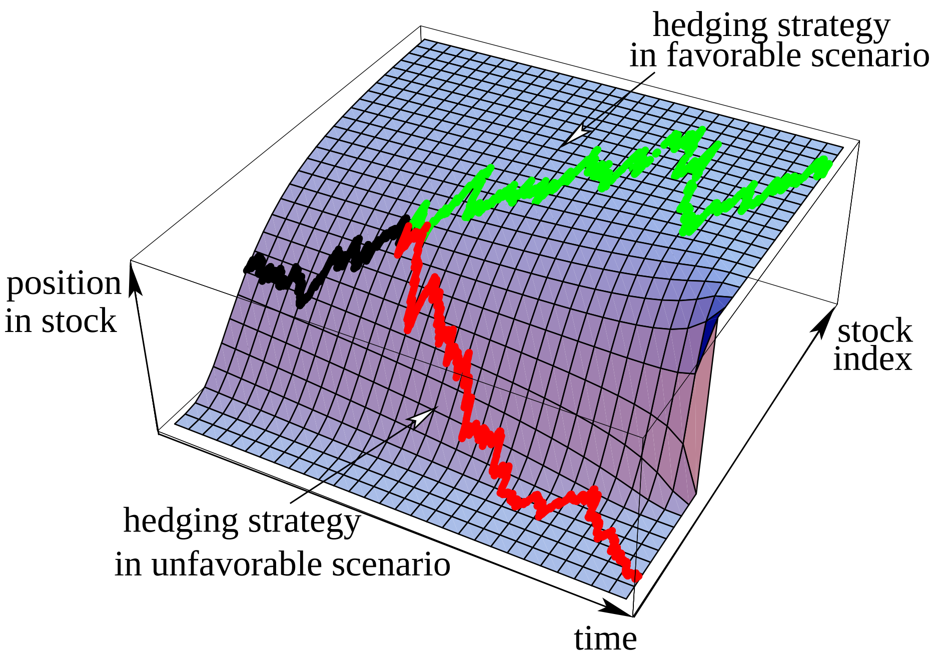

Figure 2: The evolution of a 'Δ-hedge' against a guarantee fund and the induced profits or losses in two different scenarios.

[1]Black and Scholes (1973), Merton (1969)

[2]Only M. Scholes and R. Merton became Nobel laureates, since F. Black had died already in 1995.

2 Beyond Black-Scholes-Merton

The picture emerging from the Black-Scholes-Merton theory seems to be almost too good to be true from a bank's point of view: the risk from a financial product like the guarantee fund can be removed completely by using a dynamic hedging strategy whose proceeds exactly cover the incurred payment obligations. Hence, the bank could in principle take as large a position as desired in such an instrument — its risk exposure would (seemingly) not be affected, while the profits from selling these products could be enormous. Yet, as research in Mathematical Finance has shown, this comfortable picture very much relies on a number of restrictive idealizations in the Black-Scholes-Merton model of a financial market: transaction costs and taxes are neglected, the future volatility of stock price fluctuations is assumed to be known in advance, asset prices evolve independently of traders' actions, asset returns follow a log-normal distribution .... Relaxing any of these assumptions typically leads to models where financial products bear an intrinsic risk which cannot be hedged away completely.

Hence, the existence of a perfect hedge as in the Black-Scholes-Merton model is the exception rather than the rule. However, even though in 'typical' situations a perfect hedge might not be available, it is always possible to look for strategies which reduce the risks as much as possible. In this context, one natural idea is to determine a portfolio strategy which minimizes the probability of a loss. As shown by Föllmer and Leukert (1999) this idea leads to 'partial hedges' which in most scenarios cover the payment obligations, but which in some unfavorable events result in a short fall. In our example of a guarantee fund such a partial hedge will generate a payoff described by the profile in Figure 3. The figure shows that the hedge works perfectly well as long as stock prices do not rise or fall 'too far'. If, however, stock prices pass beyond certain thresholds, then the partial hedge fails and the bank is left completely unprotected against the obligations from the guarantee fund. These loss scenarios may have a very small probability, yet, if they occur, the resulting losses are substantial. Thus, one may argue that minimizing the probability of a loss is not a suitable criterion for controlling financial risk.

This calls for more appropriate criteria to assess the quality of a hedging strategy. For instance, one may seek to minimize the expected quadratic short fall, i.e., the second lower partial moment of the loss; see Föllmer and Leukert (2000). In our guarantee fund example this criterion leads to a payoff profile for optimal hedging strategies described by Figure 4. Obviously, this kind of hedging strategy avoids extremal losses and performs reasonably well in most scenarios. Hence, from the point of view of a risk manager, the quadratic short fall-criterion proves significantly more appropriate than the previously

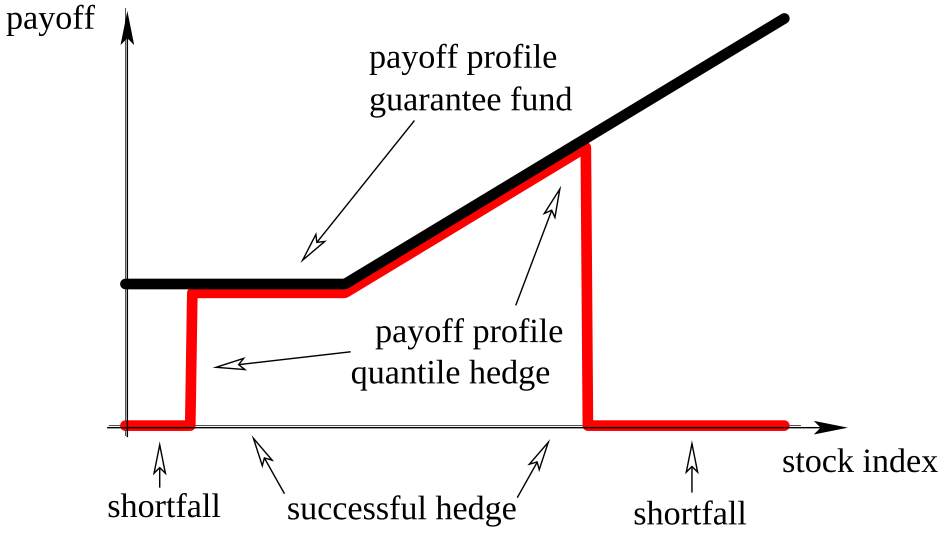

Figure 3: Payoff profile of a guarantee fund (black line) and the associated hedging profile with minimal shortfall probability (grey line).

discussed loss probability-criterion.

Of course, many other performance criteria could be (and have been) considered, and so it becomes an issue to systematically investigate which criteria are really suitable to manage risk associated to financial derivatives. Such criteria will be discussed in the next section taking a more general point of view.

3 Quantifying financial risk

Financial institutions can be viewed as companies dealing with a single good: Risk. They act as risk taker, they restructure risks, and they transfer risk from one party to another. Hence, controlling risks obviously is of central importance for every single financial institution. At the same time, it is also of vital importance for the financial industry, and thus for an economy as a whole. Therefore, beside the internal controlling units, also state-run supervising agencies such as the 'Bundesanstalt für Finanzdienstleistungsaufsicht' (BAFin) monitor the risk exposure of banks and insurance companies and, where necessary, enforce a suitable re-structuring of hazardous portfolios. The decision whether or not a position is acceptable from this point of view is based on a quantitative assessment of an institution's risk portfolio via *risk measures*.

The current industry standard for such a risk measure is the so-called *Value at Risk*. Value at Risk describes a loss threshold which will — with some high probability — not

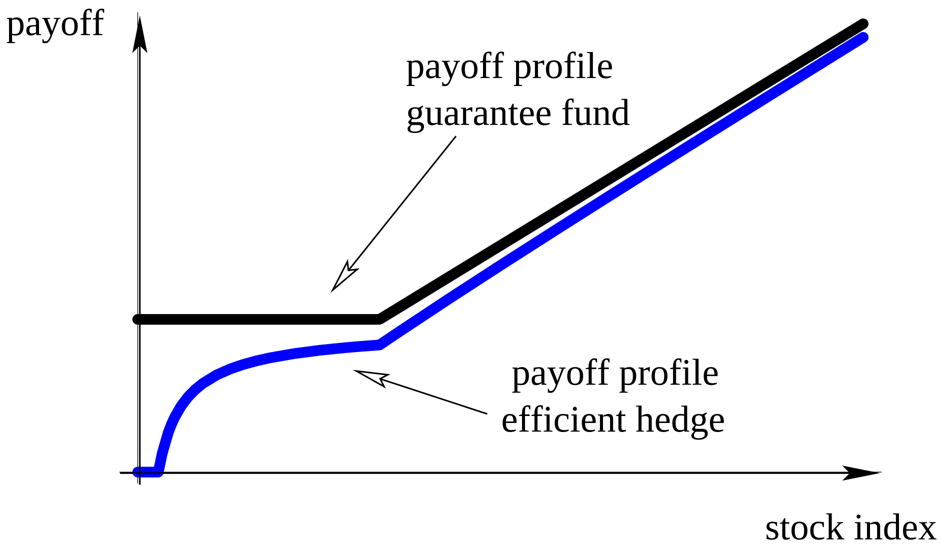

Figure 4: Payoff profile of a guarantee fund (black line) and the associated hedging profile with minimal expected quadratic shortfall (grey line).

be exceeded in the near future. Computing the Value at Risk for a given portfolio is a delicate mathematical problem. For instance, one often only has a vague idea about the loss distribution of each single position. Inferring the distribution of aggregate losses from these typically turns out to be very involved, due to highly complex interdependencies between the portfolio's positions. One approach to this aggregation problem is via *Monte Carlo Simulation* where one tests a portfolio's performance in a huge number of possible future scenarios. The statistical investigation of this very practical approach will be a central topic in project E5 (V. Spokoinyi, J. Schönmakers) of our research center. Another, more probabilistic approach will be pursued in this context by project E1 (A. Bovier) where qualitative properties of the 'macroscopic' total loss distribution will be derived from the 'microscopic' dependence structure between single losses and their distributions.

Even though being the current industry standard, Value at Risk suffers from several conceptual deficiencies. Most notably, it takes into account only the probability of a loss, and not its actual size. Hence, Value at Risk gives rise to essentially the same caveat as the loss probability-criterion for hedging strategies which has been discussed in the previous section. In order to develop more appropriate measures of risk, recent research in Mathematical Finance[3] has taken an axiomatic approach in which the structure of reasonable risk measures is derived from a set of economically desirable properties. Cur-

[3]See, for instance, Artzner, Eber, Delbaen, and Heath (1999) and Föllmer and Schied (2002)

rent research is concerned with the development of efficient methods to compute these advanced risk measures as well as with the investigation of their practical implications. A further topic is to devise risk measures which appropriately take into account the intertemporal structure of a risky portfolio, an important aspect which so far has not been incorporated. How to choose such a *dynamic risk measure* is to date a largely open question which will be a central topic in project E4 (H. Föllmer, A. Schied).

Of course, any measure of risk requires a sound modelling of the basic risk factors a portfolio depends on. Such models will be discussed in the next section.

4 Modelling financial risk

Financial risk can arise from a variety of different sources, the most important one being financial price fluctuations. In the standard economic paradigm, these fluctuations are explained as a change in the market's perception of the fundamental values underlying the considered financial assets. But usually there is no rigorous mathematical modelling of the price formation mechanism which leads to such an aggregate assessment. In fact, in typical models of Mathematical Finance, the fluctuation of asset prices is simply given as an exogenous stochastic process, neglecting crucial aspects of price formation like the interaction of market participants. The recent 'internet bubble' and the subsequent 'dotcom crash' may be viewed as being driven my such market interactions: initially, the price rally of 'dotcom'-shares was fuelled by ever more optimistic assessments of their future prospects which contagiously diffused among traders and their customers. When, finally, firms turned out not to be able to keep up with markets' expectations, the same contagion mechanism propagated harsh disappointment among market participants causing the 'dotcom crash'. Despite their obvious economic relevance, these contagion effects cannot be included in the highly aggregated models typically considered in Mathematical Finance. Contagion effects, however, are well-known in another research field: *statistical mechanics*, the physical theory of interacting particles which tries to explain macroscopic phenomena such as ferromagnetism by investigating the microscopic behavior of individual particles. It is a challenging task to extend methods and concepts developed in this area of mathematical physics and to derive new microeconomic foundations for models of asset price fluctuations.

Of course, there is a large number of other risk factors apart from financial price fluctuation. One example is weather and climate, a risk factor whose relevance is underlined by the apparent effects which are attributed to phenomena like the El Niño Southern Oscillation (ENSO). In recent years, new financial products have been introduced which

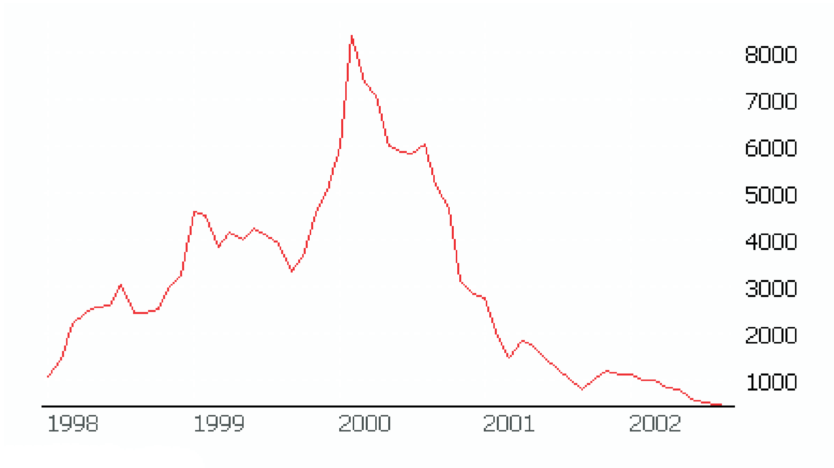

Figure 5: The 'internet bubble' of the NEMAX 50.

directly refer to this source of risk. The payoff of such a *weather derivative* can, for example, be determined by the average deviation of daily temperatures during a winter month from a pre-specified mean. This type of financial product appears to be unusual at first sight. Yet, it is of special interest to energy firms who face the risk of suffering losses when they cannot sell as much energy as expected in a surprisingly warm winter, and therefore may use weather derivatives as an 'insurance against good weather'. More generally *securitization*, that is the idea to shift certain risks to the capital markets, is attractive for a wide range of firms. Especially insurance companies have a mayor interest in the development of these markets for non-financial risk since the losses from floods, earthquakes and other catastrophes tend more and more to exceed their financial capabilities. On the other hand, securitization yields new investment opportunities which allow traditional investors to efficiently diversify their portfolio. Hence, despite its seemingly exotic nature, the market for weather derivatives has a great potential, and in fact weather derivatives are already being traded both on American and European exchanges. A mayor issue for research in securitization is to bring together models of external risk factors with financial and actuarial pricing and hedging principles. This will be a topic in project E2 (Deuschel-Imkeller).

5 A case study on intertemporal risk in Economic Theory

From a slightly different perspective, decision making under uncertainty has been a mayor topic also in Economic Theory. There one tries to explain, for instance, the *choice of life time consumption plans* by economic agents. These agents can be assumed to discriminate between different consumption plans by assigning to each plan a certain 'utility'. The utility is the higher the better the consumption plan in question is regarded by the agent. Very much like risk measures, the mathematical structure of such utilities can be sustained (at least to some extent) by a certain set of intuitively plausible axioms. One suggestion to describe utilities in an intertemporal framework is due to Hindy, Huang, and Kreps (1992). Their approach starts with a simple model for the level of satisfaction induced by a time- and scenario-dependent consumption plan $C = C(t, \omega)$. This level is modelled at each time $t \geq 0$ as an exponentially weighted average

$$Y^C(t) = \int_0^t e^{-\beta(t-s)}\, dC(s)$$

of the past consumption $dC(s) \geq 0$ $(0 \leq s \leq t)$; the parameter $\beta > 0$ describes the exponentially declining influence of past consumption on current satisfaction. Given the evolution of satisfaction Y^C and the agent's time horizon $T > 0$, one can then describe the utility derived from the consumption plan C as

$$U(C) = \mathbb{E}\left[\int_0^T u\left(t, Y^C(t)\right)\, dt\right]$$

where $u(t, .)$ is a time-dependent increasing and concave real function. Monotonicity of u reflects the unrefined, yet basic economic principle that 'more is better'; concavity captures the idea of risk-aversion, i.e., the tendency of agents to prefer sure outcomes over uncertain ones.

As a last ingredient to the intertemporal decision problem, economists introduce a price functional which assigns a unique price to each possible consumption plan. Such a functional may take the form

$$\Psi(C) = \mathbb{E}\left[\int_0^T \psi(t)\, dC(t)\right]$$

where $\psi = \psi(t, \omega)$ describes the price to be paid at time 0 for delivery of one unit of the consumption good at time t if scenario ω obtains.

The agent's decision problem then amounts to determine the consumption plan C^* which yields the maximal utility among all consumption plans C he can afford for a given budget $w > 0$. Mathematically, this amounts to maximizing the concave functional $U(C)$ subject to the linear budget-constraint $\Psi(C) \leq w$. This is a singular stochastic control problem in a non-Markovian setting which, under some technical assumptions, can be addressed by methods of convex analysis[4].

It turns out that the optimal consumption behavior can be described in terms of a 'minimal level process' L. For each point in time t and each scenario ω, this process indicates an optimal minimum level of satisfaction for the agent: if the agent's current level of satisfaction $Y^C(t,\omega)$ is above the minimum level $L(t,\omega)$ then there is no need for further consumption; otherwise, the agent should consume just enough to raise his satisfaction to the given level. The interplay between consumption prices ψ and the minimal level of satisfaction L is illustrated in Figure 6. The minimal level process can

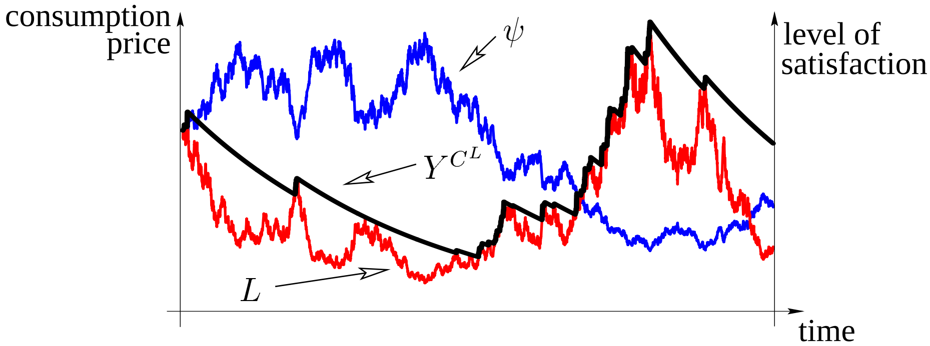

Figure 6: The evolution of consumption prices ψ (dark grey line), the associated minimal level process L (light grey line), and the optimal level of satisfaction Y^{C^L} (black line).

be characterized as the unique solution to the equation

$$\mathbb{E}\left[\int_t^T \partial_y u\left(s, \sup_{t\leq v\leq s}\left\{L(v)e^{\beta(v-t)}\right\}\right) e^{-\beta(s-t)}\,ds \,\middle|\, \mathscr{F}_t\right] = \lambda\psi(t) \quad (0 \leq t < T)$$

for some suitable Lagrange multiplier $\lambda > 0$. The construction of a solution to this equation reveals a surprising connection to the theory of *Gittins indices*. Gittins indices have been introduced in dynamic allocation problems under uncertainty. A famous example is the problem of a gambler who enters a casino and is confronted with a number of one-armed bandits to play with. Each bandit has a specific probability to

[4]See Bank and Riedel (2001)

yield a certain gain when its arm is pulled. This success probability is a priori unknown to the gambler, but of course it can be inferred by playing with the bandit. Now, the gambler's problem is to choose a suitable sequence of arms to pull so as to maximize the expected discounted gains. Clearly, the gambler has to find an optimal trade off between

Figure 7: Which arm to pull next?

the myopic incentive to pull arms with presumably high payoff probability and the strategic aspect to learn more about the success probability of other arms. To solve this intriguing problem Gittins (1979) introduced a dynamic performance measure — later called Gittins index — which at each stage allows to rank the possible action choices. The optimal gambling strategy then consists in always pulling the arm of the bandit with currently highest rank. It happens that these Gittins indices can be described by equations which are of the same type as the minimal level equation above. This alternative characterization of Gittins indices yields new algorithms for their efficient computation. In this sense, the above utility maximization provides new insights into the classical theory of Gittins indices; see Bank (2002).

6 Conclusion

The preceding sections have illustrated how advanced mathematical methods, in particular those from probability theory and statistics, are becoming important tools in the financial industry: *Mathematics emerges as a key technology in Finance.* We also have describe how expertise from a whole variety of research areas ranging from statistical mechanics to economic decision making, from stochastic climate models to statistics of simulation algorithms is brought together in our new research center in order to meet *new challenges in Mathematical Finance.* Finally, an attempt was made to illustrate how other, seemingly unrelated fields in mathematics can in turn benefit from the *new perspectives on classical topics* arising from problems of Mathematical Finance.

References

Artzner, P., J.-M. Eber, F. Delbaen, and D. Heath (1999): "Coherent Measures of Risk," *Mathematical Finance*, 9, 203–228.

Bank, P. (2002): "An Efficient Algorithm for Computing Gittins Indices," in preparation, Humboldt University Berlin.

Bank, P., and F. Riedel (2001): "Optimal Consumption Choice with Intertemporal Substitution," *Annals of Applied Probability*, 3, 750–788.

Black, F., and M. Scholes (1973): "The Pricing of Options and Corporate Liabilities," *Journal of Polictical Economy*, 81, 637–654.

Föllmer, H., and P. Leukert (1999): "Quantile Hedging," *Finance and Stochastics*, 3, 251–273.

——— (2000): "Efficient Hedging: Cost versus Shortfall Risk," *Finance and Stochastics*, 4, 117–146.

Föllmer, H., and A. Schied (2002): *Stochastic Finance: An Introduction in Discrete Time*, no. 27 in de Gruyter Studies in Mathematics. de Gruyter.

Gittins, J. C. (1979): "Bandit processes and dynamic allocation indices (with discussion)," *J. Roy. Statist. Soc. Ser. B*, 41, 148–164.

Hindy, A., C.-f. Huang, and D. Kreps (1992): "On intertemporal preferences in continuous time: the case of certainty," *J. Math. Econom.*, 21(5), 401–440.

Merton, R. C. (1969): "Lifetime Portfolio Selection under Uncertainty: the Continuous Case," *Review of Economics and Statistics*, 51, 247–257.