Technische Universität Berlin

SS 2017

Institut für Mathematik
Prof. Dr. G. Bärwolff
Sekr. MA 4-5

7.06.2017

2. Exercise sheet - solutions

FV/FD-Methods for the solution of pde's

Discussion: 12.6.15-16.6.16

1) Exercise
Show the benefit of the Kronecker product of matrices to implement a coefficient/stiffness matrix for the Poisson problems of the exercises of exercise sheet Nr. 1.

2) Exercise
Solve the equation

$$-u_{xx} - u_{yy} = (2 - \sqrt{x^2 + y^2})(1 - \sqrt{x^2 + y^2})y/\sqrt{x^2 + y^2} \; .$$

on

$$\Omega = \{(x, y) \,|\, 1 < x^2 + y^2 < 4\}$$

with homogeneous boundary conditions.

Solution:
There is a possibility to use an Ansatz

$$U(r, \varphi) = R(r) \sin \varphi \; .$$

For $R(r)$ the equation

$$-\frac{1}{r}\frac{\partial}{\partial r}[r\frac{\partial R}{\partial r}] + \frac{1}{r^2}R = r^2 - 3r + 2$$

is to solve. This is possible with the Ansatz

$$R(r) = p_4(r) + w(r) \; ,$$

where p_4 is a polynom of 4th order and w is a solution of

$$-\frac{1}{r}\frac{\partial}{\partial r}[r\frac{\partial w}{\partial r}] + \frac{1}{r^2}w = 0 \; .$$

The other way is to solve the problem numerically on an annulus circular ring with a periodicity in the φ-direction.

Listing 1: source code

```
% Exercise 3 problem 1
%
% Berechnung der Temperaturverteilung auf einer Kreisscheibe
% \Omega = \{ 1 < x^2 + y^2 < 4 \}
% -\div ( \lambda \grad T ) = f in \Omega, T = 0 auf \partial \Omega,
% f(x,y) = 1 in \Omega_0 = \{ (x-2)^2 + (y-2)^2 <= 1/4 \}, sonst =0
% \lambda = 1
n = 30;
m = 40;
```

```
R0 = 1.;
R1 = 2.;
drho = (R1-R0)/n;
dphi = 2*pi/m;
% Zentralpunkte der Finiten Zellen
rho = linspace(R0+drho/2,R1-drho/2,n);
for i=1:n
  invrho(i) = 1/rho(i);
end
phi = linspace(0+dphi/2,2*pi-dphi/2,m);
% Randpunkte der Finiten Zellen
rhop = linspace(R0,R1,n+1);
phip = linspace(0,2*pi,m+1);
% Matrixaufbau Ar
Ar = zeros(n,n);
for i=1:n
  if (i == 1)
  Ar(i,i) = rhop(1)+rhop(2);
  Ar(i,i+1) = -rhop(1);
  elseif (i == n)
  Ar(i,i) = rhop(n-1)+rhop(n);
  Ar(i,i-1) =  -rhop(n-1);
  elseif (i > 1 && i < n)
  Ar(i,i) = rhop(i-1) + rhop(i);
  Ar(i,i-1) = -rhop(i-1);
  Ar(i,i+1) = -rhop(i);
  end
end
% Matrixaufbau Aphi
Aphi = zeros(m,m);
for j= 1:m
  if (j == 1)
  Aphi(1,1) = 2;
  Aphi(1,2) = -1;
  Aphi(1,m) = -1;
  elseif (j == m)
  Aphi(m,m) = 2;
  Aphi(m,m-1) = -1;
  Aphi(m,1) = -1;
  elseif (j > 1 && j < m)
  Aphi(j,j) = 2;
  Aphi(j,j+1) = -1;
  Aphi(j,j-1) = -1;
  end
end
Irho = drho*eye(n);
Iphi = dphi*eye(m);
% Steifigkeitsmatrix
AA = kron(Iphi,Ar) + kron(Aphi,diag(invrho)*Irho);
% rechte Seite (Faktor r*drho*dphi, und Dimension)
R = kron(Iphi*ones(m,1),diag(rho)*Irho*ones(n,1));
%
% rechte Seite (Beruecksichtigung des Quellglieds)
for j=1:m
for i=1:n
   ind = i + (j-1)*n;
   f(ind) = (2-rho(i))*(1-rho(i))*sin(phi(j));
   R(ind) = R(ind)*f(ind);
end
end
%
U = AA\R;
%
%
% reshape
% Vektor U ===> Matrix X
X = reshape(U,n,m);
% Plot
for i=1:n
for j=1:m
  xk(i,j) = rho(i)*cos(phi(j));
```

```
  yk(i,j) = rho(i)*sin(phi(j));
end
xk(i,m+1) = xk(i,1);
yk(i,m+1) = yk(i,1);
end
XP = [X, X(:,1)];
figure(1);
mesh(xk,yk,XP)
title('u-Feld durch Sinus-Quelle erzeugt')
zlabel('u(\rho,\phi)')
```

3) Exercise
Solve the steady state heat conduction boundary value problem

$$
\begin{aligned}
-\Delta u &= 0\,, && \text{in } \Omega = \{(x,y,z)|x^2+y^2<1,\ 0<z<2\} \\
u &= 1\,, && \text{on } \Gamma_1 = \{(x,y,0)|x^2+y^2<1\} \\
\frac{\partial u}{\partial n} &= 0\,, && \text{on } \Gamma_2 = \{(x,y,z)|x^2+y^2=1, 0<z<2\} \\
\frac{\partial u}{\partial n} &= 1\,, && \text{on } \Gamma_3 = \{(x,y,2)|x^2+y^2<1\}\,.
\end{aligned}
$$

$\frac{\partial}{\partial n}$ means the outer normal directional derivative on Γ.

Solution:
This problem has a trivial solution $u(r,z) = z+1$.
The numerical solution is a good test for the implementation of a FD/FV-method for more complicated boundary conditions and source terms.
The way to solve the problem numerically with a finite difference scheme is a problem formulation in cylindrical coordinates. It is important to note that there is no dependance of φ and though we can solve the problem in the 2d rectangle

$$\hat{\Omega} = \{(r,z) \in [0,1] \times [0,2]\}\,.$$

The following listing contains a program of an implemented Finite-Volume method.

Listing 2: source code

```
% Exercise 3, problem 2
%
%
% Berechnung der Temperaturverteilung in einem Zylinder
% \Omega = \{ 0 < x^2 + y^2 < 4, 0 < z < 2 \}
% -\div ( \grad T ) = 0 in \Omega, RB siehe Uebungsblatt
% Waermequelle in einem Zylinderring \{  r3^2 < x^2 + y^2 < r4^2 , 0 < z < 2 \}
n = 20;
m = 30;
R0 = 0.;
R1 = 1.;
z0 = 0;
z1 = 2;
drho = (R1-R0)/n;
dz = (z1-z0)/m;
% Zentralpunkte der Finiten Zellen
rho = linspace(R0+drho/2,R1-drho/2,n);
for i=1:n
  invrho(i) = 1/rho(i);
end
z = linspace(z0+dz/2,z1-dz/2,m);
% Randpunkte der Finiten Zellen
rhop = linspace(R0,R1,n+1);
zp = linspace(z0,z1,m+1);
% Matrixaufbau Ar
Ar = zeros(n,n);
```

```
for i=1:n
  if (i == 1)
  Ar(i,i) = rhop(2);
  Ar(i,i+1) = -rhop(2);
  elseif (i == n)
  Ar(i,i) = rhop(n-1);
  Ar(i,i-1) =  -rhop(n-1);
  elseif (i > 1 && i < n)
  Ar(i,i) = rhop(i-1) + rhop(i);
  Ar(i,i-1) = -rhop(i-1);
  Ar(i,i+1) = -rhop(i);
  end
end
% Matrixaufbau Az
Az = zeros(m,m);
for j= 1:m
  if (j == 1)
  Az(1,1) = 3;
  Az(1,2) = -1;
  elseif (j == m)
  Az(m,m) = 1;
  Az(m,m-1) = -1;
  elseif (j > 1 && j < m)
  Az(j,j) = 2;
  Az(j,j+1) = -1;
  Az(j,j-1) = -1;
  end
end
Irho = drho*eye(n);
Iz = dz*eye(m);
% Steifigkeitsmatrix
AA = kron(Iz,Ar) + kron(Az,diag(rho)*Irho);
% rechte Seite (Faktor r*drho*dz, und Dimension)
R = kron(Iz*ones(m,1),diag(rho)*Irho*ones(n,1));
%
% rechte Seite (Beruecksichtigung des Quellglieds)
for j=1:m
for i=1:n
   ind = i + (j-1)*n;
% keine Waermequelle
   f(ind) = 0;
   fr(ind) = 0;
% Randbedingungen
   if (j == 1)
   fr(ind) = drho*rho(i)*2;
   elseif (j == m)
   fr(ind) = drho*rho(i)*1*dz;
   end
   R(ind) = R(ind)*f(ind) + fr(ind);
end
end
%
U = AA\R;
%
% reshape
X = reshape(U,n,m);
% exakte Loesung
for i=1:n
for j=1:m
  UE(i,j) = z(j)+1;
end
end
% Fehler
err = norm(X-UE)
% Plot
for i=1:n
for j=1:m
  xk(i,j) = rho(i);
  yk(i,j) = z(j);
end
end
```

```
figure(1);
mesh(xk,yk,X)
title('u-Feld Zylinderschnitt')
zlabel('u(\rho,z)')
```

4) Exercise
Construct the mapping $\Psi : \{1, 2, \ldots, N\} \times \{1, 2, \ldots, M\} \rightarrow \{1, 2, \ldots, N * M\}$ which maps the position (i, j) of a rectangular matrix A of type $N \times M$ to the relevant index ind of a vector $\vec{a} \in \mathbb{R}^{N \times M}$ which contains row by row of A, and its inverse Ψ^{-1}.

Solution:

Listing 3: source code

```
% exercise 3.3a
%
function ind = index_gb(i,j,n,m)

% n Zeilenzahl
% m Spaltenzahl
% (i,j) ---> ind

ind = j + (i-1)*m;
```

Listing 4: source code

```
% exercise 3.3b
%
function [i,j] = invindex_gb(ind,n,m)

% n Zeilenzahl
% m Spaltenzahl
% ind ---> (i,j)

if floor(ind/m)*m == ind
  i=ind/m; j = m;
else
  i = floor(ind/m)+1; j = ind - (i-1)*m;
end
```

5) Exercise
The 2d coefficient matrix A of the Poisson problem $-\Delta u = f$ on a quadratic domain Ω with Dirichlet boundary conditions we can create using the Kronecker-product

$$A_2 = I \otimes A + A \otimes I$$

with $A \in \mathbb{R}^{(N-1)\times(N-1)}$

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \ldots & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 \\ & & & \ddots & & \\ 0 & \ldots & 0 & -1 & 2 & -1 \\ 0 & \ldots & 0 & 0 & -1 & 2 \end{pmatrix}$$

and the identity $I \in \mathbb{R}^{(N-1)\times(N-1)}$.
Show the formula

$$(C \otimes B)(v \otimes w) = Cv \otimes Bw$$

and use it to determine the eigenvalues of A_2 when A has the eigenvalues $\lambda_1, \ldots, \lambda_{N-1}$.
Solution:
For

$$C = (c_{ij})_{\substack{i=1,\ldots,N \\ j=1,\ldots,M}} \quad \text{and} \quad B \in \mathbb{R}^{P\times K}$$

we get

$$C \otimes B = (c_{ij}B)_{\substack{i=1,\ldots,N \\ j=1,\ldots,M}} \in \mathbb{R}^{N\,M\times N\,M} \; .$$

Be that

$$v = [v_1 \; v_2 \ldots \; v_M]^T \; , \quad w \in \mathbb{R}^K$$

than we have

$$v \otimes w = (v_i w)_{i=1,\ldots,M} = \begin{pmatrix} v_1 w \\ v_2 w \\ \vdots v_M w \end{pmatrix} \in \mathbb{R}^{M\,K} \; .$$

With the Einstein-sum-convention $(C\,v)_i = c_{ij}v_j \; \forall i = 1, \ldots, N$ we get

$$\begin{aligned} C\,v \otimes B\,w &= (C\,v)_i B\,w \\ &= c_{ij}v_j\,B\,w = c_{ij}B\,v_j w \\ &= (C \otimes B)(v \otimes w) \; . \end{aligned}$$

The second part of this exercise is trivial by using the Eigenvektors u_l of A and the application of the proved rule to

$$A_2(u_l \otimes u_s) = [I \otimes A + A \otimes I](u_l \otimes u_s) \; .$$