Technische Universität Berlin SS 2017
Institut für Mathematik
Prof. Dr. G. Bärwolff
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1. Exercise sheet - Solutions

FV/FD-Methods for the solution of pde's

Discussion: 5.6.17-9.6.17

1) Exercise
a) Show that the Laplace operator Δ in the polar coordinate system (r, φ) is of the form

$$\Delta_{(r,\varphi)} u = \frac{1}{r} u_r + u_{rr} + \frac{1}{r^2} u_{\varphi\varphi} \ .$$

b) Find a solution of the problem

$$\begin{aligned} -\Delta\, u &= f\ , \quad \in \Omega, \qquad (1)\\ u &= g\ , \quad \text{on } \Gamma = \partial\Omega\ , \qquad (2) \end{aligned}$$

with

$$\Omega = \{(x, y) \mid x = r\cos\varphi,\ y = r\sin\varphi,\ 0 < r < \rho,\ 0 < \varphi < \pi/\alpha\,,\ \alpha \geq 1\},$$

$f = 0$ and

$$g(r, \varphi) = \begin{cases} 0 & \text{for } 0 \leq r < \rho \text{ and } \varphi = 0,\ \varphi = \pi/\alpha, \\ \varphi(\pi/\alpha - \varphi) & \text{for } 0 < \varphi < \pi/\alpha \text{ and } r = \rho \end{cases} \quad ,$$

by a separation Ansatz with polar coordinates.

Solution:
a) With the consequent use of the chain rule applied to

$$U(r, \varphi) = U(r(x, y), \varphi(x, y)) := u(x, y)$$

one can show the formula for the Laplacian in polar coordinates. For the gradient in the polar coordinate system we get

$$\nabla_{r\varphi} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\varphi \frac{1}{r} \frac{\partial}{\partial \varphi}$$

and for the divergence we get

$$\nabla \cdot \vec{v} = \frac{1}{r} \frac{\partial (r\, v_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\varphi}{\partial \varphi}$$

and with

$$\vec{v} = \begin{pmatrix} v_r \\ v_\varphi \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial r} \\ \frac{1}{r} \frac{\partial u}{\partial \varphi} \end{pmatrix}$$

we find for the Laplacian

$$\begin{aligned}\Delta_{r\varphi}u &= \nabla\cdot\nabla_{r\varphi}u = \\ &= \frac{1}{r}\frac{\partial}{\partial r}[r\frac{\partial u}{\partial r}] + \frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2}\ .\end{aligned}$$

b) With the separation ansatz

$$U(r,\varphi) = R(r)\Phi(\varphi)$$

we get 2 ode's

$$\begin{aligned}(i) \qquad & rR'(r) + r^2R''(r) - \mu R(r) = 0 \\ (ii) \qquad & \Phi''(\varphi) + \Phi(\varphi)\mu = 0\ .\end{aligned}$$

The boundary condition $g(r,0) = 0$ gives $U(r,0) = R(r)\Phi(0) = 0$, which implies $\Phi(0) = 0$ because $R(r) = 0$ means $U(r,\varphi) = 0$ which would contradict to the bc $g(\rho,\varphi) = \varphi(\frac{\pi}{\alpha} - \varphi)$. $g(r,\frac{\pi}{\alpha}) = 0$ leads to $\Phi(\frac{\pi}{\alpha}) = 0$.
The solution of (ii) is

$$\Phi(\varphi) = c\, e^{\sqrt{-\mu}\varphi}\ . \tag{3}$$

By a case distinction we find, that μ must be real $\mu = \omega^2,\ \Omega \neq 0$. Thus we find from (3)

$$\Phi(\varphi) = c_1\, e^{i\omega\varphi} + c_2\, e^{-i\omega\varphi}\ ,\ c_1, c_2 \in \mathbb{R}. \tag{4}$$

The first boundary condition gives

$$\begin{aligned}\Phi(0) = c_1 + c_2 = 0 &\Longleftrightarrow -c_1 = c_2 \\ \Phi(\varphi) &= c_1 2\sin(\omega\varphi)\ .\end{aligned}$$

The second bc gives

$$\Phi(\frac{\pi}{\alpha}) = c_1 2\sin(\omega\frac{\pi}{\alpha}) = 0 \iff \omega = \alpha\, n,\ n \in \mathbb{Z},\ c \neq 0\ .$$

The Φ-solution is now

$$\Phi(\varphi) = c_n \sin(\alpha\, n\, \varphi),\ c_n \in \mathbb{R}, n \in \mathbb{N}\setminus\{0\}\ ,$$

and ω is equal to $\alpha^2 n^2$.
For $R(r)$ with the ansatz $R(r) = r^k$ we find

$$R'(r) = k\, r^{k-1}\ ,\ \ R''(r) = (k-1)k\, r^{k-2} \implies (k + (k-1)k - \alpha^2 n^2)r^k = 0$$

and for $r > 0$ follows $(k + (k-1)k - \alpha^2 n^2) = 0$ or

$$k_{1,2} = \pm\alpha n\ ,\ \ n \in \mathbb{N}\setminus\{0\}, \implies R_n(r) = a_n r^{\alpha n} + b_b r^{-\alpha n},\ a_n, b_n \in \mathbb{R}\,.$$

To get a finite solution we must set $b_n = 0$. The overall solution for $n \in \mathbb{N}$ reads

$$u_n(r,\varphi) = c_n r^{\alpha n}\sin(\alpha n\varphi)$$

and the superposition gives

$$u(r,\varphi) = \sum_{i=1}^{\infty} c_n r^{\alpha n}\sin(\alpha n\varphi)\ .$$

With the last bc $u(\rho,\varphi) = \varphi(\frac{\pi}{\alpha} - \varphi)$ and the orthogonality relation

$$\int_0^{\frac{\pi}{\alpha}} \sin(\alpha n\varphi)\sin(\alpha l\varphi)d\varphi = \frac{\pi}{2\alpha}\delta_{n,l}$$

follows

$$\begin{aligned} \int_0^{\frac{\pi}{\alpha}} u(\rho,\varphi)\sin(\alpha l\varphi)d\varphi &= \sum_{i=1}^{\infty} c_n\rho^{n\alpha}\int_0^{\frac{\pi}{\alpha}} \sin(\alpha n\varphi)\sin(\alpha l\varphi)d\varphi && (5)\\ &= c_l\rho^{l\alpha}\frac{\pi}{2\alpha} && (6) \end{aligned}$$

and

$$c_l = \frac{2\alpha}{\pi}\frac{1}{\rho^{l\alpha}}\int_0^{\frac{\pi}{\alpha}} \varphi(\frac{\pi}{\alpha} - \varphi)\, d\varphi = \frac{1}{3}(\frac{\pi}{\alpha})^2\frac{1}{\rho^{l\alpha}} \ .$$

2) Exercise

We consider the Poisson equation

$$\begin{aligned} -\Delta\, u &= f\ , \quad \in \Omega =]0,1[\times]0,1[, && (7)\\ u &= 0\ , \quad \text{on } \Gamma = \partial\Omega\ , && \\ & && (8) \end{aligned}$$

with

$$f(x,y) = -\sin(2\pi x)\sin(2\pi y)\ .$$

(a) Verify

$$u(x,y) = -\frac{1}{8\pi^2}\sin(2\pi x)\sin(2\pi y)$$

as the exact solution of (7).

(b) Construct a FD-method to solve the problem (7) numerically. Use an equidistant grid with the uniform grid space h. Realize the method by a Matlab/Octave-program and demonstrate the convergence behavior for $h = 1/2,\ 1/4,\ 1/8,\ 1/16,\ 1/32$ with a plot of the error e_h vs. grid space (maybe with double logarithmic scale of the h- and the e_h-axis).
Solve the linear equation system $A_h u_h = f_h$ with the Matlab/Octave backslash-command.

(c) Plot the exact solution and the numerical solution for $h = 1/4,\ 1/16,\ 1/32$.

Solution:

The coefficient matrix of the linear equation system we get with the Kronecker-product $A \otimes B$, with

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \\ 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix} \in \mathbb{R}^{M\times M}$$

and

$$B = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \\ 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix} \in \mathbb{R}^{N\times N}$$

(in our case is $N = M$). The solution and the r.h.s. are the column vectors

$$U = [u_1\ u_2\ \dots\ u_M]^T\ , \quad F = h^2[f_1\ f_2\ \dots\ f_M]^T$$

with the entries

$$u_k = [u_{k1}\ u_{k2}\ \dots\ u_{kN}]^T\ , \quad f_k = [f_{k1}\ f_{k2}\ \dots\ f_{kN}]^T\ ,\ \ k = 1, \dots, M\ .$$

In the case of other then homogeneous dirichlet data you have to consider the boundary conditions in f_1 and f_M and in f_{k1} and f_{kN} for all k.

3) Exercise
Solve the problem (1) with

$$\Omega = \{(x, y) \mid x = r\cos\varphi,\ y = r\sin\varphi,\ 1 < r < 2,\ 0 < \varphi \leq 2\pi\}\ ,$$

and

$$f(r, \varphi) = (2 - r)(1 - r) \text{ on } \Omega\ ,$$

and homogeneous Dirichlet boundary conditions.

Solution:
Because of the independence of φ we have only to solve a 1d problem in the r-direction. Starting with

$$u'(r) + r\,u''(r) = [r\,u'(r)]_r = -2r + 3r^2 - r^3$$

we get by integration

$$r\,u'(r) = -r^2 + r^3 - \frac{1}{4}r^4 + c_1 \iff u'(r) = -r + r^2 - \frac{1}{4}r^3 + \frac{1}{r}c_1$$

and further

$$u(r) = -\frac{r^2}{2} + \frac{1}{3}r^3 - \frac{1}{16}r^4 + c_1\ln(r) + c_2\ .$$

The evaluation of the bc's $u(1) = u(2) = 0$ leads to

$$c_2 = \frac{11}{48} \quad \text{and} \quad c_1 = \frac{5}{48\ln(2)}\ .$$